Electromagnetic Wave Propagation
Lecture 1: Maxwell's equations

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Outline

1. **Maxwell's equations**
2. **Vector analysis**
3. **Boundary conditions**
4. **Conservation laws**
5. **Conclusions**

Outline

A long time ago, in a galaxy quite near us...

VIII. *A Dynamical Theory of the Electromagnetic Field.* *By* J. Clerk Maxwell, *F.R.S.*

Received October 27,—Read December 8, 1864.

A few of the equations back then:	Nowadays:
$p' = p + \frac{df}{dt},\ q' = q + \frac{dg}{dt},\ r' = r + \frac{dh}{dt}$	$\boldsymbol{J}_{\text{tot}} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$
$\begin{cases} \frac{d\gamma}{dy} - \frac{d\beta}{dz} = 4\pi p' \\ \frac{d\alpha}{dz} - \frac{d\gamma}{dx} = 4\pi q' \\ \frac{d\beta}{dx} - \frac{d\alpha}{dy} = 4\pi r' \end{cases}$	$\nabla \times \boldsymbol{H} = \boldsymbol{J}_{\text{tot}}$
$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0$	$\nabla \cdot \boldsymbol{D} = \rho$
$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0$	$\nabla \cdot \boldsymbol{J}_{\text{tot}} = 0$

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$e + \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0$	$\nabla \cdot \boldsymbol{D} = \rho$
$\frac{de}{dt} + \frac{dp}{dx} + \frac{dq}{dy} + \frac{dr}{dz} = 0$	$\nabla \cdot \boldsymbol{J}_{\text{tot}} = 0$

Note that the original equations are inconsistent.
The sign of e (ρ) needs to change in one of the equations.

Historical notes

- **Early 1800s:** Electricity and magnetism separate phenomena.
- **1819:** Hans Christian Ørsted discovers a linear current deflects a magnetized needle.
- **1831:** Michael Faraday demonstrates that a changing magnetic field can induce electric voltage.
- **1830s:** Electrical wire telegraphs come into use.
- **1864:** James Clerk Maxwell presents his paper *A Dynamical Theory of the Electromagnetic Field*, joining electricity and magnetism and predicting the existence of waves.
- **1887:** Heinrich Hertz proves experimentally the existence of electromagnetic waves.
- **1897:** Marconi founds the *Wireless Telegraph & Signal Company*
- **1905:** Albert Einstein publishes his special theory of relativity, emphasizing the role of the speed of light in vacuum.

Historical notes, continued

- 1930s: The first radio telescopes are built.
- Early 1940s: The MIT Radiation Laboratory substantially advances the knowledge of control of electromagnetic waves while developing radar technology.
- Late 1940s: The Quantum Electrodynamics theory (QED) is developed by Richard Feynman, Freeman Dyson, Julian Schwinger, and Sin-Itiro Tomonaga.
- 1957: Sputnik 1 transmits the first signal from a satellite to earth.
- 1970s: Low loss optical fibers are developed.
- 1981: The NMT system goes online.
- 2000: Mobile phones connect to internet.
- 2016: Pokémon Go demonstrates augmented reality.
- 2020: Initial observations by Square Kilometre Array(?)

Field equations and the electromagnetic fields

Faraday's law: $$\nabla \times \boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r},t)}{\partial t}$$

Ampère's law: $$\nabla \times \boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \boldsymbol{D}(\boldsymbol{r},t)}{\partial t}$$

Conservation of charge: $$\nabla \cdot \boldsymbol{J}(\boldsymbol{r},t) + \frac{\partial \rho(\boldsymbol{r},t)}{\partial t} = 0$$

Symbol	Name	Unit
$\boldsymbol{E}(\boldsymbol{r},t)$	Electric field	[V/m]
$\boldsymbol{H}(\boldsymbol{r},t)$	Magnetic field	[A/m]
$\boldsymbol{D}(\boldsymbol{r},t)$	Electric flux density	[As/m^2]
$\boldsymbol{B}(\boldsymbol{r},t)$	Magnetic flux density	[Vs/m^2]
$\boldsymbol{J}(\boldsymbol{r},t)$	Current density	[A/m^2]
$\rho(\boldsymbol{r},t)$	Charge density	[As/m^3]

