Electromagnetic Wave Propagation
Lecture 2: Uniform plane waves

Daniel Sjöberg

Department of Electrical and Information Technology
Outline

1. Plane waves in lossless media
   General time dependence
   Time harmonic waves

2. Polarization

3. Propagation in lossy media

4. Oblique propagation and complex waves

5. Paraxial approximation: beams

6. Doppler effect and negative index media

7. Conclusions
1. **Plane waves in lossless media**
   - General time dependence
   - Time harmonic waves

2. **Polarization**

3. **Propagation in lossy media**

4. **Oblique propagation and complex waves**

5. **Paraxial approximation: beams**

6. **Doppler effect and negative index media**

7. **Conclusions**
Plane electromagnetic waves

In this lecture, we dive a bit deeper into the familiar right-hand rule.

This is the building block for most of the course.
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Propagation in source free, isotropic, non-dispersive media

The electromagnetic field is assumed to depend only on one coordinate, $z$. This implies

\[
E(x, y, z, t) = E(z, t) \\
D(x, y, z, t) = \epsilon E(z, t) \\
H(x, y, z, t) = H(z, t) \\
B(x, y, z, t) = \mu H(z, t)
\]

and by using $\nabla \rightarrow \hat{z} \frac{\partial}{\partial z}$ we have

\[
\nabla \times E = -\frac{\partial B}{\partial t} \\
\nabla \times H = \frac{\partial D}{\partial t} \\
\nabla \cdot D = 0 \\
\n\nabla \cdot B = 0
\]

\[
\hat{z} \times \frac{\partial E}{\partial z} = -\mu \frac{\partial H}{\partial t} \\
\hat{z} \times \frac{\partial H}{\partial z} = \epsilon \frac{\partial E}{\partial t} \\
\frac{\partial E_z}{\partial z} = 0 \Rightarrow E_z = 0 \\
\frac{\partial H_z}{\partial z} = 0 \Rightarrow H_z = 0
\]
Rewriting the equations

The electromagnetic field has only $x$ and $y$ components, and satisfy

$$\hat{z} \times \frac{\partial E}{\partial z} = -\mu \frac{\partial H}{\partial t}$$

$$\hat{z} \times \frac{\partial H}{\partial z} = \epsilon \frac{\partial E}{\partial t}$$

Using $(\hat{z} \times F) \times \hat{z} = F$ for any vector $F$ orthogonal to $\hat{z}$, and

$c = 1/\sqrt{\epsilon \mu}$ and $\eta = \sqrt{\mu/\epsilon}$, we can write this as

$$\frac{\partial}{\partial z} E = -\frac{1}{c} \frac{\partial}{\partial t} (\eta H \times \hat{z})$$

$$\frac{\partial}{\partial z} (\eta H \times \hat{z}) = -\frac{1}{c} \frac{\partial}{\partial t} E$$

which is a symmetric hyperbolic system

$$\frac{\partial}{\partial z} \begin{pmatrix} E \\ \eta H \times \hat{z} \end{pmatrix} = -\frac{1}{c} \frac{\partial}{\partial t} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} E \\ \eta H \times \hat{z} \end{pmatrix}$$
Wave splitting

We could eliminate the magnetic field to find the wave equation

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E(z, t) = 0
\]

but it is often more convenient to keep the system formulation. The change of variables

\[
\begin{pmatrix}
E_+ \\
E_-
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
E + \eta H \times \hat{z} \\
E - \eta H \times \hat{z}
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
1 & 1 \\
1 & -1
\end{pmatrix} \begin{pmatrix}
E \\
\eta H \times \hat{z}
\end{pmatrix}
\]

is called a wave splitting and diagonalizes the system to

\[
\begin{align*}
\frac{\partial E_+}{\partial z} &= -\frac{1}{c} \frac{\partial E_+}{\partial t} & \Rightarrow & & E_+(z, t) = F(z - ct) \\
\frac{\partial E_-}{\partial z} &= \frac{1}{c} \frac{\partial E_-}{\partial t} & \Rightarrow & & E_-(z, t) = G(z + ct)
\end{align*}
\]
The total fields can now be written

\[ E(z, t) = F(z - ct) + G(z + ct) \]

\[ H(z, t) = \frac{1}{\eta} \hat{z} \times \left( F(z - ct) - G(z + ct) \right) \]

A graphical interpretation of the forward and backward waves is
Energy density in one single wave

A wave propagating in positive $z$ direction satisfies $\mathbf{H} = \frac{1}{\eta} \hat{z} \times \mathbf{E}$

$x$: Electric field  
$y$: Magnetic field  
$z$: Propagation direction

The Poynting vector and energy densities are (using the BAC-CAB rule $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$)

$$\mathbf{P} = \mathbf{E} \times \mathbf{H} = \frac{1}{\eta} \mathbf{E} \times (\hat{z} \times \mathbf{E}) = \frac{1}{\eta} \hat{z} |\mathbf{E}|^2$$

$$w_e = \frac{1}{2} \varepsilon |\mathbf{E}|^2$$

$$w_m = \frac{1}{2} \mu |\mathbf{H}|^2 = \frac{1}{2} \mu \frac{1}{\eta^2} |\hat{z} \times \mathbf{E}|^2 = \frac{1}{2} \varepsilon |\mathbf{E}|^2 = w_e$$

The transport velocity of electromagnetic energy is then

$$v = \frac{\mathbf{P}}{w_e + w_m} = \frac{1}{\eta} \frac{1}{\varepsilon} \hat{z} = c \hat{z}$$
Energy density in forward and backward wave

If the wave propagates in negative $z$ direction, we have $H = \frac{1}{\eta}(-\hat{z}) \times E$ and

$$\mathcal{P} = E \times H = \frac{1}{\eta} E \times (-\hat{z} \times E) = -\frac{1}{\eta} \hat{z}|E|^2$$

