Electromagnetic Wave Propagation
Lecture 5: Propagation in birefringent media

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Outline

1 Introduction

2 Wave propagation in isotropic media

3 Wave propagation in bianisotropic media

4 Interpretation of the fundamental equation

5 Maxwell’s equations for linear and circular polarization

6 Uniaxial and biaxial media

7 Chiral media

8 Gyrotropic media

9 Oblique propagation in biaxial media
Key questions

- How can we compute the wave number and wave impedance in bianisotropic materials?
- What is the effect if a material has different properties for different polarizations?
- How can we convert between linear and circular polarization?
- What are the effects when a wave is not propagating along an optical axis?
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Propagation in isotropic media

Maxwell’s equations for nondispersive, isotropic media are

\[ \nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \]
\[ \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \]

Assuming the fields do not depend on \( x \) or \( y \) implies

\[ \nabla \times \mathbf{E} = \hat{z} \times \frac{\partial \mathbf{E}}{\partial z} \]

\[ \begin{align*}
\hat{z} \times \frac{\partial \mathbf{E}}{\partial z} &= -\mu \frac{\partial \mathbf{H}}{\partial t} \\
\hat{z} \times \frac{\partial \mathbf{H}}{\partial z} &= \epsilon \frac{\partial \mathbf{E}}{\partial t}
\end{align*} \]

\[ \Rightarrow \frac{\partial}{\partial z} \left( \mathbf{E} \mathbf{H} \times \hat{z} \right) = -\frac{\partial}{\partial t} \left( \begin{bmatrix} 0 & \mu \\ \epsilon & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \times \hat{z} \end{bmatrix} \right) \]

The cross products with \( \hat{z} \) make the \( z \)-components vanish.
The wave equation

Eliminating the magnetic field implies

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E(z, t) = 0
\]

with the solution

\[
E(z, t) = F_+(z - ct) + F_-(z + ct)
\]

The corresponding magnetic field is

\[
H(z, t) = \frac{1}{\eta} \hat{z} \times F_+(z - ct) - \frac{1}{\eta} \hat{z} \times F_-(z + ct)
\]

The minus sign for the wave in negative \( z \)-direction is due to that both \((E_+, H_+, \hat{z})\) and \((E_-, H_-, -\hat{z})\) must be right-handed triples. The wave speed is \( c = 1/\sqrt{\epsilon\mu} \) and the wave impedance is \( \eta = \sqrt{\mu/\epsilon} \).
Some fundamental properties are observed:

- The wave speed $c = 1/\sqrt{\varepsilon \mu}$ and wave impedance $\eta = \sqrt{\mu/\varepsilon}$ depend on the material properties.
- Waves can propagate in the positive or negative $z$-direction.
- For each propagation direction, there are two possible polarizations ($\hat{x}$ and $\hat{y}$, or RCP and LCP etc).
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Wave propagation in general media

We now generalize to arbitrary materials in the frequency domain. Our plan is the following:

1. Write up the full Maxwell's equations in the frequency domain.
2. Assume the dependence on $x$ and $y$ appear at most through a factor $e^{-j(k_x x + k_y y)}$.
3. Separate the transverse components ($x$ and $y$) from the $z$ components of the fields.
4. Eliminate the $z$ components.
5. Identify the resulting differential equation as a dynamic system for the transverse components,

$$\frac{\partial}{\partial z} \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix} = -j\omega \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \cdot \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}$$
Maxwell’s equations in the frequency domain are

\[ \nabla \times H = j\omega D = j\omega \left[ \epsilon(\omega) \cdot E + \xi(\omega) \cdot H \right] \]

\[ \nabla \times E = -j\omega B = -j\omega \left[ \zeta(\omega) \cdot E + \mu(\omega) \cdot H \right] \]

The bianisotropic material is described by the dyadics \( \epsilon(\omega), \xi(\omega), \zeta(\omega), \) and \( \mu(\omega) \). The frequency dependence is suppressed in the following.
Assume that the fields depend on $x$ and $y$ only through a factor $e^{-j(k_xx+k_yy)}$ (corresponding to a Fourier transform in $x$ and $y$)

$$E(x, y, z) = E(z)e^{-jk_t \cdot r} \quad \text{where} \quad k_t = k_x \hat{x} + k_y \hat{y}$$

Since $\frac{\partial}{\partial x} e^{-jk_xx} = -jk_x e^{-jk_xx}$, the action of the curl operator is then

$$\nabla \times E(x, y, z) = e^{-jk_t \cdot r} \left( -jk_t + \frac{\partial}{\partial z} \hat{z} \right) \times E(z)$$
3) Separate the components, curls

Split the fields according to

\[ E = E_t + \hat{z}E_z \]

where \( E_t \) is in the \( xy \)-plane, and \( \hat{z}E_z \) is the \( z \)-component. In the curls, we have

\[ \left( -jk_t + \frac{\partial}{\partial z} \hat{z} \right) \times E(z) = - jk_t \times E_t - jk_t \times \hat{z}E_z + \frac{\partial}{\partial z} \hat{z} \times E_t \]

parallel to \( \hat{z} \)

orthogonal to \( \hat{z} \)

orthogonal to \( \hat{z} \)

There is no term with \( \frac{\partial E_z}{\partial z} \), since \( \hat{z} \times \hat{z} = 0 \).
3) Separate the components, fluxes

Split the material dyadics as

\[
\begin{pmatrix}
D_t \\
\hat{z}D_z
\end{pmatrix}
= 
\begin{pmatrix}
\epsilon_{tt} & \epsilon_t \hat{z} \\
\hat{z} \epsilon_z & \epsilon_{zz} \hat{z} \hat{z}
\end{pmatrix}
\cdot 
\begin{pmatrix}
E_t \\
\hat{z} E_z
\end{pmatrix}
+ 
\begin{pmatrix}
\xi_{tt} & \xi_t \hat{z} \\
\hat{z} \xi_z & \xi_{zz} \hat{z} \hat{z}
\end{pmatrix}
\cdot 
\begin{pmatrix}
H_t \\
\hat{z} H_z
\end{pmatrix}
\]

\(\epsilon_{tt}\) can be represented as a 2 \(\times\) 2 matrix operating on \(xy\) components, \(\epsilon_t\) and \(\epsilon_z\) are vectors in the \(xy\)-plane, and \(\epsilon_{zz}\) is a scalar.

