Simulation

Lecture O2

Optimization: Integer Programming

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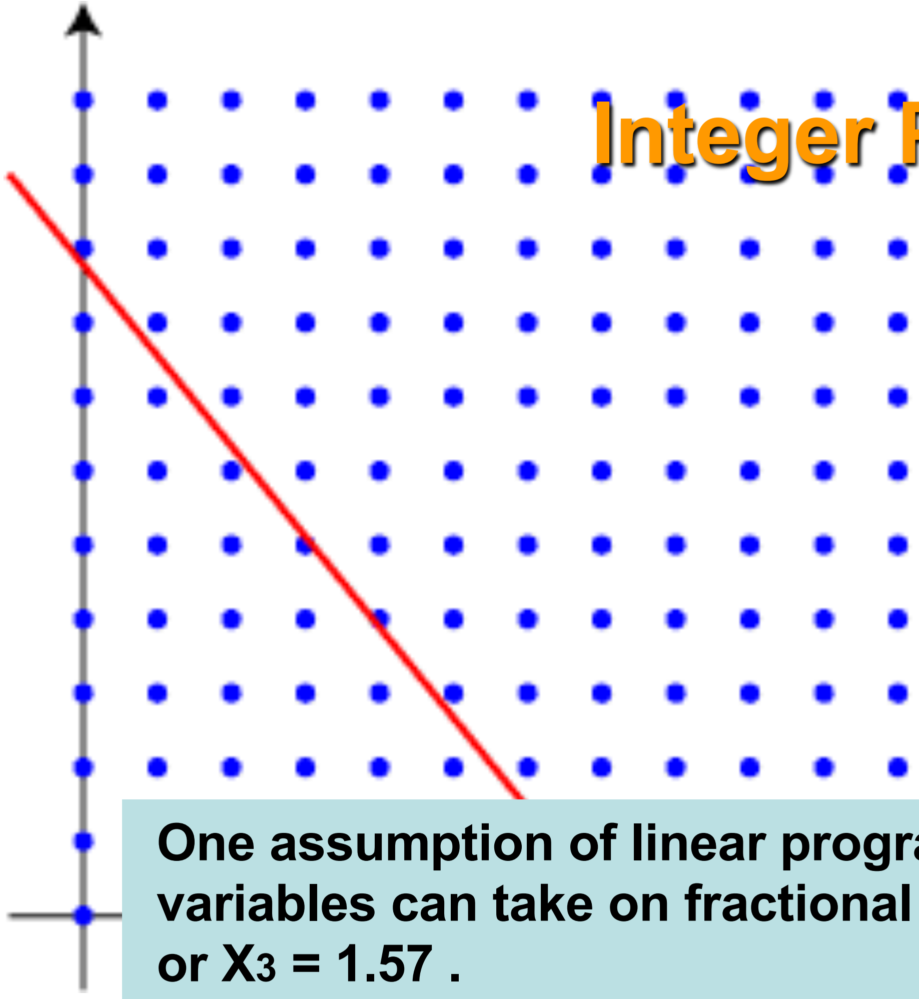
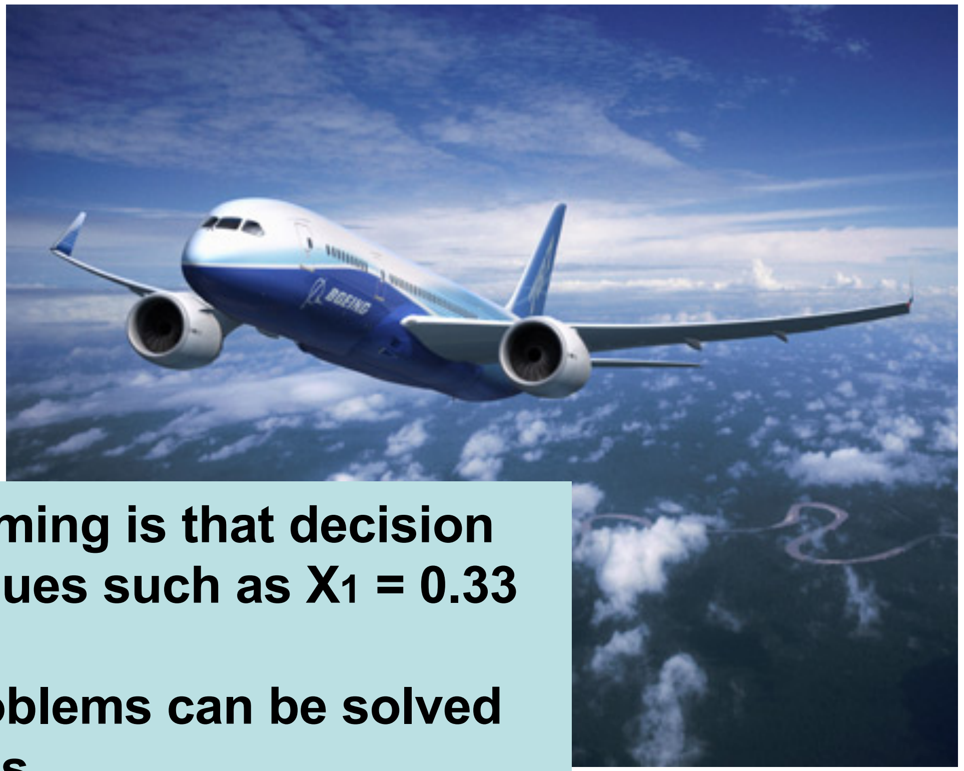
Outline

- ✓ Introduction to Integer Programming (IP)
- ✓ Examples of IP
- ✓ Developing IP
- ✓Branch and Bound (B&B) Method
- ✓B&B Example (Minimization Problem)
- ✓B&B Example (Maximization Problem)
- ✓Solving IPs in MATLAB



Integer Programming



One assumption of linear programming is that decision variables can take on fractional values such as $X_1 = 0.33$ or $X_3 = 1.57$.
Yet a large number of business problems can be solved only if variables have *integer* values.

When an airline decides how many planes to purchase, it cannot place an order for 5.38 aircraft ; it must order 4, 5, 6, or some other integer amount.

