Electromagnetic Wave Propagation
Lecture 12: Oblique incidence I

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Department of Electrical and Information Technology
1 Introduction
2 Snel’s law
3 Transverse impedance and propagation
4 Critical angle, Brewster angle
5 Evanescent and complex waves
6 Zenneck surface wave
7 Conclusions
Outline

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Key questions

- How to analyze the oblique incidence of waves on an interface?
- What are typical results?
- What are special characteristics of lossy media?
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Willebrord Snel van Royen (1580–1626)
Oblique incidence

The waves can be written

\[ E_+ e^{-j \mathbf{k}_+ \cdot \mathbf{r}} , \quad E_- e^{-j \mathbf{k}_- \cdot \mathbf{r}} , \quad E'_+ e^{-j \mathbf{k}'_+ \cdot \mathbf{r}} , \quad E'_- e^{-j \mathbf{k}'_- \cdot \mathbf{r}} \]
Matching tangential fields on the boundary implies

\[
E_{T^+}e^{-jk_+ \cdot r} + E_{T^-}e^{-jk_- \cdot r} = E'_{T^+}e^{-jk'_+ \cdot r} + E'_{T^-}e^{-jk'_- \cdot r}
\]

\[
E_{T^+}e^{-jkx+x} + E_{T^-}e^{-jkx-x} = E'_{T^+}e^{-jk'_x+x} + E'_{T^-}e^{-jk'_x-x}
\]

Since this applies for all \(x\) on the boundary \(z = 0\), we must have

\[
k_{x+} = k_{x-} = k'_{x+} = k'_{x-}
\]

and similarly for any \(y\) components. Since \(k_x = k \sin \theta = k_0 n \sin \theta\), this implies

\[
\begin{align*}
\theta_+ &= \theta_- = \theta \\
\theta'_+ &= \theta'_- = \theta'
\end{align*}
\]

\[
\text{where we used } k = nk_0 \text{ and } k' = n'k_0. \text{ Since } k \cdot k = k^2 = \omega^2 \epsilon \mu \text{ this implies}
\]

\[
k_z = \sqrt{k^2 - k_{x}^2 - k_{y}^2}
\]
A graphical argument

The condition $k^2 = \omega^2 \epsilon \mu$ describes a sphere (or circle) in $k$-space.

Can also be used for photonic crystals.
Sometimes no solutions!

When the wave is incident from a denser medium, it may not be possible to satisfy the phase matching with real wave vectors.

\[ k^2 = \omega^2 \varepsilon \mu \]

\[ k_0^2 = \omega^2 \varepsilon_0 \mu_0 \]

Corresponds to total internal reflection.

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Transverse impedance

The complete components of the forward field is

\[ E_+(r) = [(\hat{x} \cos \theta - \hat{z} \sin \theta)A_+ + \hat{y} B_+]e^{-j\mathbf{k}_+ \cdot \mathbf{r}} \]

\[ H_+(r) = \frac{1}{\eta}[\hat{y} A_+ - (\hat{x} \cos \theta - \hat{z} \sin \theta) B_+]e^{-j\mathbf{k}_+ \cdot \mathbf{r}} \]

The transverse components can then be written

\[ E_{T+}(r) = [\hat{x} C_+ + \hat{y} B_+]e^{-j\mathbf{k}_+ \cdot \mathbf{r}} \]

\[ H_{T+}(r) = [\hat{y} \frac{C_+}{\eta_{TM}} - \hat{x} \frac{B_+}{\eta_{TE}}]e^{-j\mathbf{k}_+ \cdot \mathbf{r}} \]

where \( C_+ = \cos \theta A_+ \) and

\[ \eta_{TM} = \eta \cos \theta \] TM, parallel, p-polarization

\[ \eta_{TE} = \frac{\eta}{\cos \theta} \] TE, perpendicular, s-polarization
Transverse refractive index

For *dielectric* media, that is, $\mu = \mu_0$, we have

$$\eta = \sqrt{\frac{\mu_0}{\epsilon}} = \frac{\eta_0}{\sqrt{\epsilon_r}} = \frac{\eta_0}{n}$$

This motivates the introduction of the transverse refractive index via $\eta_T = \eta_0/n_T$, or

$$n_{TM} = \frac{n}{\cos \theta} \quad \text{TM, parallel, p-polarization}$$
$$n_{TE} = n \cos \theta \quad \text{TE, perpendicular, s-polarization}$$
Similarity with normal incidence

