Electromagnetic Wave Propagation
Lecture 3: Plane waves in isotropic and bianisotropic media

Daniel Sjöberg

Department of Electrical and Information Technology
1. **Plane waves in lossless media**
   - General time dependence
   - Time harmonic waves

2. **Polarization**

3. **Wave propagation in bianisotropic media**

4. **Interpretation of the fundamental equation**
1 Plane waves in lossless media
   General time dependence
   Time harmonic waves

2 Polarization

3 Wave propagation in bianisotropic media

4 Interpretation of the fundamental equation
In this lecture, we dive a bit deeper into the familiar right-hand rule, $\hat{x} = \hat{y} \times \hat{z}$.

This is the building block for most of the course.
1 Plane waves in lossless media
   General time dependence
   Time harmonic waves

2 Polarization

3 Wave propagation in bianisotropic media

4 Interpretation of the fundamental equation
The electromagnetic field is assumed to depend only on one coordinate, \( z \). This implies

\[
E(x, y, z, t) = E(z, t) \\
D(x, y, z, t) = \varepsilon E(z, t) \\
H(x, y, z, t) = H(z, t) \\
B(x, y, z, t) = \mu H(z, t)
\]

and by using \( \nabla = \hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \rightarrow \hat{z} \frac{\partial}{\partial z} \) we have

\[
\nabla \times E = -\frac{\partial B}{\partial t} \\
\nabla \times H = \frac{\partial D}{\partial t} \\
\n\nabla \cdot D = 0 \\
\n\nabla \cdot B = 0
\]

\[
\hat{z} \times \frac{\partial E}{\partial z} = -\mu \frac{\partial H}{\partial t} \\
\hat{z} \times \frac{\partial H}{\partial z} = \varepsilon \frac{\partial E}{\partial t} \\
\frac{\partial E_z}{\partial z} = 0 \quad \Rightarrow \quad E_z = 0 \\
\frac{\partial H_z}{\partial z} = 0 \quad \Rightarrow \quad H_z = 0
\]
Rewriting the equations

The electromagnetic field has only $x$ and $y$ components, and satisfy

$$\hat{z} \times \frac{\partial E}{\partial z} = -\mu \frac{\partial H}{\partial t}$$

$$\hat{z} \times \frac{\partial H}{\partial z} = \epsilon \frac{\partial E}{\partial t}$$

Using $(\hat{z} \times F) \times \hat{z} = F$ for any vector $F$ orthogonal to $\hat{z}$, and $c = 1/\sqrt{\epsilon \mu}$ and $\eta = \sqrt{\mu/\epsilon}$, we can write this as

$$\frac{\partial}{\partial z} E = -\frac{1}{c} \frac{\partial}{\partial t} (\eta H \times \hat{z})$$

$$\frac{\partial}{\partial z} (\eta H \times \hat{z}) = -\frac{1}{c} \frac{\partial}{\partial t} E$$

which is a symmetric hyperbolic system (note $E$ and $\eta H$ have the same units)

$$\frac{\partial}{\partial z} \left( \begin{array}{c} E \\ \eta H \times \hat{z} \end{array} \right) = -\frac{1}{c} \frac{\partial}{\partial t} \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \left( \begin{array}{c} E \\ \eta H \times \hat{z} \end{array} \right)$$
Wave splitting

The change of variables

\[
\begin{pmatrix}
E_+ \\
E_-
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
E + \eta H \times \hat{z} \\
E - \eta H \times \hat{z}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
E \\
\eta H \times \hat{z}
\end{pmatrix}
\]

with inverse

\[
\begin{pmatrix}
E \\
\eta H \times \hat{z}
\end{pmatrix} = \begin{pmatrix}
E_+ + E_- \\
E_+ - E_-
\end{pmatrix} = \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
E_+ \\
E_-
\end{pmatrix}
\]

is called a wave splitting and diagonalizes the system to

\[
\begin{align*}
\frac{\partial E_+}{\partial z} &= -\frac{1}{c} \frac{\partial E_+}{\partial t} \\
&\Rightarrow \\
E_+(z, t) &= F(z - ct)
\end{align*}
\]

\[
\begin{align*}
\frac{\partial E_-}{\partial z} &= \frac{1}{c} \frac{\partial E_-}{\partial t} \\
&\Rightarrow \\
E_-(z, t) &= G(z + ct)
\end{align*}
\]
Forward and backward waves