Introduction

- Integer Programs (IP) : $X \subseteq \mathbb{Z}^n$
    - (NP-hard) computational complexity
- Mixed Integer Linear Program (MILP)
    - Generally (NP-hard)

$$\begin{array}{rc} \min\limits_{x \in X} & c^T x \\ \text{s.t.} & Ax \le b \\ \text{where,} & X \subseteq \mathbb{Z}^{n_i} \times \mathbb{R}^{n_r} \end{array}$$

- However, many problems can be solved surprisingly quickly!

(Mixed) Integer Programming

- Integer Programming:
  - ➢all variables must have Integer values
- Mixed Integer Programming :
  - ➢ some variables have integer values

Exponential solution times!

IP Examples (1)

Example IP formulation

The Knapsack problem:

$$\text{Let } x_i = \begin{cases} 1 & \text{if I select item } i \\ 0 & \text{otherwise} \end{cases}$$

I wish to select items to put in my backpack.

- There are *m* items available.
- Item *i* weights w_i kg,
- Item *i* has value v_i.
- I can carry Q kg.

$$\max \quad \sum_{\mathrm{i}} x_i v_i$$

$$\text{s.t.} \quad \sum_{\mathrm{i}} x_i w_i \le Q$$

$$x_i \in \{0,1\}$$

IP Examples (2)

Task Allocation

- *n* jobs, *m* machines
- Job *i* has a load of q_i (e.g. amount of CPU resource)
- The cost of doing job *i* by machine *j* is c_{ij}
- The load capacity of machine *j* is Q_j

Objective: assign all jobs with a minimum total cost

Formulation

$$x_{ij} = \begin{cases} 1 & \text{if task } i \text{ is assigned to machine } j \\ 0 & \text{otherwise} \end{cases}$$

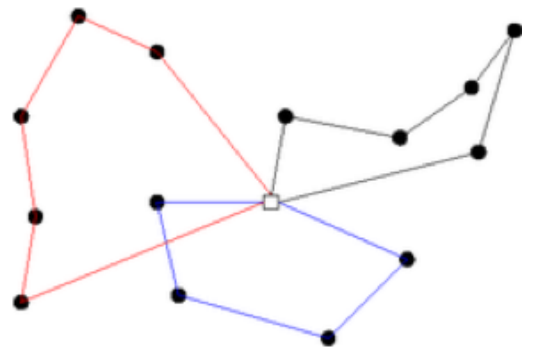
$$\min \sum_{i,j} c_{ij} x_{ij}$$

$$\sum_{j} x_{ij} = 1 \quad \forall i$$

$$\sum_{i} x_{ij} q_i \leq Q_j \quad \forall j$$

Vehicle Routing Problem (VRP)

What is the optimal set of routes for a fleet of **vehicles** to traverse in order to deliver to a given set of customers?



- n customers and m vehicles
- $c_{i,j}$ – the distance or cost of travel from i to j
- q_j – load at j
- Q_k – capacity of vehicle k

What vehicle should visit each customer, and in what order, to minimize costs?

If m =1 vehicle → Travel Salesman Problem (TSP)

Traditional formulation

$$x_{ijk} = \begin{cases} 1 & \text{if } i \text{ precedes } j \text{ on vehicle } k \\ 0 & \text{otherwise} \end{cases}$$

$$\text{minimize} \sum_{\mathrm{i,j,k}} \mathrm{c_{ij}} x_{ijk}$$

$$\sum_j \sum_k x_{ijk} = 1 \quad \forall i$$

$$\sum_j x_{ijk} = \sum_i x_{ijk} \quad \forall k$$

$$\sum_i \sum_j x_{ijk} q_j \le Q_k \quad \forall k$$

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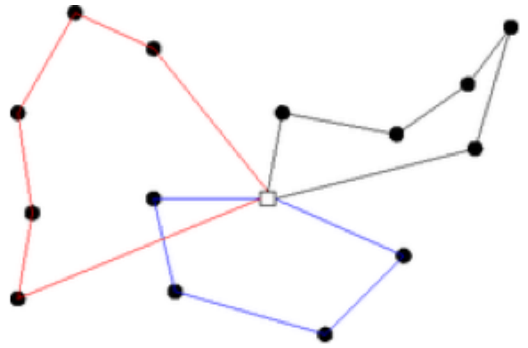


IP Formulation Tricks (1)

Logical constraints in IP

- If *x* then not *y* (*assume* x , y ϵ {0 , 1}): $(1 - x) M \geq y$

  (*M* is "big M" – a large value – larger than any feasible value for *y*)

- x or y or both (x , y ϵ {0 , 1}): $x + y \geq 1$

- $x \leq 1$ or $x \geq 5$ (x is real number):
  - define a binary variable w ϵ {0 , 1}

    if w = 1 → $x \leq 1 + M(1-w)$

    if w = 0 → $x \geq 5 - Mw$

- $x + 2y \geq 10$ or $4x - 10y \leq 2$ (x and y are real numbers) :
  - define a binary variable w ϵ {0 , 1} and big M

    if w = 1 → $x + 2y \geq 10 - M(1-w)$

    if w = 0 → $4x - 10y \leq 2 + Mw$

IP Formulation Tricks (2)

- For the purpose of this course, LP formulation is highly crucial. In your homework, you will be asked to do the formulation.
- So, start learning the tricks by practice!
- Find out interesting tricks here:

http://mixedintegerprogramming.weebly.com/uploads/1/4/1/8/14181742/integer_programming_tricks_-_aimms_modeling_guide.pdf

And here

http://ocw.mit.edu/courses/sloan-school-of-management/15-053-optimization-methods-in-management-science-spring-2013/lecture-notes/MIT15_053S13_lec11.pdf

Solving IPs

How can we solve IPs problems?

