ETEN05 Electromagnetic Wave Propagation
Handin 1

Deadline September 8, 2016

Instructions

The solution should be handed in at the web site http://elearning.eit.lth.se/moodle/ in PDF format. You may work in groups of two, and hand in a joint report (write both names clearly). You can prepare your solution using LaTeX or a word processor (like LibreOffice.org or MS Word). If you run into trouble writing the mathematics on a computer, you can do a handwritten report, but it is good exercise to use the computer; you will need it for the final project.

The problem is designed so that most calculations should be relatively short. If you end up with very long calculations, you can consider putting some of them in an appendix. The solution should be self-contained, but if you want to use results established in other sources that is perfectly OK, and even encouraged, as long as you give suitable references.

All reports will be checked through the system Urkund, which compares the text with existing texts on the web and reports when the overlap seems obvious. Please make sure you use your own words, this part of the course is intended to help you write better.

Problem: constitutive relation for a ferromagnetic material

A ferromagnetic material can be seen as a collection of magnetic dipoles, precessing around an applied magnetic field. Three elemental metals exhibit ferromagnetism: iron, nickel, and cobalt. In this assignment, we look at a phenomenological model that applies to many different materials if the constants below are chosen appropriately.

Assume the material is magnetized by the static magnetic flux $\boldsymbol{H}_0 = H_0\hat{\boldsymbol{z}}$, such that the static magnetization is $\boldsymbol{M}_0 = M_0\hat{\boldsymbol{z}}$. Further assume that a time varying magnetic field $\boldsymbol{H}_1(t) = H_{1x}(t)\hat{\boldsymbol{x}} + H_{1y}(t)\hat{\boldsymbol{y}}$ is applied. This induces a time varying magnetization $\boldsymbol{M}_1(t) = M_{1x}(t)\hat{\boldsymbol{x}} + M_{1y}(t)\hat{\boldsymbol{y}}$ described by the Landau-Lifshitz-Gilbert equation,

$$\frac{\partial \boldsymbol{M}}{\partial t} = -\gamma\mu_0 \boldsymbol{M} \times \boldsymbol{H} + \alpha \frac{\boldsymbol{M}}{|\boldsymbol{M}|} \times \frac{\partial \boldsymbol{M}}{\partial t}$$

where $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1(t)$, $\boldsymbol{M} = \boldsymbol{M}_0 + \boldsymbol{M}_1(t)$, $\gamma = 1.76 \cdot 10^{11}\,\mathrm{C\,kg^{-1}}$ is the gyromagnetic ratio, $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{Vs/Am}$ is the permeability of vacuum, and α is a phenomenological dimensionless constant in the order of 0.1 or less. The chosen polarization corresponds to a wave propagating along the z-direction, and we assume that $\boldsymbol{H}_1$ and $\boldsymbol{M}_1$ are much smaller than the static values $\boldsymbol{H}_0$ and $\boldsymbol{M}_0$.

The Landau-Lifshitz-Gilbert equation is nonlinear, and we want to find the relation between the small perturbations $\boldsymbol{H}_1$ and $\boldsymbol{M}_1$ by linearization around the large static values $\boldsymbol{H}_0$ and $\boldsymbol{M}_0$. As an example of linearization, consider the nonlinear function $f(x) = ax + bx^2$. The linearization around x_0 is found by considering $f(x_0 + \delta x) = a(x_0+\delta x)+b(x_0+\delta x)^2 = ax_0+a\delta x+bx_0^2+b2x_0\delta x+b(\delta x)^2$. By setting $f(x_0+\delta x) = f_0+\delta f$ and ignoring terms of order $(\delta x)^2$ and higher, we identify terms of corresponding order as $f_0 = ax_0 + bx_0^2 = f(x_0)$ and $\delta f = (a + b2x_0)\delta x$.

(a) Show that to order one in the fields $\boldsymbol{M}_1$ and $\boldsymbol{H}_1$, the above equations of motion read:

$$\frac{\partial \boldsymbol{M}_1}{\partial t} = -\omega_M \hat{\boldsymbol{z}} \times \boldsymbol{H}_1 - \omega_H \boldsymbol{M}_1 \times \hat{\boldsymbol{z}} + \alpha \hat{\boldsymbol{z}} \times \frac{\partial \boldsymbol{M}_1}{\partial t} \qquad \text{where} \quad \begin{cases} \omega_M = \gamma\mu_0 M_0 \\ \omega_H = \gamma\mu_0 H_0 \end{cases}$$

What are the two frequencies in Hz ($f = \omega/2\pi$) for iron, where the saturation magnetization is $M_0 = 1.7\,\mathrm{MA/m}$, subjected to a magnetic field of $H_0 = 400\,\mathrm{kA/m}$?