Thus, when the wave consists of one forward wave $F(z - ct)$ and one backward wave $G(z + ct)$, we have

$$\mathcal{P} = E \times H = \frac{1}{\eta} \hat{z} \left(|F|^2 - |G|^2\right)$$

$$w = \frac{1}{2} \epsilon|E|^2 + \frac{1}{2} \mu|H|^2 = \epsilon|F|^2 + \epsilon|G|^2$$
Generalization for nonlinear media: simple waves

A generalization of the plane wave concept is to look for solutions depending only on a scalar function of space and time

\[ \mathbf{E}(x, y, z, t) = \mathbf{E}(\varphi(x, y, z, t)) \]  
(linear media: \( \varphi = z - ct \))

This can help analyze nonlinear material models with instantaneous response (for instance a Kerr model \( \mathbf{D} = \varepsilon_0 (\mathbf{E} + \alpha |\mathbf{E}|^2 \mathbf{E}) \))

\[ \mathbf{D}(x, y, z, t) = \mathbf{D}(\mathbf{E}(x, y, z, t)) \]

Maxwell’s equations can then be written

\[ \nabla \varphi \times \mathbf{E}' = - \frac{\partial \varphi}{\partial t} \frac{\partial \mathbf{B}}{\partial \mathbf{H}} \cdot \mathbf{H}' \]

\[ \nabla \varphi \times \mathbf{H}' = \frac{\partial \varphi}{\partial t} \frac{\partial \mathbf{D}}{\partial \mathbf{E}} \cdot \mathbf{E}' \]
The wavefronts and wave slowness are defined by

\[ \varphi(x, y, z, t) = \text{constant} \]

\[ s = -\frac{\nabla \varphi}{\frac{\partial \varphi}{\partial t}} \] (unit s/m)

Maxwell’s equations become a symmetric algebraic problem

\[
\begin{pmatrix}
0 & -s \times \mathbf{I} \\
-s \times \mathbf{I} & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
E' \\
H'
\end{pmatrix}
=
\begin{pmatrix}
\frac{\partial D}{\partial E} & 0 \\
0 & \frac{\partial B}{\partial H}
\end{pmatrix}
\cdot
\begin{pmatrix}
E' \\
H'
\end{pmatrix}
\]

where \( \frac{\partial D}{\partial E} = \epsilon(E) \) is the differential permittivity and \( \frac{\partial B}{\partial H} = \mu(H) \) is the differential permeability. The “eigenvalue” \( s \) (and hence propagation speed) depends on field strengths \( E \) and \( H \).
Shock waves

Initial data propagate with a speed depending on field strength.

- Rarefaction wave
  - Fast wave outruns the slow wave

- Shock wave
  - Fast wave overtakes the slow wave
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Time harmonic waves

We assume harmonic time dependence

\[ \mathbf{E}(x, y, z, t) = \mathbf{E}(z) e^{j\omega t} \]
\[ \mathbf{H}(x, y, z, t) = \mathbf{H}(z) e^{j\omega t} \]

Using the same wave splitting as before implies

\[ \frac{\partial \mathbf{E}_\pm}{\partial z} = \mp \frac{1}{c} \frac{\partial \mathbf{E}_\pm}{\partial t} = \mp \frac{j \omega}{c} \mathbf{E}_\pm \quad \Rightarrow \quad \mathbf{E}_\pm(z) = \mathbf{E}_{0\pm} e^{\mp jkz} \]

where \( k = \omega/c \) is the wave number in the medium. Thus, the general solution for time harmonic waves is

\[ \mathbf{E}(z) = \mathbf{E}_{0+} e^{-jkz} + \mathbf{E}_{0-} e^{jkz} \]
\[ \mathbf{H}(z) = \mathbf{H}_{0+} e^{-jkz} + \mathbf{H}_{0-} e^{jkz} = \frac{1}{\eta} \hat{z} \times \left( \mathbf{E}_{0+} e^{-jkz} - \mathbf{E}_{0-} e^{jkz} \right) \]

The triples \( \{\mathbf{E}_{0+}, \mathbf{H}_{0+}, \hat{z}\} \) and \( \{\mathbf{E}_{0-}, \mathbf{H}_{0-}, -\hat{z}\} \) are right-handed systems.
A time harmonic wave propagating in the forward \( z \) direction has the space-time dependence \( E(z, t) = E_0 e^{j(\omega t - kz)} \) and \( H(z, t) = \frac{1}{\eta} \hat{z} \times E(z, t) \).

The wavelength corresponds to the spatial periodicity according to \( e^{-jk(z+\lambda)} = e^{-jkz} \), meaning \( k\lambda = 2\pi \) or

\[
\lambda = \frac{2\pi}{k} = \frac{2\pi c}{\omega} = \frac{c}{f}
\]
Refractive index

The wavelength is often compared to the corresponding wavelength in vacuum

$$\lambda_0 = \frac{c_0}{f}$$

The refractive index is

$$n = \frac{\lambda_0}{\lambda} = \frac{k}{k_0} = \frac{c_0}{c} = \sqrt{\frac{\epsilon\mu}{\epsilon_0\mu_0}}$$

Important special case: non-magnetic media, where $\mu = \mu_0$ and $n = \sqrt{\epsilon/\epsilon_0}$.

$$c = \frac{c_0}{n}, \quad \eta = \frac{\eta_0}{n}, \quad \lambda = \frac{\lambda_0}{n}, \quad k = nk_0$$

only for $\mu = \mu_0$!

