

ETEN05 Electromagnetic Wave Propagation, 2015

Workshop 2: Propagation of plane waves

1 Aims

In this workshop we study the propagation of plane waves in general bianisotropic linear media. The aim is to demonstrate that it is straight-forward to calculate the wave parameters for an arbitrary material model, at least numerically. You will compute the refractive indices and polarizations of the plane waves, interpret the results in terms of wave impedance and polarization state, and create small animations of the waves.

The polarization and wave impedance are calculated by solving the eigenvalue problem (derived in Lecture 3)

$$\begin{aligned} \frac{\beta}{\omega} \begin{pmatrix} \mathbf{E}_{0t} \\ \mathbf{H}_{0t} \times \hat{\mathbf{z}} \end{pmatrix} &= \mathbf{W} \cdot \begin{pmatrix} \mathbf{E}_{0t} \\ \mathbf{H}_{0t} \times \hat{\mathbf{z}} \end{pmatrix} \\ &= \begin{pmatrix} \mathbf{0} & -\hat{\mathbf{z}} \times \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix} \cdot \left[\begin{pmatrix} \boldsymbol{\epsilon}_{tt} & \boldsymbol{\xi}_{tt} \\ \boldsymbol{\zeta}_{tt} & \boldsymbol{\mu}_{tt} \end{pmatrix} - \mathbf{A} \right] \cdot \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{z}} \times \mathbf{I} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_{0t} \\ \mathbf{H}_{0t} \times \hat{\mathbf{z}} \end{pmatrix} \end{aligned}$$

This is what Maxwell's equations boil down to when looking for solutions on the form $\mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\beta z} e^{-j(k_x x + k_y y)}$ and $\mathbf{H}(\mathbf{r}) = \mathbf{H}_0 e^{-j\beta z} e^{-j(k_x x + k_y y)}$. Note that $\beta/\omega = n/c_0$ where n is the refractive index, and $\mathbf{E}_{0t} = \mathbf{E}_{0xy} = E_{0x}\hat{\mathbf{x}} + E_{0y}\hat{\mathbf{y}}$, *i.e.*, we only consider the components transverse to the propagation direction z .

In this workshop, we do not consider oblique propagation and set the transverse wave vector $\mathbf{k}_t = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}} = \mathbf{0}$, but feel free to experiment, both with this parameter and other.

2 Propagation in arbitrary materials

Download the script `propconst.m` from the course web site to your working directory. If you haven't done it already, try out the script by running it, and compare the code with the slides from lecture 3. You should be able to identify the decomposition in transverse (x and y) and normal (z) components, how the \mathbf{A} -matrix is computed using this decomposition, and how the final \mathbf{W} -matrix is assembled. Also

pay attention to how the electric and magnetic field amplitudes are extracted from the eigenvectors at the end of the script.

When `propconst.m` has been executed, you will have two 2×4 matrices in matlab's workspace:

$$\mathbf{E} = \begin{pmatrix} E_{x1} & E_{x2} & E_{x3} & E_{x4} \\ E_{y1} & E_{y2} & E_{y3} & E_{y4} \end{pmatrix}, \quad \mathbf{H} = \begin{pmatrix} H_{x1} & H_{x2} & H_{x3} & H_{x4} \\ H_{y1} & H_{y2} & H_{y3} & H_{y4} \end{pmatrix}$$

Write your own script that uses these variables to compute (for each of the four propagating waves)

- Wave impedance $Z_x = \frac{E_x}{H_y}$ and $Z_y = -\frac{E_y}{H_x}$. You can compute it for all eigenvectors at once by `Zx = E(1,:)./H(2,:)` and `Zy = -E(2,:)./H(1,:)`.
- Time averaged power flow in the positive z -direction $P = \frac{1}{2} \hat{\mathbf{z}} \cdot \text{Re}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2} \text{Re}(E_x H_y^* - E_y H_x^*)$. The matlab code is `P = 0.5*real(E(1,:).*conj(H(2,:)) - E(2,:).*conj(H(1,:)))`.
- Polarization state with respect to the propagation direction $\hat{\mathbf{e}}$, $-j\hat{\mathbf{e}} \cdot (\mathbf{E} \times \mathbf{E}^*)$. The propagation direction is given by the sign of the power flow as $\hat{\mathbf{e}} = \text{sign}(P)\hat{\mathbf{z}}$, and $-j\hat{\mathbf{z}} \cdot (\mathbf{E} \times \mathbf{E}^*) = -j(E_x E_y^* - E_y E_x^*) = 2 \text{Im}(E_x E_y^*)$. The matlab code is `pol = 2*sign(P).*imag(E(1,:).*conj(E(2,:)))`.

