Electromagnetic Wave Propagation
Lecture 2: Time harmonic dependence, constitutive relations

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Outline

1 Harmonic time dependence
2 Constitutive relations, time domain
3 Constitutive relations, frequency domain
4 Examples of material models
5 Composite materials
6 Conclusions
1. Harmonic time dependence
2. Constitutive relations, time domain
3. Constitutive relations, frequency domain
4. Examples of material models
5. Composite materials
6. Conclusions
The theory given in this lecture (and the entire course) is applicable to the whole electromagnetic spectrum.

However, different processes are dominant in different bands, making the material models different.

Today, you learn what restrictions are imposed by the requirements

1. Linearity
2. Causality
3. Time translational invariance
4. Passivity
Electromagnetic spectrum, $c_0 = f\lambda \approx 3 \cdot 10^8 \text{ m/s}$

<table>
<thead>
<tr>
<th>Band</th>
<th>Frequency</th>
<th>Wavelength</th>
</tr>
</thead>
<tbody>
<tr>
<td>ELF Extremely Low Frequency</td>
<td>30–300 Hz</td>
<td>1–10 Mm</td>
</tr>
<tr>
<td>VF Voice Frequency</td>
<td>300–3000 Hz</td>
<td>100–1000 km</td>
</tr>
<tr>
<td>VLF Very Low Frequency</td>
<td>3–30 kHz</td>
<td>10-100 km</td>
</tr>
<tr>
<td>LF Low Frequency</td>
<td>30–300 kHz</td>
<td>1–10 km</td>
</tr>
<tr>
<td>MF Medium Frequency</td>
<td>300–3000 kHz</td>
<td>100–1000 m</td>
</tr>
<tr>
<td>HF High Frequency</td>
<td>3–30 MHz</td>
<td>10–100 m</td>
</tr>
<tr>
<td>VHF Very High Frequency</td>
<td>30–300 MHz</td>
<td>1–10 m</td>
</tr>
<tr>
<td>UHF Ultra High Frequency</td>
<td>300–3000 MHz</td>
<td>10–100 cm</td>
</tr>
<tr>
<td>SHF Super High Frequency</td>
<td>3–30 GHz</td>
<td>1–10 cm</td>
</tr>
<tr>
<td>EHF Extremely High Frequency</td>
<td>30–300 GHz</td>
<td>1–10 mm</td>
</tr>
<tr>
<td>Submillimeter</td>
<td>300–3000 GHz</td>
<td>100–1000 µm</td>
</tr>
<tr>
<td>Infrared</td>
<td>3–30 THz</td>
<td>1–100 µm</td>
</tr>
<tr>
<td>Visible</td>
<td>385–789 THz</td>
<td>380–780 nm</td>
</tr>
<tr>
<td>Ultraviolet</td>
<td>750 THz–30 PHz</td>
<td>10–400 nm</td>
</tr>
<tr>
<td>X-ray</td>
<td>30 PHz–3 EHz</td>
<td>10 nm–100 pm</td>
</tr>
<tr>
<td>γ-ray</td>
<td>&gt;3 EHz</td>
<td>&lt;100 pm</td>
</tr>
</tbody>
</table>
Three ways of introducing time harmonic fields

- Fourier transform (finite energy fields, $\omega = 2\pi f$)
  \[
  E(\mathbf{r}, \omega) = \int_{-\infty}^{\infty} E(\mathbf{r}, t) e^{-j\omega t} \, dt
  \]
  \[
  E(\mathbf{r}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(\mathbf{r}, \omega) e^{j\omega t} \, d\omega
  \]

- Laplace transform (causal fields, zero for $t < 0$, $s = \alpha + j\omega$)
  \[
  E(\mathbf{r}, s) = \int_{0}^{\infty} E(\mathbf{r}, t) e^{-st} \, dt
  \]
  \[
  E(\mathbf{r}, t) = \frac{1}{2\pi j} \int_{\alpha-j\infty}^{\alpha+j\infty} E(\mathbf{r}, s) e^{st} \, ds
  \]

- Real-value convention (purely harmonic $\cos \omega t$, preserves units)
  \[
  E(\mathbf{r}, t) = \text{Re}\{E(\mathbf{r}, \omega)e^{j\omega t}\} 
  \]
Some examples