The divergence equations

Often the equations

$$\nabla \cdot \boldsymbol{D} = \rho \quad \text{and} \quad \nabla \cdot \boldsymbol{B} = 0 \tag{$*$}$$

are considered as a part of Maxwell's equations, but they can be derived as follows. Since $\nabla \cdot (\nabla \times \boldsymbol{F}) = 0$ for any vector function $\boldsymbol{F}$, taking the divergence of Faraday's and Ampère's laws imply

$$0 = \nabla \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$

$$0 = \nabla \cdot \left(\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t} \right) = -\frac{\partial \rho}{\partial t} + \nabla \cdot \frac{\partial \boldsymbol{D}}{\partial t}$$

Thus $\nabla \cdot \boldsymbol{B} = f_1$ and $\nabla \cdot \boldsymbol{D} - \rho = f_2$, where f_1 and f_2 are independent of t. Assuming the fields are zero at $t = -\infty$, we have $f_1 = f_2 = 0$ and the divergence equations $(*)$ follow.

Materials

Maxwell's equations give $2 \times 3 = 6$ equations, but the fields $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{H}$, $\boldsymbol{D}$ represent $4 \times 3 = 12$ unknowns. The remaining equations are given by the models of the material behavior. In vacuum we have

$$\boldsymbol{D} = \epsilon_0 \boldsymbol{E}, \quad \boldsymbol{B} = \mu_0 \boldsymbol{H}$$

where the speed of light in vacuum is
$\mathrm{c}_0 = 1/\sqrt{\epsilon_0 \mu_0} = 299\,792\,458\,\mathrm{m/s}$ (exact) and

$$\epsilon_0 = \frac{1}{\mathrm{c}_0^2 \mu_0} \approx 8.854 \cdot 10^{-12}\,\mathrm{As/Vm}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{Vs/Am} \quad \text{(exact)}$$

In a material, there is in addition polarization and magnetization:

$$\boldsymbol{P} = \boldsymbol{D} - \epsilon_0 \boldsymbol{E}, \quad \boldsymbol{M} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{H}$$

The physics of the material in question determine the fields $\boldsymbol{P}$ and $\boldsymbol{M}$ as functions of the electromagnetic fields (next lecture!).

Outline

Literature

Here, we give a brief overview of vector analysis used in the course.

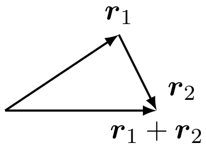
A summary of vector relations can be found in Orfanidis, Appendices C–E. If you want more in-depth, you can consult a typical first text book on electromagnetism, like Griffiths or Cheng.

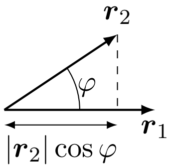
Vector analysis

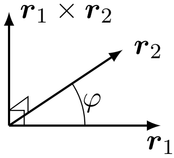
The vectors have three components, one for each spatial direction:

$$\boldsymbol{E} = E_x\hat{\boldsymbol{x}} + E_y\hat{\boldsymbol{y}} + E_z\hat{\boldsymbol{z}}$$

In particular, the position vector is $\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$. Vector addition, scalar product and vector product are







- Addition: $\boldsymbol{r}_1 + \boldsymbol{r}_2 = (x_1 + x_2)\hat{\boldsymbol{x}} + (y_1 + y_2)\hat{\boldsymbol{y}} + (z_1 + z_2)\hat{\boldsymbol{z}}$
- Scalar product: $\boldsymbol{r}_1 \cdot \boldsymbol{r}_2 = |\boldsymbol{r}_1|\,|\boldsymbol{r}_2|\cos\varphi = x_1x_2 + y_1y_2 + z_1z_2$.
- Vector product: orthogonal to both vectors, with length $|\boldsymbol{r}_1 \times \boldsymbol{r}_2| = |\boldsymbol{r}_1|\,|\boldsymbol{r}_2|\sin\varphi$, and $\boldsymbol{r}_1 \times \boldsymbol{r}_2 = -\boldsymbol{r}_2 \times \boldsymbol{r}_1$.