Use your script to investigate the different material models below. In particular, pay attention to how the polarization depends on the material model.

Given a complex vector $\mathbf{E} = E_x \hat{\mathbf{x}} + E_y \hat{\mathbf{y}}$ and refractive index n , where E_x , E_y , and n may be complex valued, you can do a simple animation (similar to the EMANIM program) in matlab as follows (also available on the course web site):

```
E0 = [2
      -1j];
n = 2-0.05j;

f = 1e9;
c0 = 299792458;
omega = 2*pi*f;
k0 = omega/c0;
lambda0 = 2*pi/k0;

Nz = 100;
z = linspace(0, 3*lambda0, Nz); % Row vector
E = E0*exp(-1j*n*k0*z); % 2xNz matrix

Nphi = 100;
phi = linspace(0, 2*pi, Nphi); % phi = omega*t

while 1, % Interrupt by a ctrl-c
```

```

for k=1:Nphi
    Et = real(E*exp(1j*phi(k)))/norm(E0);
    plot3(Et(1,:), Et(2,:), z, '-');
    xlabel('Ex');
    ylabel('Ey');
    zlabel('z');
    grid on;
    pause(1e-2);
end
end

```

You can use this animation to check your understanding of linear and circular polarization in the models below.

3 Material models

In the constitutive relations below, we use square brackets $[\epsilon]$ to emphasize that the matrix expression following it is a representation of the material dyadic ϵ in a particular coordinate system. Dyadics proportional to the identity matrix \mathbf{I} do not require this distinction.

3.1 Isotropic

An isotropic material is the standard model, used for most materials.

$$\epsilon = \epsilon_0 \epsilon_r \mathbf{I}, \quad \mu = \mu_0 \mu_r \mathbf{I}, \quad \xi = \zeta = \mathbf{0} \quad (3.1)$$

Try first a lossless case (real positive parameters, use $\epsilon_r = 2$ and $\mu_r = 1$). Apply your script to compute the numbers (a–c) above. Try the animation script for each of the four waves, and check that your interpretation of the numbers agrees with the animation. [Answers: $Z_x = (266.3886, -266.3886, \text{NaN}, \text{NaN})$; $Z_y = (\text{NaN}, \text{NaN}, 266.3886, -266.3886)$; $P = (0.0019, -0.0019, 0.0019, -0.0019)$; $\text{pol} = (0, 0, 0, 0)$]

For the isotropic material model, try also a lossy case (negative imaginary parts in the parameters, use $\epsilon_r = 2 - 0.1j$ and $\mu_r = 1$). Repeat the investigations for the lossless case. [Answers: $Z_x = 1.0e+02*(2.6614 + 0.0665i, -2.6614 - 0.0665i, 2.6614 + 0.0665i, -2.6614 - 0.0665i)$; $Z_y = 1.0e+02*(\text{NaN}, \text{NaN}, 2.6614 + 0.0665i, -2.6614 - 0.0665i)$; $P = (0.0019, -0.0019, 0.0019, -0.0019)$; $\text{pol} = (0, 0, 0.0126, -0.0126)$] Also compute the attenuation:

- For $f = 1$ GHz, how much is the power attenuated after propagating 1 m and 2 m in this material? What is this number in dB-scale?
[Answers: 0.23, 0.052, -6.4 dB, -13 dB]
- Repeat the calculations for $f = 10$ GHz.
[Answers: $3.7 \cdot 10^{-7}$, $1.4 \cdot 10^{-13}$, -64 dB, -130 dB]

For the remaining material models (anisotropic etc), no attenuation calculations are necessary since the numbers suggested in the text correspond to lossless models.