Unit step function:  \( u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \)

<table>
<thead>
<tr>
<th>( E(r, t) )</th>
<th>Fourier</th>
<th>Laplace</th>
<th>Real-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e^{-\alpha t^2} )</td>
<td>( \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\omega^2}{4\alpha}} )</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>( \cos(\omega_0 t) )</td>
<td>( \pi(\delta(\omega + \omega_0) + \delta(\omega - \omega_0)) )</td>
<td>—</td>
<td>1</td>
</tr>
<tr>
<td>( \sin(\omega_0 t) )</td>
<td>( j\pi(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)) )</td>
<td>—</td>
<td>( -j )</td>
</tr>
<tr>
<td>( e^{-at} u(t) )</td>
<td>( \frac{1}{j\omega + a} )</td>
<td>( \frac{1}{s + a} )</td>
<td>—</td>
</tr>
<tr>
<td>( e^{-at} \cos(\omega_0 t) u(t) )</td>
<td>( \frac{j\omega + a}{(j\omega + a)^2 + \omega_0^2} )</td>
<td>( \frac{s + a}{(s + a)^2 + \omega_0^2} )</td>
<td>—</td>
</tr>
<tr>
<td>( e^{-at} \sin(\omega_0 t) u(t) )</td>
<td>( \frac{\omega_0}{(j\omega + a)^2 + \omega_0^2} )</td>
<td>( \frac{\omega_0}{(s + a)^2 + \omega_0^2} )</td>
<td>—</td>
</tr>
<tr>
<td>( \delta(t) )</td>
<td>1</td>
<td>1</td>
<td>—</td>
</tr>
</tbody>
</table>
Different time conventions

Different traditions:

**Engineering:** Time dependence $e^{j\omega t}$, plane wave factor $e^{j(\omega t - k \cdot r)}$.

**Physics:** Time dependence $e^{-i\omega t}$, plane wave factor $e^{i(k \cdot r - \omega t)}$.

If you use $j$ and $i$ *consistently*, all results can be translated between conventions using the simple rule

\[
    j = -i
\]

In this course we follow Orfanidis’ choice $e^{j\omega t}$. 
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1. Harmonic time dependence

2. Constitutive relations, time domain

3. Constitutive relations, frequency domain

4. Examples of material models

5. Composite materials

6. Conclusions
The need for material models

Maxwell’s equations are

\[
\nabla \times E(r, t) = -\frac{\partial B(r, t)}{\partial t}
\]

\[
\nabla \times H(r, t) = J(r, t) + \frac{\partial D(r, t)}{\partial t}
\]

This is \(2 \times 3 = 6\) equations for at least \(4 \times 3 = 12\) unknowns. Something is needed! We choose \(E\) and \(H\) as our fundamental fields (partly due to conformity with boundary conditions), and search for constitutive relations on the form

\[
\begin{pmatrix}
D \\
B
\end{pmatrix} = F \left( \begin{pmatrix}
E \\
H
\end{pmatrix} \right)
\]

This would provide the missing \(6\) equations.
Examples of models

- Linear, isotropic materials ("standard media"): \[ D = \epsilon E, \quad B = \mu H \]

- Linear, anisotropic materials (different in different directions): \[ D = (\epsilon_x \hat{x} \hat{x} + \epsilon_y \hat{y} \hat{y} + \epsilon_z \hat{z} \hat{z}) \cdot E \quad \Leftrightarrow \quad \begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \epsilon_x & 0 & 0 \\ 0 & \epsilon_y & 0 \\ 0 & 0 & \epsilon_z \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \]

- Linear, dispersive materials (depends on the history): Debye material
  \[ D = \epsilon_0 E + P \]
  \[ \frac{\partial P}{\partial t} = \epsilon_0 \alpha E - P / \tau \]

  Gyrotropic material
  \[ B = \mu_0 (H + M) \]
  \[ \frac{\partial M}{\partial t} = \omega_S \hat{z} \times (\beta M - H) \]

The models correspond to the physical processes in the material.
Basic assumptions

To simplify, we formulate our assumptions for a non-magnetic material where \( D = F(E) \) and \( B = \mu_0 H \). We require the mapping \( F \) to satisfy four basic physical principles:

**Linearity:** For each \( \alpha, \beta, E_1, \) and \( E_2 \) we have

\[
F(\alpha E_1 + \beta E_2) = \alpha F(E_1) + \beta F(E_2)
\]

**Causality:** For all fields \( E \) such that \( E(t) = 0 \) when \( t < \tau \), we have

\[
F(E)(t) = 0 \quad \text{for} \quad t < \tau
\]

**Time translational invariance:** If \( D_1 = F(E_1), D_2 = F(E_2), \) and \( E_2(t) = E_1(t - \tau) \), we have

\[
D_2(t) = D_1(t - \tau)
\]

**Passivity:** The material is not a source of electromagnetic energy, that is, \( \nabla \cdot \langle \mathcal{P} \rangle \leq 0 \).
The general linear model

The result of the assumptions is that all such materials can be modeled as

\[
D(t) = \varepsilon_0 \left[ E(t) + \int_{-\infty}^{t} \chi_e(t - t') \cdot E(t') \, dt' \right] \\
B(t) = \int_{-\infty}^{t} \zeta(t - t') \cdot E(t') \, dt' \\
+ \mu_0 \left[ H(t) + \int_{-\infty}^{t} \chi_m(t - t') \cdot H(t') \, dt' \right]
\]

The dyadic convolution kernels \(\chi_e(t), \xi(t), \zeta(t),\) and \(\chi_m(t)\) model the induced polarization and magnetization.

**Linearity, causality, time translational invariance by construction.** Passivity will be seen in frequency domain.
Some physical processes in the material may be considerably faster than the others.

This means the susceptibility function can be split in two parts:

\[ \chi(t) = \chi_1(t) + \chi_2(t) \]
Assume that $E(t)$ does not vary considerably on the time scale of $\chi_1(t)$. We then have

$$D(t)/\varepsilon_0 = E(t) + \int_{-\infty}^{t} [\chi_1(t - t') + \chi_2(t - t')] E(t') \, dt'$$

$$= E(t) + \left[ \int_{-\infty}^{t} \chi_1(t - t') \, dt' \right] E(t) + \int_{-\infty}^{t} \chi_2(t - t') E(t') \, dt'$$

$$= E(t) + \left[ \int_{0}^{\infty} \chi_1(t') \, dt' \right] E(t) + \int_{-\infty}^{t} \chi_2(t - t') E(t') \, dt'$$

The quantity $\varepsilon_\infty = 1 + \int_{0}^{\infty} \chi_1(t') \, dt'$ is called the instantaneous response (or momentaneous response, or optical response).

Thus, there is some freedom of choice how to model the material, depending on the time scale!
Instantaneous response, example
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Constitutive relations in the frequency domain

Applying a Fourier transform to the convolutions implies

\[
D(\omega) = \epsilon_0 \left[ \epsilon_{\infty} \cdot E(\omega) + \chi_e(\omega) \cdot E(\omega) \right] + \xi(\omega) \cdot H(\omega)
\]

\[
B(\omega) = \zeta(\omega) \cdot E(\omega) + \mu_0 \left[ \mu_{\infty} \cdot H(\omega) + \chi_m(\omega) \cdot \eta_0 H(\omega) \right]
\]

or

\[
\begin{pmatrix}
D(\omega) \\
B(\omega)
\end{pmatrix} = \begin{pmatrix}
\epsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix} \cdot \begin{pmatrix}
E(\omega) \\
H(\omega)
\end{pmatrix}
\]

where we introduced the permittivity and permeability dyadics

\[
\epsilon(\omega) = \epsilon_0 \left[ \epsilon_{\infty} + \chi_e(\omega) \right] \quad \mu(\omega) = \mu_0 \left[ \mu_{\infty} + \chi_m(\omega) \right]
\]

This is a fully **bianisotropic** material model.
## Classification of materials

<table>
<thead>
<tr>
<th>Type</th>
<th>$\epsilon, \mu$</th>
<th>$\xi, \zeta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Isotropic</td>
<td>Both $\sim I$</td>
<td>Both $0$</td>
</tr>
<tr>
<td>An-isotropic</td>
<td>Some not $\sim I$</td>
<td>Both $0$</td>
</tr>
<tr>
<td>Bi-isotropic</td>
<td>Both $\sim I$</td>
<td>Both $\sim I$</td>
</tr>
<tr>
<td>Bi-an-isotropic</td>
<td>All other cases</td>
<td></td>
</tr>
</tbody>
</table>