$$\hat{\boldsymbol{x}} \times \hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}, \quad \hat{\boldsymbol{y}} \times \hat{\boldsymbol{z}} = \hat{\boldsymbol{x}}, \quad \hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}} = \hat{\boldsymbol{y}}$$

Vector analysis, continued

The nabla operator is

$$\nabla = \hat{\boldsymbol{x}}\frac{\partial}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z}$$

The divergence and curl operations are

$$\nabla \cdot \boldsymbol{E} = \frac{\partial}{\partial x}E_x + \frac{\partial}{\partial y}E_y + \frac{\partial}{\partial z}E_z$$

$$\nabla \times \boldsymbol{E} = \frac{\partial}{\partial x}\hat{\boldsymbol{x}} \times \boldsymbol{E} + \frac{\partial}{\partial y}\hat{\boldsymbol{y}} \times \boldsymbol{E} + \frac{\partial}{\partial z}\hat{\boldsymbol{z}} \times \boldsymbol{E}$$

Cartesian representation: $[\boldsymbol{E}] = \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$ and

$$[\nabla \times \boldsymbol{E}] = \frac{\partial}{\partial x} \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}}_{=[\hat{\boldsymbol{x}} \times \boldsymbol{E}]} + \frac{\partial}{\partial y} \underbrace{\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}}_{=[\hat{\boldsymbol{y}} \times \boldsymbol{E}]} + \frac{\partial}{\partial z} \underbrace{\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}}_{=[\hat{\boldsymbol{z}} \times \boldsymbol{E}]}$$

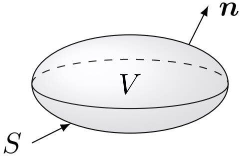
Integral theorems

Generalizations of the fundamental theorem of calculus:

$$\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$$

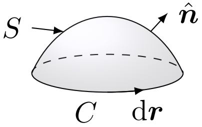
Gauss' theorem:

$$\iiint_V \nabla \cdot \boldsymbol{F}\,\mathrm{d}V = \iint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{F}\,\mathrm{d}S$$



Stokes' theorem:

$$\iint_S (\nabla \times \boldsymbol{F}) \cdot \hat{\boldsymbol{n}}\,\mathrm{d}S = \int_C \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{r}$$



Typically: a higher order integral of a derivative can be turned into a lower order integral of the function.

Dyadic products

The projection of a vector on the x-direction is defined by

$$\mathbf{P}_x \cdot \boldsymbol{E} = \hat{\boldsymbol{x}} E_x$$

Since $E_x = \hat{\boldsymbol{x}} \cdot \boldsymbol{E}$, we can write this

$$\mathbf{P}_x \cdot \boldsymbol{E} = \hat{\boldsymbol{x}} E_x = \hat{\boldsymbol{x}}(\hat{\boldsymbol{x}} \cdot \boldsymbol{E}) = (\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}) \cdot \boldsymbol{E}$$

Thus $\mathbf{P}_x = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}$, which is a dyadic product. In particular, the identity operator $\mathbf{I} \cdot \boldsymbol{E} = \boldsymbol{E}$ can be written

$$\mathbf{I} = \hat{\boldsymbol{x}}\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}\hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}\hat{\boldsymbol{z}}$$

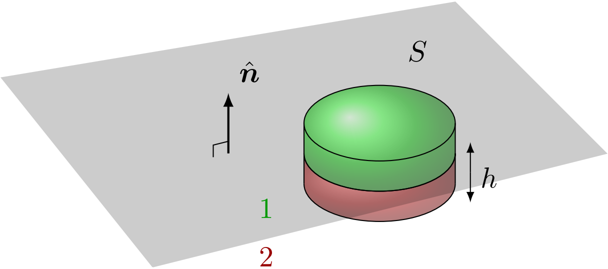
with the Cartesian representation

$$[\mathbf{I}\cdot] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\substack{= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \\ = [\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}\cdot]}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\substack{= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\ = [\hat{\boldsymbol{y}}\hat{\boldsymbol{y}}\cdot]}} + \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\substack{= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ = [\hat{\boldsymbol{z}}\hat{\boldsymbol{z}}\cdot]}}$$

Outline

Material interfaces

Consider the interface between two media, 1 and 2:



What are the relations between field quantities at different sides of the interface?