Solving IP

- Some problem classes have the “Integrality Property”: all solution naturally fall on integer points e.g.
  - Maximum Flow problems
  - Assignment problems

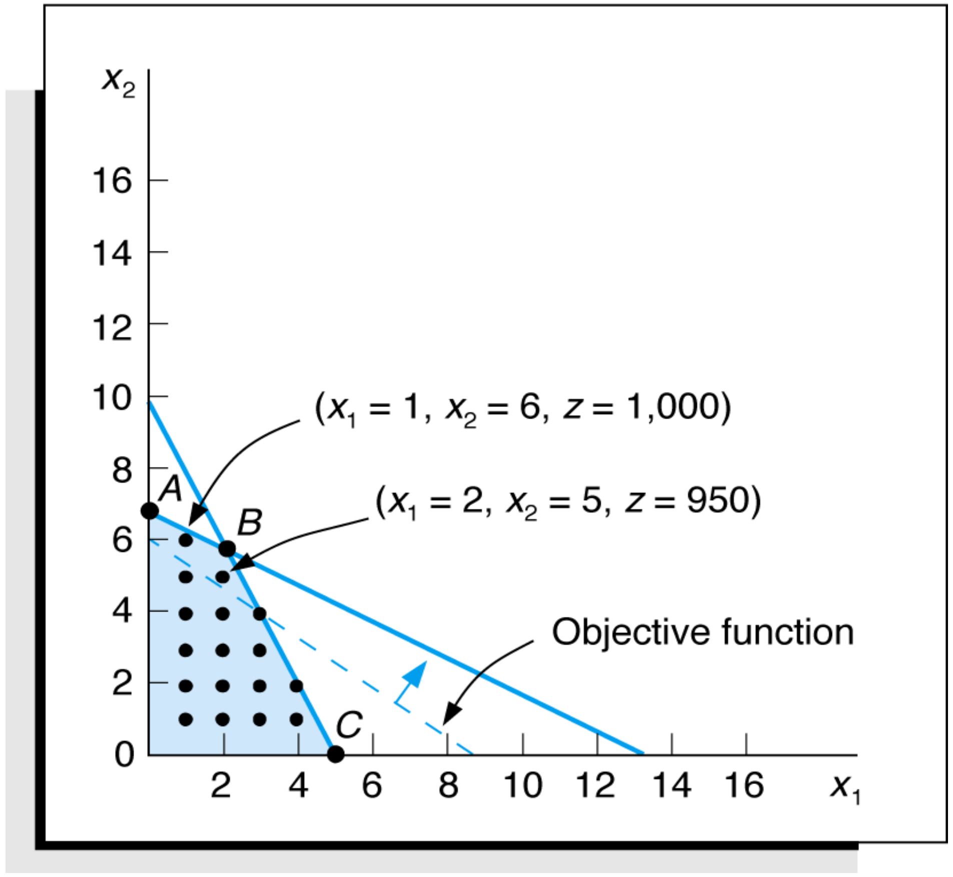
- If the constraint matrix has a special form, it will have the Integrality Property:
  - Totally unimodular
  - Balanced
  - Perfect
- But, not all problems have such properties

Solving IP by relaxing to LP

Maximize $Z = 100x_1 + 150x_2$
subject to:
$8{,}000x_1 + 4{,}000x_2 \leq 40{,}000$
$15x_1 + 30x_2 \leq 200$
$x_1, x_2 \geq 0$ and integer

