1 Introduction

2 Multiple dielectric slabs

3 Dielectric mirrors

4 Synthesis of frequency response, inverse scattering

5 Conclusions
Outline

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2 Multiple dielectric slabs

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5 Conclusions
Key questions

▶ How can we analyze multilayer structures?
▶ What can we build with them?
▶ How can we construct a multilayer structure with a given frequency response?
▶ What does it “cost” to do good designs?
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Scattering from multilayer structure

\[
\begin{pmatrix}
E_{1+} \\
E_{1-}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\eta_a & 1 \\
1 & -\eta_a
\end{pmatrix} \mathbf{P}_1 \cdots \mathbf{P}_M \begin{pmatrix}
\frac{1}{\eta_b} & 0 \\
0 & -\frac{1}{\eta_b}
\end{pmatrix} \begin{pmatrix}
E'_{M+1,+} \\
0
\end{pmatrix}
\]

\[
\mathbf{P}_i = \begin{pmatrix}
\cos(k_i \ell_i) & j \eta_i \sin(k_i \ell_i) \\
-j \eta_i^{-1} \sin(k_i \ell_i) & \cos(k_i \ell_i)
\end{pmatrix}
\]
Scattering from multilayer structure

\[
\left( \begin{array}{c} E_{1+} \\ E_{1-} \end{array} \right) = P'_1 \cdots P'_M \frac{1}{\tau_{M+1}} \left( \begin{array}{cc} 1 & \rho_{M+1} \\ \rho_{M+1} & 1 \end{array} \right) \left( \begin{array}{c} E'_{M+1,+} \\ 0 \end{array} \right)
\]

\[
P'_i = \frac{1}{\tau_i} \left( \begin{array}{cc} e^{jk_i \ell_i} & \rho_i e^{-jk_i \ell_i} \\ \rho_i e^{jk_i \ell_i} & e^{-jk_i \ell_i} \end{array} \right)
\]

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Scattering parameters

The total transfer matrix relation is

\[
\begin{pmatrix}
E_{1+} \\
E_{1-}
\end{pmatrix} = \begin{pmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{pmatrix} \begin{pmatrix}
E_{M+1,+}' \\
0
\end{pmatrix}
\]

which implies

\[
T = \frac{E_{M+1,+}'}{E_{1+}} = \frac{1}{T_{11}}
\]

\[
\Gamma_1 = \frac{E_{1-}}{E_{1+}} = \frac{T_{21} E_{M+1,+}'}{E_{1+}} = \frac{T_{21}}{T_{11}}
\]

Thus, having computed the total transfer matrix, the reflection and transmission coefficients correspond to simple rearrangements.
Layer recursion for reflection

The reflection coefficient at interface $i$ can be found by recursion

$$\frac{E_{i-}}{E_{i+}} = \Gamma_i = \frac{\rho_i + \Gamma_{i+1}e^{-2jk_i\ell_i}}{1 + \rho_i\Gamma_{i+1}e^{-2jk_i\ell_i}}, \quad \Gamma_{M+1} = \rho_{M+1}$$

and the impedance at interface $i$ in the same way

$$\frac{E_i}{H_i} = Z = \frac{Z_{i+1} + j\eta_i \tan(k_i\ell_i)}{\eta_i + jZ_{i+1} \tan(k_i\ell_i)}, \quad Z_{M+1} = \eta_b$$

These are equivalent. Thus, the reflection properties can be found from a one-pass calculation, iterating from $M + 1$ to 1.
In general, to find also the transmission coefficient, the full cascading technique must be employed. However, in the important case of no losses energy conservation gives us the result

$$\frac{1 - |\Gamma|^2}{\eta_a} = \frac{|T|^2}{\eta_b}$$

Thus, for lossless structures the number $1 - |\Gamma|^2$ represents the transmittance.
From now on, we assume the medium is non-magnetic, that is,

\[ k = \omega \sqrt{\epsilon \mu} = \frac{\omega}{c_0} \sqrt{\epsilon_r} = k_0 n \]

and

\[ \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0} \frac{1}{\sqrt{\epsilon_r}}} = \frac{\eta_0}{n} \]