![Venn diagram](image)

- General media (including nonlinear)
- Linear media (bi-an-isotropic)
Modeling arbitrariness

Assume the models

\[ J(\omega) = \sigma(\omega)E(\omega), \quad D(\omega) = \epsilon(\omega)E(\omega) \]

The total current in Maxwell’s equations can then be written

\[ J(\omega) + j\omega D(\omega) = [\sigma(\omega) + j\omega\epsilon(\omega)] E(\omega) = J'(\omega) \]

\[ = \sigma'(\omega) \]

\[ = j\omega \left[ \frac{\sigma(\omega)}{j\omega} + \epsilon(\omega) \right] E(\omega) = j\omega D'(\omega) \]

\[ = \epsilon'(\omega) \]

where \( \sigma'(\omega) \) and \( \epsilon'(\omega) \) are equivalent models for the material.
Thus, there is an arbitrariness in how to model dispersive materials, either by a conductivity model \( (\sigma'(\omega)) \) or by a permittivity model \( (\epsilon'(\omega)) \), or any combination.
Consider the total current as a sum of a conduction current $J_c$ and a displacement current $J_d$:

$$J_{\text{tot}}(\omega) = \sigma_c(\omega)E + j\omega \varepsilon_d(\omega)E$$

The ratio can take many different values (using $f = 1 \text{ GHz}$)

$$\frac{|J_c(\omega)|}{|J_d(\omega)|} = \frac{\sigma_c(\omega)}{|\omega \varepsilon_d(\omega)|} = \begin{cases} 10^9 & \text{copper (}\sigma = 5.8 \cdot 10^7 \text{ S/m and } \varepsilon = \varepsilon_0\text{)} \\ 1 & \text{seawater (}\sigma = 4 \text{ S/m and } \varepsilon = 72\varepsilon_0\text{)} \\ 10^{-9} & \text{glass (}\sigma = 10^{-10} \text{ S/m and } \varepsilon = 2\varepsilon_0\text{)} \end{cases}$$

18 orders of magnitude in difference! Conductivity model good when $|J_c| \gg |J_d|$, permittivity model good when $|J_d| \gg |J_c|$. 
Poynting’s theorem in the frequency domain

In the time domain we had

$$\nabla \cdot (E(t) \times H(t)) + H(t) \cdot \frac{\partial B(t)}{\partial t} + E(t) \cdot \frac{\partial D(t)}{\partial t} + E(t) \cdot J(t) = 0$$

For time harmonic fields, we consider the time average over one period (where $\langle f(t) \rangle = \frac{1}{T} \int_t^{t+T} f(t') \, dt'$):

$$\nabla \cdot \langle E(t) \times H(t) \rangle + \langle H(t) \cdot \frac{\partial B(t)}{\partial t} \rangle + \langle E(t) \cdot \frac{\partial D(t)}{\partial t} \rangle + \langle E(t) \cdot J(t) \rangle = 0$$

The time average of a product of two harmonic signals is $\langle f(t)g(t) \rangle = \frac{1}{2} \text{Re}\{f(\omega)g(\omega)^*\}$. 
Poynting’s theorem, continued

The different terms are

\[
\langle E(t) \times H(t) \rangle = \frac{1}{2} \text{Re}\{E(\omega) \times H(\omega)^*\}
\]

\[
\langle H(t) \cdot \frac{\partial B(t)}{\partial t} \rangle = \frac{1}{2} \text{Re}\{-j\omega H(\omega) \cdot B(\omega)^*\}
\]

\[
\langle E(t) \cdot \frac{\partial D(t)}{\partial t} \rangle = \frac{1}{2} \text{Re}\{-j\omega E(\omega) \cdot D(\omega)^*\}
\]

\[
\langle E(t) \cdot J(t) \rangle = \frac{1}{2} \text{Re}\{E(\omega) \cdot J(\omega)^*\}
\]