Optimal Solution:
$Z = \$1{,}055.56$
$x_1 = 2.22$
$x_2 = 5.55$

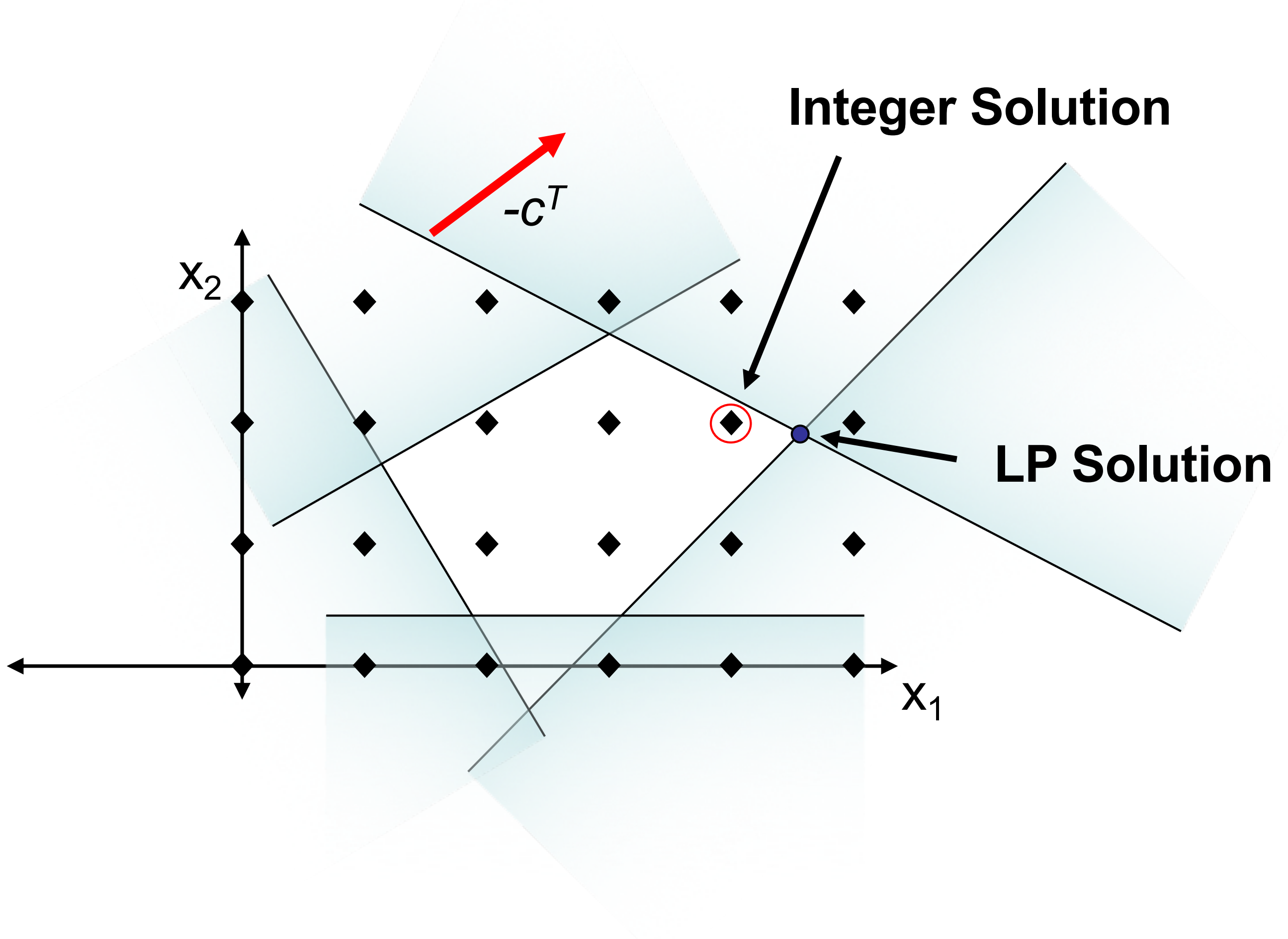
OBS! We get non-integer solution



Feasible Solution Space with Integer Solution Points

