LUND UNIVERSITY

EITP30 Modern Wireless Systems - 5G and Beyond

Lecture 3 – OFDM + FFT

ALEKSEI FEDOROV
EIT, LUND UNIVERSITY, SWEDEN

Find your partner to work together!
Measurements day is at 15 Sep at 2 pm E: 2316

Quick Test

1. 2D vectors: $a = \begin{pmatrix} 1 \\ 1 \end{pmatrix},\ b = \begin{pmatrix} -1 \\ 1 \end{pmatrix},\ c = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, which of them are orthogonal?

2. 3D vectors: $a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix},\ c = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, which of them are orthogonal?

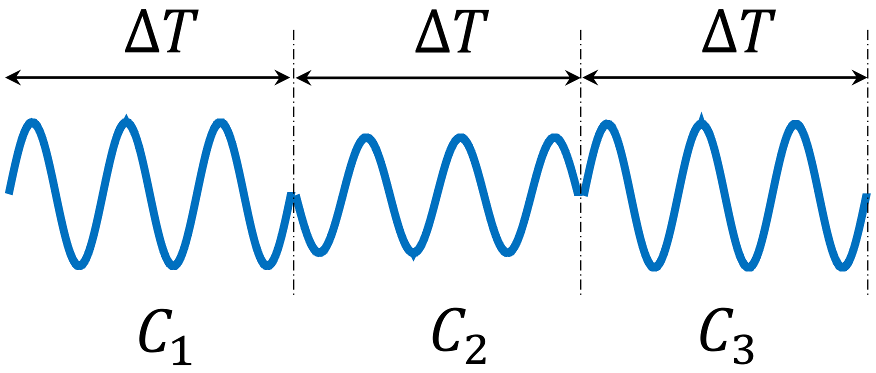
3. Are the given two functions $f(t), g(t)$ orthogonal if

   $f(t) = \sin\left(\frac{2\pi}{N}(k+1)\,t\right)$, and $g(t) = \sin\left(\frac{2\pi}{N}k\,t\right)$, $N > 2^5, 0 < k < N, \mathrm{t} \in [0, 2\pi]$?

4. Are the given two functions $f(t), g(t)$ orthogonal if

   $f(t) = \sin\left(\frac{2\pi}{N}(k+1)\,t\right)$, and $g(t) = \sin\left(\frac{2\pi}{N}k\,t\right)$, $N > 2^5, 0 < k < N,$ $\mathrm{t} \in [0, N]$?

Lecture 2 recap



Single carrier transmission

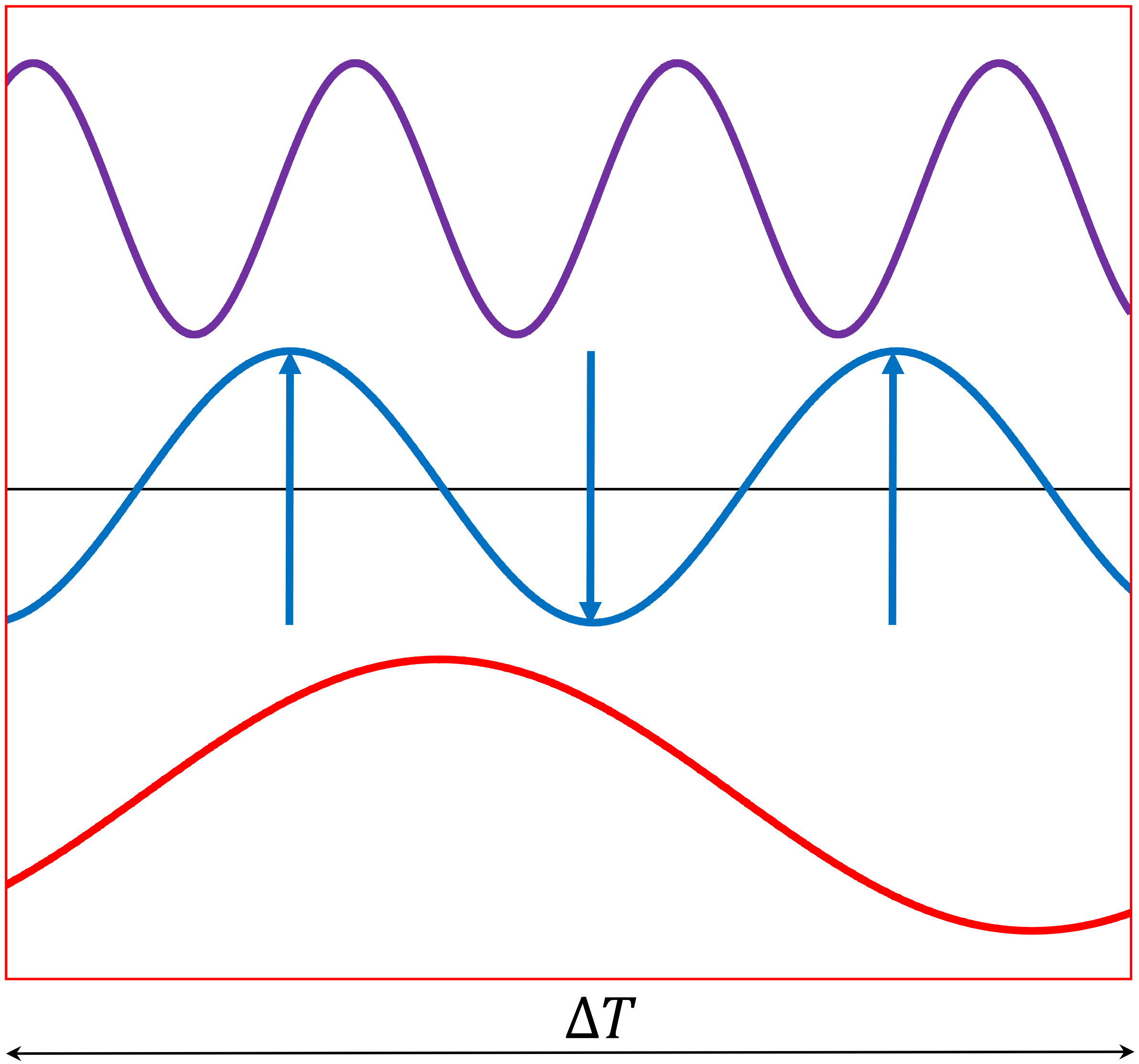
ΔT – symbol duration;
$\Delta T = M \cdot \Delta t$, where Δt – system's sampling rate

The data rate (throughput) can be roughly estimated as

$$\text{Throughput} = \frac{1}{\Delta T} \log_2 N$$

- We cannot infinitely increase the size of the constellation
- We cannot infinitely decrease the symbol's duration (sampling rate)
- But we can combine a number of waveforms