Integral form of Maxwell's equations

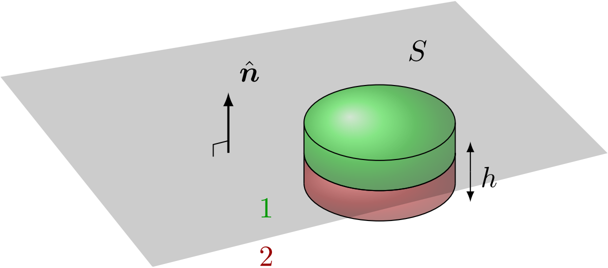
Integrate the field equations over the volume V (red **and** green):

$$\iiint_V \nabla \times \boldsymbol{E} \, \mathrm{d}V = -\iiint_V \frac{\partial \boldsymbol{B}}{\partial t} \, \mathrm{d}V$$

$$\iiint_V \nabla \times \boldsymbol{H} \, \mathrm{d}V = \iiint_V \boldsymbol{J} \, \mathrm{d}V + \iiint_V \frac{\partial \boldsymbol{D}}{\partial t} \, \mathrm{d}V$$

$$\iiint_V \nabla \cdot \boldsymbol{B} \, \mathrm{d}V = 0$$

$$\iiint_V \nabla \cdot \boldsymbol{D} \, \mathrm{d}V = \iiint_V \rho \, \mathrm{d}V$$



Integral form of Maxwell's equations, continued

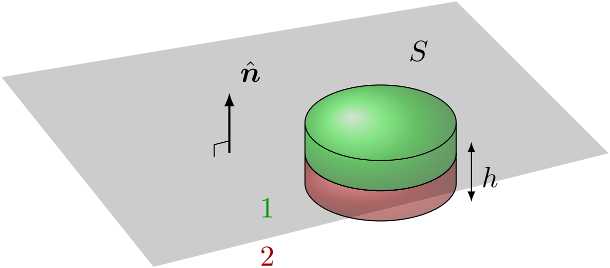
Use the integral theorems to find

$$\iint_S \hat{\boldsymbol{n}} \times \boldsymbol{E} \, \mathrm{d}V = -\frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \boldsymbol{B} \, \mathrm{d}V$$

$$\iint_S \hat{\boldsymbol{n}} \times \boldsymbol{H} \, \mathrm{d}V = \iiint_V \boldsymbol{J} \, \mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t} \iiint_V \boldsymbol{D} \, \mathrm{d}V$$

$$\iint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{B} \, \mathrm{d}V = 0$$

$$\iint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{D} \, \mathrm{d}V = \iiint_V \rho \, \mathrm{d}V$$

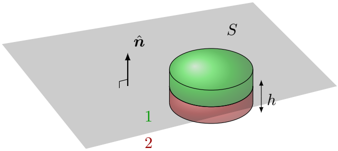


Pillbox volume

Let the height of the pillbox volume V_h become small, $h \to 0$. On metal surfaces, the current and charge *densities* can become very large. Introduce the surface current and surface charge as

$$\lim_{h\to 0} \iiint_{V_h} \boldsymbol{J}\,\mathrm{d}V = \iint_S \boldsymbol{J}_\mathrm{S}\,\mathrm{d}S$$

$$\lim_{h\to 0} \iiint_{V_h} \rho\,\mathrm{d}V = \iint_S \rho_\mathrm{S}\,\mathrm{d}S$$



The limit of the flux integrals on the other hand is zero,

$$\lim_{h\to 0} \iiint_{V_h} \boldsymbol{D}\,\mathrm{d}V = \lim_{h\to 0} \iiint_{V_h} \boldsymbol{B}\,\mathrm{d}V = 0$$