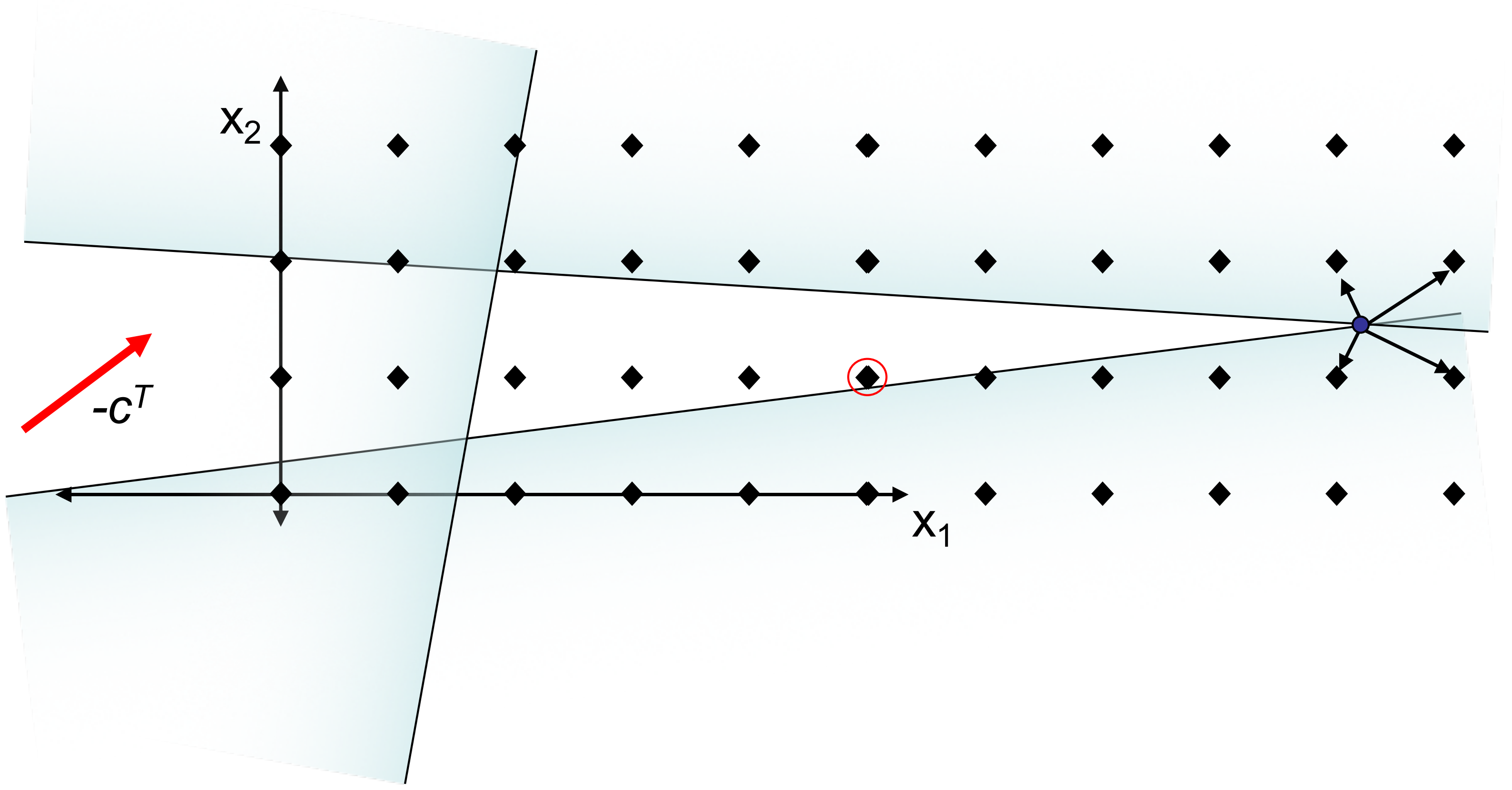
Solving IP

- How about solving LP Relaxation followed by rounding?



Solving IP

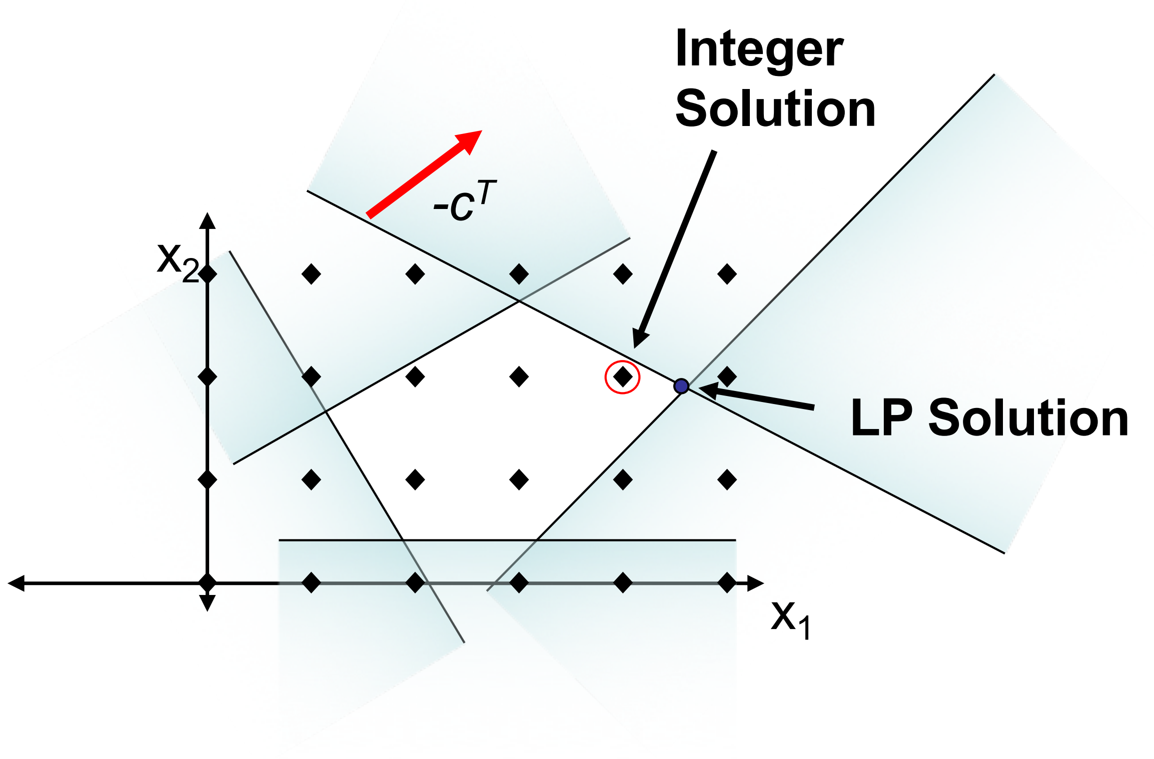
- In general, rounding does not work!