The transverse components of the electric flux are (vector equation)

\[
D_t = \epsilon_{tt} \cdot E_t + \epsilon_t E_z + \xi_{tt} \cdot H_t + \xi_t H_z
\]

and the \(z\) component is (scalar equation)

\[
D_z = \epsilon_z \cdot E_t + \epsilon_{zz} E_z + \xi_z \cdot H_t + \xi_{zz} H_z
\]
4) Eliminate the $z$ components

We first write Maxwell’s equations using matrices

$$\frac{\partial}{\partial z} \begin{pmatrix} 0 & -\hat{z} \times \mathbf{I} \\ \hat{z} \times \mathbf{I} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \begin{pmatrix} 0 & -j \mathbf{k}_t \times \mathbf{I} \\ j \mathbf{k}_t \times \mathbf{I} & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} - j \omega \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}$$

The $z$ components of these equations are (using $\hat{z} \cdot (\mathbf{k}_t \times \hat{z} E_z) = 0$ and $\hat{z} \cdot (\mathbf{k}_t \times \mathbf{E}_t) = (\hat{z} \times \mathbf{k}_t) \cdot \mathbf{E}_t$)

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & -j \hat{z} \times \mathbf{k}_t \\ j \hat{z} \times \mathbf{k}_t & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} - j \omega \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} - j \omega \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} \begin{pmatrix} \mathbf{E}_z \\ \mathbf{H}_z \end{pmatrix}$$

from which we solve for the $z$ components of the fields:

$$\begin{pmatrix} \mathbf{E}_z \\ \mathbf{H}_z \end{pmatrix} = \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \left[ \begin{pmatrix} 0 & -\omega^{-1} \hat{z} \times \mathbf{k}_t \\ \omega^{-1} \hat{z} \times \mathbf{k}_t & 0 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} - \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}_t \\ \mathbf{H}_t \end{pmatrix} \right]$$
The transverse part of Maxwell’s equations are

\[
\frac{\partial}{\partial z} \begin{pmatrix}
0 & -\hat{z} \times I \\
\hat{z} \times I & 0
\end{pmatrix} \cdot \begin{pmatrix}
E_t \\
H_t
\end{pmatrix} = \begin{pmatrix}
0 & -jk_t \times \hat{z} \\
jk_t \times \hat{z} & 0
\end{pmatrix} \begin{pmatrix}
E_z \\
H_z
\end{pmatrix}
- j\omega \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\zeta_{tt} & \mu_{tt}
\end{pmatrix} \cdot \begin{pmatrix}
E_t \\
H_t
\end{pmatrix} - j\omega \begin{pmatrix}
\epsilon_t & \xi_t \\
\zeta_t & \mu_t
\end{pmatrix} \begin{pmatrix}
E_z \\
H_z
\end{pmatrix}
\]

Inserting the expression for \([E_z, \eta_0 H_z]\) previously derived implies

\[
\frac{\partial}{\partial z} \begin{pmatrix}
0 & -\hat{z} \times I \\
\hat{z} \times I & 0
\end{pmatrix} \cdot \begin{pmatrix}
E_t \\
H_t
\end{pmatrix} = -j\omega \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\zeta_{tt} & \mu_{tt}
\end{pmatrix} \cdot \begin{pmatrix}
E_t \\
H_t
\end{pmatrix}
+ j\omega A \cdot \begin{pmatrix}
E_t \\
H_t
\end{pmatrix}
\]

where the matrix \(A\) is due to the \(z\) components.
4) The $\mathbf{A}$ matrix

The matrix $\mathbf{A}$ is a dyadic product

$$\mathbf{A} = \left[ \begin{pmatrix} 0 & -\omega^{-1} k_t \times \hat{z} \\ \omega^{-1} k_t \times \hat{z} & 0 \end{pmatrix} - \begin{pmatrix} \epsilon_t & \xi_t \\ \zeta_t & \mu_t \end{pmatrix} \right]$$

$$\left( \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} \right)^{-1} \left[ \begin{pmatrix} 0 & -\omega^{-1} \hat{z} \times k_t \\ \omega^{-1} \hat{z} \times k_t & 0 \end{pmatrix} - \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \right]$$

In particular, when $k_t = 0$ we have

$$\mathbf{A} = \begin{pmatrix} \epsilon_t & \xi_t \\ \zeta_t & \mu_t \end{pmatrix} \left( \begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix} \right)^{-1} \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix}$$

and for a uniaxial material with the axis of symmetry along the $\hat{z}$ direction, where $\epsilon_t = \xi_t = \zeta_t = \mu_t = 0$ and $\epsilon_z = \xi_z = \zeta_z = \mu_z = 0$, we have $\mathbf{A} = 0$. 
4) The $\mathbf{A}$ matrix, oblique propagation

Assuming $k_t \neq 0$ and isotropic media, implies (writing $a = \frac{k_t}{k_0} \times \hat{z} = c_0 \omega^{-1} k_t \times \hat{z}$)

$$\mathbf{A} = \begin{bmatrix} \begin{pmatrix} 0 & -\omega^{-1} k_t \times \hat{z} \\ \omega^{-1} k_t \times \hat{z} & 0 \end{pmatrix} & \begin{pmatrix} \epsilon_t & \xi_t \\ \zeta_t & \mu_t \end{pmatrix} \end{pmatrix} - \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix}$$

$$\begin{pmatrix} \epsilon_{zz} & \xi_{zz} \\ \zeta_{zz} & \mu_{zz} \end{pmatrix}^{-1} \begin{pmatrix} \begin{pmatrix} 0 & -\omega^{-1} \hat{z} \times k_t \\ \omega^{-1} \hat{z} \times k_t & 0 \end{pmatrix} & \begin{pmatrix} \epsilon_z & \xi_z \\ \zeta_z & \mu_z \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -c_0^{-1} a \\ c_0^{-1} a & 0 \end{pmatrix} \begin{pmatrix} \epsilon^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \begin{pmatrix} 0 & c_0^{-1} a \\ -c_0^{-1} a & 0 \end{pmatrix}$$

$$= \frac{1}{c_0^2} \begin{pmatrix} \mu^{-1} a a & 0 \\ 0 & \epsilon^{-1} a a \end{pmatrix}$$

The vector $a$ has amplitude $|a| = |k_t|/k_0 = \sin \theta$, and indicates the direction of a TE-polarized electric field.
5) Dynamical system

Using $-\hat{z} \times (\hat{z} \times \mathbf{E}_t) = \mathbf{E}_t$ for any transverse vector $\mathbf{E}_t$, it is seen that

\[
\begin{pmatrix}
0 & -\hat{z} \times \mathbf{I} \\
\mathbf{I} & 0
\end{pmatrix} \cdot \begin{pmatrix}
0 & -\hat{z} \times \mathbf{I} \\
\hat{z} \times \mathbf{I} & 0
\end{pmatrix} \cdot \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t
\end{pmatrix} = \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t \times \hat{z}
\end{pmatrix}
\]