Thus, a slab is characterized solely by its physical length \( \ell \) and its refractive index \( n = \sqrt{\epsilon_r} \).
% multidiel.m - reflection response of isotropic or birefringent multilayer structure

% na | n1 | n2 | ... | nM | nb
% left medium | L1 | L2 | ... | LM | right medium
% interface 1 2 3 M M+1

% Usage: [Gamma, Z] = multidiel(n, L, lambda, theta, pol)
% [Gamma, Z] = multidiel(n, L, lambda, theta) (equivalent to pol='te')
% [Gamma, Z] = multidiel(n, L, lambda) (equivalent to theta=0)

% n = isotropic 1x(M+2), uniaxial 2x(M+2), or biaxial 3x(M+2), matrix of refractive indices
% L = vector of optical lengths of layers, in units of lambda_0
% lambda = vector of free-space wavelengths at which to evaluate the reflection response
% theta = incidence angle from left medium (in degrees)
% pol = for 'tm' or 'te', parallel or perpendicular, p or s, polarizations

% Gamma = reflection response at interface-1 into left medium evaluated at lambda
% Z = transverse wave impedance at interface-1 in units of eta_a (left medium)

% notes: M is the number of layers (M >= 0)
% n = [na, n1, n2, ..., nM, nb] = 1x(M+2) row vector of isotropic indices
% [ na1 n11 n12 ... n1M nb1 ] 3x(M+2) matrix of birefringent indices,
% [ na2 n21 n22 ... n2M nb2 ] = if 2x(M+2), it is extended to 3x(M+2)
% [ na3 n31 n32 ... n3M nb3 ] by repeating the top row

% optical lengths are in units of a reference free-space wavelength lambda_0:
% for i=1,2,...,M, L(i) = n(1,i) * l(i), for TM,
% L(i) = n(2,i) * l(i), for TE,
% TM and TE L(i) are the same in isotropic case. If M=0, use L=[].
% lambda is in units of lambda_0, that is, lambda/lambda_0 = f_0/f

% reflectance = |Gamma|^2, input impedance = Z = (1+Gamma)./(1-Gamma)
% delta(i) = 2*pi*[n(1,i) * l(i) * sqrt(1 - (Na*sin(theta))^2 ./ n(3,i).^2))]/lambda, for TM
% delta(i) = 2*pi*[n(2,i) * l(i) * sqrt(1 - (Na*sin(theta))^2 ./ n(2,i).^2))]/lambda, for TE
% if n(3,i)=n(3,i+1)=Na, then will get NaN's at theta=90 because of 0/0, (see also FRESNEL)
% it uses SQRTE, which is a modified version of SQRT appropriate for evanescent waves
% see also MULTIDIEL1, MULTIDIEL2

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Application: antireflection coating

\[ n_a = 1 \quad E_i \quad E_r \quad n_a = 1 \quad E_i \quad E_r = 0 \quad n_a = 1 \quad E_i \quad E_r = 0 \]

\[ n_b = 1.5 \]

\[ n_1 = 1.22 \quad n_b = 1.5 \]

\[ n_2 = 1.38 \quad n_3 = 2.45 \]

\[ n_b = 1.5 \]

Antireflection Coatings on Glass

\[ |\Gamma_1(\lambda)|^2 \quad (\text{percent}) \]

\[ \lambda (\text{nm}) \]

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For a quarter wavelength slab, \( n_i \ell_i = \frac{\lambda_0}{4} \), we have

\[
Z_i = \eta_i \frac{Z_{i+1} + j\eta_i \tan(k_i \ell_i)}{\eta_i + jZ_{i+1} \tan(k_i \ell_i)} = \frac{\eta_i^2}{Z_{i+1}}
\]

and for a half wavelength slab, \( n_i \ell_i = \frac{\lambda_0}{2} \), we have

\[
Z_i = \eta_i \frac{Z_{i+1} + j\eta_i \tan(k_i \ell_i)}{\eta_i + jZ_{i+1} \tan(k_i \ell_i)} = Z_{i+1}
\]