- LP solution provides lower bound (for minimization) and upper bound (for maximization) on IP
- But, rounding can be arbitrarily far away from integer solution

Solving IP

- Combine both approaches
  - Solve LP Relaxation to get fractional solutions
  - Create two sub-branches by adding constraints



Solving IP

- Combine both approaches
  - ➢Solve LP Relaxation to get fractional solutions
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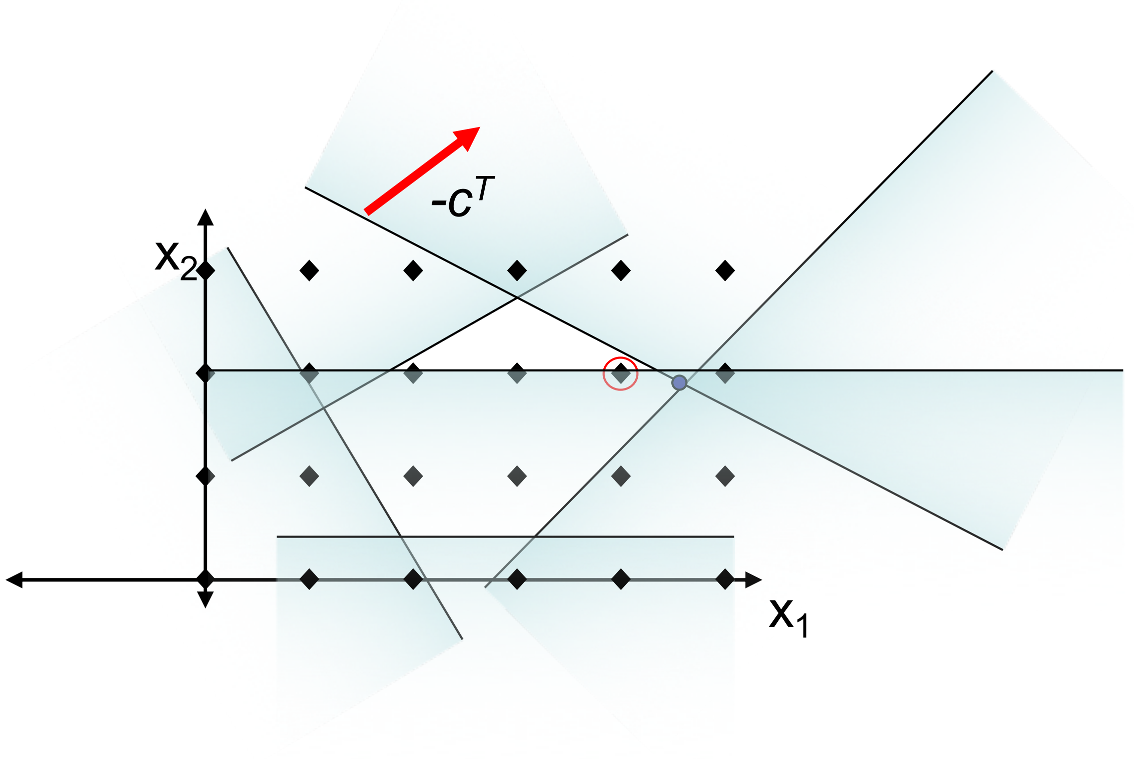
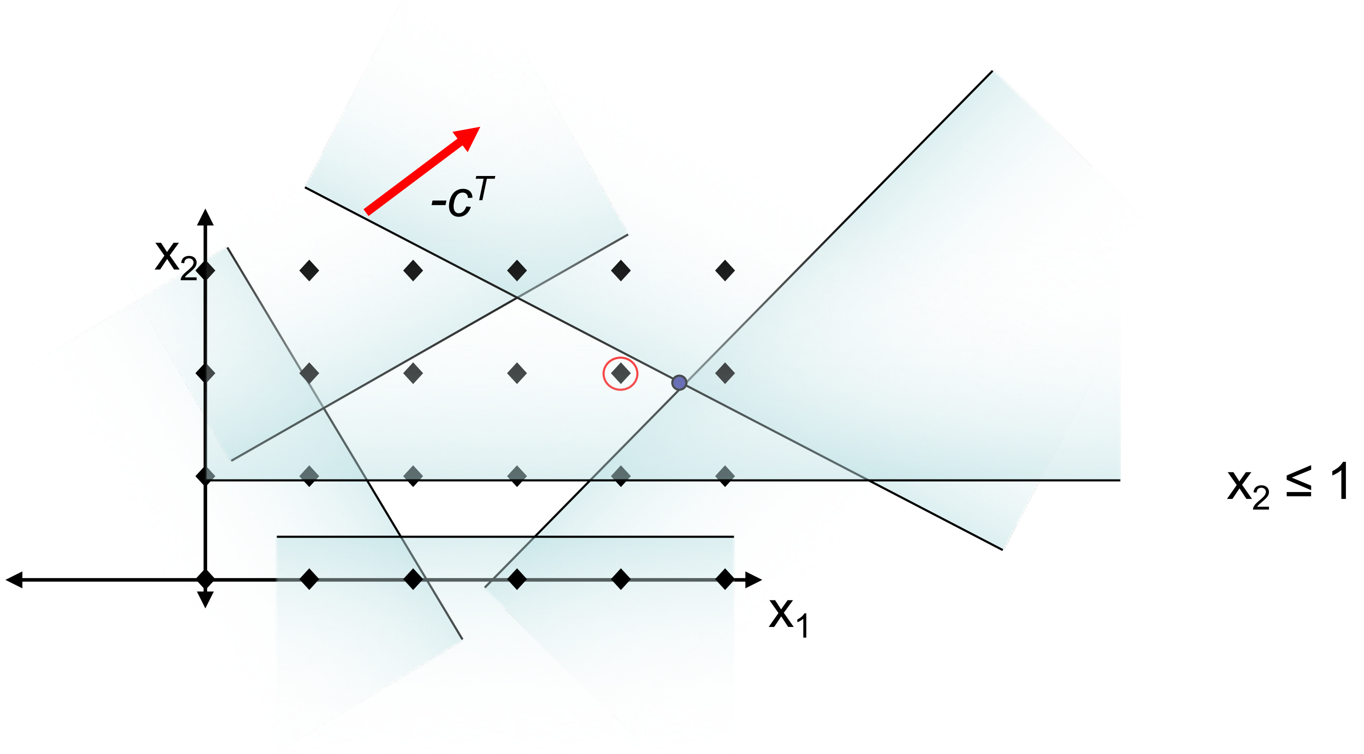


