Outline

1. Introduction
2. Interface
3. Slab
4. Applications
5. Time-domain response
6. Cascading
7. Moving boundary
8. Conclusions
Key questions

- What is the reflection and transmission from a slab?
  - Interface between two materials.
  - Propagation inside a material.
- Can we reduce the reflection from a surface?
- How do the results change if the boundary is moving?
Some typical applications

- Anti-reflective coating
- Doppler shift
- Material measurements
- Radomes
Consider wave propagation along the \( z \) direction in an isotropic medium. The electric and magnetic fields are in the \( xy \)-plane. Choosing linear polarization \( E = E \hat{x} \) and \( H = H \hat{y} \), the fields can be represented as

\[
E = E_+ e^{-jkz} + E_- e^{jkz}
\]
\[
H = \frac{1}{\eta} E_+ e^{-jkz} - \frac{1}{\eta} E_- e^{jkz}
\]

or

\[
\begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1}{\eta} & -\frac{1}{\eta} \end{pmatrix} \begin{pmatrix} E_+ \\ E_- \end{pmatrix} \quad \Leftrightarrow \quad \begin{pmatrix} E_+ \\ E_- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & \eta \\ 1 & -\eta \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix}
\]

These are completely equivalent representations. The \([E, H]\) representation is continuous across interfaces, whereas \([E_+, E_-]\) is easy to propagate in the material.
Scattering at an interface

The matching condition at an interface can be expressed as

\[
\begin{pmatrix}
1 & 1 \\
\frac{1}{\eta} & -\frac{1}{\eta}
\end{pmatrix}
\begin{pmatrix}
E_+ \\
E_-
\end{pmatrix} = \begin{pmatrix}
E \\
H
\end{pmatrix} = \begin{pmatrix}
E' \\
H'
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
\frac{1}{\eta'} & -\frac{1}{\eta'}
\end{pmatrix}
\begin{pmatrix}
E'_+ \\
E'_-
\end{pmatrix}
\]

\[
\frac{(E_+)}{(E_-)} = \frac{1}{2} \begin{pmatrix}
1 & \eta \\
1 & -\eta
\end{pmatrix}
\begin{pmatrix}
\frac{1}{\eta} & 1 \\
\frac{1}{\eta'} & -\frac{1}{\eta'}
\end{pmatrix}
\begin{pmatrix}
E'_+ \\
E'_-
\end{pmatrix} = \frac{1}{\tau} \begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix}
\begin{pmatrix}
E'_+ \\
E'_-
\end{pmatrix}
\]

\[
\tau = \frac{2\eta'}{\eta' + \eta} = \eta' \tau', \quad \rho = \frac{\eta' - \eta}{\eta' + \eta} = -\rho'
\]
Why is $\rho$ the reflection coefficient?

We can choose a situation where one of the fields is zero:

With $E'_- = 0$, we have

\[
\begin{pmatrix} E_+ \\ E_- \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \begin{pmatrix} E'_+ \\ 0 \end{pmatrix} = \frac{1}{\tau} \begin{pmatrix} E'_+ \\ \rho E'_+ \end{pmatrix}
\Rightarrow \begin{cases} E'_+ = \tau E_+ \\ E_- = \frac{1}{\tau} \rho E'_+ = \rho E_+ \end{cases}
\]

This justifies $\tau$ for transmission coefficient and $\rho$ for reflection coefficient.
The **matching matrix** is given by (relating fields on either sides to each other)

\[
\begin{pmatrix}
E_+ \\
E_-
\end{pmatrix} = \frac{1}{\tau} \begin{pmatrix}
1 & \rho \\
\rho & 1
\end{pmatrix} \begin{pmatrix}
E'_+ \\
E'_-
\end{pmatrix}
\]

and the **scattering matrix** is (relating incident waves \(E_+\) and \(E'_-\) to scattered waves \(E'_+\) and \(E'_-\))

\[
\begin{pmatrix}
E'_+ \\
E'_-
\end{pmatrix} = \begin{pmatrix}
\tau & \rho' \\
\rho & \tau'
\end{pmatrix} \begin{pmatrix}
E_+ \\
E_-
\end{pmatrix}
\]