The final form of Maxwell’s equations is then

\[
\frac{\partial}{\partial z} \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t \times \hat{z}
\end{pmatrix} = \begin{pmatrix}
0 & -\hat{z} \times \mathbf{I} \\
\mathbf{I} & 0
\end{pmatrix} \cdot \left[ -j\omega \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\zeta_{tt} & \mu_{tt}
\end{pmatrix} + j\omega \mathbf{A} \right] \cdot \begin{pmatrix}
\mathbf{I} & 0 \\
0 & \hat{z} \times \mathbf{I}
\end{pmatrix} \cdot \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t \times \hat{z}
\end{pmatrix}
\]

or

\[
\frac{\partial}{\partial z} \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t \times \hat{z}
\end{pmatrix} = -j\omega \mathbf{W} \cdot \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t \times \hat{z}
\end{pmatrix}
\]

This can also be written as (just a matter of convention)

\[
\frac{\partial}{\partial z} \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t
\end{pmatrix} = \begin{pmatrix}
0 & -\hat{z} \times \mathbf{I} \\
\hat{z} \times \mathbf{I} & 0
\end{pmatrix} \cdot \left[ -j\omega \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\zeta_{tt} & \mu_{tt}
\end{pmatrix} + j\omega \mathbf{A} \right] \cdot \begin{pmatrix}
\mathbf{E}_t \\
\mathbf{H}_t
\end{pmatrix}
\]
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If the material parameters are constant, the dynamical system

$$\frac{\partial}{\partial z} \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix} = -j \omega W \cdot \begin{pmatrix} E_t \\ H_t \times \hat{z} \end{pmatrix}$$

has the formal solution

$$\begin{pmatrix} E_t(z_2) \\ H_t(z_2) \times \hat{z} \end{pmatrix} = \exp \left(-j \omega (z_2 - z_1) W\right) \cdot \underbrace{\begin{pmatrix} E_t(z_1) \\ H_t(z_1) \times \hat{z} \end{pmatrix}}_{P(z_1, z_2)}$$

The matrix $P(z_1, z_2)$ is called a propagator. It maps the transverse fields at the plane $z_1$ to the plane $z_2$. The propagator exists even if $W$ does depend on $z$, due to the linearity of the problem.
For media with infinite extension in $z$, it is natural to look for solutions on the form

$$
\begin{pmatrix}
E_t(z) \\
H_t(z) \times \hat{z}
\end{pmatrix}
= \begin{pmatrix}
E_t(0) \\
H_t(0) \times \hat{z}
\end{pmatrix} e^{-j\beta z}
$$

This implies $\frac{\partial}{\partial z}[E_t(z), H_t(z) \times \hat{z}] = -j\beta[E_t(0), H_t(0) \times \hat{z}] e^{-j\beta z}$. We then get the algebraic eigenvalue problem

$$
\frac{\beta}{\omega} \begin{pmatrix}
E_t(0) \\
H_t(0) \times \hat{z}
\end{pmatrix} = W \cdot \begin{pmatrix}
E_t(0) \\
H_t(0) \times \hat{z}
\end{pmatrix} = \begin{pmatrix}
0 & -\hat{z} \times I \\
I & 0
\end{pmatrix} \cdot \begin{pmatrix}
(\epsilon_{tt} \xi_{tt}) - A \\
(\zeta_{tt} \mu_{tt})
\end{pmatrix} \cdot \begin{pmatrix}
I & 0 \\
0 & \hat{z} \times I
\end{pmatrix} \cdot \begin{pmatrix}
E_t(0) \\
H_t(0) \times \hat{z}
\end{pmatrix}
$$
Interpretation of the eigenvalue problem

Since the wave is multiplied by the exponential factor
\[ e^{j(\omega t - \beta z)} = e^{j\omega(t - z\beta/\omega)} = e^{j\omega(t - z/c)} \]
we identify the eigenvalue as the inverse of the phase velocity

\[ \frac{\beta}{\omega} = \frac{1}{c} = \frac{n}{c_0} \]

which defines the \textit{refractive index} of the wave. The transverse \textit{wave impedance} is defined through the relation

\[ \mathbf{E}_t(0) = \mathbf{Z} \cdot (\mathbf{H}_t(0) \times \hat{z}) \]

where \([\mathbf{E}_t(0), \mathbf{H}_t(0) \times \hat{z}]\) is the corresponding eigenvector.
Example: “standard” media

An isotropic medium is described by

\[
\begin{pmatrix}
\epsilon & \xi \\
\zeta & \mu
\end{pmatrix} =
\begin{pmatrix}
\epsilon I & 0 \\
0 & \mu I
\end{pmatrix}
\]

For \( k_t = 0 \), we have \( A = 0 \) and

\[
W = \begin{pmatrix}
0 & -\hat{z} \times I \\
I & 0
\end{pmatrix} \cdot \left[ \begin{pmatrix}
\epsilon_{tt} & \xi_{tt} \\
\zeta_{tt} & \mu_{tt}
\end{pmatrix} - A \right] \cdot \begin{pmatrix}
I & 0 \\
0 & \hat{z} \times I
\end{pmatrix}
\]

\[
= \begin{pmatrix}
0 & \mu I \\
\epsilon I & 0
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & \mu & 0 \\
0 & 0 & 0 & \mu \\
\epsilon & 0 & 0 & 0 \\
0 & \epsilon & 0 & 0
\end{pmatrix}
\]
The eigenvalues are found from the characteristic equation
\[
\det(\lambda \mathbf{I} - \mathbf{W}) = 0
\]

Thus, the eigenvalues are (with double multiplicity)
\[
\frac{\beta}{\omega} = \lambda = \pm \sqrt{\epsilon \mu}
\]
Eigenvectors

Using that $H_t \times \hat{z} = \hat{x} H_y - \hat{y} H_x$, the eigenvectors are (top row corresponds to $\beta/\omega = \sqrt{\epsilon \mu}$, bottom row to $\beta/\omega = -\sqrt{\epsilon \mu}$):

$$
\begin{pmatrix}
E_x \\
0 \\
H_y \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
E_y \\
0 \\
-H_x
\end{pmatrix}, \quad Z = \frac{E_x}{H_y} = \frac{E_y}{-H_x} = \sqrt{\frac{\mu}{\epsilon}} = \eta
$$

$$
\begin{pmatrix}
E_x \\
0 \\
H_y \\
0
\end{pmatrix}, \quad \begin{pmatrix}
0 \\
E_y \\
0 \\
-H_x
\end{pmatrix}, \quad Z = \frac{E_x}{H_y} = \frac{E_y}{-H_x} = -\sqrt{\frac{\mu}{\epsilon}} = -\eta
$$
Oblique propagation