The total fields can now be written

\[
E(z, t) = F(z - ct) + G(z + ct)
\]

\[
H(z, t) = \frac{1}{\eta} \mathbf{\hat{z}} \times \left( F(z - ct) - G(z + ct) \right)
\]

A graphical interpretation of the forward and backward waves is

(Fig. 2.1.1 in Orfanidis)
Energy density in one single wave

A wave propagating in positive $z$ direction satisfies $\mathbf{H} = \frac{1}{\eta} \mathbf{\hat{z}} \times \mathbf{E}$

$x$: Electric field  
$y$: Magnetic field  
$z$: Propagation direction

The Poynting vector and energy densities are (using the BAC-CAB rule $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$)

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = \mathbf{E} \times \left( \frac{1}{\eta} \mathbf{\hat{z}} \times \mathbf{E} \right) = \frac{1}{\eta} \mathbf{\hat{z}} |\mathbf{E}|^2$$

$$w_e = \frac{1}{2} \epsilon |\mathbf{E}|^2$$

$$w_m = \frac{1}{2} \mu |\mathbf{H}|^2 = \frac{1}{2} \mu \frac{1}{\eta^2} |\mathbf{\hat{z}} \times \mathbf{E}|^2 = \frac{1}{2} \epsilon |\mathbf{E}|^2 = w_e$$

Thus, the wave carries equal amounts of electric and magnetic energy.
Energy density in forward and backward wave

If the wave propagates in negative $z$ direction, we have

$$H = \frac{1}{\eta}(-\hat{z}) \times E$$

and

$$P = E \times H = \frac{1}{\eta} E \times (-\hat{z} \times E) = -\frac{1}{\eta} \hat{z} |E|^2$$

Thus, when the wave consists of one forward wave $F(z - ct)$ and one backward wave $G(z + ct)$, we have

$$P = E \times H = \frac{1}{\eta} \hat{z} \left(|F|^2 - |G|^2\right)$$

$$w = \frac{1}{2} \epsilon |E|^2 + \frac{1}{2} \mu |H|^2 = \epsilon |F|^2 + \epsilon |G|^2$$
Outline

1 Plane waves in lossless media
   General time dependence
   Time harmonic waves

2 Polarization

3 Wave propagation in bianisotropic media

4 Interpretation of the fundamental equation
Time harmonic waves

We assume harmonic time dependence

\[ E(x, y, z, t) = E(z) e^{j\omega t} \]

\[ H(x, y, z, t) = H(z) e^{j\omega t} \]

Using the same wave splitting as before implies

\[ \frac{\partial E_{\pm}}{\partial z} = \mp \frac{1}{c} \frac{\partial E_{\pm}}{\partial t} = \mp \frac{j\omega}{c} E_{\pm} \Rightarrow E_{\pm}(z) = E_{0\pm} e^{\mp jkz} \]

where \( k = \omega/c \) is the wave number in the medium. Thus, the general solution for time harmonic waves is

\[ E(z) = E_{0+} e^{-jkz} + E_{0-} e^{jkz} \]

\[ H(z) = H_{0+} e^{-jkz} + H_{0-} e^{jkz} = \frac{1}{\eta} \hat{z} \times \left( E_{0+} e^{-jkz} - E_{0-} e^{jkz} \right) \]

The triples \( \{ E_{0+}, H_{0+}, \hat{z} \} \) and \( \{ E_{0-}, H_{0-}, -\hat{z} \} \) are right-handed systems.
A time harmonic wave propagating in the forward $z$ direction has the space-time dependence $E(z, t) = E_0 e^{j(\omega t - kz)}$ and $H(z, t) = \frac{1}{\eta} \hat{z} \times E(z, t)$.