(Fig. 5.2.1 in Orfanidis)
Reflection at a distance from an interface

Assume we define the reflection coefficient at a different reference plane than at the interface (typical situation in measurements):

\[
\begin{align*}
E_+ & \quad \eta' \\
E_- & \quad \eta
\end{align*}
\]

\[
\begin{align*}
z = z_0 & \quad \Gamma(z_0) \\
\Gamma(0) & \quad z = 0
\end{align*}
\]

The reflection coefficient is

\[
\Gamma(z_0) = \frac{E_-(z_0)}{E_+(z_0)} = \frac{E_-(0)e^{jkz_0}}{E_+(0)e^{-jkz_0}} = \Gamma(0)e^{2jkz_0}
\]

and the impedance is

\[
Z(z_0) = \frac{E(z_0)}{H(z_0)} = \frac{E_+(0)e^{-jkz_0} + E_-(0)e^{jkz_0}}{\frac{1}{\eta}E_+(0)e^{-jkz_0} - \frac{1}{\eta}E_-(0)e^{jkz_0}} = \eta \frac{1 + \Gamma(0)e^{2jkz_0}}{1 - \Gamma(0)e^{2jkz_0}}
\]
Reflection at a distance from an interface

\[ \begin{align*}
E_+ & \quad \eta \quad \eta' \\
E_- & \quad k \quad k' \\
z = z_0 & \quad \Gamma(z_0) \\
z = 0 & \quad \Gamma(0)
\end{align*} \]

Taking into account that \( z_0 = -\ell \) implies

\[ \Gamma(z_0) = \frac{E_-(z_0)}{E_+(z_0)} = \frac{\Gamma(0)e^{2jkz_0}}{\Gamma(0)} = \Gamma(0)e^{-2jk\ell} \]

and the impedance is

\[ Z(z_0) = \eta \frac{1 + \Gamma(0)e^{2jkz_0}}{1 - \Gamma(0)e^{2jkz_0}} = \eta \frac{1 + \Gamma(0)e^{-2jk\ell}}{1 - \Gamma(0)e^{-2jk\ell}} \]
Reflection coefficient and impedance

The relation between reflection coefficient \( \Gamma(z) \) and impedance \( Z(z) \) is

\[
\Gamma(z) = \frac{Z(z) - \eta}{Z(z) + \eta} \quad \Leftrightarrow \quad Z(z) = \eta \frac{1 + \Gamma(z)}{1 - \Gamma(z)}
\]

and they are propagated as

\[
\Gamma(z) = \Gamma(0)e^{2\jmath kz}, \quad Z(z) = \eta \frac{Z(0) - \jmath \eta \tan kz}{\eta - \jmath Z(0) \tan kz}
\]

\( (E_+, E_-, \Gamma) \) Easy propagation, needs matching matrices at interfaces.

\( (E, H, Z) \) Easy to match at interfaces, but more complicated propagation.
Where are the parameters located?

$Z$ in right half plane and $\Gamma$ in unit circle

\[ Z = \eta \frac{1 + \Gamma}{1 - \Gamma} \iff \Gamma = \frac{Z - \eta}{Z + \eta} \]
A depiction of how $\text{Re}(Z/\eta) = \text{constant}$ (circles) and $\text{Im}(Z/\eta) = \text{constant}$ (circle sectors) maps on the complex reflection coefficient $\Gamma$ in the unit circle. Very common in RF electronics, more details in Section 10.14.
Reflected and transmitted power

Since tangential fields are continuous, so is the Poynting vector

\[ \mathcal{P} = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H}^* \} = \frac{1}{2} \text{Re} \{ \hat{x} \mathbf{E} \times \hat{y} \mathbf{H}^* \} = \hat{z} \frac{1}{2} \text{Re} \{ \mathbf{E} \mathbf{H}^* \} \]

Assuming \( E'_- = 0 \) and lossless left medium (\( \text{Im}(\eta) = 0 \)), we have

\[ \mathcal{P} = \frac{1}{2\eta} (1 - |\rho|^2) |E_+|^2 = \frac{1}{2} \text{Re} \left( \frac{1}{\eta'} \right) |\tau|^2 |E_+|^2 \]
Reflected and transmitted power