For oblique propagation we have

\[
\frac{\partial}{\partial z} \left( \begin{bmatrix} E_t \\ H_t \times \hat{z} \end{bmatrix} \right) =
\begin{bmatrix} 0 & -\hat{z} \times \mathbf{I} \\ \mathbf{I} & 0 \end{bmatrix} \cdot \left[ -j\omega \begin{bmatrix} \epsilon \mathbf{I} & 0 \\ 0 & \mu \mathbf{I} \end{bmatrix} + j\omega \frac{1}{c_0^2} \begin{bmatrix} \mu^{-1} aa & 0 \\ 0 & \epsilon^{-1} aa \end{bmatrix} \right] \cdot \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \hat{z} \times \mathbf{I} \end{bmatrix} \cdot \begin{bmatrix} E_t \\ H_t \times \hat{z} \end{bmatrix}
\]

\[
= \cdots = -j\omega \begin{bmatrix} 0 \\ \epsilon \mathbf{I} - c_0^{-2} \mu^{-1} aa \\ \mu \mathbf{I} - c_0^{-2} \epsilon^{-1} bb \\ 0 \end{bmatrix} \begin{bmatrix} E_t \\ H_t \times \hat{z} \end{bmatrix}
\]

where \( \mathbf{a} = \frac{k_t}{k_0} \times \hat{z} \) is the direction of TE polarized electric field and \( \mathbf{b} = \frac{k_t}{k_0} \) is the direction of TM polarized electric field, with amplitudes \( |\mathbf{a}| = |\mathbf{b}| = \sin \theta \). Wave number and wave impedance are

\[
\beta = \omega \sqrt{\epsilon \mu - c_0^{-2} \sin^2 \theta}, \quad Z = \frac{\omega \mu}{\beta} \frac{aa}{|a|^2} + \frac{\beta}{\omega \epsilon} \frac{bb}{|b|^2}
\]
The propagator matrix for isotropic media can be represented as (after some calculations)

\[
\begin{pmatrix}
E_t(z_2) \\
H_t(z_2) \times \hat{z}
\end{pmatrix} = \begin{pmatrix}
\cos(\beta \ell)I & j \sin(\beta \ell)Z \\
 j \sin(\beta \ell)Z^{-1} & \cos(\beta \ell)I
\end{pmatrix} \cdot \begin{pmatrix}
E_t(z_1) \\
H_t(z_1) \times \hat{z}
\end{pmatrix}
\]

which is recognized as the ABCD matrix for a homogeneous transmission line.
Conclusions for propagation in bianisotropic media

- We have found a way of computing time harmonic waves in any bianisotropic material.
- The transverse components ($x$ and $y$) are crucial.
- There are four fundamental waves: two propagation directions and two polarizations.
- Each wave is described by
  - The wave number $\beta$ (eigenvalue of $W$)
  - The polarization and wave impedance (eigenvector of $W$)
- The propagator $P(z_1, z_2) = \exp(-jk_0(z_2 - z_1)W)$ maps fields at $z_1$ to fields at $z_2$.
- A simple computer program is available on the course web for calculations of $W$.  

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Linear and circular polarization

To describe fields in the $xy$-plane, we can either use linear polarization

\[ \hat{x} \text{ and } \hat{y} \]

or circular polarization (since there is no propagation direction there is no connection to left or right)

\[ \hat{e}_+ = \hat{x} - j\hat{y} \text{ and } \hat{e}_- = \hat{x} + j\hat{y} \]

We typically write

\[ \mathbf{E} = E_x \hat{x} + E_y \hat{y} = E_+ \hat{e}_+ + E_- \hat{e}_- \]

where

\[ E_+ = \frac{1}{2}(E_x + jE_y) \quad E_x = E_+ + E_- \]

\[ E_- = \frac{1}{2}(E_x - jE_y) \quad E_y = \frac{1}{j}(E_+ - E_-) \]
Maxwell’s equations in one dimension

Maxwell’s equations in one dimension are

\[
\begin{align*}
\hat{z} \times \partial_z E &= -j \omega B \\
\hat{z} \times \partial_z H &= j \omega D
\end{align*}
\]
Maxwell’s equations in one dimension

Maxwell’s equations in one dimension are

\[
\begin{align*}
\hat{z} \times \partial_z E &= -j\omega B \\
\hat{z} \times \partial_z H &= j\omega D
\end{align*}
\]

With linear polarization \((E \ H) = (E_x \hat{x}) \) or \((E_y \hat{y})\)

\[
\begin{align*}
\partial_z E_x &= -j\omega B_y \\
\partial_z H_y &= -j\omega D_x \\
\partial_z E_y &= j\omega B_x \\
\partial_z H_x &= j\omega D_y
\end{align*}
\]
Maxwell’s equations in one dimension

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\[
\begin{align*}
\hat{z} \times \partial_z E &= -j \omega B \\
\hat{z} \times \partial_z H &= j \omega D
\end{align*}
\]

With linear polarization \((\mathbf{E} \mathbf{H}) = (E_x \hat{\mathbf{x}} \ H_y \hat{\mathbf{y}})\) or \((E_y \hat{\mathbf{y}} \ H_x \hat{\mathbf{x}})\)

\[
\begin{align*}
\partial_z E_x &= -j \omega B_y \\
\partial_z H_y &= -j \omega D_x
\end{align*}
\]

\[
\begin{align*}
\partial_z E_y &= j \omega B_x \\
\partial_z H_x &= j \omega D_y
\end{align*}
\]

With circular polarization \((\mathbf{E} \mathbf{H}) = (E_+ (\hat{\mathbf{x}} - j \hat{\mathbf{y}}) \ H_+ (\hat{\mathbf{x}} - j \hat{\mathbf{y}}))\) or \((E_- (\hat{\mathbf{x}} + j \hat{\mathbf{y}}) \ H_- (\hat{\mathbf{x}} + j \hat{\mathbf{y}}))\)

(using \(\hat{z} \times (\hat{\mathbf{x}} \mp j \hat{\mathbf{y}}) = \pm j (\hat{\mathbf{x}} \mp j \hat{\mathbf{y}})\))

\[
\begin{align*}
\partial_z E_+ &= -\omega B_+ \\
\partial_z H_+ &= \omega D_+
\end{align*}
\]

\[
\begin{align*}
\partial_z E_- &= \omega B_- \\
\partial_z H_- &= -\omega D_-
\end{align*}
\]
Outline

1 Introduction
2 Wave propagation in isotropic media
3 Wave propagation in bianisotropic media
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6 Uniaxial and biaxial media
7 Chiral media
8 Gyrotropic media
9 Oblique propagation in biaxial media
Anisotropic media

*Biaxial* media are described by

\[
\begin{pmatrix}
D_x \\
D_y \\
D_z
\end{pmatrix} =
\begin{pmatrix}
\epsilon_1 & 0 & 0 \\
0 & \epsilon_2 & 0 \\
0 & 0 & \epsilon_3
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\]

If two parameters are equal, \( \epsilon_2 = \epsilon_3 = \epsilon_0 \), the material is *uniaxial*. Often the refractive index \( n = \sqrt{\epsilon/\epsilon_0} \) is tabulated. Examples (at \( \lambda = 590 \text{ nm} \), stronger anisotropy possible for composite materials):