For a purely dielectric material, we have \( D(\omega) = \epsilon(\omega) \cdot E(\omega) \) and

\[
2 \text{Re}\{-j\omega E(\omega) \cdot D(\omega)^*\} = -j\omega E(\omega) \cdot [\epsilon(\omega) \cdot E(\omega)]^* + j\omega E(\omega)^* \cdot \epsilon(\omega) \cdot E(\omega)
\]

\[
= j\omega E(\omega)^* \cdot [\epsilon(\omega) - \epsilon(\omega)^\dagger] \cdot E(\omega)
\]
Using a permittivity model ($J = 0$), we have

$$\nabla \cdot \langle \mathbf{P}(t) \rangle = -\frac{j\omega}{4} \left( \begin{array}{c} E(\omega) \\ H(\omega) \end{array} \right)^\dagger \cdot \left( \begin{array}{cc} \varepsilon(\omega) - \varepsilon(\omega)^\dagger & \xi(\omega) - \zeta(\omega)^\dagger \\ \zeta(\omega) - \xi(\omega)^\dagger & \mu(\omega) - \mu(\omega)^\dagger \end{array} \right) \cdot \left( \begin{array}{c} E(\omega) \\ H(\omega) \end{array} \right)$$

Passive material: $\nabla \cdot \langle \mathbf{P}(t) \rangle \leq 0$

Active material: $\nabla \cdot \langle \mathbf{P}(t) \rangle > 0 \quad \Leftarrow \quad \text{Definitions!}$

Lossless material: $\nabla \cdot \langle \mathbf{P}(t) \rangle = 0$

This boils down to conditions on the material matrix

$$j\omega \left( \begin{array}{cc} \varepsilon(\omega) - \varepsilon(\omega)^\dagger & \xi(\omega) - \zeta(\omega)^\dagger \\ \zeta(\omega) - \xi(\omega)^\dagger & \mu(\omega) - \mu(\omega)^\dagger \end{array} \right) = -2\omega \text{Im} \left\{ \left( \begin{array}{cc} \varepsilon(\omega) & \xi(\omega) \\ \zeta(\omega) & \mu(\omega) \end{array} \right) \right\}$$

If it is positive we have a lossy material, if it is zero we have a lossless material.
Example: “Standard media”

With the material model $D = \varepsilon E$, $J = \sigma E$, $B = \mu H$ we have

$$
\begin{pmatrix}
\varepsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix} = \begin{pmatrix}
\varepsilon + \frac{\sigma}{j\omega} & I & 0 \\
0 & 0 & \mu I
\end{pmatrix}
$$

and

$$
-\omega \text{Im} \left\{ \begin{pmatrix}
\varepsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix} \right\} = \begin{pmatrix}
\sigma I & 0 \\
0 & 0
\end{pmatrix}
$$

This model is lossy with electric fields present, but not with pure magnetic fields. In wave propagation, we always have both $E$ and $H$ fields.
The condition on lossless media,

\[
\begin{pmatrix}
\epsilon(\omega) - \epsilon(\omega)^\dagger & \xi(\omega) - \zeta(\omega)^\dagger \\
\zeta(\omega) - \xi(\omega)^\dagger & \mu(\omega) - \mu(\omega)^\dagger
\end{pmatrix} = 0
\]

can also be written

\[
\begin{pmatrix}
\epsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix} = \begin{pmatrix}
\epsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix}^\dagger
\]

That is, the matrix should be hermitian symmetric.
A bi-isotropic material is described by

\[
\begin{pmatrix}
\epsilon(\omega) & \xi(\omega) \\
\zeta(\omega) & \mu(\omega)
\end{pmatrix} = \begin{pmatrix}
\epsilon(\omega)I & \xi(\omega)I \\
\zeta(\omega)I & \mu(\omega)I
\end{pmatrix}
\]