Identify the incident, reflected and transmitted power as

\[
\mathcal{P}_{\text{in}} = \frac{1}{2\eta} |E_+|^2
\]

\[
\mathcal{P}_{\text{ref}} = \frac{1}{2\eta} |E_-|^2 = \frac{1}{2\eta} |\rho|^2 |E_+|^2 = |\rho|^2 \mathcal{P}_{\text{in}}
\]

\[
\mathcal{P}_{\text{tr}} = \frac{1}{2} \text{Re} \left( \frac{1}{\eta'} \right) |E'_+|^2 = \frac{1}{2} \text{Re} \left( \frac{1}{\eta'} \right) |\tau|^2 |E_+|^2 = \text{Re} \left( \frac{\eta}{\eta'} \right) |\tau|^2 \mathcal{P}_{\text{in}}
\]

Reflectance: \[ R = \frac{\mathcal{P}_{\text{ref}}}{\mathcal{P}_{\text{in}}} = |\rho|^2 \]

Transmittance: \[ T = \frac{\mathcal{P}_{\text{tr}}}{\mathcal{P}_{\text{in}}} = \text{Re} \left( \frac{\eta}{\eta'} \right) |\tau|^2 = 1 - |\rho|^2 \]

Some example numbers (left material being air):

<table>
<thead>
<tr>
<th>Material</th>
<th>( \rho )</th>
<th>( \tau )</th>
<th>( R )</th>
<th>( T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Glass ((n \approx 1.5))</td>
<td>(-0.2)</td>
<td>0.8</td>
<td>0.04</td>
<td>0.96</td>
</tr>
<tr>
<td>Copper ((1 \text{ GHz}))</td>
<td>(-1.0 + j4.4 \cdot 10^{-5})</td>
<td>((1 + j)4.4 \cdot 10^{-5})</td>
<td>1.0</td>
<td>8.8 \cdot 10^{-5}</td>
</tr>
<tr>
<td>Copper ((600 \text{ THz}))</td>
<td>(-0.97 + j0.033)</td>
<td>0.034 + j0.033</td>
<td>0.93</td>
<td>0.066</td>
</tr>
</tbody>
</table>
What if the left medium is not lossless?

If the left medium is not lossless, we have

\[
2 \text{Re}\{EH^*\} = (E_+ + \rho E_+)[\frac{1}{\eta}(E_+ - \rho E_+)]^* + (E_+ + \rho E_+)[\frac{1}{\eta}(E_+ - \rho E_+)]
\]

\[
= \left[ (1 + \rho)(1 - \rho^*) \frac{1}{\eta^*} + (1 + \rho^*)(1 - \rho) \frac{1}{\eta} \right] |E_+|^2
\]

\[
= \left[ (1 - |\rho|^2 + 2j \text{Im}(\rho)) \frac{1}{\eta^*} + (1 - |\rho|^2 - 2j \text{Im}(\rho)) \frac{1}{\eta} \right] |E_+|^2
\]

\[
= (1 - |\rho|^2)2 \text{Re} \left( \frac{1}{\eta} \right) |E_+|^2 + 4 \text{Im}(\rho) \text{Im} \left( \frac{1}{\eta} \right) |E_+|^2
\]

The power in the left medium is then

\[
\mathcal{P} = \frac{1}{2} \text{Re}\{EH^*\} = (1 - |\rho|^2) \frac{1}{2} \text{Re} \left( \frac{1}{\eta} \right) |E_+|^2 + \text{Im}(\rho) \text{Im} \left( \frac{1}{\eta} \right) |E_+|^2
\]

\[
= \mathcal{P}_{\text{in}} - \mathcal{P}_{\text{ref}} \quad \text{cross term}
\]

Thus, the power in a lossy medium does not decouple in a pure forward-going and backward-going power, \( \mathcal{P} \neq \mathcal{P}_{\text{in}} - \mathcal{P}_{\text{ref}} \! \)
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Propagator of total fields

The relation between total fields and forward/backward fields is

\[
\begin{pmatrix}
E \\
H
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
\frac{1}{\eta} & -\frac{1}{\eta}
\end{pmatrix}
\begin{pmatrix}
E_+ \\
E_-
\end{pmatrix}
\iff
\begin{pmatrix}
E_+ \\
E_-
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
1 & \eta \\
1 & -\eta
\end{pmatrix}
\begin{pmatrix}
E \\
H
\end{pmatrix}
\]