(Fig. 2.2.1 in Orfanidis)

The wavelength corresponds to the spatial periodicity according to $e^{-jk(z+\lambda)} = e^{-jkz}$, meaning $k\lambda = 2\pi$ or

$$\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}$$
Refractive index

The wavelength is often compared to the corresponding wavelength in vacuum

$$\lambda_0 = \frac{c_0}{f}$$

The refractive index is

$$n = \frac{\lambda_0}{\lambda} = \frac{k}{k_0} = \frac{c_0}{c} = \sqrt{\frac{\varepsilon \mu}{\varepsilon_0 \mu_0}}$$

**Important special case:** non-magnetic media, where $$\mu = \mu_0$$ and $$n = \sqrt{\varepsilon/\varepsilon_0}$$.  

$$c = \frac{c_0}{n}, \quad \eta = \frac{\eta_0}{n}, \quad \lambda = \frac{\lambda_0}{n}, \quad k = nk_0$$

**Only for $$\mu = \mu_0$$!**

**Note:** We use $$c_0$$ for the speed of light in vacuum and $$c$$ for the speed of light in a medium, even though $$c$$ is the standard for the speed of light in vacuum!
EMANIM program
Energy density and power flow

For the general time harmonic solution

\[ E(z) = E_0^+ e^{-jkz} + E_0^- e^{jkz} \]

\[ H(z) = \frac{1}{\eta} \hat{z} \times \left( E_0^+ e^{-jkz} - E_0^- e^{jkz} \right) \]

the Poynting vector and energy density are

\[ \mathcal{P} = \frac{1}{2} \text{Re} \left[ E(z) \times H^*(z) \right] = \hat{z} \left( \frac{1}{2\eta} |E_0^+|^2 - \frac{1}{2\eta} |E_0^-|^2 \right) \]

\[ w = \frac{1}{4} \text{Re} \left[ \varepsilon E(z) \cdot E^*(z) + \mu H(z) \cdot H^*(z) \right] = \frac{1}{2} \varepsilon |E_0^+|^2 + \frac{1}{2} \varepsilon |E_0^-|^2 \]
Wave impedance

For forward and backward waves, the impedance $\eta = \sqrt{\mu/\epsilon}$ relates the electric and magnetic field strengths to each other.

$$E_\pm = \pm \eta H_\pm \times \hat{z}$$

When both forward and backward waves are present this is generalized as (in component form)

$$Z_x(z) = \frac{[E(z)]_x}{[H(z) \times \hat{z}]_x} = \frac{E_x(z)}{H_y(z)} = \eta \frac{E_{0+x}e^{-jkz} + E_{0-x}e^{jkz}}{E_{0+x}e^{-jkz} - E_{0-x}e^{jkz}}$$

$$Z_y(z) = \frac{[E(z)]_y}{[H(z) \times \hat{z}]_y} = \frac{E_y(z)}{-H_x(z)} = \eta \frac{E_{0+y}e^{-jkz} + E_{0-y}e^{jkz}}{E_{0+y}e^{-jkz} - E_{0-y}e^{jkz}}$$

Thus, the wave impedance is in general a non-trivial function of $z$. This will be used as a means of analysis and design in the course.
Material and wave parameters

We note that we have two material parameters

**Permittivity** \( \epsilon \), defined by \( D = \epsilon E \).

**Permeability** \( \mu \), defined by \( B = \mu H \).

But the waves are described by the wave parameters

**Wave number** \( k \), defined by \( k = \omega \sqrt{\epsilon \mu} \).

**Wave impedance** \( \eta \), defined by \( \eta = \sqrt{\mu / \epsilon} \).

This means that in a scattering experiment, where we measure wave effects, we primarily get information on \( k \) and \( \eta \), not \( \epsilon \) and \( \mu \). In order to get material data, a theoretical material model must be applied.

Often, wave number is given by transmission data (phase delay), and wave impedance is given by reflection data (impedance mismatch). More on this in future lectures!
Outline

1 Plane waves in lossless media
   General time dependence
   Time harmonic waves

2 Polarization

3 Wave propagation in bianisotropic media

4 Interpretation of the fundamental equation
Why care about different polarizations?

- Different materials react differently to different polarizations.
- Linear polarization is sometimes not the most natural.
- For propagation through the ionosphere (to satellites), or through magnetized media, often circular polarization is natural.
Complex vectors