The passivity requirement implies that all eigenvalues of the matrix

\[
j\omega \begin{pmatrix}
\epsilon(\omega) - \epsilon(\omega)^* & \xi(\omega) - \zeta(\omega)^* \\
\zeta(\omega) - \xi(\omega)^* & \mu(\omega) - \mu(\omega)^*
\end{pmatrix}
\]

are positive. If \( \xi = \zeta = 0 \) it is seen that this requires (using that \( \epsilon(\omega) - \epsilon(\omega)^* = 2j \text{ Im} \epsilon(\omega) \))

\[-\omega \text{ Im} \epsilon(\omega) > 0, \quad -\omega \text{ Im} \mu(\omega) > 0\]

and if \( \xi \) and \( \zeta \) are nonzero we also require (after more algebra)

\[|\xi(\omega) - \zeta(\omega)^*|^2 < 4 \text{ Im} \epsilon(\omega) \text{ Im} \mu(\omega)\]
The causality requirement implies the Kramers-Kronig dispersion relations (writing $\chi(\omega) = \chi_r(\omega) - j\chi_i(\omega)$ for the real and imaginary part)

$$\chi_r(\omega) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\chi_i(\omega')}{\omega' - \omega} \, d\omega'$$

$$\chi_i(\omega) = -\frac{1}{\pi} \text{p.v.} \int_{-\infty}^{\infty} \frac{\chi_r(\omega')}{\omega' - \omega} \, d\omega'$$

where the principal value of a singular integral is

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{\chi_i(\omega')}{\omega' - \omega} \, d\omega' = \lim_{\delta \to 0} \left[ \int_{-\infty}^{\omega - \delta} \frac{\chi_i(\omega')}{\omega' - \omega} \, d\omega' + \int_{\omega + \delta}^{\infty} \frac{\chi_i(\omega')}{\omega' - \omega} \, d\omega' \right]$$

The Kramers-Kronig relations prohibit the existence of a lossless frequency dependent material.
The Kramers-Kronig relations restrict the possible frequency behavior of any causal material. It requires a model $\chi(\omega)$ for all frequencies. Usually, our models are derived or measured only in a finite frequency interval, $\omega_1 < \omega < \omega_2$. We need to extrapolate the models to zero and infinite frequencies, $\omega \to 0$ and $\omega \to \infty$. Not a trivial task!
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Randomly oriented dipoles

Consider a medium consisting of randomly oriented electric dipoles, for instance water. The polarization is

\[ P = \lim_{\Delta V \to 0} \frac{\sum_i p_i}{\Delta V} \]

A typical situation is depicted below.
We now consider two processes:

1. The molecules strive to align with an imposed electric field, at the rate $\epsilon_0 \alpha E$.

2. Thermal motion tries to disorient the polarization. With $\tau$ being the relaxation time for this process, the rate of changes in $P$ are proportional to $-P/\tau$.

This results in the following differential equation:

$$\frac{\partial P(t)}{\partial t} = \epsilon_0 \alpha E(t) - \frac{P(t)}{\tau}$$
Physical processes

We now consider two processes:

1. The molecules strive to align with an imposed electric field, at the rate $\epsilon_0 \alpha E$.

2. Thermal motion tries to disorient the polarization. With $\tau$ being the relaxation time for this process, the rate of changes in $P$ are proportional to $-P/\tau$.

This results in the following differential equation:

$$\frac{\partial P(t)}{\partial t} = \epsilon_0 \alpha E(t) - \frac{P(t)}{\tau}$$

This is an ordinary differential equation with the solution (assuming $P = 0$ at $t = -\infty$)

$$P(t) = \epsilon_0 \int_{-\infty}^{t} \alpha e^{-(t-t')/\tau} E(t') \, dt'$$
Dispersion or conductivity model

From the solution we identify the susceptibility function

$$\chi(t) = u(t) \alpha e^{-t/\tau}$$

This is monotonically decaying without oscillations.
Dispersion or conductivity model

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\[ \chi(t) = u(t) \alpha e^{-t/\tau} \]

This is monotonically decaying without oscillations. Including all the effects in the \( D \)-field results in

\[
\begin{cases}
  D(t) = \varepsilon_0 \left( E(t) + \int_{-\infty}^{t} \alpha e^{-(t-t')/\tau} E(t') \, dt' \right) \\
  J(t) = 0
\end{cases}
\]
Dispersion or conductivity model

From the solution we identify the susceptibility function

\[ \chi(t) = u(t) \alpha e^{-t/\tau} \]