Thus, quarter wavelength slabs inverts the impedance, whereas half wavelength slabs preserves impedance.
An impedance approach to antireflection

The goal is to transform the impedance $\eta_b$ to $\eta_a$. Two possibilities:

First possibility (quarter-quarter):

$$\eta_a = \frac{\eta_1^2}{Z_2} = \frac{\eta_1^2}{\eta_2^2/\eta_b} = \eta_b \frac{\eta_1^2}{\eta_2^2} \quad \Rightarrow \quad \frac{n_a}{n_b} = \frac{n_1}{n_2}$$

Second possibility (quarter-half-quarter):

$$\eta_a = \frac{\eta_1^2}{Z_2} = \frac{\eta_1^2}{\eta_3^2/\eta_b} = \eta_b \frac{\eta_1^2}{\eta_3^2} \quad \Rightarrow \quad \frac{n_a}{n_b} = \frac{n_1}{n_3}$$
The design was made for the center wavelength $\lambda = 550\,\text{nm}$, the result for any wavelength is computed with `multidiel.m`:

Thicker structures are usually more broad band.
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Dielectric mirrors

Instead of anti-reflection, we can design a multilayered structure aimed at significant reflection.

Useful to avoid losses in metal structures. Typical design issues: thickness and bandwidth.
Design at center wavelength

Alternating quarter wavelength slabs of high \((n_H)\) and low \((n_L)\) refractive index (short hand notation \(AH(LH)^4G\)):

The impedance at interface 2 is

\[
Z_2 = \frac{\eta_L^2}{Z_3} = \frac{\eta_L^2}{\eta_H^2/Z_4} = \left(\frac{n_H}{n_L}\right)^2 Z_4 = \left(\frac{n_H}{n_L}\right)^4 Z_6 = \cdots = \left(\frac{n_H}{n_L}\right)^8 \eta_b
\]
Reflection

The impedance at interface 1 after $N$ bilayers is

$$Z_1 = \frac{\eta_H^2}{Z_2} = \frac{\eta_0^2/n_H^2}{\left(\frac{n_H}{n_L}\right)^{2N} \eta_b}$$

which implies the reflection coefficient

$$\Gamma_1 = \frac{Z_1 - \eta_a}{Z_1 + \eta_a} = \ldots = \frac{1 - \left(\frac{n_H}{n_L}\right)^{2N} \frac{n_H^2}{n_a n_b}}{1 + \left(\frac{n_H}{n_L}\right)^{2N} \frac{n_H^2}{n_a n_b}} \to -1, \ N \to \infty$$

Thus, for many layers, we get high reflection. Without the final $n_H$ layer, the reflection would have the limit $\Gamma \to +1$. 

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Results for different number of layers

Computed using `multidiel.m` with the parameters $n_a = 1$, $n_b = 1.52$, $n_H = 2.32$, $n_L = 1.38$. Bandwidth is relatively insensitive to the number of layers.
Bandwidth for an infinite structure

Consider the limit case \( N \to \infty \). We look for waves propagating with the effective wave number \( K \), so that there should exist solutions (where \( \delta_L = k_L \ell_L \), \( \delta_H = k_H \ell_H \), and \( \ell = \ell_L + \ell_H \))

\[
\begin{pmatrix}
E_i \\
H_i
\end{pmatrix} = \begin{pmatrix}
\cos(\delta_L) & j\eta_L \sin(\delta_L) \\
-j\eta_L^{-1} \sin(\delta_L) & \cos(\delta_L)
\end{pmatrix}
\cdot \begin{pmatrix}
\cos(\delta_H) & j\eta_H \sin(\delta_H) \\
-j\eta_H^{-1} \sin(\delta_H) & \cos(\delta_H)
\end{pmatrix}
\begin{pmatrix}
E_{i+2} \\
H_{i+2}
\end{pmatrix}
\]

\[
= e^{jK\ell} \begin{pmatrix}
E_i \\
H_i
\end{pmatrix}
\]