$x_2 \geq 2$

Solving IP

- Combine both approaches
  - ➢Solve LP Relaxation to get fractional solutions
  - ➢Create two sub-branches by adding constraints



An Example Maximization Problem

Harrison Electric Company

PURE INTEGER PROBLEM

The Harrison Electric Company produces two products: old-fashioned chandeliers and ceiling fans. Both products require a two-step process involving wiring and assembly.

It takes 2 hours to wire each chandelier and 3 hours to wire a ceiling fan.

Final assembly of the chandeliers and fans requires 6 and 5 hours, respectively.

The production capability is such that only 12 hours of wiring time and 30 hours of assembly time are available.

If each chandelier produced nets the firm $7.00 and each fan $6.00, the Production mix decision can be formulated using LP as follows:

Harrison Electric Company

Maximize profit = $\$7.00\ X_1 + \$6.00\ X_2$

subject to:

$2X_1 + 3X_2 =< 12$ (wiring hours)

$6X_1 + 5X_2 =< 30$ (assembly hours)

$X_1, X_2 => 0$

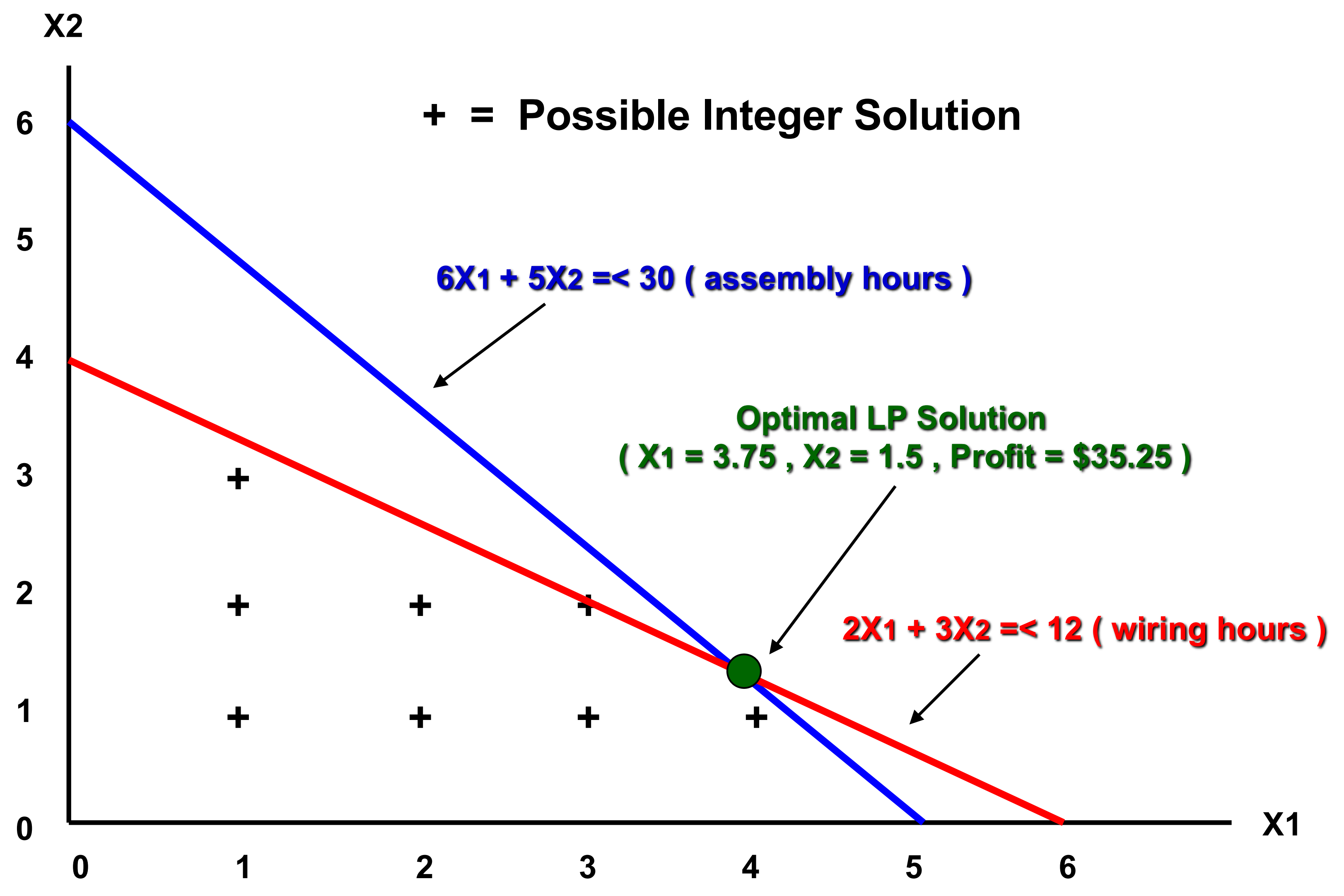
The Model

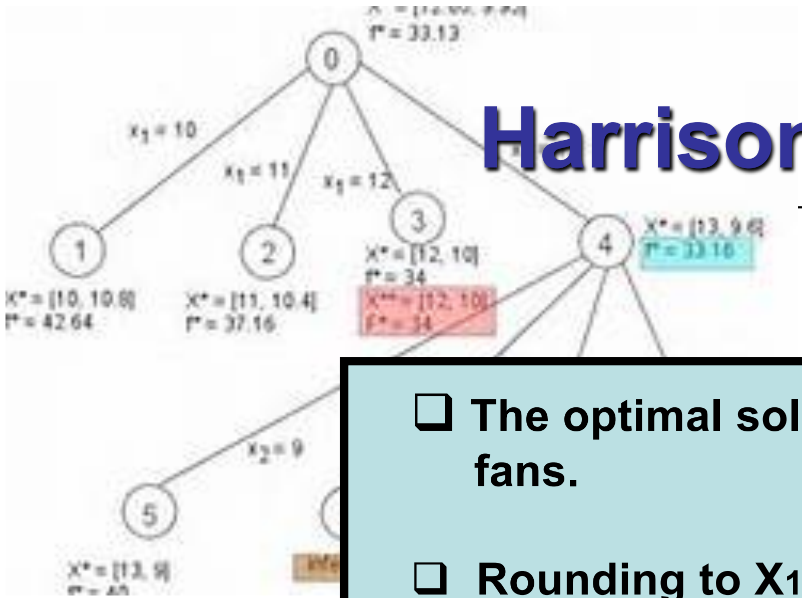
where:

X_1 = number of chandeliers produced

X_2 = number of ceiling fans produced

Harrison Electric Company





Harrison Electric Company

- The optimal solution is X_1 = 3.75 chandeliers and X_2 = 1.5 ceiling fans.
- Rounding to X_1 = 4 and X_2 = 2 makes the solution unfeasible.
- Rounding to X_1 = 4 and X_2 = 2 is probably not the optimal feasible integer solution either .
- There are 18 feasible integer solutions to this problem.
- The optimal integer solution is X_1 = 5 and X_2 = 0 , with a total profit of $35.00 .
- The integer restriction reduced profit from $35.25 to $35.00
- An integer solution can never produce a greater profit than the LP solution to the same problem.

DISCUSSION

Harrison Electric Company

Listing all feasible solutions and selecting the one with the best objective function value is called the *enumeration* method. This can be virtually impossible for large problems where the number of feasible solutions is extremely large !

Chandeliers (X1)	Ceiling Fans (X2)	Profit (Z)
0	0	$0.00
1	0	7
2	0	14
3	0	21
4	0	28
5	0	35
0	1	6
1	1	13
2	1	20

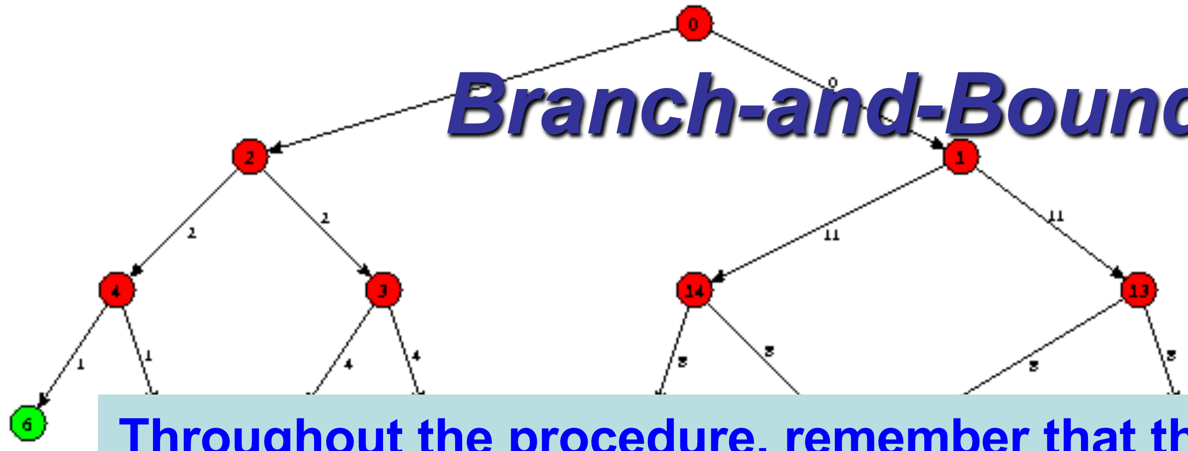
Integer optimal solution

Harrison Electric Company

Listing all feasible solutions and selecting the one with the best objective function value is called the *enumeration* method. This can be virtually impossible for large problems where the number of feasible solutions is extremely large !

Chandeliers (X1)	Ceiling Fans (X2)	Profit (Z)
3	1	27
4	1	34
0	2	12
1	2	19
2	2	26
3	2	33
0	3	18
1	3	25
0	4	24