The limit of $\boldsymbol{E}$ is $\boldsymbol{E}_1$ when coming from material 1 and $\boldsymbol{E}_2$ when coming from material 2, implying

$$\lim_{h\to 0} \iint_{S_h} \hat{\boldsymbol{n}} \times \boldsymbol{E}\,\mathrm{d}S = \iint_S \underbrace{(\hat{\boldsymbol{n}}_1 \times \boldsymbol{E}_1 + \hat{\boldsymbol{n}}_2 \times \boldsymbol{E}_2)}_{=\hat{\boldsymbol{n}}_1 \times (\boldsymbol{E}_1 - \boldsymbol{E}_2)}\,\mathrm{d}S$$

Boundary conditions

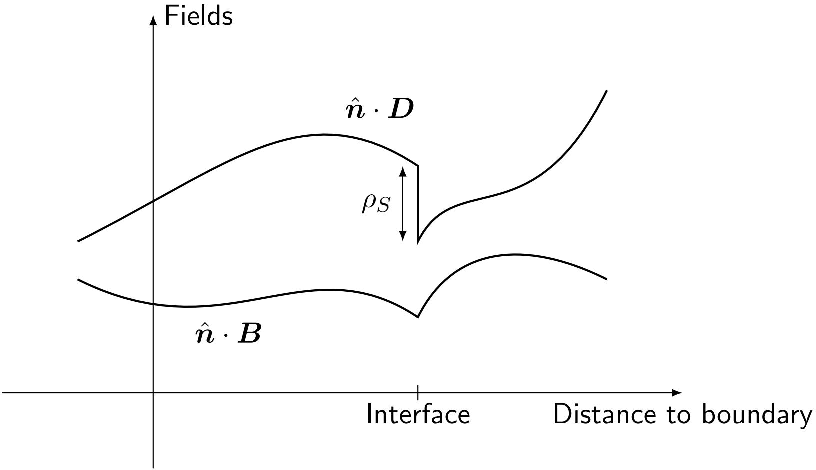
We summarize as (when material 2 is a perfect electric conductor (PEC) the fields with index 2 are zero):

$$\hat{\boldsymbol{n}}_1 \times (\boldsymbol{E}_1 - \boldsymbol{E}_2) = \boldsymbol{0}$$
$$\hat{\boldsymbol{n}}_1 \times (\boldsymbol{H}_1 - \boldsymbol{H}_2) = \boldsymbol{J}_\mathrm{S}$$
$$\hat{\boldsymbol{n}}_1 \cdot (\boldsymbol{B}_1 - \boldsymbol{B}_2) = 0$$
$$\hat{\boldsymbol{n}}_1 \cdot (\boldsymbol{D}_1 - \boldsymbol{D}_2) = \rho_\mathrm{S}$$

In words, this means:

- The tangential electric field is continuous.
- The tangential magnetic field is discontinuous if $\boldsymbol{J}_\mathrm{S} \neq \boldsymbol{0}$.
- The normal component of the magnetic flux is continuous.
- The normal component of the electric flux is discontinuous if $\rho_\mathrm{S} \neq \boldsymbol{0}$.

Example


Fields
$\hat{\boldsymbol{n}} \cdot \boldsymbol{D}$
ρ_S
$\hat{\boldsymbol{n}} \cdot \boldsymbol{B}$
Interface
Distance to boundary

Outline

Conservation of charge

The conservation of charge is postulated,

$$\nabla \cdot \boldsymbol{J} + \frac{\partial \rho}{\partial t} = 0$$

Alternatively, we could postulate Gauss' law, $\nabla \cdot \boldsymbol{D} = \rho$, and derive the conservation of charge from Ampère's law $\nabla \times \boldsymbol{H} = \boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}$.