Making the substitutions (where T can be either TM or TE)

\[ \eta \rightarrow \eta_T, \quad e^{\pm jkz} \rightarrow e^{\pm jkz} = e^{\pm jkz \cos \theta} \]

everything we derived on propagation in layered structures for normal incidence remain valid. For instance,

\[
\begin{pmatrix}
E_{T1+} \\
E_{T1-}
\end{pmatrix}
= \begin{pmatrix}
e^{jkz\ell} & 0 \\
0 & e^{-jkz\ell}
\end{pmatrix}
\begin{pmatrix}
E_{T2+} \\
E_{T2-}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
E_{T1} \\
H_{T1}
\end{pmatrix}
= \begin{pmatrix}
\cos k_z\ell & j\eta_T \sin k_z\ell \\
j\eta_T^{-1} \sin k_z\ell & \cos k_z\ell
\end{pmatrix}
\begin{pmatrix}
E_{T2} \\
H_{T2}
\end{pmatrix}
\]
In particular, at an interface we can define the matching matrix

\[
\begin{pmatrix}
E_{T+} \\
E_{T-}
\end{pmatrix}
= \frac{1}{\tau_T}
\begin{pmatrix}
1 & \rho_T \\
\rho_T & 1
\end{pmatrix}
\begin{pmatrix}
E'_{T+} \\
E'_{T-}
\end{pmatrix}
\]

where \( \rho_T \) and \( \tau_T \) are the Fresnel coefficients

\[
\rho_T = \frac{\eta'_T - \eta_T}{\eta'_T + \eta_T} = \frac{n_T - n'_T}{n_T + n'_T}
\]

\[
\tau_T = \frac{2\eta'_T}{\eta'_T + \eta_T} = \frac{2n_T}{n_T + n'_T}
\]

which take different values depending on polarization.
Writing out $n_{TM} = n / \cos \theta$ and $n_{TE} = n \cos \theta$, the explicit form of the Fresnel reflection coefficient is (after some algebra)

\[
\rho_{TM} = \frac{n \cos \theta' - n' \cos \theta}{n \cos \theta' + n' \cos \theta} = \frac{\sqrt{(n')^2 - \sin^2 \theta - (n')^2 \cos \theta}}{\sqrt{(n')^2 - \sin^2 \theta + (n')^2 \cos \theta}}
\]

\[
\rho_{TE} = \frac{n \cos \theta - n' \cos \theta'}{n \cos \theta + n' \cos \theta'} = \frac{\cos \theta - \sqrt{(n')^2 - \sin^2 \theta}}{\cos \theta + \sqrt{(n')^2 - \sin^2 \theta}}
\]

From the rightmost expressions, we find

\[
\rho_{TM} \to 1, \quad \rho_{TE} \to -1, \quad \text{as} \quad \theta \to 90^\circ
\]

regardless of $n$ and $n'$. 

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Critical angle

**Refraction**

\[ \sin \theta' = \frac{n}{n'} \]

**Reflection**

\[ \sin \theta_c = \frac{n'}{n} \]
Examples

Prism

Optical manhole

Optical fiber
Examples

Prism

Optical manhole

Optical fiber

Nobel prize!!
Optical manhole — Snel’s window

http://www.uwphotographyguide.com/snells-window-underwater, photo taken using fisheye lens to cover the angle.
Total internal reflection: numbers

Using the critical angle, the reflection coefficients can be written

\[ \rho_{TM} = \frac{\sqrt{\sin^2 \theta_c - \sin^2 \theta - \sin^2 \theta_c \cos \theta}}{\sqrt{\sin^2 \theta_c - \sin^2 \theta + \sin^2 \theta_c \cos \theta}} \]

\[ \rho_{TE} = \frac{\cos \theta - \sqrt{\sin^2 \theta_c - \sin^2 \theta}}{\cos \theta + \sqrt{\sin^2 \theta_c - \sin^2 \theta}} \]