Orthogonal EM waves



Waveforms must be orthogonal within the given time frame

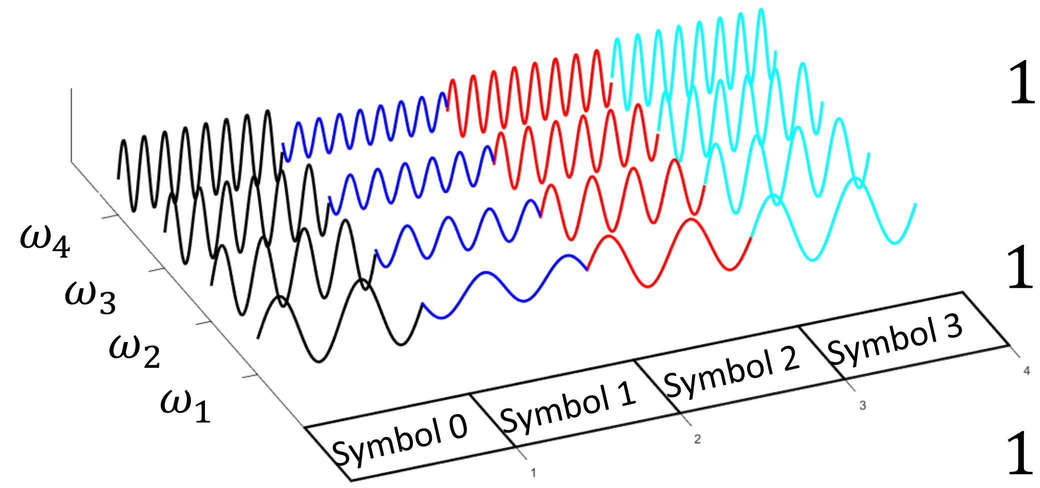
Lecture 3

- Orthogonal harmonics
- Relation between harmonics and Fourier transform
- Discrete convolution: linear and circular
- **C**yclic **P**refix (CP) insertion
- OFDM issues (**P**eak to **A**verage **P**ower **R**atio – PAPR)

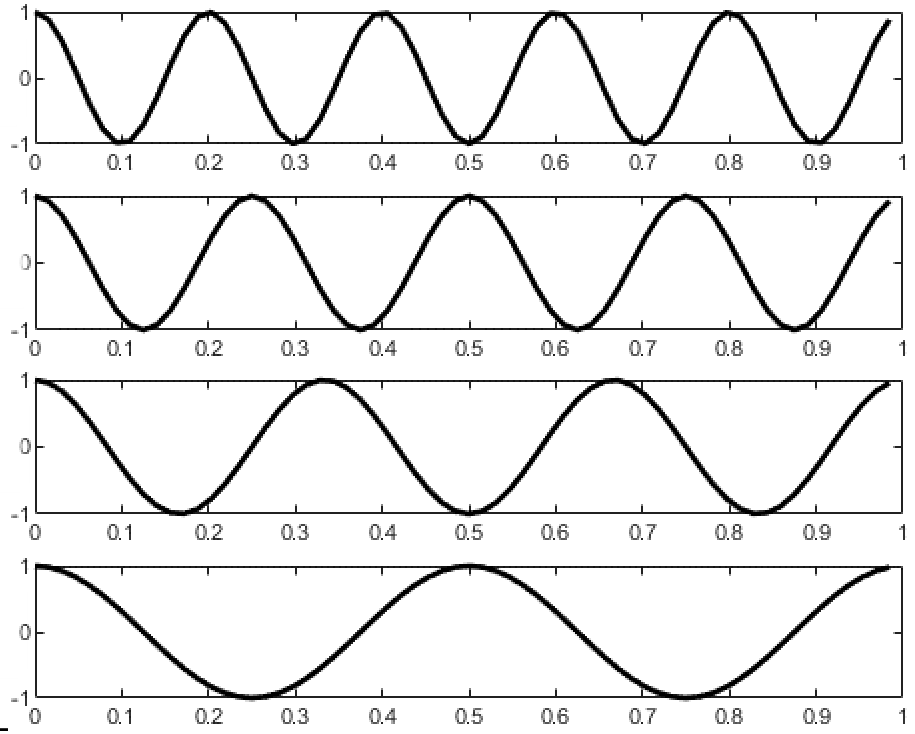
Lecture 3

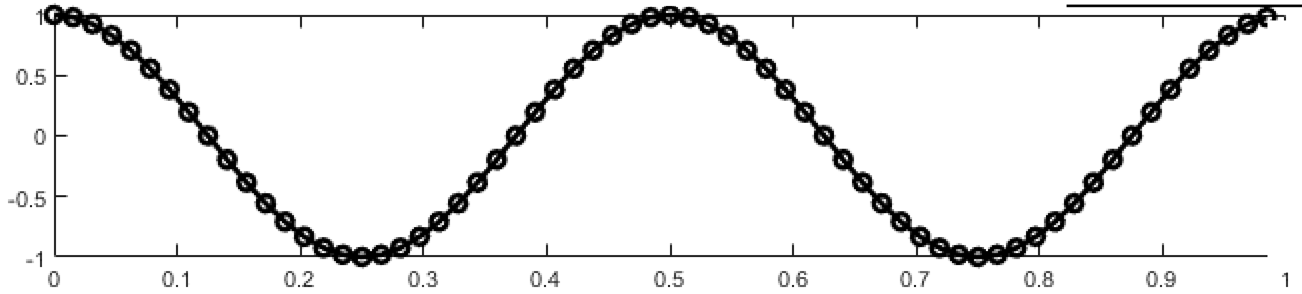
- Orthogonal harmonics
- Relation between harmonics and Fourier transform

Orthogonal Harmonics



$$1 \cdot \cos(\omega_4)$$
$$+$$
$$1 \cdot \cos(\omega_3)$$
$$+$$
$$1 \cdot \cos(\omega_2)$$
$$+$$
$$1 \cdot \cos(\omega_1)$$