Thus, the propagation of total fields can be written

\[
\begin{pmatrix}
E(z) \\
H(z)
\end{pmatrix}
= \begin{pmatrix}
1 & 1 \\
\frac{1}{\eta} & -\frac{1}{\eta}
\end{pmatrix}
\begin{pmatrix}
e^{-jkz} & 0 \\
0 & e^{jkz}
\end{pmatrix}
\frac{1}{2}
\begin{pmatrix}
1 & \eta \\
1 & -\eta
\end{pmatrix}
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix}
\]

\[
= \cdots = \begin{pmatrix}
\cos(kz) & -j\eta \sin(kz) \\
-j\eta^{-1} \sin(kz) & \cos(kz)
\end{pmatrix}
\begin{pmatrix}
E(0) \\
H(0)
\end{pmatrix}
\]

propagator matrix

With \(z_1 = -\ell\) and \(z_2 = 0\) (and hence \(\ell = z_2 - z_1\)), this is

\[
\begin{pmatrix}
E(z_1) \\
H(z_1)
\end{pmatrix}
= \begin{pmatrix}
\cos(k\ell) & j\eta \sin(k\ell) \\
\eta^{-1} \sin(k\ell) & \cos(k\ell)
\end{pmatrix}
\begin{pmatrix}
E(z_2) \\
H(z_2)
\end{pmatrix}
\]
Scattering against a slab

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix}
= \frac{1}{2} \begin{bmatrix}
1 & \eta_a \\
1 & -\eta_a
\end{bmatrix}
\begin{bmatrix}
\cos(k_1 \ell_1) & j \eta_1 \sin(k_1 \ell_1) \\
1/\eta_1^{-1} \sin(k_1 \ell_1) & \cos(k_1 \ell_1)
\end{bmatrix}
\begin{bmatrix}
1/\eta_b & 1/\eta_b \\
1/\eta_b & -1/\eta_b
\end{bmatrix}
\begin{bmatrix}
E'_{2+} \\
E'_{2-}
\end{bmatrix}
\]

This provides the transfer matrix

\[
\begin{bmatrix}
E_{1+} \\
E_{1-}
\end{bmatrix}
= \begin{bmatrix}
\mathcal{T}_{11} & \mathcal{T}_{12} \\
\mathcal{T}_{21} & \mathcal{T}_{22}
\end{bmatrix}
\begin{bmatrix}
E'_{2+} \\
E'_{2-}
\end{bmatrix}
\]
Scattering against a slab, alternative

\[ \begin{pmatrix} E_{1+} \\ E_{1-} \end{pmatrix} = \frac{1}{\tau_1} \begin{pmatrix} 1 & \rho_1 \\ \rho_1 & 1 \end{pmatrix} \begin{pmatrix} e^{jk_1\ell_1} & 0 \\ 0 & e^{-jk_1\ell_1} \end{pmatrix} \frac{1}{\tau_2} \begin{pmatrix} 1 & \rho_2 \\ \rho_2 & 1 \end{pmatrix} \begin{pmatrix} E'_{2+} \\ E'_{2-} \end{pmatrix} \]

\[ \begin{array}{c} \text{matching matrix} \\ \text{propagator matrix} \\ \text{split fields at interface 2 (left)} \\ \text{split fields at interface 1 (right)} \end{array} \]

This provides exactly the same transfer matrix

\[ \begin{pmatrix} E_{1+} \\ E_{1-} \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{11} & \mathcal{T}_{12} \\ \mathcal{T}_{21} & \mathcal{T}_{22} \end{pmatrix} \begin{pmatrix} E'_{2+} \\ E'_{2-} \end{pmatrix} \]
Scattering matrix

By rearranging the variables, we define the scattering matrix as

\[
\begin{pmatrix}
E_1^- \\
E'_2^+
\end{pmatrix} = \begin{pmatrix}
\Gamma_{aa} & T_{ab} \\
T_{ba} & \Gamma_{bb}
\end{pmatrix} \begin{pmatrix}
E_1^+ \\
E'_2^-
\end{pmatrix}
\]