With \( \theta > \theta_c \) this is (using the branch \( \sqrt{-1} = -j \))

\[ \rho_{TM} = \frac{-j\sqrt{\sin^2 \theta - \sin^2 \theta_c - \sin^2 \theta_c \cos \theta}}{-j\sqrt{\sin^2 \theta - \sin^2 \theta_c + \sin^2 \theta_c \cos \theta}} = \frac{1 + jx n^2}{1 - jx n^2} \]

\[ \rho_{TE} = \frac{\cos \theta + j\sqrt{\sin^2 \theta_c - \sin^2 \theta}}{\cos \theta - j\sqrt{\sin^2 \theta_c - \sin^2 \theta}} = \frac{1 + jx}{1 - jx} \]

where \( x = \frac{\sqrt{\sin^2 \theta - \sin^2 \theta_c}}{\cos \theta} \), and \( \sin \theta_c = 1/n \). Thus \( |\rho_{TE,TM}| = 1 \).
Phase shift at total reflection

The TM and TE polarization are reflected with different phase

\[ \rho_{TM} = -\frac{1 + jx n^2}{1 - jx n^2} = e^{j\pi + 2j\psi_{TM}} \]

\[ \rho_{TE} = \frac{1 + jx}{1 - jx} = e^{2j\psi_{TE}} \]

where

\[ \tan \psi_{TM} = xn^2, \quad \tan \psi_{TE} = x \]

The relative phase change between the polarizations is

\[ \frac{\rho_{TM}}{\rho_{TE}} = e^{j\pi + 2j\psi_{TM} - 2j\psi_{TE}} \]

Thus, if \( \theta \) is chosen so that \( \psi_{TM} - \psi_{TE} = \pi/8 \), we have

\[ \frac{\rho_{TM}}{\rho_{TE}} = e^{j\pi + j\pi/4} \Rightarrow \left( \frac{\rho_{TM}}{\rho_{TE}} \right)^2 = e^{2j\pi + j\pi/2} = e^{j\pi/2} \]
Thus, after two reflections the TM and TE polarizations differ in phase by $\pi/2$.

Using a glass with $n = 1.51$, we have $\theta_c = 41.47^\circ$. The angle $54.6^\circ$ results in $\psi_{\text{TM}} - \psi_{\text{TE}} = \pi/8$. The angle $48.6^\circ$ would also work, see Example 7.5.6.

Ideally there is no frequency dependence, that is, the Fresnel rhomb can convert linear to circular polarization in a much wider band than a quarter wavelength plate.
Goos-Hänchen shift

The phase shift in the reflection coefficient also gives rise to the Goos-Hänchen shift (see Example 7.5.7).

\[ D_{\text{TE}} = \frac{2 \sin \theta_0}{k_0 n \sqrt{\sin^2 \theta_0 - \sin^2 \theta_c}}, \quad D_{\text{TM}} = \frac{(n')^2 D_{\text{TE}}}{(n^2 + 1) \sin^2 \theta_0 - (n')^2} \]
The phase shift in the reflection coefficient also gives rise to the Goos-Hänchen shift (see Example 7.5.7).

\[
D_{\text{TE}} = \frac{2 \sin \theta_0}{k_0 n \sqrt{\sin^2 \theta_0 - \sin^2 \theta_c}}, \quad D_{\text{TM}} = \frac{(n')^2 D_{\text{TE}}}{(n^2 + 1) \sin^2 \theta_0 - (n')^2}
\]
Goos-Hänchen shift

Goos-Hänchen shift (n=1.50)

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The Brewster angle (TM polarization)

\[ \rho_{TM} = \frac{\sqrt{(n')^2 - \sin^2 \theta - (n')^2 \cos \theta}}{\sqrt{(n')^2 - \sin^2 \theta + (n')^2 \cos \theta}} \]

\[ \tan \theta_B = \frac{n'}{n} \]
\[ \theta_B + \theta'_B = \frac{\pi}{2} \]
\[ \tan \theta'_B = \frac{n}{n'} \]
Brewster angle, reflection

For glass with $n = 1.5$, we have $\theta_B = 56.3^\circ$ and $\theta'_B = 33.7^\circ$, and $\theta_c = 41.8^\circ$.

Can be used to obtain linear polarization, but loses power through partial transmission of TE component.
Measuring Brewster’s angle between classes

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What happens on the other side at total reflection?