<table>
<thead>
<tr>
<th>Uniaxial material</th>
<th>( n_0 )</th>
<th>( n_e )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ice ( \text{H}_2\text{O} )</td>
<td>1.309</td>
<td>1.313</td>
</tr>
<tr>
<td>Calcite ( \text{CaCO}_3 )</td>
<td>1.658</td>
<td>1.486</td>
</tr>
<tr>
<td>Rutile ( \text{TiO}_2 )</td>
<td>2.616</td>
<td>2.903</td>
</tr>
<tr>
<td>Sapphire ( \text{Al}_2\text{O}_3 )</td>
<td>1.768</td>
<td>1.760</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Biaxial material</th>
<th>( n_1 )</th>
<th>( n_2 )</th>
<th>( n_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Perovskite ( \text{CaTiO}_3 )</td>
<td>2.300</td>
<td>2.340</td>
<td>2.380</td>
</tr>
<tr>
<td>Topaz ( \text{Al}_2\text{SiO}_4(\text{F, OH})_2 )</td>
<td>1.618</td>
<td>1.620</td>
<td>1.627</td>
</tr>
</tbody>
</table>
Maxwell’s equations for linear polarization are

\[
\begin{aligned}
\partial_z E_x &= -j\omega\mu_0 H_y \\
\partial_z H_y &= -j\omega\epsilon_1 E_x \\
\partial_z E_y &= j\omega\mu_0 H_x \\
\partial_z H_x &= j\omega\epsilon_2 E_y
\end{aligned}
\]
Solutions in linear polarization

Maxwell’s equations for linear polarization are

\[
\begin{align*}
\partial_z E_x &= -j\omega \mu_0 H_y \\
\partial_z H_y &= -j\omega \epsilon_1 E_x \\
\partial_z E_y &= j\omega \mu_0 H_x \\
\partial_z H_x &= j\omega \epsilon_2 E_y
\end{align*}
\]

There is no coupling between the equations, and eliminating \( H_x \) and \( H_y \) implies

\[
\begin{align*}
\partial_z^2 E_x &= -\omega^2 \mu_0 \epsilon_1 E_x \\
\partial_z^2 E_y &= -\omega^2 \mu_0 \epsilon_2 E_y
\end{align*}
\]
Solutions in linear polarization

Maxwell’s equations for linear polarization are

\[
\begin{align*}
\partial_z E_x &= -j\omega \mu_0 H_y \\
\partial_z H_y &= -j\omega \epsilon_1 E_x \\
\partial_z E_y &= j\omega \mu_0 H_x \\
\partial_z H_x &= j\omega \epsilon_2 E_y
\end{align*}
\]

There is no coupling between the equations, and eliminating \( H_x \) and \( H_y \) implies

\[
\begin{align*}
\partial_z^2 E_x &= -\omega^2 \mu_0 \epsilon_1 E_x \\
\partial_z^2 E_y &= -\omega^2 \mu_0 \epsilon_2 E_y
\end{align*}
\]

with the forward moving solutions

\[
\begin{align*}
E_x(z) &= A e^{-jk_1 z} \\
E_y(z) &= B e^{-jk_2 z}
\end{align*}
\]

\[
\begin{align*}
k_1 &= \omega \sqrt{\mu_0 \epsilon_1} = k_0 n_1 \\
k_2 &= \omega \sqrt{\mu_0 \epsilon_2} = k_0 n_2
\end{align*}
\]
The total propagated field at distance $z = \ell$ is (with $\mathbf{E}(0) = A\mathbf{\hat{x}} + B\mathbf{\hat{y}}$)

$$\mathbf{E}(z) = \mathbf{\hat{x}} A e^{-jk_1\ell} + \mathbf{\hat{y}} B e^{-jk_2\ell} = \left[\mathbf{\hat{x}} A + \mathbf{\hat{y}} B e^{j(k_1-k_2)\ell}\right] e^{-j k_1 \ell}$$

The phase difference due to propagation is

$$\phi = (k_1 - k_2)\ell = (n_1 - n_2)k_0\ell = (n_1 - n_2)\frac{2\pi \ell}{\lambda_0}$$

where $\lambda_0$ is the vacuum wavelength. This changes the polarization state as the wave is propagating.
Example: linear to circular polarization

Send in a wave with equal linear polarization in $x$ and $y$

$$E(0) = A(\hat{x} + \hat{y})$$

The propagated field becomes circularly polarized at $z = \ell$

$$E(\ell) = A(\hat{x} + \hat{y}e^{j\phi})e^{-jk_1\ell} = A(\hat{x} + j\hat{y})e^{-jk_1\ell}$$

provided that

$$\phi = (n_1 - n_2)\frac{2\pi\ell}{\lambda_0} = \frac{\pi}{2} \Rightarrow (n_1 - n_2)\ell = \frac{\lambda_0}{4}$$

This is called a quarter-wavelength retarder (or quarter-wavelength plate).
General polarization change
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8. Gyrotropic media
9. Oblique propagation in biaxial media
Chirality

Some structures do not possess mirror symmetry:

These are called chiral structures, and leads to birefringence in circular polarization.
Chiral media are modeled by

\[
D = \epsilon E - j\chi H \\
B = \mu H + j\chi E
\]

This is a bi-isotropic material, with material matrix

\[
\begin{pmatrix}
\epsilon & -j\chi \\
 j\chi & \mu
\end{pmatrix}
\]

which is hermitian symmetric if \(\epsilon, \mu,\) and \(\chi\) are real, and positive definite if \(\epsilon\mu - \chi^2 > 0\). However, usually all parameters are complex and depend on frequency. Typically, \(\chi(\omega) \rightarrow 0\) as \(\omega \rightarrow 0\).
Solutions in circular polarization

Maxwell’s equations for circular polarization are (where ± refers to the polarization \( \hat{x} \mp j\hat{y} \))

\[
\begin{align*}
\partial_z E_+ &= -\omega (\mu H_+ + j\chi E_+) \\
\partial_z H_+ &= \omega (\epsilon E_+ - j\chi H_+) \\
\partial_z E_- &= \omega (\mu H_- + j\chi E_-) \\
\partial_z H_- &= -\omega (\epsilon E_- - j\chi H_-)
\end{align*}
\]
Maxwell’s equations for circular polarization are (where $\pm$ refers to the polarization $\hat{x} \mp j\hat{y}$)

$$\begin{align*}
\partial_z E_+ &= -\omega (\mu H_+ + j\chi E_+) \\
\partial_z H_+ &= \omega (\epsilon E_+ - j\chi H_+)
\end{align*}$$

$$\begin{align*}
\partial_z E_- &= \omega (\mu H_- + j\chi E_-) \\
\partial_z H_- &= -\omega (\epsilon E_- - j\chi H_-)
\end{align*}$$