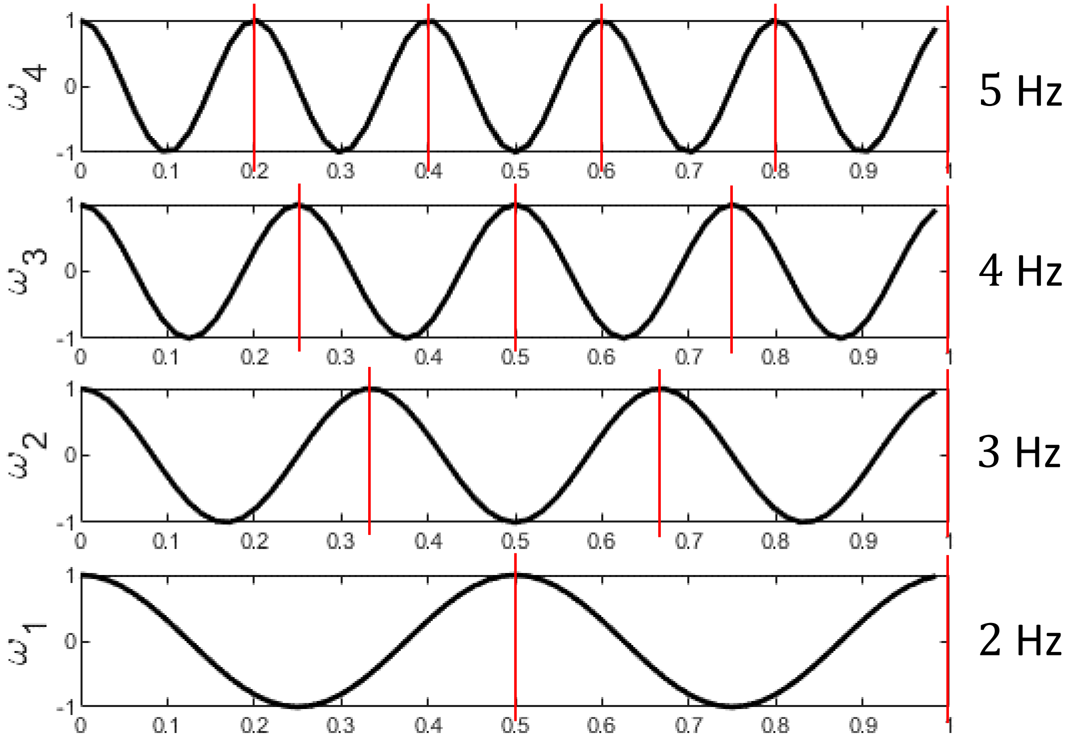


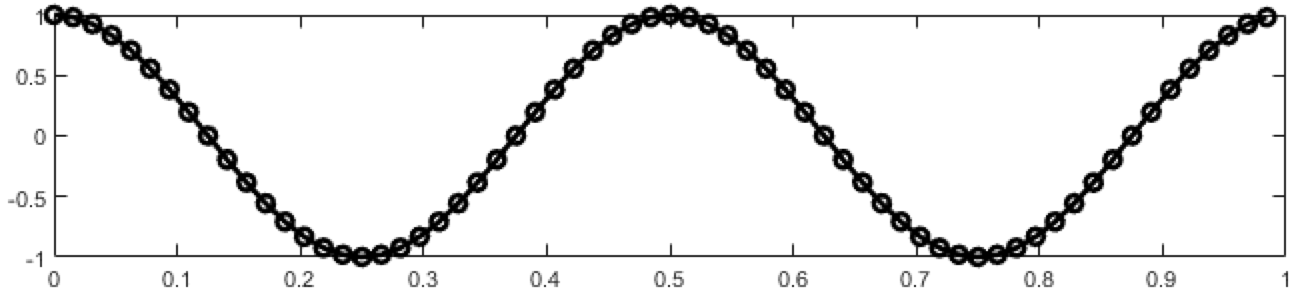


Discretized cosine 64 samples

Orthogonal Harmonics

- We have 64 samples within 1 second
- The sampling rate is 64 sps
- The duration of one sample is 1/64 sec
- The maximum frequency that can be recovered <32 Hz for real and <64 Hz for complex valued signals (Nyquist-Shannon)
- Let’s consider the size of IDFT 64



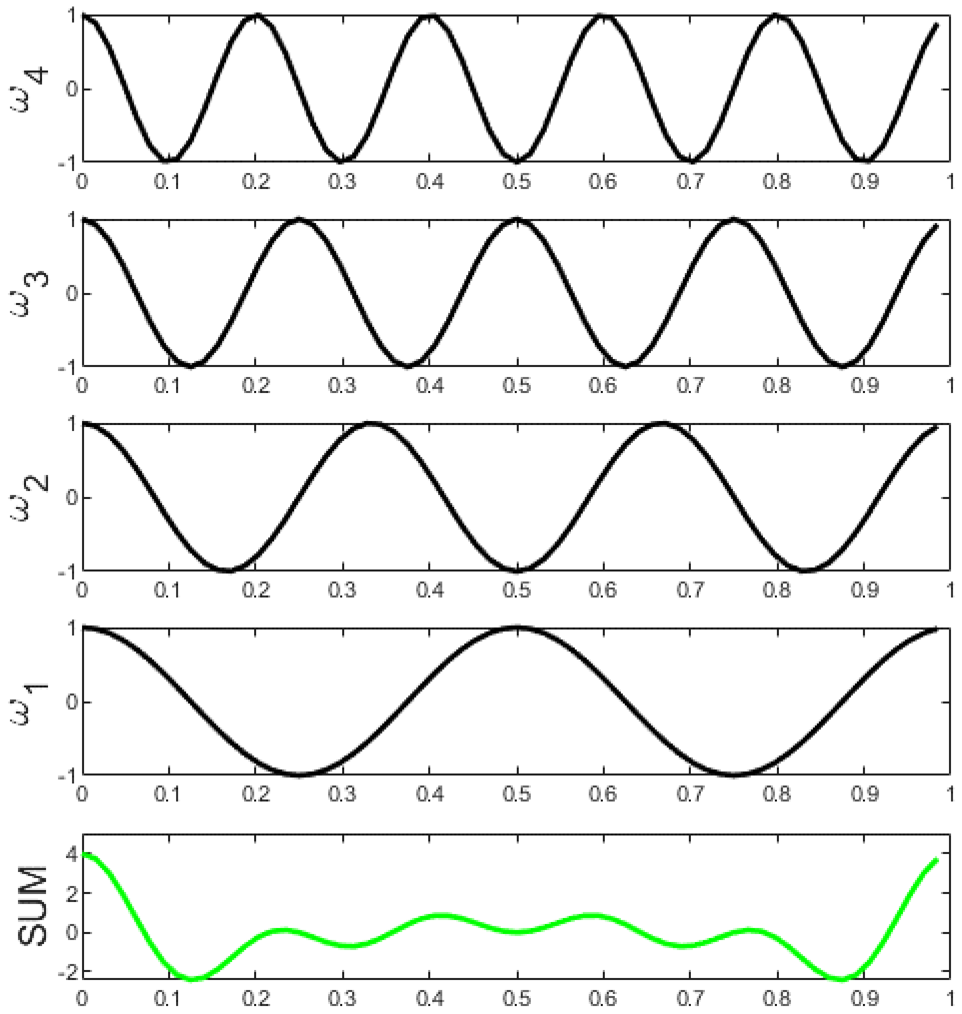


Discretized cosine 64 samples

0	1	2	3	4	5	6					64
0	0	**1**	**1**	**1**	**1**	0	0	0		**..............**	0

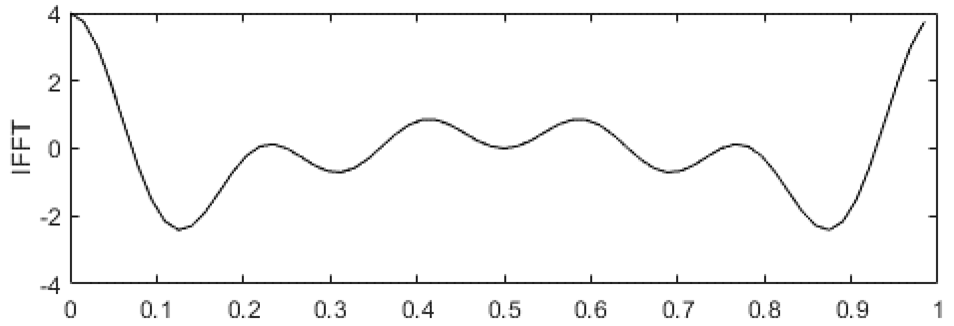
Vector S with 64 elements

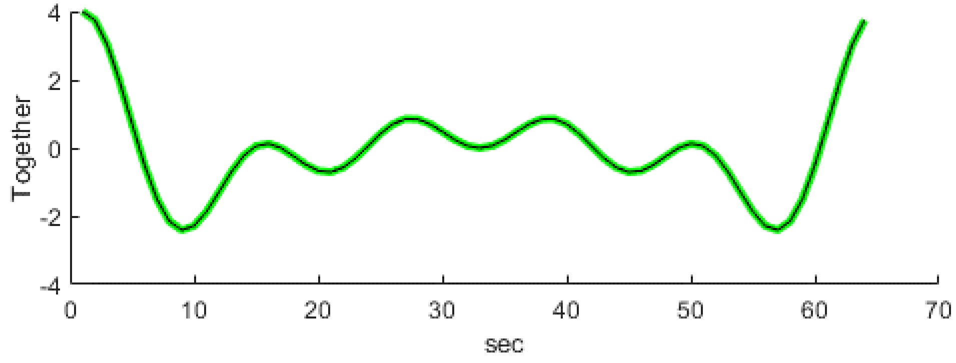
Relation between Harmonics and DFT



$$S = (0, 0, 1, 1, 1, 1, 0, \ldots, 0)$$

$S \rightarrow IFFT\ (64)$ -> plot real part of it since we're interested in cosines

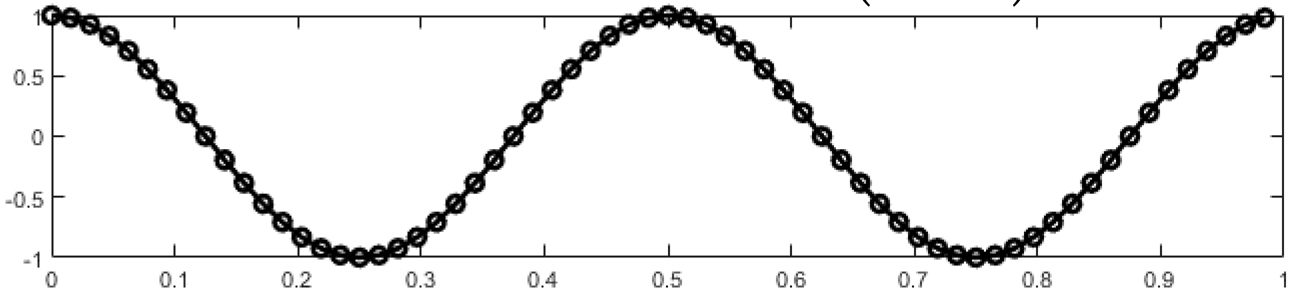




How does DFT work?