Note: We use $c_0$ for the speed of light in vacuum and $c$ for the speed of light in a medium, even though $c$ is the standard for the speed of light in vacuum!
For the general time harmonic solution

\[ E(z) = E_0^+ e^{-j kz} + E_0^- e^{j kz} \]

\[ H(z) = \frac{1}{\eta} \hat{z} \times \left( E_0^+ e^{-j kz} - E_0^- e^{j kz} \right) \]

the energy density and Poynting vector are

\[ w = \frac{1}{4} \text{Re} \left[ \epsilon E(z) \cdot E^*(z) + \mu H(z) \cdot H^*(z) \right] = \frac{1}{2} \epsilon |E_0^+|^2 + \frac{1}{2} \epsilon |E_0^-|^2 \]

\[ \mathcal{P} = \frac{1}{2} \text{Re} \left[ E(z) \times H^*(z) \right] = \hat{z} \left( \frac{1}{2\eta} |E_0^+|^2 - \frac{1}{2\eta} |E_0^-|^2 \right) \]
Wave impedance

For forward and backward waves, the impedance \( \eta = \sqrt{\mu/\epsilon} \) relates the electric and magnetic field strengths to each other.

\[
E_\pm = \pm \eta H_\pm \times \hat{z}
\]

When both forward and backward waves are present this is generalized as (in component form)

\[
Z_x(z) = \frac{[E(z)]_x}{[H(z) \times \hat{z}]_x} = \frac{E_x(z)}{H_y(z)} = \eta \frac{E_{0+}e^{-jkz} + E_{0-}e^{jkz}}{E_{0+}e^{-jkz} - E_{0-}e^{jkz}}
\]

\[
Z_y(z) = \frac{[E(z)]_y}{[H(z) \times \hat{z}]_y} = -\frac{E_y(z)}{H_x(z)} = \eta \frac{E_{0+}e^{-jkz} + E_{0-}e^{jkz}}{E_{0+}e^{-jkz} - E_{0-}e^{jkz}}
\]

Thus, the wave impedance is in general a non-trivial function of \( z \). This will be used as a means of analysis and design in the course.
We note that we have two material parameters

Permittivity $\epsilon$, defined by $D = \epsilon E$.

Permeability $\mu$, defined by $B = \mu H$.

But the waves are described by the wave parameters

Wave number $k$, defined by $k = \omega \sqrt{\epsilon \mu}$.

Wave impedance $\eta$, defined by $\eta = \sqrt{\mu/\epsilon}$.

This means that in a scattering experiment, we primarily get information on $k$ and $\eta$, not $\epsilon$ and $\mu$. In order to get material data, a theoretical material model must be applied.

Often, wave number is given by transmission data (phase delay), and wave impedance is given by reflection data (impedance mismatch). More on this in future lectures!
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Why care about different polarizations?

- Different materials react differently to different polarizations.
- Linear polarization is sometimes not the most natural.
- For propagation through the ionosphere (to satellites), or through magnetized media, often circular polarization is natural.
Complex vectors

The time dependence of the electric field is

\[ E(t) = \text{Re}\{ E_0 e^{j\omega t} \} \]

where the complex amplitude can be written

\[ E_0 = \hat{x} E_{0x} + \hat{y} E_{0y} + \hat{z} E_{0z} = \hat{x} |E_{0x}| e^{j\alpha} + \hat{y} |E_{0y}| e^{j\beta} + \hat{z} |E_{0z}| e^{j\gamma} = E_{0r} + j E_{0i} \]

where \( E_{0r} \) and \( E_{0i} \) are real-valued vectors. We then have

\[ E(t) = \text{Re}\{( E_{0r} + j E_{0i}) e^{j\omega t} \} = E_{0r} \cos(\omega t) - E_{0i} \sin(\omega t) \]

This lies in a plane with normal

\[ \hat{n} = \pm \frac{E_{0r} \times E_{0i}}{|E_{0r} \times E_{0i}|} \]
Linear polarization

The simplest case is given by $E_{0i} = 0$, implying $E(t) = E_{0r} \cos \omega t$. 

![Diagram of linear polarization with $E_{0r}$ along the x-axis and $E_{0r}$ along the y-axis.]
Circular polarization

Assume $E_{0r} = \hat{x}$ and $E_{0i} = -\hat{y}$. The total field $E(t) = \hat{x} \cos \omega t + \hat{y} \sin \omega t$ then rotates in the $xy$-plane:

The rotation direction needs to be compared to the propagation direction.
Circular polarization and propagation direction
Elliptical polarization

The general polarization is elliptical

The direction $\hat{e}$ is parallel to the Poynting vector (the power flow).
Classification of polarization

The following classification can be given:

<table>
<thead>
<tr>
<th>$-j\hat{e} \cdot (\mathbf{E}_0 \times \mathbf{E}_0^*)$</th>
<th>Polarization</th>
</tr>
</thead>
<tbody>
<tr>
<td>$= 0$</td>
<td>Linear polarization</td>
</tr>
<tr>
<td>$&gt; 0$</td>
<td>Right handed elliptic polarization</td>
</tr>
<tr>
<td>$&lt; 0$</td>
<td>Left handed elliptic polarization</td>
</tr>
</tbody>
</table>

Further, circular polarization is characterized by $\mathbf{E}_0 \cdot \mathbf{E}_0 = 0$.

Typical examples:

**Linear:** $\mathbf{E}_0 = \hat{x}$ or $\mathbf{E}_0 = \hat{y}$.

**Circular:** $\mathbf{E}_0 = \hat{x} - j\hat{y}$ (right handed) or $\mathbf{E}_0 = \hat{x} + j\hat{y}$ (left handed).

See the literature for more in depth descriptions.
To describe an arbitrary vector in the $xy$-plane, the unit vectors

$$\hat{x} \quad \text{and} \quad \hat{y}$$

are usually used. However, we could just as well use the RCP and LCP vectors

$$\hat{x} - j\hat{y} \quad \text{and} \quad \hat{x} + j\hat{y}$$

Sometimes the linear basis is preferrable, sometimes the circular.
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We study lossy isotropic media, where

\[
D = \varepsilon_d E, \quad J = \sigma E, \quad B = \mu H
\]

The conductivity is incorporated in the permittivity,

\[
J_{\text{tot}} = J + j\omega D = (\sigma + j\omega\varepsilon_d)E = j\omega \left(\varepsilon_d + \frac{\sigma}{j\omega}\right)E
\]

which implies a complex permittivity

\[
\varepsilon_c = \varepsilon_d - j\frac{\sigma}{\omega}
\]