Energy conservation

Take the scalar product of Faraday's law with $\boldsymbol{H}$ and the scalar product of Ampère's law with $\boldsymbol{E}$:

$$\boldsymbol{H} \cdot (\nabla \times \boldsymbol{E}) = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t}$$
$$\boldsymbol{E} \cdot (\nabla \times \boldsymbol{H}) = \boldsymbol{E} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}$$

Take the difference of the equations and use the identity $\nabla \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\nabla \times \boldsymbol{a}) - \boldsymbol{a} \cdot (\nabla \times \boldsymbol{b})$

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} + \boldsymbol{E} \cdot \boldsymbol{J} = 0$$

This is Poynting's theorem on differential form.

Interpretation of Poynting's theorem

Integrating over a volume V implies (where $\boldsymbol{\mathcal{P}} = \boldsymbol{E} \times \boldsymbol{H}$)

$$\iint_S \hat{\boldsymbol{n}} \cdot \boldsymbol{\mathcal{P}} \, \mathrm{d}S = \iiint_V \nabla \cdot \boldsymbol{\mathcal{P}} \, \mathrm{d}V$$
$$= -\iiint_V \left(\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t} \right) \mathrm{d}V - \iiint_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}V$$

- The first integral is the total power radiated out of the bounding surface S.
- The second integral is the rate of change of electromagnetic energy stored in the volume V (not obvious at this point).
- The last integral is the work per unit time (the power) that the field does on charges in V.

This is conservation of power (or energy).

Momentum conservation

Take the vector product of $\boldsymbol{D}$ with Faraday's law and $\boldsymbol{B}$ with Ampère's law,

$$\begin{aligned}\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) &= -\boldsymbol{D} \times \partial_t \boldsymbol{B} \\ \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) &= \boldsymbol{B} \times \boldsymbol{J} + \boldsymbol{B} \times \partial_t \boldsymbol{D} = -\boldsymbol{J} \times \boldsymbol{B} - (\partial_t \boldsymbol{D}) \times \boldsymbol{B}\end{aligned}$$

Adding the equations results in

$$\partial_t(\boldsymbol{D} \times \boldsymbol{B}) + \boldsymbol{J} \times \boldsymbol{B} + \boldsymbol{D} \times (\nabla \times \boldsymbol{E}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{H}) = \boldsymbol{0}$$

It can be shown that

$$\boldsymbol{D} \times (\nabla \times \boldsymbol{E}) = (\nabla \cdot \boldsymbol{D})\boldsymbol{E} + \sum_{i=x,y,z} D_i \nabla E_i - \nabla \cdot (\boldsymbol{D}\boldsymbol{E})$$

implying (using $\nabla \cdot \boldsymbol{D} = \rho$ and $\nabla \cdot \boldsymbol{B} = 0$)

$$\partial_t \underbrace{(\boldsymbol{D} \times \boldsymbol{B})}_{\text{momentum}} + \underbrace{\boldsymbol{J} \times \boldsymbol{B} + \rho \boldsymbol{E}}_{\text{Lorentz force}} = \nabla \cdot (\boldsymbol{D}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{H}) - \sum_{i=x,y,z} D_i \nabla E_i + B_i \nabla H_i$$

A simple material model

Often materials can be described by using an energy potential ϕ:

$$\boldsymbol{D} = \frac{\partial \phi(\boldsymbol{E}, \boldsymbol{H})}{\partial \boldsymbol{E}} \quad \text{and} \quad \boldsymbol{B} = \frac{\partial \phi(\boldsymbol{E}, \boldsymbol{H})}{\partial \boldsymbol{H}}$$

which is interpreted $D_i = \partial \phi / \partial E_i$ etc. For linear, isotropic media

$$\phi(\boldsymbol{E}, \boldsymbol{H}) = \frac{1}{2}\epsilon |\boldsymbol{E}|^2 + \frac{1}{2}\mu |\boldsymbol{H}|^2 \quad \Leftrightarrow \quad \begin{cases} \boldsymbol{D} = \epsilon \boldsymbol{E} \\ \boldsymbol{B} = \mu \boldsymbol{H} \end{cases}$$

Nonlinear materials have additional terms like $|\boldsymbol{E}|^4$, $|\boldsymbol{E}|^6$ etc. For a material described by potential ϕ, Poynting's theorem is