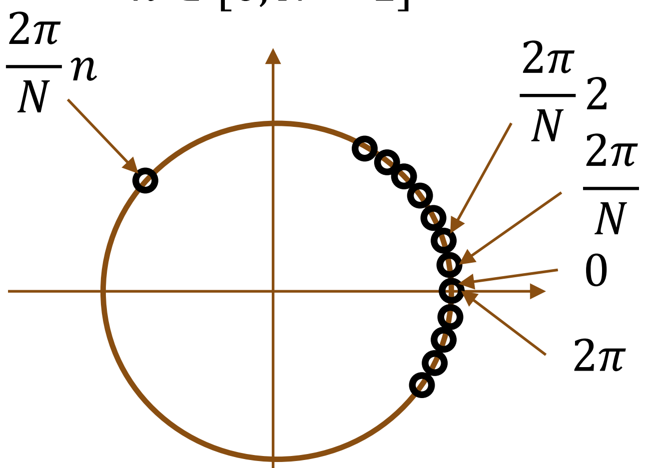
$\Delta t = \frac{1}{N}$ s, $F_s = N$ Hz

$\cos\left(\frac{2\pi}{N} 2\, k\right), k \in [0, N-1]$



N samples

$n \in [0, N-1]$



N samples

Time

Frequency

$$
\begin{array}{ll}
 & \begin{array}{cccccc} 0 & 1 & 2 & 3 & & (N-1) \end{array} \\
\begin{array}{c} \frac{2\pi}{N}0 \\ \frac{2\pi}{N}1 \\ \frac{2\pi}{N}2 \\ \frac{2\pi}{N}3 \\ \vdots \\ \frac{2\pi}{N}(N-1) \end{array} & M_{DFT} = \begin{pmatrix}
e^{-\frac{j2\pi}{N}00} & e^{-\frac{j2\pi}{N}01} & e^{-\frac{j2\pi}{N}02} & e^{-\frac{j2\pi}{N}03} & \dots\dots & e^{-\frac{j2\pi}{N}0(N-1)} \\
e^{-\frac{j2\pi}{N}10} & e^{-\frac{j2\pi}{N}11} & e^{-\frac{j2\pi}{N}12} & e^{-\frac{j2\pi}{N}13} & \dots\dots & e^{-\frac{j2\pi}{N}1(N-1)} \\
e^{-\frac{j2\pi}{N}20} & e^{-\frac{j2\pi}{N}21} & e^{-\frac{j2\pi}{N}22} & e^{-\frac{j2\pi}{N}23} & \dots\dots & e^{-\frac{j2\pi}{N}2(N-1)} \\
e^{-\frac{j2\pi}{N}30} & e^{-\frac{j2\pi}{N}31} & e^{-\frac{j2\pi}{N}32} & e^{-\frac{j2\pi}{N}33} & \dots\dots & e^{-\frac{j2\pi}{N}3(N-1)} \\
\vdots & \vdots & \vdots & \vdots & \dots & \vdots \\
e^{-\frac{j2\pi}{N}(N-1)0} & e^{-\frac{j2\pi}{N}(N-1)1} & e^{-\frac{j2\pi}{N}(N-1)2} & e^{-\frac{j2\pi}{N}(N-1)3} & \dots & e^{-\frac{j2\pi}{N}(N-1)(N-1)}
\end{pmatrix}
\end{array}
$$

HW: Generate DFT matrix (dftmtx)

$$Y = M_{DFT} \cdot y = M_{DFT} \begin{pmatrix} \cos\left(\frac{2\pi}{N} \cdot 2 \cdot 0\right) \\ \cos\left(\frac{2\pi}{N} \cdot 2 \cdot 1\right) \\ \cos\left(\frac{2\pi}{N} \cdot 2 \cdot 2\right) \\ \vdots \\ \cos\left(\frac{2\pi}{N} \cdot 2 \cdot (N-1)\right) \end{pmatrix}, y = \begin{bmatrix} \cos\left(\frac{2\pi}{N} \cdot 2 \cdot \boldsymbol{t}^{\boldsymbol{T}}\right); \\ \exp\left(1j \cdot \frac{2\pi}{N} \cdot 2 \cdot \boldsymbol{t}^{\boldsymbol{T}}\right). \end{bmatrix}$$

How to calculate inverse DFT matrix?

Hint of the Day! (What IF?)

- As we saw, with the sampling rate 64 sps and FFT size 64, the separation between harmonics, i.e. $e^{(j2\pi\, f_k\, t)}$, is equal to 1 Hz. In other words, $f_{k+1} - f_k = 1$ Hz.
- As you noticed, all these harmonics are mutually orthogonal. Why?
- In OFDM, these harmonics are called as subcarriers.
- What if we use the size of FFT to be equal to 32 instead of 64? Note, the sampling rate stays the same!
  - the separation will be 2 Hz;
- In LTE, the sampling rate is 30.72 Msps and the FFT size is 2048 (this is also the number of subcarriers).
  - What is the distance between subcarriers in the frequency domain?

Answer to why

Note that $f_k = k\,\Delta f$ and $f_l = l\,\Delta f$

$$\int_0^{N\cdot\Delta t} e^{-j\cdot 2\pi\cdot f_k\cdot t}\cdot e^{j\cdot 2\pi\cdot f_l\cdot t}\,dt = Ce^{j\cdot 2\pi\cdot(f_l-f_k)\cdot t}\Big|_0^{N\cdot\Delta t} = Ce^{j\cdot 2\pi\cdot(l-k)} - C = 0$$

if $k \neq l$

Where N is the size of DFT, $\Delta t = \frac{1}{f_s}$ is the sample duration, and the subcarriers separation can be found by $\Delta f = \frac{f_s}{N}$

To Digest Information

Summation of harmonics can become the same as operation of FFT/IFFT if proper relations between time and frequency domains are taken into account.