The total reflection and transmission coefficients for a wave incident from the left are \( \Gamma_{aa} = \Gamma_1 \) and \( T_{ba} = \mathcal{T} \), where (see Orfanidis for derivation)

\[
\Gamma_1 = \frac{E_1^-}{E_1^+} = \frac{\rho_1 + \rho_2 e^{-2j k_1 \ell_1}}{1 + \rho_1 \rho_2 e^{-2j k_1 \ell_1}}, \quad \mathcal{T} = \frac{E'_2^+}{E_1^+} = \frac{\tau_1 \tau_2 e^{-j k_1 \ell_1}}{1 + \rho_1 \rho_2 e^{-2j k_1 \ell_1}}
\]

If the slab is non-reciprocal (for instance a magnetized plasma), the scattering matrix is not necessarily symmetric. Usually requires taking care of cross-polarization terms as well.
Generalization to bianisotropic materials

For a slab of general bianisotropic material, we can write

$$\begin{pmatrix}
E(z_1) \\
H(z_1) \times \hat{z}
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \cdot \begin{pmatrix}
E(z_2) \\
H(z_2) \times \hat{z}
\end{pmatrix}$$

In the surrounding free space, we have

$$E_\pm = \pm Z_0 \cdot (H_\pm \times \hat{z}),$$

where $Z_0$ is the wave impedance dyadic. This implies

$$\begin{pmatrix}
(I + r) \cdot E_{1+} \\
Z_0^{-1} \cdot (I - r) \cdot E_{1+}
\end{pmatrix} = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix} \cdot \begin{pmatrix}
t \cdot E_{1+} \\
Z_0^{-1} \cdot t \cdot E_{1+}
\end{pmatrix}$$

Solving for the reflection and transmission dyadics implies

$$r = (A + B \cdot Z_0^{-1} - Z_0 \cdot C - Z_0 \cdot D \cdot Z_0^{-1}) \cdot (A + B \cdot Z_0^{-1} + Z_0 \cdot C + Z_0 \cdot D \cdot Z_0^{-1})^{-1}$$

$$t = 2(A + B \cdot Z_0^{-1} + Z_0 \cdot C + Z_0 \cdot D \cdot Z_0^{-1})^{-1}$$

This result includes co- and cross-polarization, and oblique incidence. It all boils down to computing the propagator $\left(\begin{array}{cc} A & B \\ C & D \end{array}\right)$!
Material measurements

By measuring $\Gamma_1$ and $\mathcal{T}$, the material parameters $\epsilon$ and $\mu$ can be determined by inverting the formulas

$$\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-2jk_1\ell_1}}{1 + \rho_1 \rho_2 e^{-2jk_1\ell_1}}, \quad \mathcal{T} = \frac{\tau_1 \tau_2 e^{-jk_1\ell_1}}{1 + \rho_1 \rho_2 e^{-2jk_1\ell_1}}$$

$$\rho_{1,2} = \rho_{1,2}(\epsilon, \mu), \quad \tau_{1,2} = \tau_{1,2}(\epsilon, \mu), \quad k_1 = k_1(\epsilon, \mu)$$
If the surrounding medium is air on both sides, we have

\[
\rho = \rho_1 = -\rho_2 = \frac{\eta - \eta_0}{\eta + \eta_0}
\]

\[
\tau_1 \tau_2 = (1 + \rho_1)(1 + \rho_2) = 1 - \rho^2
\]

\[
\Gamma_1 = \frac{\rho_1 + \rho_2 e^{-2jk_1\ell_1}}{1 + \rho_1 \rho_2 e^{-2jk_1\ell_1}} = \frac{(1 - P^2)\rho}{1 - \rho^2 P^2}
\]

\[
\mathcal{T} = \frac{\tau_1 \tau_2 e^{-jk_1\ell_1}}{1 + \rho_1 \rho_2 e^{-2jk_1\ell_1}} = \frac{(1 - \rho^2)P}{1 - \rho^2 P^2}
\]

Solving for \(\rho = \frac{\eta - \eta_0}{\eta + \eta_0}\) and \(P = e^{-jk_1\ell_1}\) it is found

\[
\rho = K \pm \sqrt{K^2 - 1}, \quad \text{where} \quad K = \frac{\Gamma_1^2 - \mathcal{T}^2 + 1}{2\Gamma_1}
\]

\[
P = \frac{\Gamma_1 + \mathcal{T} - \rho}{1 - (\Gamma_1 + \mathcal{T})\rho}, \quad (\rho, P) \Rightarrow (\eta, k_1) \Rightarrow (\epsilon, \mu)
\]
Typical behavior (X-band waveguide setup)
Reflectionless slab