Often, the dielectric permittivity \(\varepsilon_d\) is itself complex, \(\varepsilon_d = \varepsilon'_d - j\varepsilon''_d\).
Examples of lossy media

- Metals (high conductivity)
- Liquid solutions (ionic conductivity)
- Resonant media
- Just about anything!
In the previous lecture, we have shown that a passive material is characterized by

\[
\text{Re} \left[ j \omega \begin{pmatrix} \epsilon & \xi \\ \zeta & \mu \end{pmatrix} \right] \geq 0
\]

For isotropic media with \( \epsilon = \epsilon_c I \), \( \xi = \zeta = 0 \) and \( \mu = \mu_c I \), this boils down to

\[
\begin{align*}
\epsilon_c &= \epsilon_c' - j \epsilon_c'' \\
\mu_c &= \mu_c' - j \mu_c''
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\epsilon_c'' \geq 0 \\
\mu_c'' \geq 0
\end{cases}
\]
Maxwell’s equations in lossy media

Assuming dependence only on $z$ we obtain

$$\begin{align*}
\nabla \times E &= -j\omega \mu_c H \\
\nabla \times H &= j\omega \varepsilon_c E
\end{align*}$$

$$\Rightarrow \begin{align*}
\frac{\partial}{\partial z} \hat{z} \times E &= -j\omega \mu_c H \\
\frac{\partial}{\partial z} \hat{z} \times H &= j\omega \varepsilon_c E
\end{align*}$$

Nothing really changes compared to the lossless case, for instance it is seen that the fields do not have a $z$-component. This can be written as a system

$$\frac{\partial}{\partial z} \left( \eta_c H \times \hat{z} \right) = \begin{pmatrix} 0 & -j k_c \\ -j k_c & 0 \end{pmatrix} \left( \eta_c H \times \hat{z} \right)$$

where the complex wave number $k_c$ and the complex wave impedance $\eta_c$ are

$$k_c = \omega \sqrt{\varepsilon_c \mu_c}, \quad \text{and} \quad \eta_c = \sqrt{\frac{\mu_c}{\varepsilon_c}}$$
The parameters $\epsilon_c$, $\mu_c$, and $k_c = \omega \sqrt{\epsilon_c \mu_c}$ take their values in the complex lower half plane, whereas $\eta_c = \sqrt{\mu_c/\epsilon_c}$ is restricted to the right half plane.
The solution to the system

\[
\frac{\partial}{\partial z}\left(\eta_c^\textbf{H} \times \hat{z}\right) = \begin{pmatrix} 0 & -jk_c \\ -jk_c & 0 \end{pmatrix} \left(\eta_c^\textbf{H} \times \hat{z}\right)
\]

can be written (no \(z\)-components in the amplitudes \(E_+\) and \(E_-\))

\[
\textbf{E}(z) = E_+ e^{-jk_c z} + E_- e^{jk_c z}
\]

\[
\textbf{H}(z) = \frac{1}{\eta_c} \hat{z} \times \left(E_+ e^{-jk_c z} - E_- e^{jk_c z}\right)
\]

Thus, the solutions are the same as in the lossless case, as long as we “complexify” the coefficients.
Exponential damping

The dominating effect of wave propagation in lossy media is exponential damping of the amplitude of the wave:

\[ k_c = \beta - j\alpha \quad \Rightarrow \quad e^{-j k_c z} = e^{-j\beta z} e^{-\alpha z} \]

Thus, \( \alpha = -\text{Im}(k_c) \) represents the damping of the wave, whereas \( \beta = \text{Re}(k_c) \) represents the oscillations.

The exponential is sometimes written in terms of \( \gamma = j k_c = \alpha + j\beta \) as

\[ e^{-\gamma z} = e^{-j\beta z} e^{-\alpha z} \]

where \( \gamma \) corresponds to a spatial Laplace transform variable, in the same way that the temporal Laplace variable is \( s = j\omega + \nu \).
Poynting’s theorem for isotropic media \((\epsilon_c = \epsilon'_c - j\epsilon''_c \text{ and } \mu_c = \mu'_c - j\mu''_c)\) is

\[
\nabla \cdot \left( \frac{1}{2} \text{Re}\{\mathbf{E} \times \mathbf{H}^*\} \right) = -\frac{\omega}{2} \left( \epsilon''_c |\mathbf{E}|^2 + \mu''_c |\mathbf{H}|^2 \right)
\]

We will write out the left and right hand side explicitly to see if they are indeed equal.
The Poynting vector

Consider a wave propagating in the positive $z$ direction,

$$
\begin{align*}
E(z) &= E_0 e^{-jk_c z} \\
\hat{z} \times E &= E_0 e^{-jk_c z}
\end{align*}
$$

The Poynting vector is

$$
\frac{1}{2} \text{Re}\{E(z) \times H(z)^*\} = \frac{1}{2} \text{Re}\left\{E_0 \times (\hat{z} \times E_0^*) \frac{1}{\eta_c^*} e^{-j(\beta z - \alpha z) + j(\beta z - \alpha z)} \right\}
$$

$$
= \hat{z} \frac{1}{2} \text{Re}\left(\frac{1}{\eta_c^*}\right) |E_0|^2 e^{-2\alpha z}
$$

with the divergence

$$
\nabla \cdot \frac{1}{2} \text{Re}\{E \times H^*\} = \frac{1}{2} \frac{\partial}{\partial z} \text{Re}\left(\frac{1}{\eta_c^*}\right) |E_0|^2 e^{-2\alpha z} = -\alpha \text{Re}\left(\frac{1}{\eta_c^*}\right) |E_0|^2 e^{-2\alpha z}
$$