Defining $c = 1/\sqrt{\mu\epsilon}$, $\eta = \sqrt{\mu/\epsilon}$, $k = \omega/c = \omega/\sqrt{\mu\epsilon}$, and the dimensionless parameter $a = c\chi = \chi/\sqrt{\mu\epsilon}$ (so that $|a| < 1$) this is

$$\frac{\partial}{\partial z} \begin{pmatrix} E_+ \\ \eta H_+ \end{pmatrix} = \mp \begin{pmatrix} jka & k \\ -k & jka \end{pmatrix} \begin{pmatrix} E_+ \\ \eta H_+ \end{pmatrix}$$
Maxwell’s equations for circular polarization are (where ± refers to the polarization \( \hat{x} \mp j\hat{y} \))

\[
\begin{align*}
\partial_z E_+ &= -\omega(\mu H_+ + j\chi E_+ ) \\
\partial_z H_+ &= \omega(\varepsilon E_+ - j\chi H_+)
\end{align*}
\begin{align*}
\partial_z E_- &= \omega(\mu H_- + j\chi E_- ) \\
\partial_z H_- &= -\omega(\varepsilon E_- - j\chi H_-)
\end{align*}
\]

Defining \( c = 1/\sqrt{\mu\varepsilon} \), \( \eta = \sqrt{\mu/\varepsilon} \), \( k = \omega/c = \omega\sqrt{\mu\varepsilon} \), and the dimensionless parameter \( a = c\chi = \chi/\sqrt{\mu\varepsilon} \) (so that \( |a| < 1 \)) this is

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_+ \\ \eta H_+ \end{pmatrix} = \mp \begin{pmatrix} jka & k \\ -k & jka \end{pmatrix} \begin{pmatrix} E_+ \\ \eta H_+ \end{pmatrix}
\]

This is diagonalized by the change of variables

\[
E_{R\pm} = \frac{1}{2}[E_\pm - j\eta H_\pm] \quad E_\pm = E_{R\pm} + E_{L\pm}
\]
\[
E_{L\pm} = \frac{1}{2}[E_\pm + j\eta H_\pm] \quad H_\pm = \frac{-1}{j\eta}[E_{R\pm} - E_{L\pm}]
\]
Solution, continued

In the new variables we have (see Orfanidis for details)

\[ \frac{\partial}{\partial z} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix} = \begin{pmatrix} -jk_+ & 0 \\ 0 & jk_- \end{pmatrix} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix} \]

\[ \frac{\partial}{\partial z} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix} = \begin{pmatrix} -jk_- & 0 \\ 0 & jk_+ \end{pmatrix} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix} \]

where

\[ k_{\pm} = k(1 \pm a) = \omega(\sqrt{\mu\epsilon} \pm \chi) = n_{\pm} k_0 \]
Solution, continued

In the new variables we have (see Orfanidis for details)

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix} = \begin{pmatrix} -jk_+ & 0 \\ 0 & jk_- \end{pmatrix} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix}
\]

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix} = \begin{pmatrix} -jk_- & 0 \\ 0 & jk_+ \end{pmatrix} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix}
\]

where

\[
k_\pm = k(1 \pm a) = \omega(\sqrt{\mu\varepsilon} \pm \chi) = n_\pm k_0
\]

This has the solutions

\[
E_+(z) = E_{R+}(z) + E_{L+}(z) = \underbrace{A_+e^{-jk_+z}}_{\text{forward RCP}} + \underbrace{B_+e^{jk_-z}}_{\text{backward LCP}}
\]

\[
E_-(z) = E_{L-}(z) + E_{R-}(z) = \underbrace{A_-e^{-jk_-z}}_{\text{forward LCP}} + \underbrace{B_-e^{jk_+z}}_{\text{backward RCP}}
\]

RCP travel with \( k_+ \), LCP with \( k_- \), in either direction.
Natural rotation

Start with a linearly polarized field at $z = 0$

$$E(0) = \hat{x}2A = \hat{e}_+A + \hat{e}_-A$$
Natural rotation

Start with a linearly polarized field at \( z = 0 \)

\[
E(0) = \hat{x}2A = \hat{e}_+A + \hat{e}_-A
\]

Propagating the circular polarizations individually implies

\[
E(\ell) = \hat{e}_+Ae^{-jk_+\ell} + \hat{e}_-Ae^{-jk_-\ell} \\
= \left( \hat{e}_+e^{-j(k_+-k_-)\ell/2} + \hat{e}_-e^{j(k_+-k_-)\ell/2} \right) Ae^{-j(k_++k_-)\ell/2}
\]
Natural rotation

Start with a linearly polarized field at $z = 0$

$$E(0) = \hat{x}2A = \hat{e}_+A + \hat{e}_-A$$

Propagating the circular polarizations individually implies

$$E(\ell) = \hat{e}_+ Ae^{-jk+\ell} + \hat{e}_- Ae^{-jk-\ell}$$

$$= \left( \hat{e}_+ e^{-j(k+-k_-)\ell/2} + \hat{e}_- e^{j(k+-k_-)\ell/2} \right) Ae^{-j(k_+ + k_-)\ell/2}$$

Denoting $(k_+ - k_-)\ell/2 = \phi$, we go back to the linear basis (remember that $\hat{e}_\pm = \hat{x} \mp j\hat{y}$)

$$\hat{e}_+ e^{-j\phi} + \hat{e}_- e^{j\phi} = \hat{x}(e^{-j\phi} + e^{j\phi}) + \hat{y}(-je^{-j\phi} + je^{j\phi})$$

$$= 2(\hat{x} \cos \phi - \hat{y} \sin \phi)$$
Collecting the results, we see that a linearly polarized field

\[ E(0) = \hat{x}2A \]

propagates to become

\[ E(\ell) = (\hat{x}\cos\phi - \hat{y}\sin\phi)2Ae^{-j(k_+ + k_-)\ell/2} \]

where \( \phi = (k_+ - k_-)\ell/2 \). Thus, the polarization is always linear, but the plane of polarization is rotating.
Collecting the results, we see that a linearly polarized field

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\[ E(\ell) = (\hat{x}\cos\phi - \hat{y}\sin\phi)2Ae^{-j(k_+ + k_-)\ell/2} \]

where \( \phi = (k_+ - k_-)\ell/2 \). Thus, the polarization is always linear, but the plane of polarization is rotating. If the propagation is in the negative \( z \)-direction, change the roles of \( k_+ \) and \( k_- \).
Collecting the results, we see that a linearly polarized field

$$\mathbf{E}(0) = \hat{x}2A$$

propagates to become

$$\mathbf{E}(\ell) = (\hat{x}\cos\phi - \hat{y}\sin\phi)2Ae^{-j(k_+ + k_- \ell)/2}$$

where $\phi = (k_+ - k_-)\ell/2$. Thus, the polarization is always linear, but the plane of polarization is rotating. If the propagation is in the negative $z$-direction, change the roles of $k_+$ and $k_-$. This means a rotation by $\phi$ is canceled by a rotation $-\phi$ if the wave is reflected and travels back the same distance.
Collecting the results, we see that a linearly polarized field

\[ E(0) = \hat{x}2A \]

propagates to become

\[ E(\ell) = (\hat{x} \cos \phi - \hat{y} \sin \phi)2Ae^{-j(k_+ + k_-)\ell/2} \]

where \( \phi = (k_+ - k_-)\ell/2 \). Thus, the polarization is always linear, but the plane of polarization is rotating. If the propagation is in the negative \( z \)-direction, change the roles of \( k_+ \) and \( k_- \). This means a rotation by \( \phi \) is canceled by a rotation \(-\phi\) if the wave is reflected and travels back the same distance.