Even though we have total reflection, there are fields on the far side of the interface. The transmission coefficients are

\[ \tau_{TM} = 1 + \rho_{TM}, \quad \tau_{TE} = 1 + \rho_{TE} \]

which are nonzero unless \( \rho_{TM} = \rho_{TE} = -1 \). The \( z \) wavenumbers are

\[ k_z = \sqrt{\omega^2 \mu_0 \epsilon - k_x^2} \]
\[ k'_z = \sqrt{\omega^2 \mu_0 \epsilon' - k_x^2} \]

Since \( k_x = k \sin \theta \), \( k = \omega \sqrt{\mu_0 \epsilon} \), and \( \epsilon' = \epsilon \sin^2 \theta_c \), we have

\[ k'_z = k \sqrt{\sin^2 \theta_c - \sin^2 \theta} = -jk \sqrt{\sin^2 \theta - \sin^2 \theta_c} = -j\alpha' \]

Note the branch \( \sqrt{-1} = -j \) must be taken in order to have exponential decay \( e^{-jk'_z z} = e^{-\alpha' z} \).
Exponential decay

The transmitted wave has spatial dependence

\[ e^{-jk'z} e^{-jk_xx} = e^{-\alpha'z} e^{-j\beta'x} \]

Exponential attenuation in the \( z \)-direction, same transverse phase as in incident wave (\( \beta' = k_x \)).

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Evanescent waves

An *evanescent* wave oscillates so quickly in \( x (k_x > k') \) that it is exponentially attenuated in \( z (k_z = -j\alpha) \) due to \( k_x^2 + k_z^2 = (k')^2 \).

Thus, there is a region close to the surface containing *reactive* fields (non-propagating). The size of the region is on the order

\[
\delta = \frac{1}{\alpha} = \frac{1}{k \sqrt{\sin^2 \theta - \sin^2 \theta_c}} = \frac{\lambda}{2\pi \sqrt{\sin^2 \theta - \sin^2 \theta_c}}
\]
Typical field distribution
Complex waves

The generalization of evanescent waves, which are strictly defined only in lossless media, is necessary for lossy media $\epsilon = \epsilon_R - j\epsilon_I$. In order to avoid using complex angles $\theta$, use the wavenumbers

$$\eta_{TM} = \eta \cos \theta = \frac{\eta k_z}{k} = \frac{k_z}{\omega \epsilon}, \quad \eta_{TE} = \frac{\eta}{\cos \theta} = \frac{\eta k}{k_z} = \frac{\omega \mu}{k_z}$$

The wave vector $k = \beta - j\alpha$ may be complex, but must satisfy

$$k \cdot k = \omega^2 \mu \epsilon$$

The real vectors $\beta$ and $\alpha$ need not be parallel.
Evanescent square root

A recurring task is to take the square root

\[ k_z = \sqrt{\omega^2 \mu_0 \varepsilon - k_x^2} = \beta - j\alpha. \]

In order to guarantee \( \alpha > 0 \), the square root is defined as

\[
k_z = \begin{cases} 
\sqrt{\omega^2 \mu_0 (\varepsilon_R - j\varepsilon_I) - k_x^2} & \text{if } \varepsilon_I \neq 0 \\
-j \sqrt{k_x^2 - \omega^2 \mu_0 \varepsilon_R} & \text{if } \varepsilon_I = 0
\end{cases}
\]

Thus, everything works fine for complex valued material coefficients, but real valued needs some extra treatment for evanescent waves.
Oblique incidence on a lossy medium

To the left: \( k_x = k \sin \theta, \quad k_z = k \cos \theta \).

To the right: \( k'_x = k_x, \quad k'_z = \beta'_z - j\alpha'_z \).

It can be shown that \( \frac{\rho_{TM}}{\rho_{TE}} = \frac{\beta'_z - j\alpha'_z - k \sin \theta \tan \theta}{\beta'_z - j\alpha'_z + k \sin \theta \tan \theta} \), implying elliptic polarization of the reflected wave.

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No standard Brewster angle for lossy media

The TM reflection coefficient is

$$\rho_{TM} = \frac{\eta'_{TM} - \eta_{TM}}{\eta'_{TM} + \eta_{TM}} = \frac{k_z'\epsilon - k_z\epsilon'}{k_z'\epsilon + k_z\epsilon'}$$

which cannot be exactly zero when $\epsilon$, $k_z$ and $k_x$ are real and $\epsilon'$ is complex.

