Lectures 8 and 9

1 Rectangular waveguides

Consider a rectangular waveguide with $0 < x < a$, $0 < y < b$ and $a > b$. There are two types of waves in a hollow waveguide with only one conductor;

- Transverse electric waves (TE-waves). $\mathbf{E} = (E_x, E_y, 0)$ and $\mathbf{H} = (H_x, H_y, H_z)$.
- Transverse magnetic waves (TM-waves). $\mathbf{E} = (E_x, E_y, E_z)$ and $\mathbf{H} = (H_x, H_y, 0)$.

They need to satisfy the Maxwells equations and the boundary conditions. The boundary conditions are that the tangential components of the electric field and the normal derivative of the tangential components of the magnetic field are zero at the boundaries.

1.1 TE-waves

We now try to find the electromagnetic fields for TE-waves, when $E_z$ is zero. The electromagnetic fields are obtained from $H_z$. The equation to be solved is

$$\nabla^2 H_z + k^2 H_z = 0$$

$$\frac{\partial H_z}{\partial x}(0, y, z) = \frac{\partial H_z}{\partial x}(a, y, z) = \frac{\partial H_z}{\partial y}(x, a, z) = \frac{\partial H_z}{\partial y}(x, b, z) = 0$$

(1.1)

where $k = \omega/c$ is the wave number. There are infinitely many solutions to this equation

$$H_{zmn}(x, y, z) = h_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jkz}$$

The $m, n$ values can take the values $m = 0, 1, 2 \ldots$ and $n = 0, 1, 2 \ldots$, but $(m, n) \neq (0, 0)$. The corresponding transverse electric and magnetic fields are obtained from
Maxwell’s equations. The spatial dependence of these components are

\[ E_x \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jkz} \]
\[ E_y \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jkz} \]
\[ H_x \sim \sin\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jkz} \]
\[ H_y \sim \cos\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right) e^{-jkz} \]

Each of these components satisfies the Helmholtz equation and the boundary conditions.

The electromagnetic field corresponding to \((m, n)\) is called a TE\(_{mn}\) mode. Thus there are infinitely many TE\(_{mn}\) modes.

The \(k_z\) is the \(z\)-component of the wave vector. For a given frequency it is given by

\[ k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \quad (1.2) \]

This means that for \(m\) and \(n\) values such that \(k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 > 0\), or, equivalently \(f > \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\), then \(k_z\) is real and the TE\(_{mn}\) mode is propagating. One can also introduce the wavelength in the \(z\)-direction as \(\lambda_z = \frac{2\pi}{k_z}\). If we move a distance \(\lambda_z\) in the \(z\)-direction then \(e^{-jk_z(z+\lambda_z)} = e^{-jk_zz-j2\pi} = e^{-jk_zz}\) and we are back to the same phase as we started from.

For \(m\) and \(n\) values such that \(k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2 < 0\), or, equivalently \(f < \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\), then \(k_z\) is imaginary and the TE\(_{mn}\) mode is a non-propagating mode. The \(z\)-dependence of the non-propagating mode is \(e^{-\alpha z}\) where \(\alpha = |k_{zmn}|\).

### 1.1.1 Cut-off frequency

For a TE\(_{mn}\) mode the cut-off frequency is the frequency for which \(k_z = 0\). This means that the mode is in between its propagating and non-propagating stages.

The cut off frequency for the TE\(_{mn}\) mode is

\[ f_{cmn} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2} \]

### 1.1.2 The fundamental mode TE\(_{10}\)

The fundamental mode of a waveguide is the mode that has the lowest cut-off frequency. For a rectangular waveguide it is the TE\(_{10}\) mode that is the fundamental mode. It has \(f_{c10} = \frac{c}{2a}\). The electric field of the fundamental mode is \(E = E_0 \sin\left(\frac{\pi x}{a}\right) e^{-jk_zz} e_y\). It is almost always the fundamental mode that is used
in the waveguide. It is then crucial to make sure that the frequency is low enough such that only the fundamental mode can propagate. Otherwise there will be more than one mode in the waveguide and since the modes travel with different speeds, as will be seen below, one cannot control the phase of the wave.