- Optical activity is detected by rotation of \textit{transmission} polarization, not reflection.
Quarter-wavelength retarder
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7 Chiral media
8 **Gyrotropic media**
9 Oblique propagation in biaxial media
Gyrotropic media

A gyroelectric medium (plasma) is modeled by

\[
\begin{pmatrix}
  D_x \\
  D_y \\
  D_z
\end{pmatrix}
= \begin{pmatrix}
  \epsilon_1 & j\epsilon_2 & 0 \\
  -j\epsilon_2 & \epsilon_1 & 0 \\
  0 & 0 & \epsilon_3
\end{pmatrix}
\begin{pmatrix}
  E_x \\
  E_y \\
  E_z
\end{pmatrix}, \quad B = \mu H
\]

and a gyromagnetic medium (ferromagnetic) is modeled by

\[
\begin{pmatrix}
  B_x \\
  B_y \\
  B_z
\end{pmatrix}
= \begin{pmatrix}
  \mu_1 & j\mu_2 & 0 \\
  -j\mu_2 & \mu_1 & 0 \\
  0 & 0 & \mu_3
\end{pmatrix}
\begin{pmatrix}
  H_x \\
  H_y \\
  H_z
\end{pmatrix}, \quad D = \epsilon E
\]

The medium is lossless and passive if \(\epsilon_{1,2,3}\) are real, and \(\epsilon_1 > 0\), \(|\epsilon_2| < \epsilon_1\), and \(\epsilon_3 > 0\) (correspondingly for \(\mu_{1,2,3}\)).

Usually complex and frequency dependent parameters.
Circularly polarized fields

The gyrotropic medium is diagonalized by circularly polarized waves:

\[ D_{\pm} = (\epsilon_1 \pm \epsilon_2)E_{\pm}, \quad B_{\pm} = \mu H_{\pm} \]  

(gyroelectric)

\[ B_{\pm} = (\mu_1 \pm \mu_2)H_{\pm}, \quad D_{\pm} = \epsilon E_{\pm} \]  

(gyromagnetic)

and we introduce (keeping only gyroelectric case)

\[ \epsilon_{\pm} = \epsilon_1 \pm \epsilon_2, \quad k_{\pm} = \omega \sqrt{\mu \epsilon_{\pm}}, \quad \eta_{\pm} = \sqrt{\frac{\mu}{\epsilon_{\pm}}} \]

Use the corresponding change of variables as in the chiral case

\[ E_{R\pm} = \frac{1}{2}[E_{\pm} - j\eta_{\pm}H_{\pm}] \quad E_{\pm} = E_{R\pm} + E_{L\pm} \]

\[ E_{L\pm} = \frac{1}{2}[E_{\pm} + j\eta_{\pm}H_{\pm}] \quad H_{\pm} = \frac{-1}{j\eta_{\pm}}[E_{R\pm} - E_{L\pm}] \]
**Solution**

In the new variables we have (see Orfanidis for details)

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix} = \begin{pmatrix} -jk_+ & 0 \\ 0 & jk_+ \end{pmatrix} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix}
\]

\[
\frac{\partial}{\partial z} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix} = \begin{pmatrix} -jk_- & 0 \\ 0 & jk_- \end{pmatrix} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix}
\]

This has the solutions

\[
E_{R+}(z) = E_{R+} + (z) + E_{L+}(z) = A + e^{-jk_+z}
\]

**Forward RCP**

\[
E_{L-}(z) = E_{L-} - (z) + E_{R-}(z) = A - e^{-jk_-z}
\]

**Forward LCP**

\[
E_{R-}(z) = E_{R-} + (z) + E_{L-}(z) = B + e^{-jk_-z}
\]

**Backward RCP**

\[
E_{L+}(z) = E_{L+} - (z) + E_{R+}(z) = B - e^{-jk_+z}
\]

**Backward LCP**
Solution

In the new variables we have (see Orfanidis for details)

\[ \frac{\partial}{\partial z} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix} = \begin{pmatrix} -jk_+ & 0 \\ 0 & jk_+ \end{pmatrix} \begin{pmatrix} E_{R+} \\ E_{L+} \end{pmatrix} \]

\[ \frac{\partial}{\partial z} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix} = \begin{pmatrix} -jk_- & 0 \\ 0 & jk_- \end{pmatrix} \begin{pmatrix} E_{L-} \\ E_{R-} \end{pmatrix} \]

This has the solutions

\[ E_+(z) = E_{R+}(z) + E_{L+}(z) = A_+ e^{-jk_+z} + B_+ e^{jk_+z} \]

\[ E_-(z) = E_{L-}(z) + E_{R-}(z) = A_- e^{-jk_-z} + B_- e^{jk_-z} \]

Forward RCP and backward LCP travel with \( k_+ \), forward LCP and backward RCP with \( k_- \).
Calculations corresponding to the natural rotation now leads to that the initial linear polarization

\[ E(0) = \hat{x}2A \]

is propagated to

\[ E(\ell) = (\hat{x} \cos \phi - \hat{y} \sin \phi)2A e^{-j(k_+ + k_-)\ell/2} \]

where \( \phi = (k_+ - k_-)\ell/2 \). Thus, we still have rotation of the polarization plane,
Faraday rotation

Calculations corresponding to the natural rotation now leads to that the initial linear polarization

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where \( \phi = (k_+ - k_-)\ell/2 \). Thus, we still have rotation of the polarization plane, but the roles of \( k_+ \) and \( k_- \) remain the same when propagating in the negative \( z \) direction.
Faraday rotation

Calculations corresponding to the natural rotation now leads to that the initial linear polarization

\[ E(0) = \hat{x}2A \]

is propagated to

\[ E(\ell) = (\hat{x} \cos \phi - \hat{y} \sin \phi)2Ae^{-j(k_++k_-)\ell/2} \]

where \( \phi = (k_+ - k_-)\ell/2 \). Thus, we still have rotation of the polarization plane, but the roles of \( k_+ \) and \( k_- \) remain the same when propagating in the negative \( z \) direction. Thus, a reflected wave experiences \textit{twice} the rotation of the polarization plane.
Faraday rotation

Calculations corresponding to the natural rotation now leads to that the initial linear polarization

\[ E(0) = \hat{x}2A \]

is propagated to

\[ E(\ell) = (\hat{x} \cos \phi - \hat{y} \sin \phi)2Ae^{-j(k_+ + k_-)\ell/2} \]

where \( \phi = (k_+ - k_-)\ell/2 \). Thus, we still have rotation of the polarization plane, but the roles of \( k_+ \) and \( k_- \) remain the same when propagating in the negative \( z \) direction. Thus, a reflected wave experiences \textit{twice} the rotation of the polarization plane.