Matlab Code from the slides

```
% define size of IFFT, sampling rate
N1 = 64; dt1 = 1/N1; t1 = (0:N1-1)*dt1;

figure
h1 = cos(2*pi*2*t1); subplot(5,1,4), plot(t1,h1,'k','linewidth',2), ylabel('\omega_1','FontSize',16)
h2 = cos(2*pi*3*t1); subplot(5,1,3), plot(t1,h2,'k','linewidth',2), ylabel('\omega_2','FontSize',16)
h3 = cos(2*pi*4*t1); subplot(5,1,2), plot(t1,h3,'k','linewidth',2), ylabel('\omega_3','FontSize',16)
h4 = cos(2*pi*5*t1); subplot(5,1,1), plot(t1,h4,'k','linewidth',2), ylabel('\omega_4','FontSize',16)
h_sum = h1+h2+h3+h4;
subplot(5,1,5), plot(t1,h_sum,'g','linewidth',2), ylabel('SUM','FontSize',14), xlabel('Symbol 0','FontSize',14)

% preparing a vector for IFFT
H_sum = [0 0 1 1 1 1 zeros(1,58)]; % we have to have the same 64 length vector
ifft_hh= ifft(H_sum);

% plot the results of SUM, IFFT, and together
figure
title('Harmonics sum & IFFT'), hold on
subplot(3,1,1), plot(t1, h_sum,'g','linewidth',3), ylabel('SUM')
subplot(3,1,2), plot(t1, N1*real(ifft_hh),'k'), ylabel('IFFT')
subplot(3,1,3), hold on, plot(h_sum,'g','linewidth',3), plot(N1*real(ifft_hh),'k'), ylabel('Together'), xlabel('sec')
% coefficient N1 in plot(N1*real(ifft_hh),'k') comes from the normalization

% plotting single cosine to visualize discretization
figure
plot(t1,h1,'-ok','linewidth',2)

% the same test but with FFT matrix
hh_sum = dftmtx(N1)'*H_sum;
figure
title('Harmonics sum & IFFT & IFFT via FFT matrix'), hold on
plot(h_sum,'y','linewidth',4)
plot(real(hh_sum),'g','linewidth',2)
plot(N1*real(ifft_hh),'k')
```

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- Relation between harmonics and Fourier transform
- Discrete convolution: linear and circular

Discrete Convolution

- Channel: $h(m), m \in [0, L-1]$
- Signal: $x(n), n \in [0, N-1]$

- Linear convolution: $y(k) = \sum_{m=0}^{L-1} h(m)\, x(k-m)\,,\ \ k \in [0, N+L-1]$

  if $k\ -m < 0$ or $k - m \geq N$, then $x(k-m) = 0$;

- Circular convolution: $y(k) = \sum_{m=0}^{L-1} h(m)\, x\big((k\ -m)_{\mathrm{mod}\,N}\big)\,,\ \ k \in [0, N-1]$

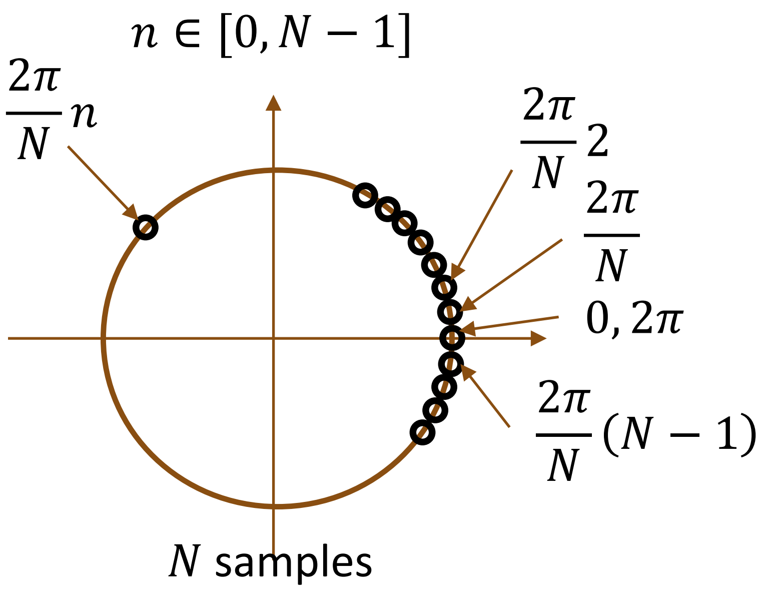
  if $k\ -m < 0$, then $x(k-m) = x(N+k-m)$;

Discrete Fourier Transform

- $F(k) = \sum_{n=0}^{N-1} f(n) \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot n}$

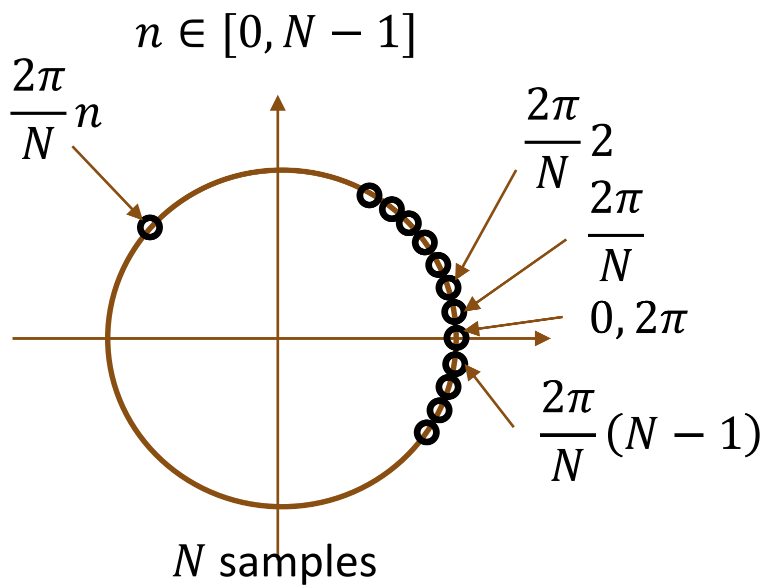
One of nice properties of DFT is

- $\boxed{Y(k) = H(k) \cdot X(k)}$



- $y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n}$

Discrete Fourier Transform



- $F(k) = \sum_{n=0}^{N-1} f(n) \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot n}$

One of nice properties of DFT is

- $\boxed{Y(k) = H(k) \cdot X(k)}$

- $y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n} = \frac{1}{N} \sum_{k=0}^{N-1} H(k) \cdot X(k) \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n} =$

$$\frac{1}{N} \sum_{k=0}^{N-1} \left[\sum_{m=0}^{L-1} h(m) \cdot e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot m} \right] \cdot X(k) \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n} =$$

$$\frac{1}{N} \sum_{m=0}^{L-1} h(m) \left[\sum_{k=0}^{N-1} X(k) \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot n} e^{-j \cdot \frac{2\pi}{N} \cdot k \cdot m} \right] =$$

$$\frac{1}{N} \sum_{m=0}^{L-1} h(m) \left[\sum_{k=0}^{N-1} X(k) \cdot e^{j \cdot \frac{2\pi}{N} \cdot k \cdot (n-m)} \right] = \sum_{m=0}^{L-1} h(m) \cdot x(n-m)$$