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For the plane wave under study we have \( \mathbf{H} = \frac{1}{\eta_c} \mathbf{\hat{z}} \times \mathbf{E}_0 e^{-j(\beta z - \alpha z)} \), which implies

\[
|\mathbf{H}|^2 = \frac{1}{|\eta_c^*|^2} |\mathbf{E}_0|^2 e^{-2\alpha z} = \frac{|\varepsilon_c|}{|\mu_c|} |\mathbf{E}_0|^2 e^{-2\alpha z}
\]

The right hand side of Poynting’s theorem is then

\[
- \frac{\omega}{2} \left( \varepsilon''_c |\mathbf{E}|^2 + \mu''_c |\mathbf{H}|^2 \right) = - \frac{\omega}{2} \left( \varepsilon''_c + \frac{\mu''_c |\varepsilon_c|}{|\mu_c|} \right) |\mathbf{E}_0|^2 e^{-2\alpha z}
\]
Poynting’s theorem

In order for Poynting’s theorem to be satisfied, this requires

\[ \alpha \Re \left( \frac{1}{\eta_c^*} \right) = \frac{\omega}{2} \left( \varepsilon'' + \frac{\mu'' |\varepsilon_c|}{|\mu_c|} \right) \]

This is not a trivial relation to prove. In the case of non-magnetic media \((\mu_c = \mu_0, \alpha = -\omega \Im(\sqrt{\varepsilon_c \mu_0}), 1/\eta_c = \sqrt{\varepsilon_c / \mu_0})\), the explicit form is

\[ -\Im(\sqrt{\varepsilon_c}) \Re(\sqrt{\varepsilon_c}) = \frac{1}{2} \varepsilon'' = -\frac{1}{2} \Im(\varepsilon_c) \]

This is true due to the general relation (valid for any complex number \(w\))

\[ \Im(w^2) = 2 \Re(w) \Im(w) \]
In conclusion, Poynting’s theorem

\[ \nabla \cdot \frac{1}{2} \text{Re}\{E \times H^*\} = -\frac{\omega}{2} \left( \epsilon'' |E|^2 + \mu'' |H|^2 \right) \]

is valid for waves propagating in lossy media, but the equality is obscured by some complex algebra.
Characterization of attenuation

The power is damped by a factor $e^{-2\alpha z}$. The attenuation is often expressed in logarithmic scale, decibel (dB).

$$A = e^{-2\alpha z} \Rightarrow A_{\text{dB}} = -10 \log_{10}(A) = 20 \log_{10}(e) \alpha z = 8.686 \alpha z$$

Thus, the attenuation coefficient $\alpha$ can be expressed in dB per meter as

$$\alpha_{\text{dB}} = 8.686 \alpha$$

Instead of the attenuation coefficient, often the skin depth (also called penetration depth)

$$\delta = 1/\alpha$$

is used. When the wave propagates the distance $\delta$, its power is attenuated a factor $e^2 \approx 7.4$, or $8.686 \text{ dB} \approx 9 \text{ dB}$. 

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A common way to characterize losses is by the loss tangent (sometimes denoted $\tan \delta$)

$$\tan \theta = \frac{\varepsilon''_c}{\varepsilon'_c} = \frac{\varepsilon''_d + \sigma/\omega}{\varepsilon'_d}$$

which usually depends on frequency. In spite of this, it is often seen that the loss tangent is given for only one frequency. This may be acceptable if the material properties vary little with frequency.
Example of material properties

From D. M. Pozar, *Microwave Engineering*:

<table>
<thead>
<tr>
<th>Material</th>
<th>Frequency</th>
<th>$\epsilon_r$</th>
<th>$\tan \theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Beeswax</td>
<td>10 GHz</td>
<td>2.35</td>
<td>0.005</td>
</tr>
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<td>Fused quartz</td>
<td>10 GHz</td>
<td>6.4</td>
<td>0.0003</td>
</tr>
<tr>
<td>Gallium arsenide</td>
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<td>13.0</td>
<td>0.006</td>
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<tr>
<td>Glass (pyrex)</td>
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<td>4.82</td>
<td>0.0054</td>
</tr>
<tr>
<td>Plexiglass</td>
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<td>2.60</td>
<td>0.0057</td>
</tr>
<tr>
<td>Silicon</td>
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<td>11.9</td>
<td>0.004</td>
</tr>
<tr>
<td>Styrofoam</td>
<td>3 GHz</td>
<td>1.03</td>
<td>0.0001</td>
</tr>
<tr>
<td>Water (distilled)</td>
<td>3 GHz</td>
<td>76.7</td>
<td>0.157</td>
</tr>
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</table>
Approximations for weak losses

In weakly lossy dielectrics, the material parameters are (where $\varepsilon''_c \ll \varepsilon'_c$)

$$\varepsilon_c = \varepsilon'_c - j\varepsilon''_c = \varepsilon'_c (1 - j \tan \theta)$$
$$\mu_c = \mu_0$$

The wave parameters can then be approximated as

$$k_c = \omega \sqrt{\varepsilon_c \mu_c} \approx \omega \sqrt{\varepsilon'_c \mu_0} \left(1 - j \frac{1}{2} \tan \theta\right)$$

$$\eta_c = \sqrt{\frac{\mu_c}{\varepsilon_c}} \approx \sqrt{\frac{\mu_0}{\varepsilon'_c}} \left(1 + j \frac{1}{2} \tan \theta\right)$$

If the losses are caused mainly by a small conductivity, we have $\varepsilon''_c = \sigma/\omega$, $\tan \theta = \sigma/(\omega \varepsilon'_c)$, and the attenuation constant

$$\alpha = - \text{Im}(k_c) = \frac{1}{2} \omega \sqrt{\varepsilon'_c \mu_0} \frac{\sigma}{\omega \varepsilon'_c} = \frac{\sigma}{2} \sqrt{\frac{\mu_0}{\varepsilon'_c}}$$

is proportional to conductivity and independent of frequency.
Example: propagation in sea water