3.2 Anisotropic dielectrics

Anisotropic materials occur whenever there is some asymmetry on a microscopic level, for instance in crystals with asymmetric unit cells. In many cases, the magnetic properties can be neglected.

$$[\boldsymbol{\epsilon}] = \epsilon_0 \begin{pmatrix} \epsilon_{r1} & 0 & 0 \\ 0 & \epsilon_{r2} & 0 \\ 0 & 0 & \epsilon_{r3} \end{pmatrix}, \quad \boldsymbol{\mu} = \mu_0 \mathbf{I}, \quad \boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0} \quad (3.2)$$

Some example values at optical frequencies ($\lambda_0 = 590 \text{ nm}$) are given below, try rutile since it has the biggest difference in parameters. [Answers: $\mathbf{Zx} = (129.7759, -129.7759, \text{NaN}, \text{NaN})$; $\mathbf{Zy} = (\text{NaN}, \text{NaN}, 144.0043, -144.0043)$; $\mathbf{P} = (0.0039, -0.0039, 0.0035, -0.0035)$; $\text{pol} = (0, 0, 0, 0)$]

Uniaxial material	ϵ_{r1}	ϵ_{r2}	ϵ_{r3}
Ice H_2O	1.724	1.714	1.714
Calcite CaCO_3	2.208	2.749	2.749
Rutile TiO_2	8.427	6.844	6.844
Sapphire Al_2O_3	3.098	3.126	3.126
Biaxial material	ϵ_{r1}	ϵ_{r2}	ϵ_{r3}
Perovskite CaTiO_3	5.290	5.476	5.664
Topaz $\text{Al}_2\text{SiO}_4(\text{F}, \text{OH})_2$	2.618	2.624	2.647

As you can see, anisotropies are often quite small, but when the wave propagates many wavelengths it is still an important effect. More on this in Chapter 4 of Orfanidis' book, and applications to giant birefringent optics in Chapter 8.

3.3 Gyrotropic

A common example of a gyrotropic material is a magnetized plasma, *i.e.*, a collection of free charges floating around subjected to a static magnetic field (for instance the ions in the ionosphere combined with the earth magnetic field, or free electrons in a conductor with an external magnetic field). The main effect is that the charges experience a spiraling movement around the magnetic field.

In the model below, keep $|\epsilon_g| < |\epsilon_r|$.

$$[\boldsymbol{\epsilon}] = \epsilon_0 \begin{pmatrix} \epsilon_r & -j\epsilon_g & 0 \\ j\epsilon_g & \epsilon_r & 0 \\ 0 & 0 & \epsilon_r \end{pmatrix}, \quad \boldsymbol{\mu} = \mu_0 \mathbf{I}, \quad \boldsymbol{\xi} = \boldsymbol{\zeta} = \mathbf{0} \quad (3.3)$$

The permittivity of a plasma has large variation with frequency and density of charges: for the ionosphere it actually varies with the time of the day. Some suitable numbers for short-wave radio at $f \approx 10 \text{ MHz}$ propagating in the ionosphere may

be $\epsilon_r = 0.18$ and $\epsilon_g = 0.11$. That these parameters can be less than one will be explored in Orfanidis' Chapters 3–4. [Answers: $\mathbf{Zx} = 1.0\text{e}+03*(-0.6996 + 0.0000i, -1.4239 - 0.0000i, 0.6996 + 0.0000i, 1.4239 - 0.0000i)$; $\mathbf{Zy} = 1.0\text{e}+03*(-0.6996 + 0.0000i, -1.4239 + 0.0000i, 0.6996 + 0.0000i, 1.4239 + 0.0000i)$; $\mathbf{P} = 1.0\text{e}-03*(-0.7147, -0.3511, 0.7147, 0.3511)$; $\text{pol} = (1.0000, -1.0000, -1.0000, 1.0000)$]

Note that the same effect occurs in the magnetic dyadic $\boldsymbol{\mu}$ when considering ferromagnetic materials, see handin 1.