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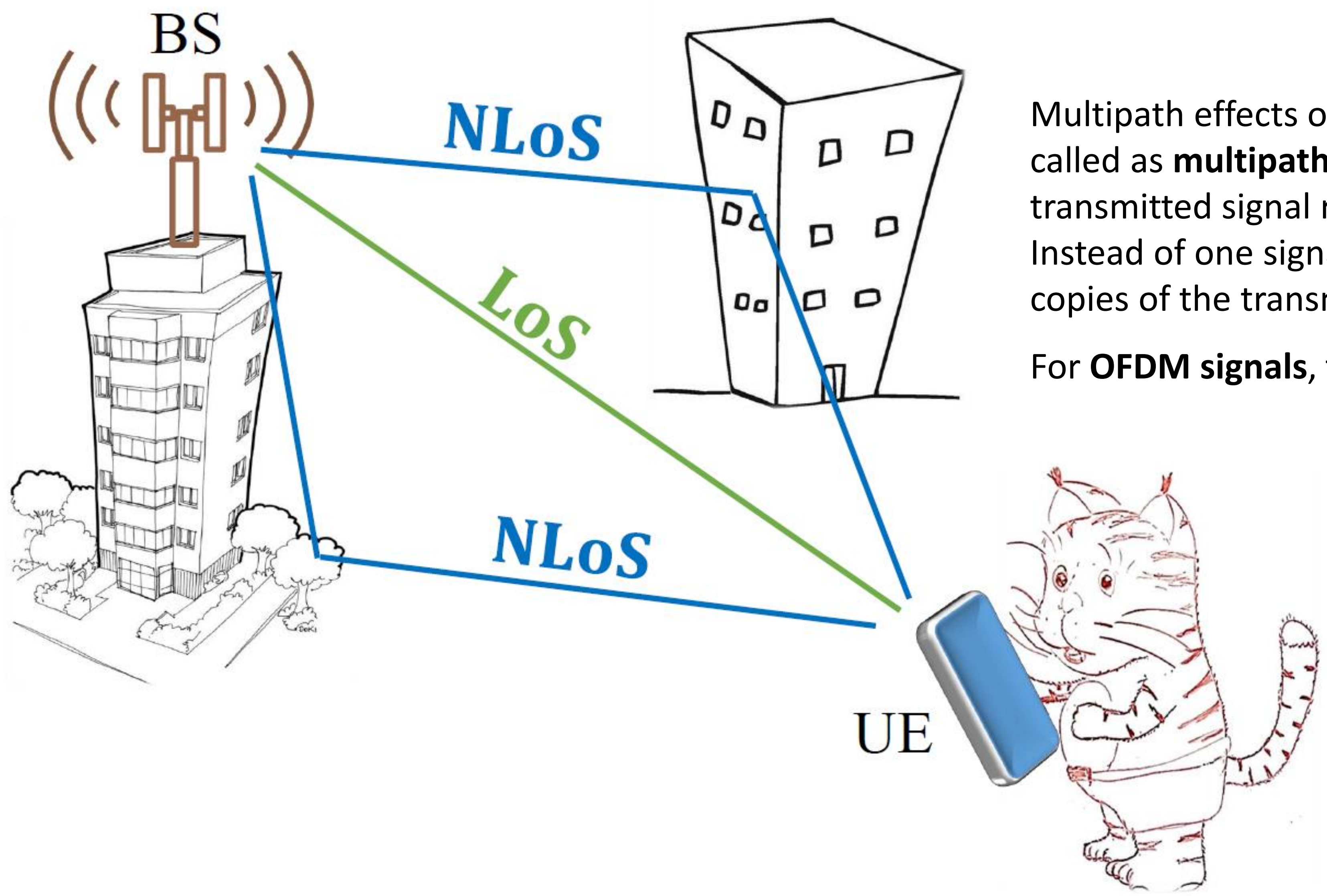
- Discrete convolution: linear and circular
- **C**yclic **P**refix (CP) insertion

Why did we start to use OFDM only in LTE?

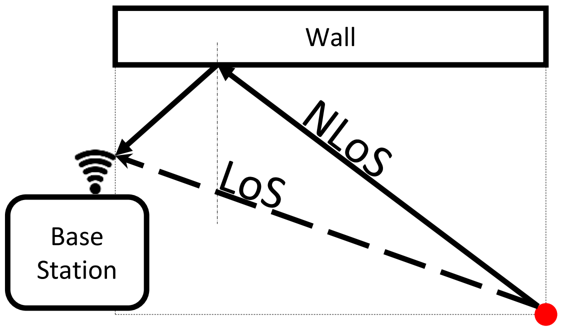
Multipath propagation



Multipath effects occur in a wireless channel, which is called as **multipath propagation** channel, where a transmitted signal reflects from different obstacles. Instead of one signal, the receiver receives several copies of the transmitted signal.

For **OFDM signals**, this becomes frequency-selective!

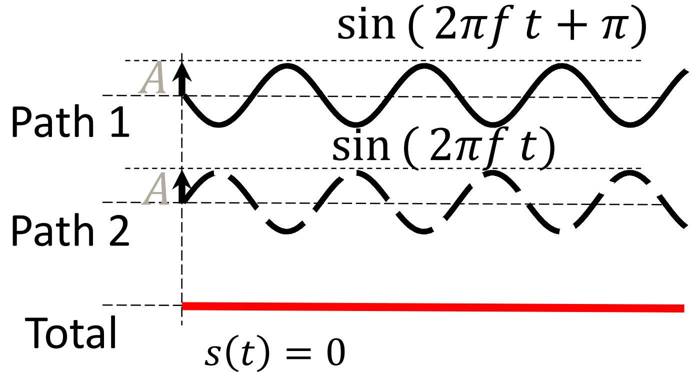
Example



D_{NLoS} - NLoS path length
D_{LoS} - LoS path

$$\tau = \frac{D_{NLoS} - D_{LoS}}{c}$$

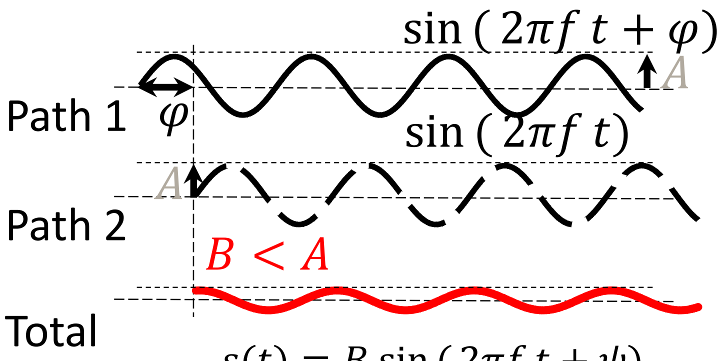
$$\varphi = \omega \cdot \tau$$



$s(t) = 0$

If the signals come in opposite phases, they can give almost zero in sum

$s(t) = s(t) + s(t + \tau)$



$s(t) = B \sin(2\pi f\, t + \psi)$

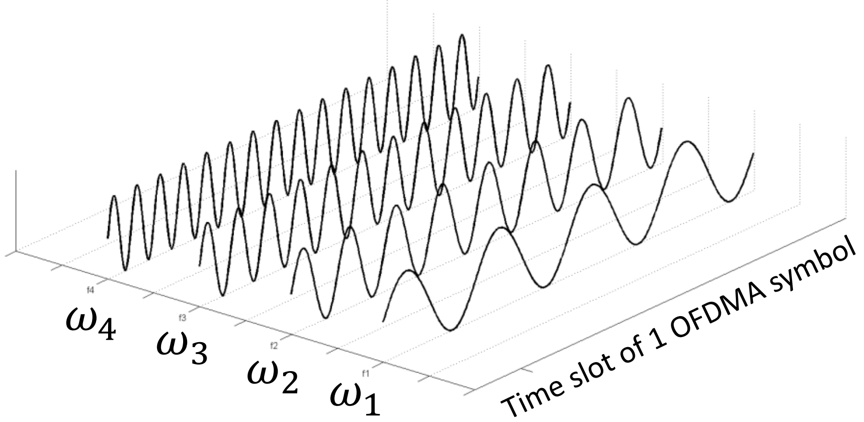
If the signals come in different phases, they can give hugely attenuated signal in sum

Find values of B, ψ –?

Significant attenuation of the received signals is called as Deep Fading Effect. The most harmful effect of the multipath propagation.