This is an eigenvalue equation, which can be put in the form

\[
\cos(K\ell) = \frac{\cos(\delta_H + \delta_L) - \rho^2 \cos(\delta_H - \delta_L)}{1 - \rho^2}
\]

where \( \rho = \frac{n_H - n_L}{n_H + n_L} \). \( K \) is called the Bloch wavenumber.
With $\delta_H = \delta_L = \delta$, the equation is

$$\cos(K\ell) = \frac{\cos(2\delta) - \rho^2}{1 - \rho^2}$$

Propagating waves correspond to real $K$, which implies $|\cos(K\ell)| \leq 1$.

Inside the band gap, the waves are exponentially attenuated.
Bandgaps generalize to 3D structures
Short- and longpass reflectors

Short-hand notation for the dielectric mirror: \( A H(LH)^8 G \), where \( A \) denotes the air medium, \( H \) the high-index medium, \( L \) the low-index medium, and \( G \) is the glass. Two variations:

**Shortpass:** \( A(0.5L)H(LH)^8(0.5L)G \)

**Longpass:** \( A(0.5H)L(HL)^8(0.5H)G \)
Having designed a dielectric mirror, we can use it to construct a Fabry-Perot resonator, which is a narrow-band transmission filter.

Since a half wavelength slab preserves impedance, we can eliminate such slabs at the design frequency. By inserting a low-index material between two bilayer stacks, we obtain

\[(HL)^N L (HL)^N = (HL)^{N-1} HLLHL (HL)^{N-1}\]
\[\rightarrow (HL)^{N-1} HHL (HL)^{N-1}\]
\[\rightarrow (HL)^{N-1} L (HL)^{N-1} \rightarrow \cdots \rightarrow L\]

Adding another outer layer \(L\), the final structure \((HL)^N L (HL)^N L \rightarrow 2L\) allows perfect transmission.
Example of a Fabry-Perot resonator design

Computed using `multidiel.m` with parameters $n_a = n_b = 1.52$, $n_L = 1.4$, and $n_H = 2.1$. 
Two FPRs

\[ G | (HL)^{N_1} L (HL)^{N_1} | (HL)^{N_2} L (HL)^{N_2} | G \]

The bandwidth is controlled by the number of layers.
Note an extra $L$ layer is added. The ripple is decreased by slight increase of middle FPR.
Four FPRs

\[ G | (HL)^{N_1} L (HL)^{N_1} | (HL)^{N_2} L (HL)^{N_2} | (HL)^{N_3} L (HL)^{N_3} | (HL)^{N_4} L (HL)^{N_4} | G \]

No extra \( L \) layer needed. The ripple is decreased by slight increase of the middle FPRs.
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Realization of filters

Often, a design specification is given as requirements on the scattering parameters (reflection level $A$, bandwidth $\Delta f$):

$$|\Gamma(f)|^2$$

The design problem consists in finding the physical structure which realizes these requirements.
Equal travel-time multilayer structures