The time dependence of the electric field is

$$\mathbf{E}(t) = \text{Re}\{E_0 e^{j\omega t}\}$$

where the complex amplitude can be written

$$E_0 = \hat{x} E_{0x} + \hat{y} E_{0y} + \hat{z} E_{0z} = \hat{x} |E_{0x}| e^{j\alpha} + \hat{y} |E_{0y}| e^{j\beta} + \hat{z} |E_{0z}| e^{j\gamma} = E_{0r} + j E_{0i}$$

where $E_{0r}$ and $E_{0i}$ are real-valued vectors. We then have

$$\mathbf{E}(t) = \text{Re}\{(E_{0r} + j E_{0i}) e^{j\omega t}\} = E_{0r} \cos(\omega t) - E_{0i} \sin(\omega t)$$

This lies in a plane with normal

$$\hat{n} = \pm \frac{E_{0r} \times E_{0i}}{|E_{0r} \times E_{0i}|}$$
Linear polarization

The simplest case is given by $E_{0i} = 0$, implying $E(t) = E_{0r} \cos \omega t$. 

![Diagram showing linear polarization with vector $E_{0r}$ along the $x$-axis and vector $E_{0y}$ along the $y$-axis.]
Circular polarization

Assume $E_{0r} = \hat{x}$ and $E_{0i} = -\hat{y}$. The total field $E(t) = \hat{x} \cos \omega t + \hat{y} \sin \omega t$ then rotates in the $xy$-plane:

The rotation direction needs to be related to the propagation direction to determine left or right.
IEEE definition of left and right

With your right hand thumb in the propagation direction and fingers in rotation direction: right hand circular.

(right-polarized forward-moving)

(left-polarized backward-moving)

(right-polarized backward-moving)

(left-polarized forward-moving)

(Fig. 2.5.1 in Orfanidis)
Elliptical polarization

The general polarization state is elliptical

The direction $\hat{e}$ is parallel to the Poynting vector (the power flow).
Classification of polarization

The following classification can be given ($\hat{e}$ is the propagation direction):

<table>
<thead>
<tr>
<th>$-j\hat{e} \cdot (E_0 \times E_0^*)$</th>
<th>Polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 0$</td>
<td>Linear polarization</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>Right handed elliptic polarization</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>Left handed elliptic polarization</td>
</tr>
</tbody>
</table>

Further, circular polarization is characterized by $E_0 \cdot E_0 = 0$. Typical examples:

**Linear:** $E_0 = \hat{x}$ or $E_0 = \hat{y}$.

**Circular:** $E_0 = \hat{x} - j\hat{y}$ (right handed for $\hat{e} = \hat{z}$) or $E_0 = \hat{x} + j\hat{y}$ (left handed for $\hat{e} = \hat{z}$).

See the literature for more in depth descriptions.
Alternative bases in the plane

To describe an arbitrary vector in the $xy$-plane, the unit vectors

\[ \hat{x} \quad \text{and} \quad \hat{y} \]

are usually used. However, we could just as well use the RCP and LCP vectors

\[ \hat{x} - j\hat{y} \quad \text{and} \quad \hat{x} + j\hat{y} \]

Sometimes the linear basis is preferrable, sometimes the circular.
Outline

1. Plane waves in lossless media
   - General time dependence
   - Time harmonic waves

2. Polarization

3. Wave propagation in bianisotropic media

4. Interpretation of the fundamental equation
Observations

Some fundamental properties are observed from the isotropic case:

- The wave speed \( c = 1/\sqrt{\epsilon \mu} \) and wave impedance \( \eta = \sqrt{\mu/\epsilon} \) depend on the material properties.
- Waves can propagate in the positive or negative \( z \)-direction.
- For each propagation direction, there are two possible polarizations (\( \hat{x} \) and \( \hat{y} \), or RCP and LCP etc).