Frequency Selective Channel

Time delay between LoS and NLoS
$\tau = 10^{-6}$ sec ~ 300 m



$\Delta f = \frac{\pi}{3} \cdot 10^6$ Hz ~ 1047197 Hz

$$\omega_1 = 1 \cdot \Delta f, \qquad s_1(t) = \sin(\omega_1 t)$$
$$\omega_2 = 2 \cdot \Delta f, \qquad s_2(t) = \sin(\omega_2 t)$$
$$\omega_3 = \boxed{3 \cdot \Delta f,} \qquad s_3(t) = \sin(\omega_3 t)$$
$$\omega_4 = 4 \cdot \Delta f, \qquad s_4(t) = \sin(\omega_4 t)$$

$$s(t) = s(t) + s(t + \tau)$$

$$\boxed{\omega_3 t + \omega_3 \tau = \omega_3 t + \pi\, 10^6 \cdot 10^{-6} = \omega_3 t + \pi}$$

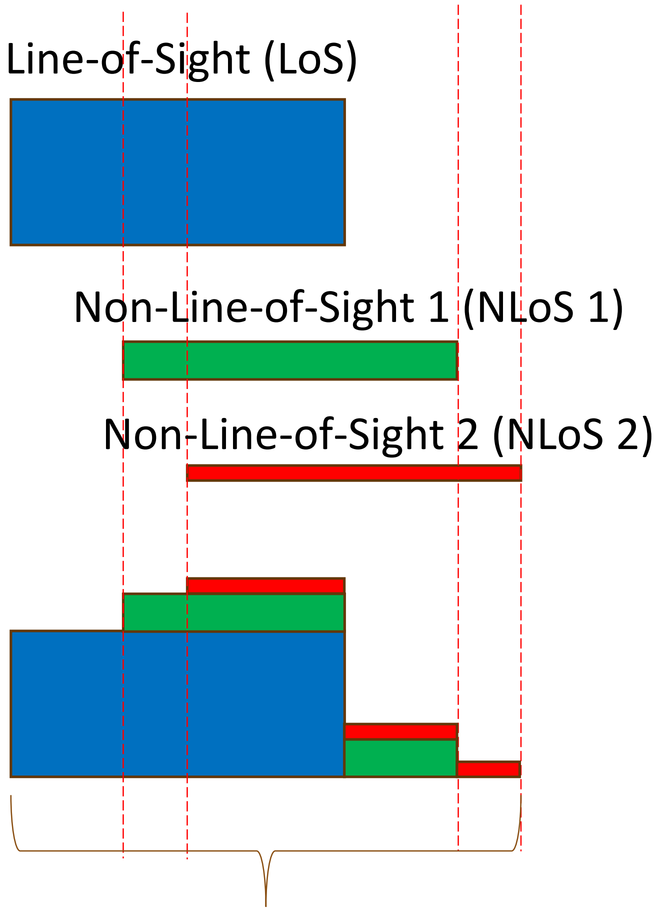
$$\boxed{\begin{aligned} \omega_1 &= \frac{\pi}{3} 10^6 \\ \omega_2 &= \frac{2\pi}{3} 10^6 \\ \omega_3 &= \pi\, 10^6 \\ \omega_4 &= \frac{4\pi}{3} 10^6 \end{aligned}}$$

$$\sin(\omega_1 t) + \sin(\omega_1 t + \omega_1 \tau) = B_1 \sin(\omega_1 t + \psi_1)$$
$$\sin(\omega_2 t) + \sin(\omega_2 t + \omega_2 \tau) = B_2 \sin(\omega_2 t + \psi_2)$$
$$\sin(\omega_3 t) + \sin(\boxed{\omega_3 t + \omega_3 \tau}) = \sin(\omega_3 t) - \sin(\omega_3 t) = 0$$
$$\sin(\omega_4 t) + \sin(\omega_4 t + \omega_4 \tau) = B_4 \sin(\omega_4 t + \psi_4)$$

Multipath channel becomes frequency selective for OFDM signals

Inter-Symbol-Interference

Let us consider three paths channel



A received signal becomes elongated!

Let's consider a channel as a filter $h(t)$

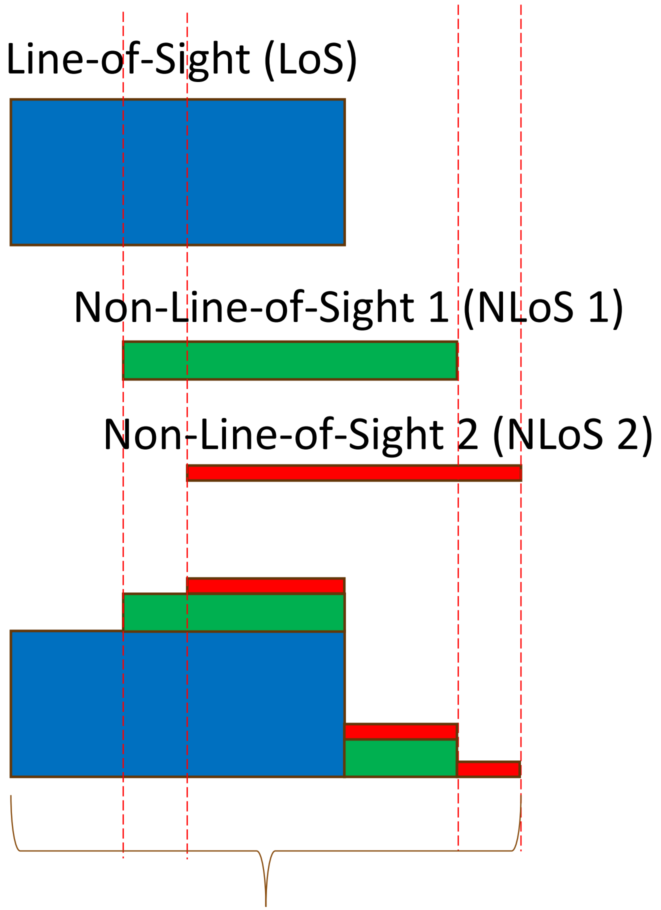
The impact of a multipath channel then can be formulated as
$s'(t) = (h * s)(t) + n(t)$

Fourier transform has a nice property to transform convolution in time domain into multiplication in frequency domain:
$(f * g)(t) \xLeftrightarrow{Fourier\,Transform} F(\omega) \cdot G(\omega)$

However, in DFT with length N, it is NOT necessarily a case!
Let us consider a channel filter $\mathrm{h}[n]$ with length L, then convolution will have length $N + L - 1 > N$. In that case, $s'[n] = (h * s)[n]$ cannot be represented as multiplication of $S'[k] \neq H[k] \cdot S[k]$!

Cyclic Prefix Insertion Eliminates ISI

Let us consider three paths channel



A received signal becomes elongated!