The two-way travel-time delay is the same in all layers

\[
\frac{2n_1 \ell_1}{c_0} = \frac{2n_2 \ell_2}{c_0} = \ldots = \frac{2n_M \ell_M}{c_0} = T_s
\]

and we define the \( z \)-domain variable (an alternative to \( \omega \))

\[
z = e^{j \omega T_s} = e^{2jk_i \ell_i}
\]
Using the $z = e^{2jk_i\ell_i}$ variable, the propagation

\[
\begin{pmatrix}
E_{i+} \\
E_{i-}
\end{pmatrix}
= \frac{1}{\tau_i}
\begin{pmatrix}
e^{jk_i\ell_i} & \rho_i e^{-jk_i\ell_i} \\
\rho_i e^{jk_i\ell_i} & e^{-jk_i\ell_i}
\end{pmatrix}
\begin{pmatrix}
E_{i+1,+} \\
E_{i+1,-}
\end{pmatrix}
\]

can be written

\[
\begin{pmatrix}
E_{i+} \\
E_{i-}
\end{pmatrix}
= \frac{z^{1/2}}{\tau_i}
\begin{pmatrix}
1 & \rho_i z^{-1} \\
\rho_i & z^{-1}
\end{pmatrix}
\begin{pmatrix}
E_{i+1,+} \\
E_{i+1,-}
\end{pmatrix}
\]
Cascading

The fields at the $i$th interface are then described by

$$
\begin{pmatrix}
E_{i+} \\
E_{i-}
\end{pmatrix}
= \frac{z^{(M+1-i)/2}}{\nu_i}
\begin{pmatrix}
1 & \rho_i z^{-1} \\
\rho_i & z^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & \rho_{i+1} z^{-1} \\
\rho_{i+1} & z^{-1}
\end{pmatrix}
\cdots
\begin{pmatrix}
1 & \rho_M z^{-1} \\
\rho_M & z^{-1}
\end{pmatrix}
\begin{pmatrix}
1 & \rho_{M+1} \\
\rho_{M+1} & 1
\end{pmatrix}
\begin{pmatrix}
E'_{M+1,+}
\end{pmatrix}
$$

$$
= \frac{z^{(M+1-i)/2}}{\nu_i}
\begin{pmatrix}
A_i(z) & C_i(z) \\
B_i(z) & D_i(z)
\end{pmatrix}
\begin{pmatrix}
E'_{M+1,+}
\end{pmatrix}
$$

where $\nu_i = \tau_i \tau_{i+1} \cdots \tau_M \tau_{M+1}$. The matrix elements are polynomials of order $M + 1 - i$ in $z^{-1}$, meaning the reflection coefficient

$$
\Gamma_i(z) = \frac{E_{i-}}{E_{i+}} = \frac{B_i(z)}{A_i(z)}
$$

is then the quotient between two polynomials of order $M + 1 - i$. 

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Recursion

Considering only factors relevant to reflection, the forward recursion (order-increasing in $z^{-1}$) is

$$
\begin{pmatrix}
A_i(z) \\
B_i(z)
\end{pmatrix} =
\begin{pmatrix}
1 & \rho_i z^{-1} \\
\rho_i & z^{-1}
\end{pmatrix}
\begin{pmatrix}
A_{i+1}(z) \\
B_{i+1}(z)
\end{pmatrix}
$$

and the backward recursion (order-decreasing in $z^{-1}$) is

$$
\begin{pmatrix}
A_{i+1}(z) \\
B_{i+1}(z)
\end{pmatrix} = \frac{1}{1 - \rho_i^2}
\begin{pmatrix}
1 & -\rho_i \\
-\rho_i z & z
\end{pmatrix}
\begin{pmatrix}
A_i(z) \\
B_i(z)
\end{pmatrix}
$$

The polynomials are

$$
B_i(z) = \sum_{m=0}^{M+1-i} b_i(m) z^{-m}, \quad A_i(z) = \sum_{m=0}^{M+1-i} a_i(m) z^{-m}
$$

with

$$
b_i(0) = \rho_i, \quad a_i(0) = 1,
\quad b_i(M + 1 - i) = \rho_{M+1}, \quad a_i(M + 1 - i) = \rho_{M+1} \rho_i
$$
A closer look at the recursion