A simple model of the dielectric properties of sea water is

$$\varepsilon_c = \varepsilon_0 \left( 81 - j \frac{4 \text{S/m}}{\omega \varepsilon_0} \right)$$

that is, it has a relative permittivity of 81 and a conductivity of $\sigma = 4 \text{S/m}$. The imaginary part is much smaller than the real part for frequencies

$$f \gg \frac{4 \text{S/m}}{81 \cdot 2\pi \varepsilon_0} = 888 \text{MHz}$$

for which we have $\alpha = 728 \text{dB/m}$. For lower frequencies, the exact calculations give

- $f = 50 \text{Hz}$ \quad $\alpha = 0.028 \text{dB/m}$ \quad $\delta = 35.6 \text{m}$
- $f = 1 \text{kHz}$ \quad $\alpha = 1.09 \text{dB/m}$ \quad $\delta = 7.96 \text{m}$
- $f = 1 \text{MHz}$ \quad $\alpha = 34.49 \text{dB/m}$ \quad $\delta = 25.18 \text{cm}$
- $f = 1 \text{GHz}$ \quad $\alpha = 672.69 \text{dB/m}$ \quad $\delta = 1.29 \text{cm}$
Approximations for good conductors

In good conductors, the material parameters are (where $\sigma \gg \omega \epsilon$)

$$\epsilon_c = \epsilon - j\sigma/\omega = \epsilon \left(1 - j\frac{\sigma}{\omega \epsilon}\right)$$

$$\mu_c = \mu$$

The wave parameters can then be approximated as

$$k_c = \omega \sqrt{\epsilon_c \mu_c} \approx \omega \sqrt{-j\frac{\sigma}{\omega} \mu} = \sqrt{\frac{\omega \mu \sigma}{2}} (1 - j)$$

$$\eta_c = \sqrt{\frac{\mu_c}{\epsilon_c}} \approx \sqrt{\frac{\mu}{-j\sigma/\omega}} = \sqrt{\frac{\omega \mu}{2\sigma}} (1 + j)$$

This demonstrates that the wave number is proportional to $\sqrt{\omega}$ rather than $\omega$ in a good conductor, and that the real and imaginary part have equal amplitude.
The skin depth of a good conductor is

$$\delta = \frac{1}{\alpha} = \sqrt{\frac{2}{\omega \mu \sigma}} = \frac{1}{\sqrt{\pi f \mu \sigma}}$$

For copper, we have $\sigma = 5.8 \cdot 10^7$ S/m. This implies

- $f = 50$ Hz  \hspace{1cm} $\delta = 9.35$ mm
- $f = 1$ kHz  \hspace{1cm} $\delta = 2.09$ mm
- $f = 1$ MHz \hspace{1cm} $\delta = 0.07$ mm
- $f = 1$ GHz \hspace{1cm} $\delta = 2.09$ $\mu$m

This effectively confines all fields in a metal to a thin region near the surface.
Surface impedance

Integrating the currents near the surface $z = 0$ implies (with $\gamma = \alpha + j\beta$)

$$J_s = \int_{0}^{\infty} J(z) \, dz = \int_{0}^{\infty} \sigma E_0 e^{-\gamma z} \, dz = \frac{\sigma E_0}{\gamma}$$

Thus, the surface current can be expressed as

$$J_s = \frac{1}{Z_s} E_0$$

where the surface impedance is

$$Z_s = \frac{\gamma}{\sigma} = \frac{\alpha + j\beta}{\sigma} = \frac{\alpha}{\sigma} (1 + j) = \frac{1}{\sigma \delta} (1 + j) = \sqrt{\frac{\omega \mu}{2\sigma}} (1 + j) = \eta_c$$

Since $H_0 \times \hat{z} = E_0 / \eta_c$, this can also be written

$$J_s = H_0 \times \hat{z} = -\hat{z} \times H_0 = \hat{n} \times H_0.$$
1. Plane waves in lossless media
   General time dependence
   Time harmonic waves

2. Polarization

3. Propagation in lossy media

4. Oblique propagation and complex waves

5. Paraxial approximation: beams

6. Doppler effect and negative index media

7. Conclusions
Generalized propagation factor

For a wave propagating in an arbitrary direction, the propagation factor is generalized as

\[ e^{-jkz} \rightarrow e^{-j\mathbf{k} \cdot \mathbf{r}} \]

Assuming this as the only spatial dependence, the nabla operator can be replaced by \(-jk\) since

\[ \nabla (e^{-jk\cdot\mathbf{r}}) = -j\mathbf{k} (e^{-jk\cdot\mathbf{r}}) \]

Writing the fields as \( \mathbf{E}(\mathbf{r}) = \mathbf{E}_0 e^{-j\mathbf{k} \cdot \mathbf{r}} \), Maxwell’s equations for isotropic media can then be written

\[
\begin{align*}
- j \mathbf{k} \times \mathbf{E}_0 &= -j \omega \mu H_0 \\
- j \mathbf{k} \times \mathbf{H}_0 &= j \omega \epsilon \mathbf{E}_0
\end{align*}
\]  
\[ \Rightarrow \]

\[
\begin{align*}
\mathbf{k} \times \mathbf{E}_0 &= \omega \mu \mathbf{H}_0 \\
\mathbf{k} \times \mathbf{H}_0 &= -\omega \epsilon \mathbf{E}_0
\end{align*}
\]
Eliminating the magnetic field, we find

\[ \mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) = -\omega^2 \epsilon \mu \mathbf{E}_0 \]

This shows that \( \mathbf{E}_0 \) does not have any components parallel to \( \mathbf{k} \), and the BAC-CAB rule implies \( \mathbf{k} \times (\mathbf{k} \times \mathbf{E}_0) = -\mathbf{E}_0 (\mathbf{k} \cdot \mathbf{k}) \). Thus,

\[ k^2 = \mathbf{k} \cdot \mathbf{k} = \omega^2 \epsilon \mu \]

It is further clear that \( \mathbf{E}_0, \mathbf{H}_0 \) and \( \mathbf{k} \) constitute a right-handed triple since \( \mathbf{k} \times \mathbf{E}_0 = \omega \mu \mathbf{H}_0 \), or

\[ \mathbf{H}_0 = \frac{k}{\omega \mu} \frac{k}{k} \times \mathbf{E}_0 = \frac{1}{\eta} \hat{k} \times \mathbf{E}_0 \]
What happens when \( \hat{k} \) is not along the \( z \)-direction (which could be the normal to a plane surface)?