This is monotonically decaying without oscillations. Including all the effects in the \( D \)-field results in

\[
\begin{cases}
D(t) = \epsilon_0 \left( E(t) + \int_{-\infty}^{t} \alpha e^{-(t-t')/\tau} E(t') \, dt' \right) \\
J(t) = 0
\end{cases}
\]

and shifting it to \( J \) results in

\[
\begin{cases}
D(t) = \epsilon_0 E(t) \\
J(t) = \epsilon_0 \alpha E(t) - \epsilon_0 \frac{\alpha}{\tau} \int_{-\infty}^{t} e^{-(t-t')/\tau} E(t') \, dt'
\end{cases}
\]

Both versions have the same total current \( J(t) + \frac{\partial D(t)}{\partial t} \).
Consider two different excitation functions:

- One square pulse $E(t) = 1$ for $0 < t < T$, and zero elsewhere.
- A damped sine function, $E(t) = e^{-t\nu} \sin(\omega t)$.

The response $P(t)$ can be calculated by numerically performing the convolution integral

$$P(t) = \epsilon_0 \int_{-\infty}^{t} \chi(t - t') E(t') \, dt'$$
Debye model, square pulse excitation
Debye model, sine excitation
Debye material in frequency domain

The susceptibility kernel is \( \chi(t) = \alpha e^{-t/\tau}u(t) \), with the Fourier transform

\[
\chi(\omega) = \int_0^\infty \alpha e^{-t/\tau}e^{-j\omega t} \, dt = \frac{\alpha}{j\omega + 1/\tau} = \frac{\alpha \tau}{1 + j\omega \tau}
\]

The frequency dependent relative permittivity is \( \epsilon_r(\omega) = 1 + \chi(\omega) = \epsilon' - j\epsilon'' \), with typical behavior as below:
Harmonic oscillator

The archetypical material behavior is derived from an electron orbiting a positively charged nucleus. The typical forces on the electron are:

1. An electric force \( F_1 = qE \) from the applied electric field.
2. A restoring force proportional to the displacement \( F_2 = -m\omega_0^2 r \), where \( \omega_0 \) is the harmonic frequency.
3. A frictional force proportional to the velocity, \( F_3 = -mv\partial r/\partial t \).
Newton’s acceleration law now gives

\[ m \frac{\partial^2 r}{\partial t^2} = F_1 + F_2 + F_3 = qE - m\omega_0^2r - m\nu \frac{\partial r}{\partial t} \]

Introducing the polarization as \( P = Nq r \), where \( N \) is the number of charges per unit volume, we have

\[ \frac{\partial^2 P(t)}{\partial t^2} + \nu \frac{\partial P(t)}{\partial t} + \omega_0^2 P(t) = \frac{Nq^2}{m} E(t) \]

This is an ordinary differential equation, with the solution (assuming \( P = 0 \) for \( t = -\infty \))

\[ P(t) = \varepsilon_0 \frac{\omega_p^2}{\nu_0} \int_{-\infty}^{t} \text{e}^{-(t-t')\nu/2} \sin(\nu_0(t - t')) E(t') \, dt' \]

with \( \omega_p = Nq^2/(m\varepsilon_0) \) and \( \nu_0^2 = \omega_0^2 - \nu^2/4 \).
The susceptibility function of the Lorentz model is

$$\chi(t) = u(t) \frac{\omega_p^2}{\nu_0} e^{-t\nu/2} \sin(\nu_0 t)$$

with typical behavior as below.
Lorentz model, square pulse excitation

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Lorentz model, sine excitation

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Lorentz model, resonant excitation
Lorentz material in frequency domain

The susceptibility kernel is \( \chi(t) = \frac{\omega_p^2}{\nu_0} e^{-\nu t/2} \sin(\nu_0 t) u(t) \), with the Fourier transform

\[
\chi(\omega) = \int_0^\infty \frac{\omega_p^2}{\nu_0} e^{-\nu t/2} \sin(\nu_0 t) e^{-j\omega t} dt = \frac{\omega_p^2}{-\omega^2 + \omega_0^2 + j\omega\nu}
\]

The frequency dependent relative permittivity is

\( \epsilon_r(\omega) = 1 + \chi(\omega) = \epsilon' - j\epsilon'' \), with typical behavior as below:
Example: permittivity of water

Microwave properties (one Debye model):

Light properties (many Lorentz resonances):
In the first handin, you will model and interpret the response of a ferromagnetic material, when subjected to a magnetic field.