Writing the polynomial coefficients as a vector

\[
\mathbf{a}_i = \begin{pmatrix}
a_i(0) \\
a_i(1) \\
\vdots \\
a_i(M + 1 - i)
\end{pmatrix} \quad \mathbf{b}_i = \begin{pmatrix}
b_i(0) \\
b_i(1) \\
\vdots \\
b_i(M + 1 - i)
\end{pmatrix}
\]

the forward and backward recursion can be written

\[
\mathbf{a}_i = \begin{pmatrix}
\mathbf{a}_{i+1} \\
0
\end{pmatrix} + \rho_i \begin{pmatrix}
0 \\
\mathbf{b}_{i+1}
\end{pmatrix} \quad \begin{pmatrix}
\mathbf{a}_{i+1} \\
0
\end{pmatrix} = \frac{\mathbf{a}_i - \rho_i \mathbf{b}_i}{1 - \rho_i^2}
\]

\[
\mathbf{b}_i = \rho_i \begin{pmatrix}
\mathbf{a}_{i+1} \\
0
\end{pmatrix} + \begin{pmatrix}
0 \\
\mathbf{b}_{i+1}
\end{pmatrix} \quad \begin{pmatrix}
0 \\
\mathbf{b}_{i+1}
\end{pmatrix} = \frac{-\rho_i \mathbf{a}_i + \mathbf{b}_i}{1 - \rho_i^2}
\]
Filter realization

The procedure for designing a structure which realizes the reflection coefficient $\Gamma(f)$ can be summarized as:

1. Change variables to $z = e^{j\omega T_s}$, where $T_s = \frac{2k_i}{c_0}$ determines the optical thickness of the layers (center wavelength).
2. Approximate $\Gamma(z)$ with a polynomial ratio $\frac{B(z)}{A(z)}$, where the order of the polynomials determines the number of layers.
3. Use backwards recursion to find $A_i(z)$ and $B_i(z)$. The coefficients give reflection coefficients $\rho_1, \rho_2, \ldots, \rho_{M+1}$.
4. Calculate refractive indices $n_1, n_2, \ldots, n_{M+1}$ from the reflection coefficients.

The polynomial $A(z)$ must have all its zeros inside the unit circle in the $z$-plane (minimum-phase polynomial in $z^{-1}$, corresponding to stability and causality).
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The polynomial $A(z)$ must have all its zeros inside the unit circle in the $z$-plane (minimum-phase polynomial in $z^{-1}$, corresponding to stability and causality).

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Example: 6.6.1 in Orfanidis

Determine the number of layers $M$, the reflection coefficients at the $M + 1$ interfaces, and the refractive indices of the $M + 2$ media for a multilayer structure whose overall reflection response is given by:

$$\Gamma(z) = \frac{B(z)}{A(z)} = \frac{-0.1 - 0.188z^{-1} - 0.35z^{-2} + 0.5z^{-3}}{1 - 0.1z^{-1} - 0.064z^{-2} - 0.05z^{-3}}$$
Example: 6.6.1 in Orfanidis, solution

**Solution:** From the degree of the polynomials, the number of layers is $M = 3$. The starting polynomials in the backward recursion (6.6.50) are:

$$a_1 = a = \begin{bmatrix} 1.000 \\ -0.100 \\ -0.064 \\ -0.050 \end{bmatrix}, \quad b_1 = b = \begin{bmatrix} -0.100 \\ -0.188 \\ -0.350 \\ 0.500 \end{bmatrix}$$

From the first and last coefficients of $b_1$, we find $\rho_1 = -0.1$ and $\rho_4 = 0.5$. Setting $i = 1$, the first step of the recursion gives:

$$\begin{bmatrix} a_2 \\ 0 \end{bmatrix} = \frac{a_1 - \rho_1 b_1}{1 - \rho_1^2} = \begin{bmatrix} 1.000 \\ -0.120 \\ -0.100 \\ 0.000 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ b_2 \end{bmatrix} = \frac{-\rho_1 a_1 + b_1}{1 - \rho_1^2} = \begin{bmatrix} 0.000 \\ -0.200 \\ -0.360 \\ 0.500 \end{bmatrix}$$