- There are then two preferred directions, \( \hat{k} \) and \( \hat{z} \).
- These span a plane, the plane of incidence.
- It is natural to specify the polarizations with respect to that plane.
- When the \( H \)-vector is orthogonal to the plane of incidence, we have *transverse magnetic* polarization (TM).
- When the \( E \)-vector is orthogonal to the plane of incidence, we have *transverse electric* polarization (TE).
From these figures it is clear that the transverse impedance is

\[ \eta_{\text{TM}} = \frac{E_x}{H_y} = \frac{A \cos \theta}{\frac{1}{\eta} A} = \eta \cos \theta \]

\[ \eta_{\text{TE}} = -\frac{E_y}{H_x} = \frac{B}{\frac{1}{\eta} B \cos \theta} = \frac{\eta}{\cos \theta} \]
Interpretation of transverse wave vector

The transverse wave vector \( k_t = k_x \hat{x} \) corresponds to the angle of incidence \( \theta \) as

\[
[k_x = k \sin \theta]
\]

The transverse impedance is

\[
E_t = Z_r \cdot (H_t \times \hat{z}), \quad Z_r = \eta \cos \theta \hat{x} \hat{x} + \frac{\eta}{\cos \theta} \hat{y} \hat{y}
\]

isotropic case

In a coming lecture, we will see how to generalize the transverse wave impedance for oblique propagation in bianisotropic materials.
Complex waves

When the material parameters are complexified, we still have

\[ k_c^2 = k \cdot k = \omega^2 \varepsilon c \mu c \]

with a complex wave vector

\[ k = \beta - j\alpha \quad \Rightarrow \quad e^{-jk \cdot r} = e^{-j\beta \cdot r} e^{-\alpha \cdot r} \]

The real vectors \( \alpha \) and \( \beta \) do not need to be parallel.

<table>
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<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \alpha_z )</th>
<th>( \alpha_x )</th>
<th>( \beta_z )</th>
<th>( \beta_x )</th>
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<td>0</td>
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<td>-</td>
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<tr>
<td>↑ ( \rightarrow )</td>
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<td>( \uparrow ) ( \downarrow )</td>
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<td>+</td>
<td>+</td>
<td>-</td>
<td>Zennecke surface wave</td>
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<tr>
<td>( \downarrow ) ( \uparrow )</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>leaky wave</td>
<td></td>
</tr>
</tbody>
</table>
Outline

1. Plane waves in lossless media
   - General time dependence
   - Time harmonic waves

2. Polarization

3. Propagation in lossy media

4. Oblique propagation and complex waves

5. Paraxial approximation: beams

6. Doppler effect and negative index media

7. Conclusions
So far we have treated plane waves, which have a serious drawback:

- Due to the infinite extent of $e^{-j\beta z}$ in the $xy$-plane, the plane wave has infinite energy.

However, the plane wave is a useful object with which we can build other, more physically reasonable, solutions.
Finite extent in the $xy$-plane

We can represent a field distribution with finite extent in the $xy$-plane using a Fourier transform:

$$
E_t(r, \omega) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_t(z, k_t, \omega) e^{-jk_t \cdot r} \, dk_x \, dk_y
$$

$$
E_t(z, k_t, \omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E_t(r, \omega) e^{jk_t \cdot r} \, dx \, dy
$$

The $z$ dependence in $E_t(z, k_t, \omega)$ corresponds to a plane wave

$$
E_t(z, k_t, \omega) e^{-jk_t \cdot r} = E_t(0, k_t, \omega) e^{-jk_t \cdot r} e^{-j\beta z}
$$

The total wavenumber for each $k_t$ is given by $k^2 = \omega^2 \varepsilon \mu$ and

$$
k^2 = |k_t|^2 + \beta^2 = k_x^2 + k_y^2 + \beta^2 \quad \Rightarrow \quad \beta(k_t) = (k^2 - |k_t|^2)^{1/2}
$$
Initial distribution

Assume a Gaussian distribution in the plane $z = 0$

$$E_t(x, y, z = 0, \omega) = A(\omega)e^{-(x^2+y^2)/(2b^2)},$$

The transform is itself a Gaussian

$$E_t(z = 0, k_t, \omega) = A(\omega)2\pi b^2 e^{-(k_x^2+k_y^2)b^2/2}$$
Paraxial approximation

The field in $z \geq 0$ is then

$$E_t(r, \omega) = \frac{1}{2\pi} Ab^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(k^2_x+k^2_y)b^2/2-j(k_x x+k_y y)-j\beta(k_t)z} dk_x dk_y$$

The exponential makes the main contribution to come from a region close to $k_t \approx 0$. This justifies the paraxial approximation

$$\beta(k_t) = (k^2 - |k_t|^2)^{1/2} = k(1 - |k_t|^2/k^2)^{1/2}$$

$$= k \left( 1 - \frac{1}{2} \frac{|k_t|^2}{k^2} + O\left(\frac{|k_t|^4}{k^4}\right) \right) = k - \frac{|k_t|^2}{2k} + \cdots$$
Inserting the paraxial approximation in the Fourier integral implies
(details can be found in Kristensson 5.3)

\[ E_t(r) \approx \frac{1}{2\pi} Ab^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\left(k_x^2 + k_y^2\right)\left(\frac{b^2}{2} - \frac{z}{2k}\right) - j(k_xx + k_y y) - jkz} \, dk_x \, dk_y \]

\[ = \cdots = \frac{A}{1 - j\xi(z, \omega)} e^{-\frac{(x^2 + y^2)}{2F^2} - jkz} \]