There is no reflection if $\Gamma = 0$, that is,

$$\frac{\rho_1 + \rho_2 e^{-2j k_1 \ell_1}}{1 + \rho_1 \rho_2 e^{-2j k_1 \ell_1}} = 0 \quad \Rightarrow \quad e^{2j k_1 \ell_1} = -\frac{\rho_2}{\rho_1}$$

For lossless media (where $\rho_{1,2}$ are real), this can be true only in the following cases:

- $e^{2j k_1 \ell_1} = 1$ \hspace{1cm} $\rho_2 = -\rho_1$ \hspace{1cm} (half wavelength)
- $e^{2j k_1 \ell_1} = -1$ \hspace{1cm} $\rho_2 = \rho_1$ \hspace{1cm} (quarter wavelength)

With

$$\rho_1 = \frac{\eta_1 - \eta_a}{\eta_1 + \eta_a} \quad \text{and} \quad \rho_2 = \frac{\eta_b - \eta_1}{\eta_b + \eta_1}$$

we have

$$\rho_2 = -\rho_1 \quad \Rightarrow \quad \eta_a = \eta_b$$

$$\rho_2 = \rho_1 \quad \Rightarrow \quad \eta_1 = \sqrt{\eta_a \eta_b}$$
Reflectionless slab

Maximum reflection \( |\Gamma_1|^2_{\text{max}} = \frac{4\rho_1^2}{(1+\rho_1^2)^2} \), bandwidth \( \frac{\Delta \omega}{\omega_0} \approx \frac{1-\rho_1^2}{\pi} \).
Half wave radome

An antenna may need some physical shielding from wear and tear.

With a dielectric wall of thickness $n_1 \ell_1 = \lambda_0 / 2$, we have perfect transmission at normal incidence at the center wavelength.
Antireflective coating

The reflection against a glass surface (refractive index $n = 1.5$) is

$$\rho = \frac{1 - 1.5}{1 + 1.5} = -0.2 \quad \Rightarrow \quad \rho^2 = 4\%$$

How can we reduce this reflection? With a quarter wavelength slab of refractive index

$$n_1 = \sqrt{n_a n_b} = \sqrt{1.5} = 1.22$$

the reflection is zero at the design wavelength. But maybe the closest available material has $n_1 = 1.38$.

(Fig. 5.5.3 in Orfanidis)
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Slab scattering in the time domain

The impulse response of a lossless slab can be written

\[ \Gamma_1(t) = \rho_1 \delta(t) + \sum_{n=1}^{\infty} \tau_1 \tau'_1 (\rho'_1)^{n-1} \rho_2^n \delta(t - nT) \]

(Fig. 5.6.1 in Orfanidis)
To obtain the reflected field for a general time-dependence, convolve with the impulse response:

\[
E_{1-}(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Gamma_1(\omega) E_{1+}(\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \Gamma_1(t') E_{1+}(t-t') dt'
\]

\[
= \rho_1 E_{1+}(t) + \sum_{n=1}^{\infty} \tau_1 \tau_1' (\rho_1')^{n-1} \rho_2^n E_{1+}(t - nT')
\]

For causal waveforms, where \( E_{1+}(t) = 0 \) if \( t < t_0 \), the sum is finite for each finite \( t \).
Demo
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Two slabs

Assume several slabs are stacked on each other

The scattering is given by

\[
\begin{pmatrix}
E_{1+} \\
E_{1-}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & \eta_a \\
1 & -\eta_a
\end{pmatrix} \begin{pmatrix}
\cos(k_1 \ell_1) & j \eta_1 \sin(k_1 \ell_1) \\
j \eta_1^{-1} \sin(k_1 \ell_1) & \cos(k_1 \ell_1)
\end{pmatrix} \begin{pmatrix}
\cos(k_2 \ell_2) & j \eta_2 \sin(k_2 \ell_2) \\
j \eta_2^{-1} \sin(k_2 \ell_2) & \cos(k_2 \ell_2)
\end{pmatrix} \begin{pmatrix}
\frac{1}{\eta_b} & 1 \\
1 & -\frac{1}{\eta_b}
\end{pmatrix} \begin{pmatrix}
E_{3+}' \\
E_{3-}'
\end{pmatrix}
\]
Cascading