Thus,

$$a_2 = \begin{bmatrix} 1.000 \\ -0.120 \\ -0.100 \end{bmatrix}, \quad b_2 = \begin{bmatrix} -0.200 \\ -0.360 \\ 0.500 \end{bmatrix}$$
The first coefficient of $b_2$ is $\rho_2 = -0.2$ and the next step of the recursion gives:

$$\begin{bmatrix} a_3 \\ 0 \end{bmatrix} = \frac{a_2 - \rho_2 b_2}{1 - \rho_2^2} = \begin{bmatrix} 1.0 \\ -0.2 \\ 0.0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ b_3 \end{bmatrix} = \frac{-\rho_2 a_2 + b_2}{1 - \rho_2^2} = \begin{bmatrix} 0.0 \\ -0.4 \\ 0.5 \end{bmatrix}$$

Thus,

$$a_3 = \begin{bmatrix} 1.0 \\ -0.2 \end{bmatrix}, \quad b_3 = \begin{bmatrix} -0.4 \\ 0.5 \end{bmatrix} \Rightarrow \rho_3 = -0.4$$

The last step of the recursion for $i = 3$ is not necessary because we have already determined $\rho_4 = 0.5$. Thus, the four reflection coefficients are:

$$[\rho_1, \rho_2, \rho_3, \rho_4] = [-0.1, -0.2, -0.4, 0.5]$$

The corresponding refractive indices can be obtained by solving Eq. (6.1.1), that is, $n_i = n_{i-1}(1 - \rho_i) / (1 + \rho_i)$. Starting with $i = 1$ and $n_0 = n_a = 1$, we obtain:

$$[n_a, n_1, n_2, n_3, n_b] = [1, 1.22, 1.83, 4.28, 1.43]$$

The same results can be obtained by working with the polynomial version of the recursion, Eq. (6.6.46).

$\square$
Results

In the green curve, one of the refractive indices have been slightly changed from the computed values.

The whole recursion procedure can be done in matlab by

```matlab
a = [1, -0.1, -0.064, -0.05];
b = [-0.1, -0.188, -0.35, 0.5];
[r,A,B] = bkwrec(a,b);
n = r2n(r);
```
Applications

Impedance matching

Acoustic matching

Oil prospecting

Bragg grating
The design method assumes

\[ |\Gamma(f)|^2 = \frac{e_1^2 T_M^2(x)}{1 + e_1^2 T_M^2(x)} \]

\[ x = x_0 \cos \delta = x_0 \cos \left( \frac{\pi f}{2f_0} \right) \]

where \( T_M(x) = \cos(M \arccos(x)) \), and the design parameters are \( e_1 \) (attenuation), and \( x_0 \) (bandwidth). \( M \) must be “large enough”.
The Chebyshev design is well established, and is coded in

\[
\begin{align*}
[n,a,b] &= \text{chebtr}(na,nb,A,DF); \quad \% \text{Chebyshev multilayer design} \\
[n,a,b,A] &= \text{chebtr2}(na,nb,M,DF); \quad \% \text{specify order and bandwidth} \\
[n,a,b,DF] &= \text{chebtr3}(na,nb,M,A); \quad \% \text{specify order and attenuation}
\end{align*}
\]

Note that only two of the design parameters (attenuation, bandwidth, order) are set at once. The remaining parameter is a consequence of the others.
Outline

1 Introduction
2 Multiple dielectric slabs
3 Dielectric mirrors
4 Synthesis of frequency response, inverse scattering
5 Conclusions
Conclusions

- Multilayer structures are easily analyzed by cascading techniques.
- Typical designs are based on quarter wavelength and half wavelength slabs.
- A stack of high/low index slabs can form a dielectric mirror.
- An arbitrary frequency response can be synthesized in a structured design process.
- The order of the response function typically corresponds to the number of quarter wavelength layers.
- High bandwidth and high attenuation requires many layers.