We will generalize this to bianisotropic materials, where

\[
\begin{pmatrix}
D(\omega) \\
B(\omega)
\end{pmatrix} =
\begin{pmatrix}
\epsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix}
\begin{pmatrix}
E(\omega) \\
H(\omega)
\end{pmatrix}
\]

See sections 1 and 2 in the book chapter “Circuit analogs for wave propagation in stratified structures” by Sjöberg. The rest of the paper is interesting but not essential to this course.
Wave propagation in general media

We now generalize to arbitrary materials in the frequency domain. Our plan is the following:

1. Write up the full Maxwell’s equations in the frequency domain.

2. Assume the dependence on $x$ and $y$ appear at most through a factor $e^{-j(k_x x + k_y y)}$.

3. Separate the transverse components ($x$ and $y$) from the $z$ components of the fields.

4. Eliminate the $z$ components.

5. Identify the resulting differential equation as a dynamic system for the transverse components,

$$\frac{\partial}{\partial z} \left( \begin{array}{c} E_t \\ H_t \times \hat{z} \end{array} \right) = -j \omega \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \cdot \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}$$
Maxwell’s equations in the frequency domain are

\[ \nabla \times \mathbf{H} = j\omega \mathbf{D} = j\omega \left[ \epsilon(\omega) \cdot \mathbf{E} + \xi(\omega) \cdot \mathbf{H} \right] \]

\[ \nabla \times \mathbf{E} = -j\omega \mathbf{B} = -j\omega \left[ \zeta(\omega) \cdot \mathbf{E} + \mu(\omega) \cdot \mathbf{H} \right] \]

The bianisotropic material is described by the dyadics \( \epsilon(\omega) \), \( \xi(\omega) \), \( \zeta(\omega) \), and \( \mu(\omega) \). The frequency dependence is suppressed in the following.
2) Transverse behavior

Assume that the fields depend on $x$ and $y$ only through a factor $e^{-j(k_x x + k_y y)}$ (corresponding to a Fourier transform in $x$ and $y$)

$$E(x, y, z) = E(z)e^{-jk_t \cdot r} \text{ where } k_t = k_x \hat{x} + k_y \hat{y}$$

Since $\frac{\partial}{\partial x} e^{-jk_x x} = -jk_x e^{-jk_x x}$, the action of the curl operator is then

$$\nabla \times E(x, y, z) = e^{-jk_t \cdot r} \left( -jk_t + \frac{\partial}{\partial z} \hat{z} \right) \times E(z)$$
3) Separate the components, curls

Split the fields according to

\[ E = E_t + \hat{z}E_z \]

where \( E_t \) is in the \( xy \)-plane, and \( \hat{z}E_z \) is the \( z \)-component. By expanding the curls, we find

\[
\left(-jk_t + \frac{\partial}{\partial z}\hat{z}\right) \times E(z) = -jk_t \times E_t - jk_t \times \hat{z}E_z + \frac{\partial}{\partial z}\hat{z} \times E_t
\]

parallel to \( \hat{z} \)

orthogonal to \( \hat{z} \)

orthogonal to \( \hat{z} \)

There is no term with \( \frac{\partial E_z}{\partial z} \), since \( \hat{z} \times \hat{z} = 0 \).
3) Separate the components, fluxes

Split the material dyadics as

\[
\begin{pmatrix}
D_t \\
\hat{z}D_z
\end{pmatrix} = \begin{pmatrix}
\epsilon_{tt} & \epsilon_t\hat{z} \\
\hat{z}\epsilon_z & \epsilon_{zz}\hat{z}\hat{z}
\end{pmatrix} \cdot \begin{pmatrix}
E_t \\
\hat{z}E_z
\end{pmatrix} + \begin{pmatrix}
\xi_{tt} & \xi_t\hat{z} \\
\hat{z}\xi_z & \xi_{zz}\hat{z}\hat{z}
\end{pmatrix} \cdot \begin{pmatrix}
H_t \\
\hat{z}H_z
\end{pmatrix}
\]

\(\epsilon_{tt}\) can be represented as a 2 \(\times\) 2 matrix operating on xy components, \(\epsilon_t\) and \(\epsilon_z\) are vectors in the xy-plane, and \(\epsilon_{zz}\) is a scalar.