Example: Consider $a = 0.3 \text{ m}$ and $b = 0.15 \text{ m}$. Then $f_{c10} = 500 \text{ MHz}$, $f_{c01} = f_{c20} = 1000 \text{ MHz}$ and $f_{c11} = 1118 \text{ MHz}$. It means that for frequencies lower than 500 MHz there are no waves that can propagate through the waveguide. In the interval $500 \text{ MHz} < f < 1 \text{ GHz}$ only the TE$_{10}$ mode can propagate. One has to be in this frequency span in order to transfer a well defined signal.

1.2 TM-waves

The electromagnetic fields are obtained from

\[ \nabla^2 E_z + k^2 E_z = 0 \]
\[ E_z(0, y, z) = E_z(a, y, z) = E_z(x, 0, z) = E_z(x, b, z) = 0 \]

where $k = \omega/c$ is the wave number. There are infinitely many solutions to this equation

\[ E_{zmn}(x, y, z) = e_{mn} \sin \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-jk_z z} \]

The $m, n$ values can take the values $m = 1, 2 \ldots$ and $n = 1, 2 \ldots$. The corresponding transverse electric and magnetic fields are obtained from Maxwell’s equations. The spatial dependence of these components are the same as for the TE-waves

\[ E_x \sim \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-jk_z z} \]
\[ E_y \sim \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) e^{-jk_z z} \]
\[ H_x \sim \sin \left( \frac{m\pi x}{a} \right) \cos \left( \frac{n\pi y}{b} \right) e^{-jk_z z} \]
\[ H_y \sim \cos \left( \frac{m\pi x}{a} \right) \sin \left( \frac{n\pi y}{b} \right) e^{-jk_z z} \]

The electromagnetic field corresponding to $(m, n)$ is called a TE$_{mn}$ mode. Thus there are infinitely many TE$_{mn}$ modes.

For a given frequency $k_z$ for the TE-modes is the same as for the TM-modes

\[ k_z = \sqrt{k^2 - \left( \frac{m\pi}{a} \right)^2 - \left( \frac{n\pi}{b} \right)^2} \]

1.2.1 Cut-off frequency

For a TM$_{mn}$ mode the cut-off frequencies are the same as for the TE$_{mn}$ modes, i.e.,

\[ f_{cmn} = \frac{c}{2\pi} \sqrt{\left( \frac{m\pi}{a} \right)^2 + \left( \frac{n\pi}{b} \right)^2} \]. The only difference is that one cannot have $m = 0$ or $n = 0$.

Example: When $a = 0.3 \text{ m}$ and $b = 0.15 \text{ m}$, the lowest cut-off frequency for TM-modes is $f_{c11} = 1118 \text{ MHz}$. 
2 Derivation of the modes

In order to solve Eqs. (1.1) and (1.3) we use the method of separation of variables. Consider Eq. (1.1). We assume that all solutions to the equation can be written as a product \( H_z(x, y, z) = F(x)G(y)K(z) \). When we insert this into the equation we get

\[
GK \frac{\partial^2 F}{\partial x^2} + FK \frac{\partial^2 G}{\partial y^2} + FG \frac{\partial^2 K}{\partial z^2} + k^2 FGK = 0
\]

We divide by \( FGK \)

\[
\frac{1}{F} \frac{\partial^2 F}{\partial x^2} + \frac{1}{G} \frac{\partial^2 G}{\partial y^2} + k^2 = -\frac{1}{K} \frac{\partial^2 K}{\partial z^2}
\]

The left hand side is independent of \( z \) and the right hand side is independent of \( x,y \). The only way that this can happen is if both sides are constant. We call this constant \( k_z^2 \) and get the equation for \( K(z) \)

\[
\frac{\partial^2 K}{\partial z^2} + k_z^2 K = 0
\]

The solution is

\[ K(z) = a_p e^{-jk_z z} + a_n e^{jk_z z} \]