(b) Show that in component form, these equations are:

$$\begin{aligned} \frac{\partial M_{1x}}{\partial t} &= \omega_M H_{1y} - \omega_H M_{1y} - \alpha \frac{\partial M_{1y}}{\partial t} \\ \frac{\partial M_{1y}}{\partial t} &= -\omega_M H_{1x} + \omega_H M_{1x} + \alpha \frac{\partial M_{1x}}{\partial t} \end{aligned}$$

(c) To solve this system, start by assuming harmonic time-dependence and show that the time harmonic solution is

$$\begin{aligned} M_{1x}(\omega) &= \frac{\omega_M(\omega_H + \alpha \mathrm{j}\omega) H_{1x}(\omega) + \mathrm{j}\omega\omega_M H_{1y}(\omega)}{(\omega_H + \alpha\mathrm{j}\omega)^2 - \omega^2} \\ M_{1y}(\omega) &= \frac{-\mathrm{j}\omega\omega_M H_{1x}(\omega) + \omega_M(\omega_H + \alpha\mathrm{j}\omega) H_{1y}(\omega)}{(\omega_H + \alpha\mathrm{j}\omega)^2 - \omega^2} \end{aligned}$$

Also write this result in matrix form, that is, $\begin{pmatrix} M_{1x}(\omega) \\ M_{1y}(\omega) \end{pmatrix} = \begin{pmatrix} \chi_{xx}(\omega) & \chi_{xy}(\omega) \\ \chi_{yx}(\omega) & \chi_{yy}(\omega) \end{pmatrix} \begin{pmatrix} H_{1x}(\omega) \\ H_{1y}(\omega) \end{pmatrix}$ and identify the functions $\chi_{xx}(\omega)$, $\chi_{xy}(\omega)$ etc in the matrix.

(d) Show that this system can be written in vector form as

$$\boldsymbol{M}_1(\omega) = \frac{\omega_M}{(\omega_H + \alpha\mathrm{j}\omega)^2 - \omega^2} \Big((\omega_H + \alpha\mathrm{j}\omega) \boldsymbol{H}_1(\omega) - \mathrm{j}\omega \hat{\boldsymbol{z}} \times \boldsymbol{H}_1(\omega) \Big)$$

Hint: the cross product can be represented as $[\hat{\boldsymbol{z}} \times \boldsymbol{H}_1] = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_{1x} \\ H_{1y} \end{pmatrix}$.

(e) Instead of the representations $\boldsymbol{H}_1 = H_{1x}\hat{\boldsymbol{x}} + H_{1y}\hat{\boldsymbol{y}}$ and $\boldsymbol{M}_1 = M_{1x}\hat{\boldsymbol{x}} + M_{1y}\hat{\boldsymbol{y}}$ (corresponding to linear polarization), consider the representations $\boldsymbol{H}_1 = H_{1+}\frac{\hat{\boldsymbol{x}} - \mathrm{j}\hat{\boldsymbol{y}}}{\sqrt{2}} + H_{1-}\frac{\hat{\boldsymbol{x}} + \mathrm{j}\hat{\boldsymbol{y}}}{\sqrt{2}}$ and $\boldsymbol{M}_1 = M_{1+}\frac{\hat{\boldsymbol{x}} - \mathrm{j}\hat{\boldsymbol{y}}}{\sqrt{2}} + M_{1-}\frac{\hat{\boldsymbol{x}} + \mathrm{j}\hat{\boldsymbol{y}}}{\sqrt{2}}$ (corresponding to circular polarization representation). Determine the relation between $\boldsymbol{M}_1$ and $\boldsymbol{H}_1$ in this representation, *i.e.*, determine the matrix elements $\chi_{++}(\omega)$, $\chi_{+-}(\omega)$ etc in

$$\begin{pmatrix} M_{1+}(\omega) \\ M_{1-}(\omega) \end{pmatrix} = \begin{pmatrix} \chi_{++}(\omega) & \chi_{+-}(\omega) \\ \chi_{-+}(\omega) & \chi_{--}(\omega) \end{pmatrix} \begin{pmatrix} H_{1+}(\omega) \\ H_{1-}(\omega) \end{pmatrix}$$

Is there a significant difference with the results in (c)?

Hint: Start by evaluating the cross products $\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{x}} - \mathrm{j}\hat{\boldsymbol{y}})$ and $\hat{\boldsymbol{z}} \times (\hat{\boldsymbol{x}} + \mathrm{j}\hat{\boldsymbol{y}})$.