The transverse components of the electric flux are (vector equation)

\[D_t = \epsilon_{tt} \cdot E_t + \epsilon_t E_z + \xi_{tt} \cdot H_t + \xi_t H_z\]

and the \(z\) component is (scalar equation)

\[D_z = \epsilon_z \cdot E_t + \epsilon_{zz} E_z + \xi_z \cdot H_t + \xi_{zz} H_z\]
4) Eliminate the $z$ components

We first write Maxwell’s equations using matrices (all components)

$$\frac{\partial}{\partial z} \begin{pmatrix} 0 & -\hat{z} \times I \\ \hat{z} \times I & 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ H \end{pmatrix} = \begin{pmatrix} 0 & -jk_t \times I \\ jk_t \times I & 0 \end{pmatrix} \cdot \begin{pmatrix} E \\ H \end{pmatrix} - j\omega \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \cdot \begin{pmatrix} E \\ H \end{pmatrix}$$

The $z$ components of these equations are (take scalar product with $\hat{z}$ and use $\hat{z} \cdot (k_t \times \hat{z} E_z) = 0$ and $\hat{z} \cdot (k_t \times E_t) = (\hat{z} \times k_t) \cdot E_t$)

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -j\hat{z} \times k_t \\ j\hat{z} \times k_t & 0 \end{pmatrix} \cdot \begin{pmatrix} E_t \\ H_t \end{pmatrix} - j\omega \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \cdot \begin{pmatrix} E_t \\ H_t \end{pmatrix} - j\omega \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} \begin{pmatrix} E_z \\ H_z \end{pmatrix}$$

from which we solve for the $z$ components of the fields:

$$\begin{pmatrix} E_z \\ H_z \end{pmatrix} = \left( \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} \right)^{-1} \left[ \begin{pmatrix} 0 & -\omega^{-1}\hat{z} \times k_t \\ \omega^{-1}\hat{z} \times k_t & 0 \end{pmatrix} - \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \right] \cdot \begin{pmatrix} E_t \\ H_t \end{pmatrix}$$
4) Insert the $z$ components

The transverse part of Maxwell’s equations are

\[
\frac{\partial}{\partial z} \begin{pmatrix} 0 & -\hat{z} \times \mathbf{I} \\ \hat{z} \times \mathbf{I} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} = \begin{pmatrix} 0 & -j\mathbf{k}_t \times \hat{z} \\ j\mathbf{k}_t \times \hat{z} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{E}_z \\ \mathbf{H}_z \end{pmatrix} - j\omega \begin{pmatrix} \epsilon_{tt} & \xi_{tt} \\ \zeta_{tt} & \mu_{tt} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} - j\omega \begin{pmatrix} \epsilon_t & \xi_t \\ \zeta_t & \mu_t \end{pmatrix} \begin{pmatrix} \mathbf{E}_z \\ \mathbf{H}_z \end{pmatrix}
\]

Inserting the expression for $[\mathbf{E}_z, \mathbf{H}_z]$ previously derived implies

\[
\frac{\partial}{\partial z} \begin{pmatrix} 0 & -\hat{z} \times \mathbf{I} \\ \hat{z} \times \mathbf{I} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} = -j\omega \begin{pmatrix} \epsilon_{tt} & \xi_{tt} \\ \zeta_{tt} & \mu_{tt} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} + j\omega \mathbf{A} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix}
\]

where the matrix $\mathbf{A}$ is due to the $z$ components.
4) The $A$ matrix

The matrix $A$ is a dyadic product

$$A = \left[ \begin{pmatrix} 0 & -\omega^{-1}k_t \times \hat{z} \\ \omega^{-1}k_t \times \hat{z} & 0 \end{pmatrix} - \begin{pmatrix} \epsilon_t & \xi_t \\ \zeta_t & \mu_t \end{pmatrix} \right]$$

$$\left( \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} \right)^{-1} \left[ \begin{pmatrix} 0 & -\omega^{-1}\hat{z} \times k_t \\ \omega^{-1}\hat{z} \times k_t & 0 \end{pmatrix} - \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \right]$$

In particular, when $k_t = 0$ we have

$$A = \begin{pmatrix} \epsilon_t & \xi_t \\ \zeta_t & \mu_t \end{pmatrix} \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix}$$

and for a uniaxial material with the axis of symmetry along the $\hat{z}$ direction, where $\epsilon_t = \xi_t = \zeta_t = \mu_t = 0$ and $\epsilon_z = \xi_z = \zeta_z = \mu_z = 0$, we have $A = 0$. 