These solutions correspond to waves traveling in the positive \( z \) direction \( (e^{-jk_z z}) \) and waves in the negative \( z \)-direction \( (e^{jk_z z}) \). The equation for \( FG \) can now be written as

\[
\frac{1}{F} \frac{\partial^2 F}{\partial x^2} + k^2 - k_z^2 = -\frac{1}{G} \frac{\partial^2 G}{\partial y^2}
\]

Also here both sides have to be constant and we call this constant \( k_y^2 \). Then the equation for \( G \) is the following eigenvalues problem

\[
\frac{\partial^2 G(y)}{\partial y^2} + k_y^2 G(y)
\]

\[ G'(0) = G'(b) = 0 \]

The solution to the equation is

\[ G(y) = \alpha_1 \cos(k_y y) + \alpha_2 \sin(k_y y) \]

The solution has to satisfy the boundary conditions \( G'(0) = G'(b) = 0 \), where \( G'(y) = -\alpha_1 k_y \sin(k_y y) + \alpha_2 k_y \cos(k_y y) \). The condition \( G'(0) = 0 \) gives \( \alpha_2 = 0 \) and the condition \( G'(b) = 0 \) gives \( \alpha_1 k_y \sin(k_y b) = 0 \). We must have \( \alpha_1 \neq 0 \), otherwise \( H_z = 0 \). This means that \( k_y b = n\pi \) and \( k_y = \frac{n\pi}{b} \). Thus the eigenvalue problem has the eigenfunctions \( G(y) = \alpha_1 \cos \left( \frac{n\pi y}{b} \right) \) and eigenvalues \( k_y^2 = \left( \frac{n\pi}{b} \right)^2 \).

The equation for \( F \) is

\[
\frac{\partial^2 F(x)}{\partial x^2} + (k^2 - (n\pi/b)^2 - k_z^2) F(x) = 0
\]
By introducing \( k_x^2 = k^2 - (n\pi/b)^2 - k_z^2 \) then
\[
\frac{\partial^2 F(x)}{\partial x^2} + k_x^2 F(x) = 0
\]
\[
F'(0) = F'(a) = 0
\]
This gives the eigenfunctions \( F(x) = \beta_1 \cos\left(\frac{m\pi x}{a}\right) \) and eigenvalues \( k_y^2 = \left(\frac{m\pi}{a}\right)^2 \).

We have seen that the solutions to Eq. (1.1) are
\[
H_{zmn}(x, y, z) = h_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{-jk_z z}
\]
where \( k_z = \sqrt{k^2 - (m\pi/a)^2 - (n\pi/b)^2} \). There is also a corresponding field traveling in the negative \( z \)-direction
\[
H_{zmn}(x, y, z) = h_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) e^{jk_z z}
\]
It turns out that when both \( m \) and \( n \) are zero, \((m, n) = (0, 0)\), then the field contradicts the Maxwell equations and for this reason \((m, n) \neq (0, 0)\).

The solutions have the important property that they are orthogonal over the cross section of the waveguide, i.e.,
\[
\int_0^a \int_0^b H_{zmn}(x, y, z) H_{zm'n'}^* dxdy = \begin{cases} 
\frac{1}{2} h_{mn}^2 \delta_{mm'} \delta_{nn'} & \text{if } m = 0 \text{ or } n = 0 \\
\frac{1}{4} h_{mn}^2 \delta_{mm'} \delta_{nn'} & \text{if } m > 0, \ n > 0
\end{cases}
\]
where \( * \) denotes complex conjugate and \( \delta_{mm'} \) is one if \( m = m' \) and zero if \( m \neq m' \).

In general the total field in a waveguide is a superposition of waveguide modes traveling in both directions.

## 2.1 Phase and group speeds

The phase speed is the speed that the pattern of the wave moves. The group speed is the speed that the energy is traveling with.

For a plane wave that travels in vacuum the phase and group speeds are the same and equals the speed of light \( c = 1/\sqrt{\varepsilon_0 \mu_0} = 299792458 \) m/s. Also for a transmission line the two speeds are the same and equals the speed of light in the material between the conductors.