The magnetic moment of each atom precesses around the applied magnetic field, described by the Landau-Lifshitz-Gilbert equation

\[ \frac{\partial M}{\partial t} = -\gamma \mu_0 M \times H + \alpha \frac{M}{|M|} \times \frac{\partial M}{\partial t} \]
Outline

1. Harmonic time dependence
2. Constitutive relations, time domain
3. Constitutive relations, frequency domain
4. Examples of material models
5. Composite materials
6. Conclusions
Composite materials offer

- Low weight, high mechanical strength
- Increased control of material properties

Some products where control of electromagnetic properties is essential:

- Power transformers
- Antennas
- Radomes
- Integrated circuits

New possibilities of nanotechnology offer means of controlling material properties on an even finer scale.
Long wavelength limit

Often, the microstructure is small compared to the wavelength and we cannot “see” the details:

\[ \varepsilon_{\text{eff}} \]

\[ \varepsilon \]

\[ \delta \]

\[ \lambda >> \delta \]

\[ \frac{\varepsilon_1}{\varepsilon_2} \]

Exact solution (red) and homogenized solution (black):

But how to compute \( \varepsilon_{\text{eff}} \)?
Homogenization

In the limit of long wavelength $\lambda \to \infty$, we have $\omega \to 0$. Thus, the source free Maxwell’s equations imply

$$\nabla \times \mathbf{E} = -j\omega \mathbf{B} \to 0 \quad \nabla \cdot \mathbf{D} = 0$$
$$\nabla \times \mathbf{H} = j\omega \mathbf{D} \to 0 \quad \nabla \cdot \mathbf{B} = 0$$

Thus, both the electric and the magnetic fields solve the same kind of equations:

$$\nabla \times \mathbf{E} = 0, \quad \nabla \cdot \mathbf{D} = 0, \quad \mathbf{D}(\mathbf{r}) = \varepsilon(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r})$$

In a periodic structure, all unit cells are equivalent. Macroscopic fields are given by mean values (\(|U| = \text{volume of unit cell } U\)):

$$\langle \mathbf{E} \rangle = \frac{1}{|U|} \iiint_U \mathbf{E}(\mathbf{r}) \, dx \, dy \, dz$$
$$\langle \mathbf{D} \rangle = \varepsilon_{\text{eff}} \cdot \langle \mathbf{E} \rangle$$
\( \nabla \times \mathbf{E} = 0 \) implies \( \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 - \nabla V(\mathbf{r}) \), where \( \mathbf{E}_0 \) is constant and \( V(\mathbf{r}) \) is periodic.

For given \( \epsilon(\mathbf{r}) \) and \( \mathbf{E}_0 \), solve for \( V \) in

\[

\nabla \cdot [\epsilon(\mathbf{r}) \cdot (\mathbf{E}_0 - \nabla V(\mathbf{r}))] = 0

\]

and compute \( \langle \mathbf{D} \rangle = \langle \epsilon \cdot \mathbf{E} \rangle \).

The mean value of \( \mathbf{E} \) is \( \langle \mathbf{E} \rangle = \mathbf{E}_0 \).

The effective permittivity (corresponding to the direction of \( \mathbf{E}_0 \)) is then given by

\[

\epsilon_{\text{eff}} = \frac{\mathbf{E}_0 \cdot \epsilon_{\text{eff}} \cdot \mathbf{E}_0}{|\mathbf{E}_0|^2} = \frac{\mathbf{E}_0 \cdot \langle \mathbf{D} \rangle}{|\mathbf{E}_0|^2}

\]
Task at the workshop

$E_0 = \hat{z}$

$V = -z$

Compute $\epsilon_{\text{eff}}$ for different sphere permittivities $\epsilon_r$. The simulation model is provided, you need to set up a parameter sweep and export and postprocess data.
Outline

1 Harmonic time dependence
2 Constitutive relations, time domain
3 Constitutive relations, frequency domain
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6 Conclusions
Conclusions

- Constitutive relations are necessary in order to fully solve Maxwell’s equations.
- Their form is restricted by physical principles such as linearity, causality, time translational invariance, and passivity.
- A Debye model is suitable for dipoles aligning with an imposed field (relaxation model).
- A Lorentz model is suitable for bound charges (resonance model).
- A composite material can be modeled with homogeneous effective material parameters if the microstructure is small compared to the wavelength.