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H}) + \frac{\partial}{\partial t}(\boldsymbol{D} \cdot \boldsymbol{E} + \boldsymbol{B} \cdot \boldsymbol{H} - \phi) = -\boldsymbol{E} \cdot \boldsymbol{J}$$

and the conservation of momentum is

$$\frac{\partial}{\partial t}(\boldsymbol{D} \times \boldsymbol{B}) + \rho \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} = \nabla \cdot (\boldsymbol{D}\boldsymbol{E} + \boldsymbol{B}\boldsymbol{H} - \phi \mathbf{I})$$

The EM momentum is still debated!

Resource Letter EM-1: Electromagnetic Momentum

David J. Griffiths
Department of Physics, Reed College, Portland, Oregon 97202

(Received 29 July 2011; accepted 1 September 2011)

This Resource Letter surveys the literature on momentum in electromagnetic fields, including the general theory, the relation between electromagnetic momentum and vector potential, "hidden" momentum, the 4/3 problem for electromagnetic mass, and the Abraham–Minkowski controversy regarding the field momentum in polarizable and magnetizable media.

[DOI: 10.1119/1.3641979]

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ELSEVIER

Resolution of the Abraham–Minkowski controversy

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ABSTRACT

The momentum of light inside ponderable media has an electromagnetic part and a mechanical part. The local and instantaneous density of the electromagnetic part of the momentum is given by the Poynting vector divided by the square of the speed of light in vacuum, irrespective of the nature of the electromagnetic fields or the local or global properties of the material media. The mechanical part of the momentum is associated with the action of the electromagnetic field on the atomic constituents of the media, as spec-

PRL **101,** 243601 (2008) PHYSICAL REVIEW LETTERS week ending 12 DECEMBER 2008

Observation of a Push Force on the End Face of a Nanometer Silica Filament Exerted by Outgoing Light

Weilong She,* Jianhui Yu,† and Raohui Feng
State Key Laboratory of Optoelectronic Materials and Technologies, Sun Yat-Sen University, Guangzhou 510275, China
(Received 12 February 2008; revised manuscript received 15 September 2008; published 8 December 2008)

There are two different proposals for the momentum of light in a transparent dielectric of refractive index n: Minkowski's version nE/c and Abraham's version $E/(nc)$, where E and c are the energy and vacuum speed of light, respectively. Despite many tests and debates over nearly a century, momentum of light in a transparent dielectric remains controversial. In this Letter, we report a direct observation of the inward push force on the free end face of a nanometer silica filament exerted by the outgoing light. Our results suggest that Abraham's momentum is correct.

DOI: 10.1103/PhysRevLett.101.243601 PACS numbers: 42.50.Wk, 03.50.De

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The observable pressure of light in dielectric fluids

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By considering a perfect reflector submerged in a dielectric fluid, we show that the Minkowski formulation describes the optical momentum transfer to submerged objects. This result is required by global energy conservation, regardless of the phase of the reflected wave. While the electromagnetic pressure on a submerged reflector can vary with phase of the mirror reflection coefficient between twice the Abraham momentum and twice the Minkowski momentum, the Minkowski momentum is always restored due to the additional pressure on the dielectric surface. This analysis also gives further evidence for use of the Minkowski stress tensor at the boundary of a dielectric interface, which has been the subject of a long-standing debate in physics and the source of uncertainty in the modeling of optical forces on submerged particles.
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TUTORIAL REVIEW

Radiation pressure and the photon momentum in dielectrics

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Radiation force and balance of electromagnetic momentum

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The total momentum is well defined, but not its division into EM and mechanical momentum when inside materials.

Outline

Conclusions

- Maxwell's equations describe the dynamics of the electromagnetic fields.
- Boundary conditions relate the values of the fields on different sides of material boundaries to each other.
- Conservation laws can be derived to describe the conservation of physical entities such as power and momentum; these are usually products of two fields.
- Constitutive relations are necessary in order to fully solve Maxwell's equations. Topic of next lecture!