But when $k_x = \beta_x - j\alpha_x$ and $k_z = \beta_z - j\alpha_z$, we can achieve $\rho_{TM} = 0$. This is the Zenneck surface wave.
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The Zenneck surface wave

\[ \rho_{TM} = 0 \]

phase planes

amplitude planes

\[ \psi \]

\[ \phi \]

\[ \alpha \]

\[ \alpha' \]

\[ \beta \]

\[ \beta' \]

\[ x \]

\[ z \]

\[ \epsilon \]

\[ \epsilon' \]
Conditions for the Zenneck wave

The TM reflection coefficient is

\[ \rho_{TM} = \frac{k'_z \epsilon - k_z \epsilon'}{k'_z \epsilon + k_z \epsilon'} = 0 \quad \Rightarrow \quad k'_z \epsilon = k_z \epsilon' \]

Using \( k'^2_x + k'^2_z = \omega^2 \mu_0 \epsilon \) and \( (k'_x)^2 + (k'_z)^2 = \omega^2 \mu_0 \epsilon' \) and \( k_x = k'_x \)
we find

\[ k_x = \omega \sqrt{\mu_0} \frac{\sqrt{\epsilon \epsilon'}}{\sqrt{\epsilon + \epsilon'}}, \quad k_z = \omega \sqrt{\mu_0} \frac{\epsilon}{\sqrt{\epsilon + \epsilon'}}, \quad k'_z = \omega \sqrt{\mu_0} \frac{\epsilon'}{\sqrt{\epsilon + \epsilon'}} \]

This results in complex wave vectors on both sides of the interface. For weakly lossy media (\( \epsilon' = \epsilon_R - j\epsilon_I \) where \( \epsilon_I/\epsilon_R \ll 1 \), we can estimate

\[ \frac{\alpha_x}{|\alpha_z|} = \sqrt{\frac{\epsilon}{\epsilon_R}} \]

Thus, the attenuation in the \( z \)-direction (\( \alpha_z \)) is larger than in the \( x \)-direction (\( \alpha_x \)) if \( \epsilon_R > \epsilon \).
Consider an interface between air \((\epsilon = \epsilon_0)\) and sea water:

\[ \epsilon' = 81\epsilon_0 - j\sigma/\omega, \quad \sigma = 4 \text{ S/m} \]

The wave numbers at 1 GHz and 100 MHz are

<table>
<thead>
<tr>
<th>( f = 1 \text{ GHz} )</th>
<th>( f = 100 \text{ MHz} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( k = \beta - j\alpha = 20.94 )</td>
<td>( k = 2.094 )</td>
</tr>
<tr>
<td>( k' = \beta' - j\alpha' = 203.76 - 77.39j )</td>
<td>( k' = \beta' - j\alpha' = 42.01 - 37.54j )</td>
</tr>
<tr>
<td>( k_x = \beta_x - j\alpha_x = 20.89 - 0.064j )</td>
<td>( k_x = \beta_x - j\alpha_x = 2.1 - 0.001j )</td>
</tr>
<tr>
<td>( k_z = \beta_z - j\alpha_z = 1.88 + 0.71j )</td>
<td>( k_z = \beta_z - j\alpha_z = 0.06 + 0.05j )</td>
</tr>
<tr>
<td>( k'_z = \beta'_z - j\alpha'_z = 202.97 - 77.80j )</td>
<td>( k'_z = \beta'_z - j\alpha'_z = 42.01 - 37.59j )</td>
</tr>
</tbody>
</table>

Thus, the attenuation in the \( z \)-direction is much larger than in the \( x \)-direction, and the wave can propagate relatively freely along the interface.
Typical field distribution, Zenneck wave
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The tangential wavenumber $k_x$ is the same in both mediums due to phase matching.

All standard formulas for normal incidence are valid when considering tangential field components, and splitting the field into TM and TE polarizations.

The normal wavenumber $k_z = \sqrt{\omega^2 \mu \epsilon - k_x^2}$ may be real or imaginary in the lossless case, or complex in the lossy case.

The critical angle is the largest angle of refraction, or the smallest angle of total reflection.

There is a phase shift at total internal reflection, which is different for different polarizations.

Complex wave vectors $k = \beta - j\alpha$ are necessary for lossy media.