where \( F^2(z, \omega) = b^2 - jz/k = b^2(1 - j\xi(z, \omega)) \) and \( \xi = z/(kb^2) \) is a real quantity.
The power density of the beam is proportional to

$$e^{-(x^2+y^2) \text{Re}(1/F^2)}$$

and the beam width is then

$$B(z) = \frac{1}{\sqrt{\text{Re} \frac{1}{F^2}}} = \cdots = b\xi \sqrt{1 + \xi^{-2}}$$

where $\xi = z/(kb^2)$. For large $z$, the beam width is

$$B(z) \to b\xi(z) = \frac{z}{kb}, \quad z \to \infty$$
The beam angle $\theta_{b}$ is characterized by

$$\tan \theta_{b} = \frac{B(z)}{z} = \frac{1}{kb}$$

Small initial width compared to wavelength implies large beam angle.
Outline

1 Plane waves in lossless media
   General time dependence
   Time harmonic waves

2 Polarization

3 Propagation in lossy media

4 Oblique propagation and complex waves

5 Paraxial approximation: beams

6 Doppler effect and negative index media

7 Conclusions
Classical Doppler effect

\[ f_b = \frac{c}{\lambda_b} = f_a \frac{c}{c - v_a} \]

\[ f_b = \frac{c - v_b}{\lambda_a} = f_a \frac{c - v_b}{c} \]

\[ f_b = f_a \frac{c - v_b}{c - v_a} \approx f_a \left(1 + \frac{v_a - v_b}{c}\right) \]

Lots more on relativistic Doppler effect in Orfanidis.
Passivity requires the material parameters $\epsilon_c$ and $\mu_c$ to be in the lower half plane,

This means we could very well have $\epsilon_c/\epsilon_0 \approx -1$ and $\mu_c/\mu_0 \approx -1$ for some frequency. What is then the appropriate value for $k = \omega \sqrt{\epsilon_c \mu_c} = k_0 \sqrt{(\epsilon_c/\epsilon_0)(\mu_c/\mu_0)} = k_0 \sqrt{(-1)(-1)}$,

$k = +k_0 \text{ or } k = -k_0$?
Negative refractive index

The product of two complex valued parameters, both in the lower half plane, can be located anywhere in the complex plane.

By defining the argument as negative, the square root operation brings back the lower half plane (the square root reduces the argument of a complex number by a factor of 2). Thus, the refractive index is

\[ n = \sqrt{\frac{\varepsilon_c}{\varepsilon_0}}\left(\frac{\mu_c}{\mu_0}\right) = \sqrt{(-1)(-1)} = -1 \]

Alternatively, use the standard square root and

\[ j\omega n = \sqrt{(j\omega \varepsilon_c/\varepsilon_0)(j\omega \mu_c/\mu_0)} \]
Consequences of negative refractive index

With a negative refractive index, the exponential factor

\[ e^{j(\omega t - k z)} = e^{j(\omega t + |k| z)} \]

represents a phase traveling in the negative \( z \)-direction, even though the Poynting vector \( \frac{1}{2} \text{Re}\{E \times H^*\} = \hat{z} \frac{1}{2} \text{Re}\left(\frac{1}{\eta_c^*}\right)|E_0|^2 \) is still pointing in the positive \( z \)-direction.

- The power flow is in the opposite direction of the phase velocity!
- Snel’s law has to be “inverted”, the rays are refracted in the wrong direction.

First investigated by Veselago in 1967. Enormous scientific interest since about a decade, since the materials can now (to some extent) be fabricated.
Negative refraction
Realization of negative refractive index

In order to obtain $\epsilon_c(\omega)/\epsilon_0 = -1$ and $\mu_c(\omega)/\mu_0 = -1$, artificial materials are necessary. Common strategy:

- Periodic structure with inclusions small compared to wavelength.
- To obtain negative properties, the inclusions are usually resonant.

Drawbacks:

- The inclusions are not always small, putting the material point of view in doubt.
- The interesting applications require very small losses.
- The negative properties require frequency variation, making it difficult to maintain negative properties in a large frequency band.
If the negative properties are realized with passive, causal materials, they must satisfy Kramers-Kronig’s relations ($\epsilon_\infty = \lim_{\omega \to \infty} \epsilon(\omega)$)

$$
\epsilon'(\omega) - \epsilon_\infty = \frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\epsilon''(\omega')}{\omega' - \omega} \, d\omega'
$$

$$
\epsilon''(\omega) = -\frac{1}{\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\epsilon'(\omega') - \epsilon_\infty}{\omega' - \omega} \, d\omega'
$$

These relations represent restriction on the possible frequency behavior, and can be used to derive bounds on the bandwidth where the material parameters can be negative.
An $\epsilon_m$ between $\epsilon_s$ and $\epsilon_\infty$ is easily realized for a large bandwidth, whereas an $\epsilon_m < \epsilon_\infty$ is not. With fractional bandwidth $B$:

$$\max_{\omega \in B} |\epsilon(\omega) - \epsilon_m| \geq \frac{B}{1 + B/2} (\epsilon_\infty - \epsilon_m) \begin{cases} 1/2 & \text{lossy case} \\ 1 & \text{lossless case} \end{cases}$$
Outline

1 Plane waves in lossless media
   General time dependence
   Time harmonic waves

2 Polarization

3 Propagation in lossy media

4 Oblique propagation and complex waves

5 Paraxial approximation: beams

6 Doppler effect and negative index media

7 Conclusions
Conclusions

- Plane waves are characterized by wave vector $k$ and wave impedance $\eta$.
- Polarization is deduced from time domain electric field.
- Lossy media leads to complex material parameters, but plane wave formalism remains the same.
- At oblique propagation, the transverse fields are most important.
- The paraxial approximation can be used to describe beams. The beam angle depends on the original beam width in terms of wavelengths.
- The Doppler effect can be used to detect motion.
- Negative refractive index is possible, but only for very narrow frequency band.