4 Rotation of axes

The material models above were given in a coordinate system where the z direction is highly symmetric. We will now investigate what happens if this symmetry is broken.

A rotation of axes transforms the cartesian representation of a material dyadic in the following way:

$$[\boldsymbol{\epsilon}'] = [\mathbf{R}] [\boldsymbol{\epsilon}] [\mathbf{R}]^{-1} \quad (4.1)$$

The rotation around each principal axis are

$$[\mathbf{R}_x(\varphi)] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}, \quad [\mathbf{R}_y(\varphi)] = \begin{pmatrix} \cos \varphi & 0 & \sin \varphi \\ 0 & 1 & 0 \\ -\sin \varphi & 0 & \cos \varphi \end{pmatrix}, \quad [\mathbf{R}_z(\varphi)] = \begin{pmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.2)$$

Use these rotation matrices together with the anisotropic material model in Section 3.2 in the following. Choose values $\epsilon_{r1} = 1$, $\epsilon_{r2} = 2$, and $\epsilon_{r3} = 3$, and follow the instructions below.

4.1 Rotation around the z -axis

When rotating the coordinate system around the z -axis, the propagation direction is still an axis of symmetry. Verify this by applying a rotation of 30° around the z -axis and confirm that you get the same values for refractive index, wave impedance, power flow, and polarization state as before the rotation (except possibly the order in which they are computed). [Answers: $\mathbf{Zx} = (-266.3886, -376.7303, 266.3886, 376.7303)$; $\mathbf{Zy} = (-266.3886, -376.7303, 266.3886, 376.7303)$; $\mathbf{P} = (-0.0019, -0.0013, 0.0019, 0.0013)$; $\text{pol} = (0, 0, 0, 0)$]

4.2 Rotation around the y -axis

When rotating the coordinate system around the y -axis, we break the symmetry with respect to the propagation direction z . Apply a rotation of 30° around the y -axis, and pay close attention to what changes in the results that are introduced by this. [Answers: $\mathbf{Zx} = (343.9062, -343.9062, 308.4858, -308.4858)$; $\mathbf{Zy} = (\text{NaN}, \text{NaN}, 266.3886, -266.3886)$; $\mathbf{P} = (0.0015, -0.0015, 0.0019, -0.0019)$; $\text{pol} = (0, 0, 0, 0)$]

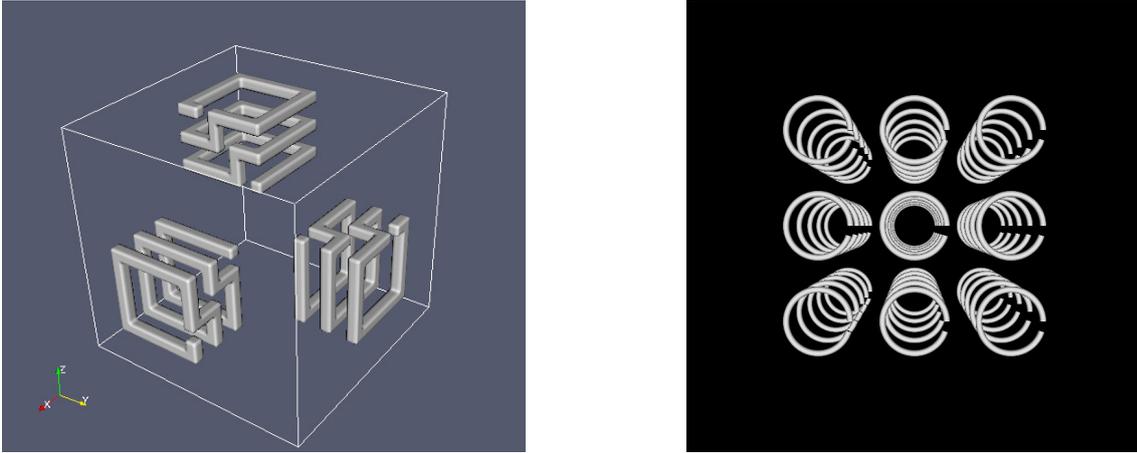


Figure 1: Microstructure of two artificial materials. To the left, a biisotropic material, where the chirality comes from the spirals. To the right, a fully bianisotropic material, consisting of metal split rings arranged in a regular lattice.

