The paraxial approximation: beams

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Abstract

We show how the paraxial approximation gives a description of beams of electromagnetic radiation, i.e., “plane waves” with finite extent. This is essentially a brief translation of Section 5.3 in Kristensson’s Swedish book.

1 Introduction

We have seen how we can derive solutions \([E(x, y, z), \eta_0 H(x, y, z)]^T\) on the form \([E(z), \eta_0 H(z)]^T e^{-jk_0 \cdot r}\), where \(k_t = k_x \hat{x} + k_y \hat{y}\) and \(k_t \cdot r = k_x x + k_y y\), for any material model as the solutions of the fundamental equation

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_t(z) \\ \eta_0 H_t(z) \end{pmatrix} = \begin{pmatrix} 0 & -\hat{z} \times \hat{I} \\ \hat{z} \times \hat{I} & 0 \end{pmatrix} \cdot \left[ -jk_0 \begin{pmatrix} \epsilon_{tt} & \xi_{tt} \\ \eta_{tt} & \mu_{tt} \end{pmatrix} \right] \cdot \begin{pmatrix} E_t(z) \\ \eta_0 H_t(z) \end{pmatrix} - jk_0 W \cdot \begin{pmatrix} E_t(z) \\ \eta_0 H_t(z) \end{pmatrix} \tag{1.1}
\]

By assuming waves on the special form

\[
\begin{pmatrix} E_t(z) \\ \eta_0 H_t(z) \end{pmatrix} = \begin{pmatrix} E_t(0) \\ \eta_0 H_t(0) \end{pmatrix} e^{-j\beta z} \tag{1.2}
\]

this turns into the eigenvalue problem

\[
\frac{\beta}{k_0} \begin{pmatrix} E_t(0) \\ \eta_0 H_t(0) \end{pmatrix} = \begin{pmatrix} 0 & -\hat{z} \times \hat{I} \\ \hat{z} \times \hat{I} & 0 \end{pmatrix} \cdot \left[ \begin{pmatrix} \epsilon_{tt} & \xi_{tt} \\ \eta_{tt} & \mu_{tt} \end{pmatrix} - A \right] \cdot \begin{pmatrix} E_t(0) \\ \eta_0 H_t(0) \end{pmatrix} \tag{1.3}
\]

which helps to define the refractive index \(n = \beta/k_0\) and the relative wave impedance \(Z_r\) via \(E_t = Z_r \cdot (\eta_0 H_t \times \hat{z})\).

We now drop the magnetic field and focus on the electric field. We can sum up all possible transverse behavior via the Fourier transform pair

\[
\left\{ \begin{array}{l}
E_t(r, \omega) = \frac{1}{(2\pi)^2} \int\int E_t(k_t, \omega) e^{-jk_0 \cdot r - j\beta z} \, dk_x \, dk_y \\
E_t(k_t, \omega) = \int\int E_t(r, \omega) e^{jk_0 \cdot r} \, dr \, dy
\end{array} \right. \tag{1.4}
\]
Defining the (square of the) total wavenumber as $$k^2 = |k_t|^2 + \beta^2 = k_x^2 + k_y^2 + \beta^2$$ implies
$$\beta = (k^2 - |k_t|^2)^{1/2}$$ (1.5)

For isotropic media it is straightforward to show that $$k = k_0\sqrt{\varepsilon\mu}$$, where $$k_0 = \omega/c_0$$. For each Fourier amplitude $$E_t(k_t, \omega)$$, it is then clear that we can associate a wave propagating with wave factor $$\exp(-j\beta(k_t, \omega)z)$$. Thus, if we know the field distribution at $$z = 0$$, we can represent the field with its Fourier transform and propagate each amplitude separately, and then add the results at a different value of $$z$$ via the inverse transform.

2 Initial distribution

Assume now that the field distribution at $$z = 0$$ is given by a Gaussian distribution as in Figure 1
$$E_t(x, y, z = 0, \omega) = A(\omega)e^{-(x^2+y^2)/(2b^2)},$$ (2.1)

The Fourier transform of a Gaussian is itself a Gaussian, and we have
$$E_t(k_t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} A(\omega)e^{-(x^2+y^2)/(2b^2)+j(k_xx+k_yy)} \, dx \, dy = A(\omega)2\pi b^2e^{-(k_x^2+k_y^2)b^2/2}$$ (2.2)

where we used the general identity
$$\int_{-\infty}^{\infty} e^{-\alpha t^2+\beta t} \, dt = \sqrt{\pi/\alpha}e^{\beta^2/(4\alpha)}, \quad \text{Re} \, \alpha > 0$$ (2.3)
The field in $z \geq 0$ is then given by
\[ E_t(r) = \frac{1}{2\pi} Ab^2 \int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2)/2-\beta(\xi_x+k_y)+j\beta z} \, dk_x \, dk_y \] (2.4)

It is seen that the exponential damping in $k_x$ and $k_y$ causes the integral to be dominated by values of $k_t$ close to zero. This corresponds to waves propagating close to the $z$ axis.