Stacking $N$ slabs on each other, simply means

$$
\begin{pmatrix}
E_{a+} \\
E_{a-}
\end{pmatrix}
= \frac{1}{2}
\begin{pmatrix}
1 & \eta_a \\
1 & -\eta_a
\end{pmatrix}
P_1 \cdots P_N
\begin{pmatrix}
\frac{1}{\eta_a} & -\frac{1}{\eta_a} \\
\frac{1}{\eta_b} & -\frac{1}{\eta_b}
\end{pmatrix}
\begin{pmatrix}
E_{b+} \\
E_{b-}
\end{pmatrix}
$$

where $P_n$ is the propagator matrix for layer $n$. Thus, any layered structure can be analyzed in the same way as the single slab. In order to extract the reflection and transmission coefficients, the transfer matrix can be put in the form of a scattering matrix

$$
\begin{pmatrix}
E_{a-} \\
E_{b+}
\end{pmatrix}
= \begin{pmatrix}
\Gamma_{aa} & T_{ab} \\
T_{ba} & \Gamma_{bb}
\end{pmatrix}
\begin{pmatrix}
E_{a+} \\
E_{b-}
\end{pmatrix}
$$

Note that the procedure is completely analogous for bianisotropic materials. Topic of next lecture!
The following material is not central to the course, it is provided to help you see the broad lines in Orfanidis 5.8.
Strategy: Solve scattering equations in the moving frame, where the boundary is stationary. Then transform back to the stationary frame.
Sketch of analysis

The detailed calculations are in Orfanidis, only a brief outline is given here. In the fixed frame, the fields are written

\[
\begin{align*}
E_x &= E_i e^{i(\omega t - k_i z)} + E_r e^{i(\omega_r t - k_r z)} \\
H_x &= H_i e^{i(\omega t - k_i z)} - H_r e^{i(\omega_r t - k_r z)}
\end{align*}
\]

\[
\begin{align*}
E_x &= E_t e^{i(\omega t - k_t z)} \\
H_x &= H_t e^{i(\omega t - k_t z)}
\end{align*}
\]

The phases are the same in both frames (see Appendix H)

\[
\begin{align*}
\phi_i &= \omega t - k_i z = \omega' t' - k_i' z' = \phi_i' \\
\phi_r &= \omega_r t - k_r z = \omega' t' - k_r' z' = \phi_r' \\
\phi_t &= \omega t - k_t z = \omega' t' - k_t' z' = \phi_t'
\end{align*}
\]

The frequencies and wavenumbers transform according to (Lorentz transformation, Appendix H)

\[
\omega' = \gamma(\omega - \beta c_0 k_t), \quad k_t' = \gamma(k_t - \beta \omega / c_0)
\]

where \( \gamma = 1 / \sqrt{1 - \beta^2} \) and \( \beta = v / c_0 \).
Results

The transformed frequencies are

\[ \omega_r = \omega \frac{1 - \beta}{1 + \beta} \approx \omega \left( 1 - 2 \frac{v}{c_0} \right), \quad \omega_t = \omega \frac{1 + \beta n}{1 + \beta} \approx \omega \left( 1 + (n - 1) \frac{v}{c_0} \right) \]

with phase velocities

\[ v_r = \frac{\omega_r}{k_r} = c_0, \quad v_t = \frac{\omega_t}{k_t} = c_0 \frac{1 + \beta n}{n + \beta} \approx \frac{c_0}{n} + v \left( 1 - \frac{1}{n^2} \right) \]

The scattering coefficients are modified as

\[ \frac{E_r}{E_i} = \rho \frac{1 - \beta}{1 + \beta}, \quad \frac{E_t}{E_i} = \tau \frac{1 + \beta n}{1 + \beta}, \]

Fresnel drag
Conclusions

- The scattering problem boils down to
  - Interface matching (continuous tangential fields)
  - Propagation (exponential factors for forward/backward fields)
- A half wavelength slab has perfect transmission if $\eta_a = \eta_b$.
- A quarter wavelength slab is a perfect transformer if $\eta = \sqrt{\eta_a \eta_b}$.
- Stacked slabs can be treated by cascading.
- A moving slab implies frequency shifts proportional to relative velocity.