Cyclic prefix insertion makes the linear convolution CIRCULAR!

$$s' = h * s + r \quad \longleftrightarrow \quad s' = h \circledast s + r$$

Due to CP insertion

Hence, due to $s' = h \circledast s + r$, in the frequency domain

$$S'[k] = H[k] \cdot S[k]$$

The length of CP depends on the environment (on the channel response filter)

Nice explanation of this effect is here: https://www.youtube.com/watch?v=xWEHnFEY_yE&t=219s

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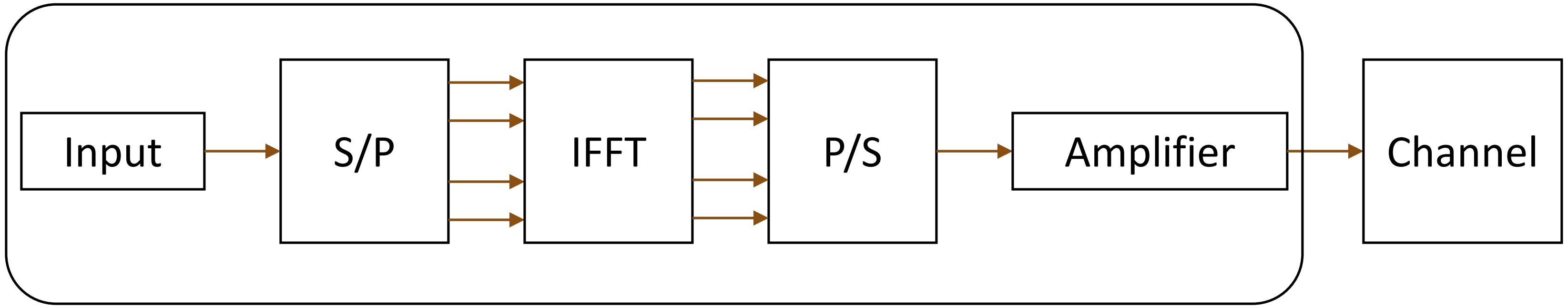
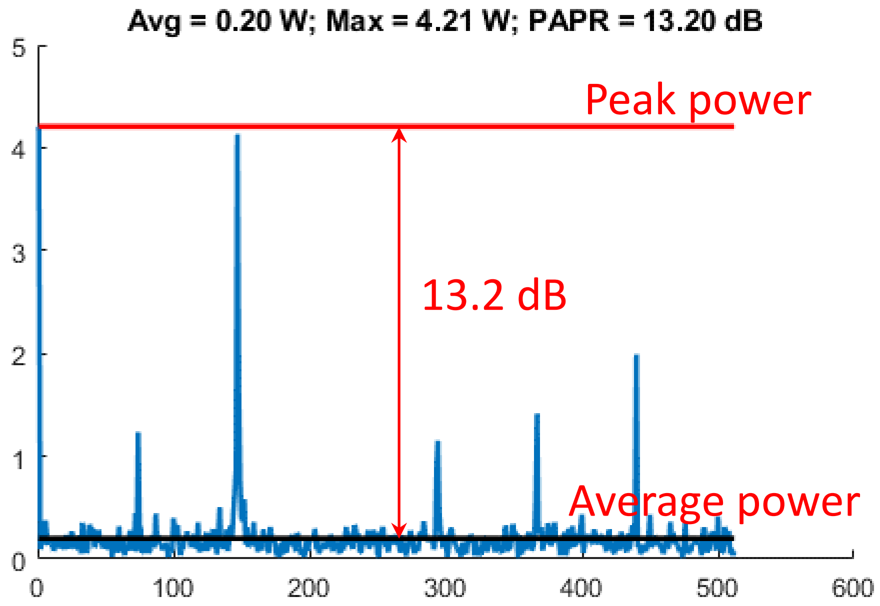
- OFDM issues (**P**eak to **A**verage **P**ower **R**atio – PAPR)

Peak to Average Power Ratio Issue in OFDM

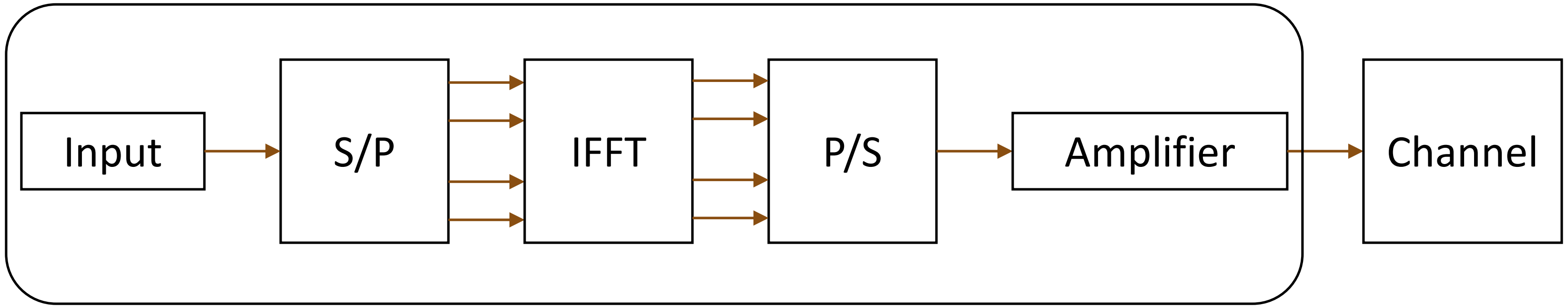


OFDM Transmitter Scheme

Text: Usually the information is “structured” meaning that it is far from being considered as noise/white noise. This means that information represented as zeros and ones has dominating frequencies almost all the time on which its FFT will be spiking.



Peak to Average Power Ratio Issue in OFDM

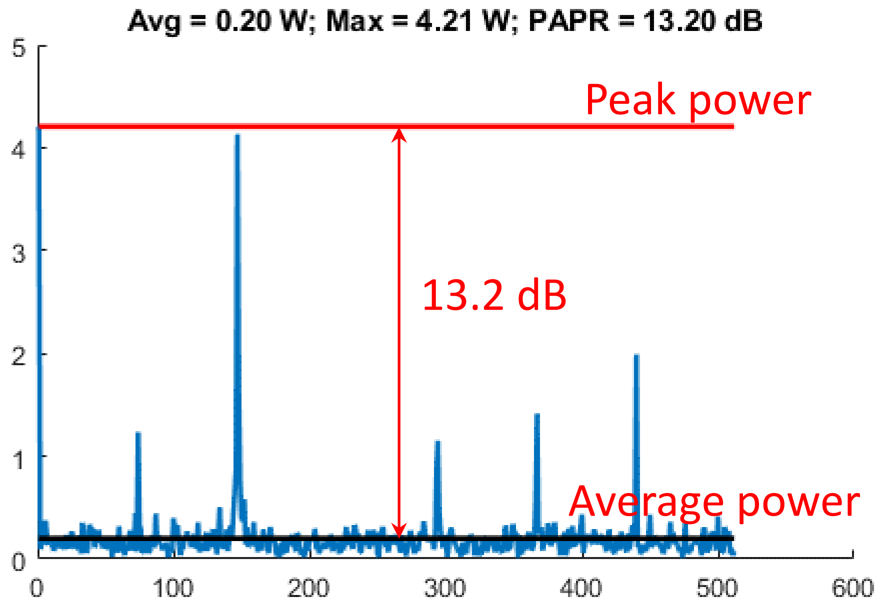


OFDM Transmitter Scheme

- PAPR of 13 dB that for a cellphone with 200 mW in average, its transmitter should be able to handle power peak 4 W (20 times higher).

- High battery consumption!

- Also, the non-linearity effect may be an issue.



Matlab Code from the slides

```
[binV, binS] = text2bin(Text); % https://github.com/Nikeshbajaj/Text-to-Binary/blob/master/bin2text.m

M = 16;
qam_01text = qammod(binV(1:2048)', M, 'InputType','bit');

figure
plot(binV)

% powers are calculated incorrectly, for presentation purposes
fft_text = .2*abs(fft(qam_01text))/sqrt(2048);
mean_01text = mean(fft_text);
max_01text = max(fft_text);

title_text = ['Avg = ', num2str(mean_01text,'%10.2f'),' W; Max = ', num2str(max_01text,'%10.2f'), ' W; PAPR = ',
num2str(10*log10(max_01text/mean_01text),'%10.2f'), ' dB'];

figure
hold on
title(title_text)
plot(fft_text, 'LineWidth', 2)
line([1,2048/log2(M)], [mean_01text,mean_01text], 'Color','black','LineWidth',2);
line([1,2048/log2(M)], [max_01text,max_01text], 'Color','red','LineWidth',2);
ylabel('Watts')
```

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