### 3 The paraxial approximation

Observing that most of the contributions come from $k_t \approx 0$, we make the paraxial approximation
\[ \beta = (k^2 - |k_t|^2)^{1/2} = k(1 - |k_t|^2/k^2)^{1/2} = k \left(1 - \frac{1}{2} \frac{|k_t|^2}{k^2} + O(|k_t|^4/k^4)\right) = k - \frac{|k_t|^2}{2k} + \cdots \] (3.1)

The integral can now be computed as
\[ E_t(r) \approx \frac{1}{2\pi} Ab^2 \int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2)/(2\beta^2)-\beta(\xi_x+k_y)-j\beta z} \, dk_x \, dk_y \] (3.2)

Introduce $F^2(z, \omega) = b^2 - j\zeta/2k = b^2(1 - j\xi(z, \omega))$ where $\xi = z/(kb^2)$ is a real quantity. The electric field is then (using (2.3))
\[ E_t(r) = \frac{1}{2\pi} Ab^2 e^{-j\beta z} \frac{2}{F^2} e^{-(x^2+y^2)/(2F^2)} = \frac{A}{1 - j\xi(z, \omega)} e^{-(x^2+y^2)/(2F^2)-j\beta z} \] (3.3)

The corresponding magnetic field is found from Faraday’s law
\[ \nabla \times E = -j\omega B \] (3.4)

The curl of the electric field is
\[
\nabla \times E = -\frac{(x/F^2)\hat{x} \times A - (y/F^2)\hat{y} \times A}{1 - j\xi(z, \omega)} e^{-(x^2+y^2)/(2F^2)-j\beta z} \\
+ \hat{z} \times A e^{-(x^2+y^2)/(2F^2)-j\beta z} \left[ -\frac{j\beta}{1 - j\xi(z, \omega)} + \frac{j/(kb^2)}{(1 - j\xi(z, \omega))^2} \right] \\
\approx -\frac{j\beta \hat{z} \times A}{1 - j\xi(z, \omega)} e^{-(x^2+y^2)/(2F^2)-j\beta z} = -j\beta \hat{z} \times E \] (3.5)

where we used the approximation that $x/F^2$, $y/F^2$, and $1/(kb^2)$ are all much smaller than $k$, which is the same order of approximation as used previously. Assuming standard media with $B = \mu_0 \mu H$ and $D = \epsilon_0 \epsilon E$, we now find
\[ \eta_0 H_t(r) = \sqrt{\frac{\epsilon}{\mu}} \hat{z} \times E_t(r) = \frac{1}{\eta} \hat{z} \times E_t(r) \] (3.6)
which is simply the standard relation for a wave propagating purely in the $z$ direction.

4 Power and beam width

The time average of the Poynting vector is

$$\langle \mathbf{S}(t) \rangle = \frac{1}{2} \text{Re}\{\mathbf{E} \times \mathbf{H}\} = \frac{\hat{z}}{2\eta_0\eta} |\mathbf{E}_i(\mathbf{r})|^2$$  (4.1)

The intensity of the beam is proportional to

$$|\mathbf{E}_i(\mathbf{r})|^2 = \frac{|A|^2}{|1-j(\xi(z))|^2} e^{-(x^2+y^2)\text{Re}(\frac{1}{F})}$$  (4.2)

The width of the beam depends on the $z$ coordinate as

$$B(z) = \frac{1}{\sqrt{\text{Re}(\frac{1}{F})}} = \frac{1}{\sqrt{\text{Re}(\frac{1}{1-j\xi})}} = \frac{b}{\sqrt{1+\xi^2}} = b\xi\sqrt{1+\xi^{-2}}$$  (4.3)

where $\xi = z/(kb^2)$. For large $z$, the beam width is

$$B(z) \rightarrow b\xi(z) = \frac{z}{kb}, \quad z \rightarrow \infty$$  (4.4)

The beam angle $\theta_b$ is thus characterized by

$$\tan \theta_b = \frac{B(z)}{z} = \frac{1}{kb}$$  (4.5)

This means that the smaller the startvalue $b$, the wider the beam angle $\theta_b$. Also, the higher the frequency $\omega \sim k$, the smaller the beam angle $\theta_b$.

5 Conclusions

• There exist solutions where a narrow beam of electromagnetic radiation can propagate close to a preferred direction ($z$ in these notes).

• The width of these beams grow linearly with propagation distance.

• The beam angle is $1/(kb)$, where $b$ is the width of the beam at the origin, and $k$ is the wavenumber in the material.

• A beam with large initial width compared to wavelength ($kb \gg 1$) has small beam angle.
Figure 2: The beam width grows as $z$ increases.