Daniel Sjöberg, Department of Electrical and Information Technology
5) Dynamical system

Using \(-\hat{z} \times (\hat{z} \times E_t) = E_t\) for any transverse vector \(E_t\), it is seen that

\[
\begin{pmatrix}
0 & -\hat{z} \times I \\
I & 0
\end{pmatrix} \cdot \begin{pmatrix}
0 & -\hat{z} \times I \\
\hat{z} \times I & 0
\end{pmatrix} \cdot \begin{pmatrix} E_t \\ H_t \end{pmatrix} = \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}
\]

The final form of Maxwell’s equations is then

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix} = \begin{pmatrix} 0 & -\hat{z} \times I \\
I & 0 \end{pmatrix} \cdot \left[ -j\omega \begin{pmatrix} \epsilon_{tt} & \xi_{tt} \\ \zeta_{tt} & \mu_{tt} \end{pmatrix} + j\omega A \right] \cdot \begin{pmatrix} I & 0 \\ 0 & \hat{z} \times I \end{pmatrix} \cdot \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}
\]
or

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix} = -j\omega W \cdot \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}
\]

Due to the formal similarity with classical transmission line formulas, the equivalent voltage and current vectors

\[
\begin{pmatrix} V \\ I \end{pmatrix} = \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}
\]

are often introduced in electrical engineering (just a relabeling).
Outline

1. Plane waves in lossless media
   - General time dependence
   - Time harmonic waves

2. Polarization

3. Wave propagation in bianisotropic media

4. Interpretation of the fundamental equation
If the material parameters are constant, the dynamical system
\[
\frac{\partial}{\partial z} \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix} = -j\omega W \cdot \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}
\]
has the formal solution
\[
\begin{pmatrix} E_t(z_2) \\ H_t(z_2) \times \hat{z} \end{pmatrix} = \exp \left(-j\omega (z_2 - z_1) W\right) \cdot \begin{pmatrix} E_t(z_1) \\ H_t(z_1) \times \hat{z} \end{pmatrix}
\]
The matrix \( P(z_1, z_2) \) is called a propagator. It maps the transverse fields at the plane \( z_1 \) to the plane \( z_2 \). Due to the linearity of the problem, the propagator exists even if \( W \) does depend on \( z \), but it cannot be represented with an exponential matrix.
Eigenvalue problem

For media with infinite extension in $z$, it is natural to look for solutions on the form

$$\begin{pmatrix}
E_t(z) \\
H_t(z) \times \hat{z}
\end{pmatrix} = \begin{pmatrix}
E_{0t} \\
H_{0t} \times \hat{z}
\end{pmatrix} e^{-j\beta z}$$

This implies $\frac{\partial}{\partial z} [E_t(z), H_t(z) \times \hat{z}] = -j\beta [E_{0t}, H_{0t} \times \hat{z}] e^{-j\beta z}$. We then get the algebraic eigenvalue problem

$$\frac{\beta}{\omega} \begin{pmatrix}
E_{0t} \\
H_{0t} \times \hat{z}
\end{pmatrix} = W \cdot \begin{pmatrix}
E_{0t} \\
H_{0t} \times \hat{z}
\end{pmatrix}$$

$$= \begin{pmatrix}
0 & -\hat{z} \times I \\
I & 0
\end{pmatrix} \cdot \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\xi_{tt} & \mu_{tt}
\end{pmatrix} - A \cdot \begin{pmatrix}
I & 0 \\
0 & \hat{z} \times I
\end{pmatrix} \cdot \begin{pmatrix}
E_{0t} \\
H_{0t} \times \hat{z}
\end{pmatrix}$$
Interpretation of the eigenvalue problem

Since the wave is multiplied by the exponential factor
\[ e^{i(\omega t - \beta z)} = e^{i\omega (t - \frac{z\beta}{\omega})} = e^{i\omega (t - \frac{z}{c})} \],
we identify the eigenvalue as the inverse of the phase velocity
\[ \frac{\beta}{\omega} = \frac{1}{c} = \frac{n}{c_0} \]

which defines the refractive index \( n \) of the wave. The transverse wave impedance dyadic \( Z \) is defined through the relation
\[ E_{0t} = Z \cdot (H_{0t} \times \hat{z}) \]

where \([E_{0t}, H_{0t} \times \hat{z}]\) is the corresponding eigenvector.
Example: “standard” media