5 Extra: Some exotic materials

This section only serves to give additional concrete examples of material models. If you are short for time, there is no need to go through it at the workshop. The following models are given as examples of more rarely encountered material models.

5.1 Biisotropic (chiral)

A biisotropic material experiences a coupling between electric and magnetic fields in the constitutive relations. This typically happens when the material has a chiral microstructure, *i.e.*, when the molecules have a “handedness” where one of the left or right hand kind dominates. Typical examples are sugar and amino acids, which can be detected with visible light.

In the model below, keep $|\kappa|^2 < |\epsilon_r \mu_r|$ (this is usually a very small parameter).

$$\boldsymbol{\epsilon} = \epsilon_0 \epsilon_r \mathbf{I}, \quad \boldsymbol{\mu} = \mu_0 \mu_r \mathbf{I}, \quad \boldsymbol{\xi} = -j\kappa/c_0 \mathbf{I}, \quad \boldsymbol{\zeta} = j\kappa/c_0 \mathbf{I} \quad (5.1)$$

The chiral parameter κ is usually very small in natural materials (say around 10^{-6} for a sugar solution at optical frequencies), but for specifically engineered materials values around $\kappa \approx 0.03$ with $\epsilon_r \approx 1.4$ and $\mu_r = 1$ can be achieved, see the structure to the left in Figure 1. [Answers: $\mathbf{Zx} = 1.0\text{e}+02 * (-3.1840 + 0.0000i, -3.1840 - 0.0000i, 3.1840 - 0.0000i, 3.1840 + 0.0000i)$; $\mathbf{Zy} = 1.0\text{e}+02 * (-3.1840 + 0.0000i, -3.1840 - 0.0000i, 3.1840 - 0.0000i, 3.1840 + 0.0000i)$; $\mathbf{P} = (-0.0016, -0.0016, 0.0016, 0.0016)$; $\mathbf{pol} = (1.0000, -1.0000, 1.0000, -1.0000)$]

5.2 Bianisotropic

An example of a fully bianisotropic material is given by an engineered material, with a microstructure consisting of split ring resonators as in Figure 1. For low

frequencies this material can be modeled as

$$[\boldsymbol{\epsilon}] = \epsilon_0 \begin{pmatrix} a_x & 0 & 0 \\ 0 & a_y & 0 \\ 0 & 0 & a_z \end{pmatrix}, \quad [\boldsymbol{\mu}] = \mu_0 \begin{pmatrix} b_x & 0 & 0 \\ 0 & b_y & 0 \\ 0 & 0 & b_z \end{pmatrix}, \quad [\boldsymbol{\xi}] = \frac{1}{c_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & jc_z \\ 0 & -jc_y & 0 \end{pmatrix}, \quad [\boldsymbol{\zeta}] = \frac{1}{c_0} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & jc_y \\ 0 & -jc_z & 0 \end{pmatrix} \quad (5.2)$$

Some example values are (depend on how densely packed the rings are) $a_x = 1.49$, $a_y = 1.43$, $a_z = 1.01$, $b_x = 0.995$, $b_y = 0.994$, $b_z = 0.991$, $c_y = 0.0003$, $c_z = 0.39$. The values b_x , b_y , and b_z are less than one since the metal in the split ring resonators is modelled as PEC, which has negative magnetic polarizability. Using a finite conductivity model, they would be identically 1. [Answers: $Z_x = (307.7022, -307.7022, \text{NaN}, \text{NaN})$; $Z_y = (\text{NaN}, \text{NaN}, 332.6048, -332.6048)$; $P = (0.0016, -0.0016, 0.0015, -0.0015)$; $\text{pol} = (0, 0, 0, 0)$]