- Faraday rotation is a non-reciprocal phenomenon, caused by the break in symmetry by an applied magnetic field.
Conclusions for propagation in birefringent media

- Different polarizations may generate different response from the material.
- Linear polarization is natural for uniaxial and biaxial media.
- Circular polarization is natural for chiral and gyrotropic media.
- In uniaxial and biaxial materials, an originally linear polarization may undergo transformations during propagation: linear – circular – linear – circular etc.
- Natural rotation (chiral media) is restored upon reflection.
- Faraday rotation (gyrotropic media) is doubled upon reflection.
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9. Oblique propagation in biaxial media
We consider the material relation (with $B = \mu_0 H$)
\[
\begin{pmatrix}
D_x \\
D_y \\
D_z
\end{pmatrix} = 
\begin{pmatrix}
\varepsilon_1 & 0 & 0 \\
0 & \varepsilon_2 & 0 \\
0 & 0 & \varepsilon_3
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix} = 
\varepsilon_0
\begin{pmatrix}
n_1^2 & 0 & 0 \\
0 & n_2^2 & 0 \\
0 & 0 & n_3^2
\end{pmatrix}
\begin{pmatrix}
E_x \\
E_y \\
E_z
\end{pmatrix}
\]
and look for plane wave solutions
\[
E(r) = E e^{-jk \cdot r}, \quad H(r) = H e^{-jk \cdot r}
\]
where $k = \hat{k} k$. The effective refractive index $N$ is defined from
\[
k = N k_0
\]
where $k_0 = \omega/c_0$. 
Maxwell’s equations for biaxial media

Looking for plane wave solutions Maxwell’s equations become

\[ \mathbf{k} \times \mathbf{E} = \omega \mu_0 \mathbf{H} \]
\[ \mathbf{k} \times \mathbf{H} = -\omega \varepsilon \cdot \mathbf{E} \]

Eliminating the magnetic field implies

\[ -\mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \omega^2 \mu_0 \varepsilon \cdot \mathbf{E} \]

Using \( \mathbf{k} = \hat{k} \mathbf{k} = \hat{k} \mathbf{N} k_0 \) implies

\[ -\hat{k} \times (\hat{k} \times \mathbf{E}) = \frac{1}{\varepsilon_0 N^2} \varepsilon \cdot \mathbf{E} \quad \Rightarrow \quad \mathbf{E} - \frac{1}{\varepsilon_0 N^2} \varepsilon \cdot \mathbf{E} = \hat{k} (\hat{k} \cdot \mathbf{E}) \]

For a given propagation direction \( \hat{k} \) this is a homogeneous system, having non-trivial solutions \( \mathbf{E} \) only if the determinant is zero.
Special geometry, $\hat{k} = \hat{x} \sin \theta + \hat{z} \cos \theta$

\[ E = E_x \hat{x} + E_z \hat{z} \]

\[
\left(1 - \frac{n_1^2}{N^2}\right) E_x = (E_x \sin \theta + E_z \cos \theta) \sin \theta \quad \left(1 - \frac{n_2^2}{N^2}\right) E_y = 0
\]

\[
\left(1 - \frac{n_3^2}{N^2}\right) E_z = (E_x \sin \theta + E_z \cos \theta) \cos \theta
\]
For each propagation direction \( \hat{k} \), there exist two solutions. This corresponds to two \( k \)-surfaces:

\[
\begin{align*}
\{ a &= \frac{\omega}{\sqrt{\varepsilon \mu / c_0}} \\
       b &= \frac{\omega}{\sqrt{\varepsilon \mu z / c_0}} \}
\end{align*}
\]

The explicit solutions are (extra-ordinary and ordinary wave)

\[
\frac{1}{N_{eo}^2} = \frac{\cos^2 \theta}{n_1^2} + \frac{\sin^2 \theta}{n_3^2}, \quad \frac{1}{N_o^2} = \frac{1}{n_2^2}
\]

Using \( k = N \frac{\omega}{c_0} (\cos \theta \hat{z} + \sin \theta \hat{x}) \) the group velocity \( v_g = \partial \omega / \partial k \)

can be calculated. We show the energy propagates in this direction.
Orfanidis makes the detailed calculations, here a quick argument is given. Poynting’s theorem for the plane wave is

$$\mathbf{k} \cdot \frac{1}{2} \text{Re}\{\mathbf{E} \times \mathbf{H}^*\} = \frac{\omega}{2} \text{Re}(\mathbf{D}^* \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}^*) = \omega w$$

where $w = \frac{1}{2} \text{Re}(\mathbf{D}^* \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{H}^*)$ is the energy density.
Energy propagation in the group velocity direction

Orfanidis makes the detailed calculations, here a quick argument is given. Poynting’s theorem for the plane wave is

\[ k \cdot \frac{1}{2} \text{Re}\{E \times H^*\} = \frac{\omega}{2} \text{Re}(D^* \cdot E + B \cdot H^*) = \omega w \]

where \( w = \frac{1}{2} \text{Re}(D^* \cdot E + B \cdot H^*) \) is the energy density. Differentiating with respect to \( k_x \) implies

\[ \hat{x} \cdot \frac{1}{2} \text{Re}\{E \times H^*\} = w \frac{\partial \omega}{\partial k_x} = w v_{g,x} \]

where \( v_{g,x} \) is the \( x \)-komponent of the group velocity
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\[ v_g = \frac{\partial \omega}{\partial k} = \frac{\frac{1}{2} \text{Re}\{\mathbf{E} \times \mathbf{H}^*\}}{w} \]

This demonstrates that the group velocity is the ratio of the Poynting vector over the energy density.
The $k$-surfaces can be interpreted as level surfaces for a fixed $\omega$:

$$\begin{cases}
a = \omega \sqrt{\varepsilon \mu / c_0} \\
b = \omega \sqrt{\varepsilon \mu_z / c_0}
\end{cases}$$

The group velocity $v_g = \partial \omega / \partial k$ is then orthogonal to this surface.

- Power is not propagating along $k$ for the extraordinary wave!!

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Conclusions for oblique propagation

- Biaxial media supports two typical waves, ordinary and extra-ordinary.
- An effective refractive index can be defined, as the solution to a matrix equation.
- The ordinary wave is polarized along an optical axis, and propagates as if in an isotropic medium.
- The extra-ordinary wave is polarized between two optical axes.
- The power of the extra-ordinary wave does not travel in the \( k \)-direction (direction of the phase).