An isotropic medium is described by

\[
\begin{pmatrix}
\epsilon & \xi \\
\zeta & \mu
\end{pmatrix} = \begin{pmatrix}
\epsilon I & 0 \\
0 & \mu I
\end{pmatrix}
\]

For \( k_t = 0 \), we have \( A = 0 \) and

\[
W = \begin{pmatrix}
0 & -\hat{z} \times I \\
I & 0
\end{pmatrix} \cdot \left[ \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\zeta_{tt} & \mu_{tt}
\end{pmatrix} - A \right] \cdot \begin{pmatrix}
I \\
0 \\
\hat{z} \times I
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & \mu I \\
0 & \epsilon I
\end{pmatrix} = \begin{pmatrix}
0 & 0 & \mu & 0 \\
0 & 0 & 0 & \mu \\
\epsilon & 0 & 0 & 0 \\
0 & \epsilon & 0 & 0
\end{pmatrix}
\]
The eigenvalues are found from the characteristic equation
\[ \det(\lambda I - W) = 0 \]

\[
0 = \begin{vmatrix}
\lambda & 0 & -\mu & 0 \\
0 & \lambda & 0 & -\mu \\
-\epsilon & 0 & \lambda & 0 \\
0 & -\epsilon & 0 & \lambda \\
\end{vmatrix} = -\begin{vmatrix}
\lambda & -\mu & 0 & 0 \\
0 & 0 & \lambda & -\mu \\
-\epsilon & \lambda & 0 & 0 \\
0 & 0 & -\epsilon & \lambda \\
\end{vmatrix} = \begin{vmatrix}
\lambda & -\mu & 0 & 0 \\
0 & 0 & \lambda & -\mu \\
-\epsilon & \lambda & 0 & 0 \\
0 & 0 & -\epsilon & \lambda \\
\end{vmatrix} = (\lambda^2 - \epsilon \mu)^2
\]

Thus, the eigenvalues are (with double multiplicity)
\[
\frac{\beta}{\omega} = \lambda = \pm \sqrt{\epsilon \mu}
\]
Using that $H_t \times \hat{z} = \hat{x} H_y - \hat{y} H_x$, the eigenvectors are (top row corresponds to $\frac{\beta}{\omega} = \sqrt{\epsilon \mu}$, bottom row to $\frac{\beta}{\omega} = -\sqrt{\epsilon \mu}$):

$$
\begin{pmatrix}
E_x \\
0 \\
H_y \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
E_y \\
0 \\
-H_x
\end{pmatrix}, \\
\begin{pmatrix}
E_x \\
0 \\
H_y \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
E_y \\
0 \\
-H_x
\end{pmatrix},
$$

$$Z = \frac{E_x}{H_y} = \frac{E_y}{-H_x} = \sqrt{\frac{\mu}{\epsilon}} = \eta$$

$$Z = \frac{E_x}{H_y} = \frac{E_y}{-H_x} = -\sqrt{\frac{\mu}{\epsilon}} = -\eta$$
The propagator matrix for isotropic media can be represented as (after some calculations)

\[
\begin{pmatrix}
E_t(z_2) \\
H_t(z_2) \times \hat{z}
\end{pmatrix}
= \begin{pmatrix}
\cos(\beta \ell) \mathbf{I} & j \sin(\beta \ell) \mathbf{Z} \\
-j \sin(\beta \ell) \mathbf{Z}^{-1} & \cos(\beta \ell) \mathbf{I}
\end{pmatrix}
\cdot
\begin{pmatrix}
E_t(z_1) \\
H_t(z_1) \times \hat{z}
\end{pmatrix}
\]

For those familiar with transmission line theory, this is the ABCD matrix for a homogeneous transmission line.
We have found a way of computing time harmonic waves in any bianisotropic material.

The transverse components ($x$ and $y$) are crucial.

There are four fundamental waves: two propagation directions and two polarizations.

Each wave is described by

- The wave number $\beta$ (eigenvalue of $W$)
- The polarization and wave impedance (eigenvector of $W$)

The propagator $P(z_1, z_2) = \exp(-jk_0(z_2 - z_1)W)$ maps fields at $z_1$ to fields at $z_2$.

A simple computer program is available on the course web for calculation of $W$, given material matrices $\epsilon$, $\xi$, $\zeta$, and $\mu$. 