In a waveguide it is different. Then the phase speed turns out to be larger than the speed of light and the group speed is smaller than the speed of light. One can show that the phase speed is given by
\[
v_f = \frac{\omega}{k_z}
\]
and that the group speed is given by the derivative
\[
v_g = \frac{d\omega}{dk_z} = \left(\frac{dk_z}{d\omega}\right)^{-1}
\]
2.1.1 Example

The TE<sub>mn</sub> and TM<sub>mn</sub> modes in a rectangular waveguide have the z—component of the wave vector given by Eq. (1.2)

\[ k_z = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2} \]  

(2.1)

The phase speed is given by

\[ v_f = \frac{\omega}{k_z} = \frac{\omega}{\sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}} \]

Since \( k = \frac{\omega}{c} \) it means that \( v_f < c \), when the mode is propagating. When we decrease the frequency such that \( f \to f_{cmn} \) then \( v_f \to \infty \). When \( f \to \infty \) then \( v_f \to c \).

The group speed is given by

\[ v_g = \left(\frac{dk_z}{d\omega}\right)^{-1} = \frac{k_z c}{k} \]

This means that \( v_g < c \) and as \( f \to f_{cmn} \) then \( v_g \to 0 \). As \( f \to \infty \) then \( v_g \to c \).
3 Lecture: Comsol for waveguides

Determination of losses in a waveguide

Consider a rectangular waveguide $0.3 \times 0.15$ m$^2$ made out of copper and filled with air. The fundamental mode propagates in the waveguide. Use Comsol to determine the real and imaginary part of $k_z$ as a function of frequency in the interval 501 MHz-1 GHz.

- Choose 2d, Electromagnetic waves and Mode analysis

- Choose Parameters under Global definitions and define a parameter called frequency. Write freq in expression since that is the predefined name of the frequency.

- Build the geometry.

- Choose air and copper as materials and let the internal domain be air and the boundary be copper.

- Choose Impedance boundary condition by a right click on Electromagnetic waves.

- Choose Parametric sweep under bf Study 1. Choose frequency as parameter by clicking on the plus sign and then click on the diagram symbol to the right of the save button. Let the frequency start at 510 MHz, stop at 1001 MHz with step 20 MHz.

- Choose Step 1 : Mode analysis and put 1 in desired number of modes and 0.7 as effective mode index in Search for modes around: This means that it will find one mode and it will start to search at 0.7 for the effective mode index ($n = c/V_f$).
• Run. Make sure that it is the fundamental mode for all frequencies. Choose Global plot under 1D Plot Group under Results. Make sure that you have the correct solution in Data set. Pick attenuation constant on the y-axis and frequency on the y-axis. Under parameter it should say expression

Propagation of waves in a system of waveguides
Consider a lossless rectangular airfilled waveguide with cross section $0.3 \times 0.15 \text{ m}^2$ and with a shape given by the figure.

a) Determine the propagation of the $\text{TE}_{10}$ mode for the frequency 704 MHz.
b) Determine the propagation of the $\text{TE}_{10}$ mode for the frequency 1500 MHz.
c) Determine the propagation of the $\text{TE}_{20}$ mode for the frequency 1500 MHz.
d) Which other modes couple to the $\text{TE}_{10}$ and $\text{TE}_{20}$ modes in the waveguide.

Choose 2d, Electromagnetic waves and Frequency domain

• Build the geometry. Remember to do Union under Boolean operations in Geometry.

• Choose port under Electromagnetic waves and add the surface that is the entrance port. Choose rectangular under Type of port. Click on in Wave excitation. Let the mode be TE and number 1 (gives $\text{TE}_{10}$).

• Choose port again and add the surface for the exit port. Wave excitation as to be off here but the mode is TE and number 1.

• Refine the mesh to finest.

• Click Frequency domain under Study and add the frequency.
• Run.

• The default plot is the norm of the electric field. Using that it is hard to see the waves. Choose the z-component of the electric field instead and take the real part of this. It should say real(Ez) under **Expression**.

• Make a movie by using **animation** under **Export**. Use **dynamic data extension** under **Sweep type**.