Rounding optimal solution



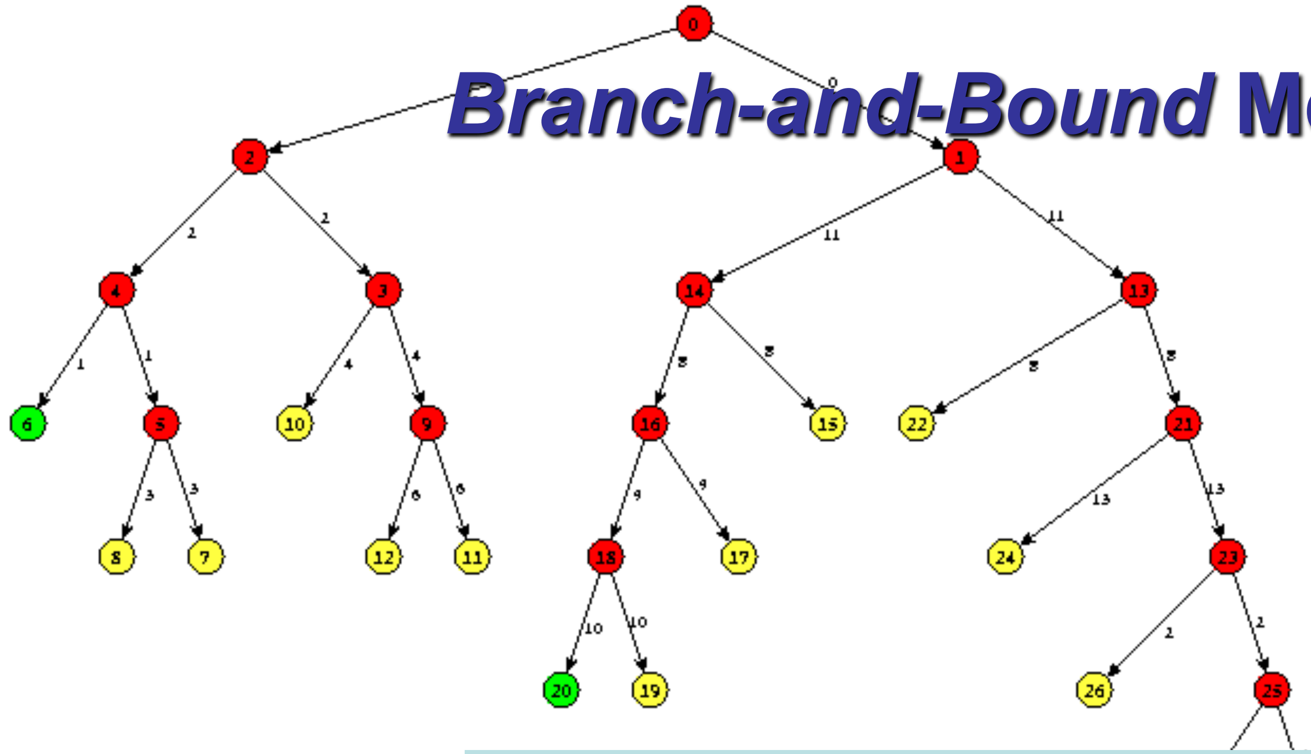
Branch-and-Bound Method

Throughout the procedure, remember that the lower bound solution is determined by feasible integer solutions. Upper bound is determined by fractional LP solutions. Define two parameters LB and UB to update the lower bound and upper bound.

1. Solve the original problem using linear programming.
   If the answer satisfies the integer constraints, we are done.
   If not, this value provides an initial *upper bound* for the objective function.

2. Find any feasible solution that meets the integer constraints for use as a *lower bound*. Usually, rounding down each variable will accomplish this.

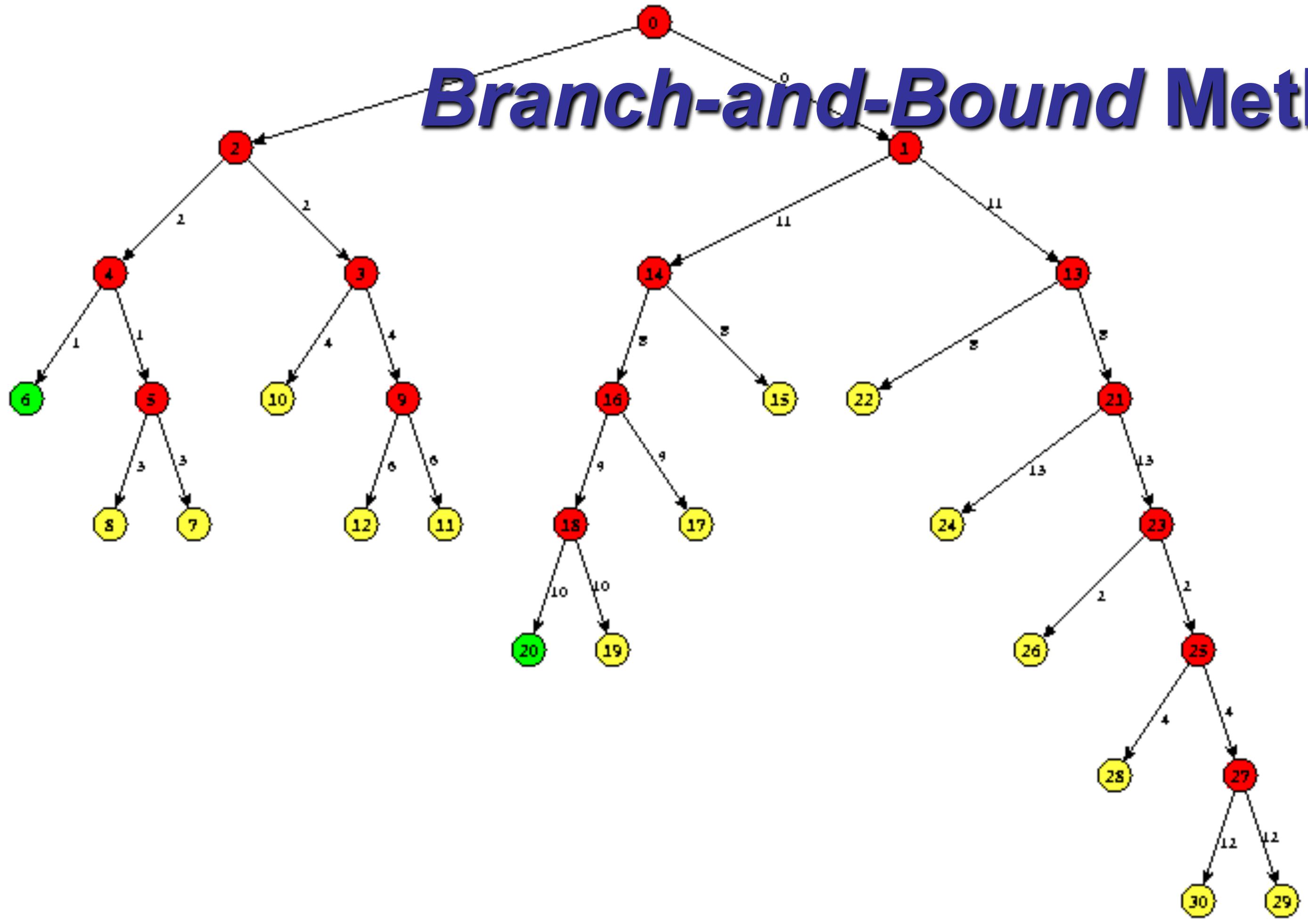
Branch-and-Bound Method



3. Branch on one variable from step 1 that does not have an integer value. Split the problem into two subproblems based on integer values that are above and below the noninteger value.

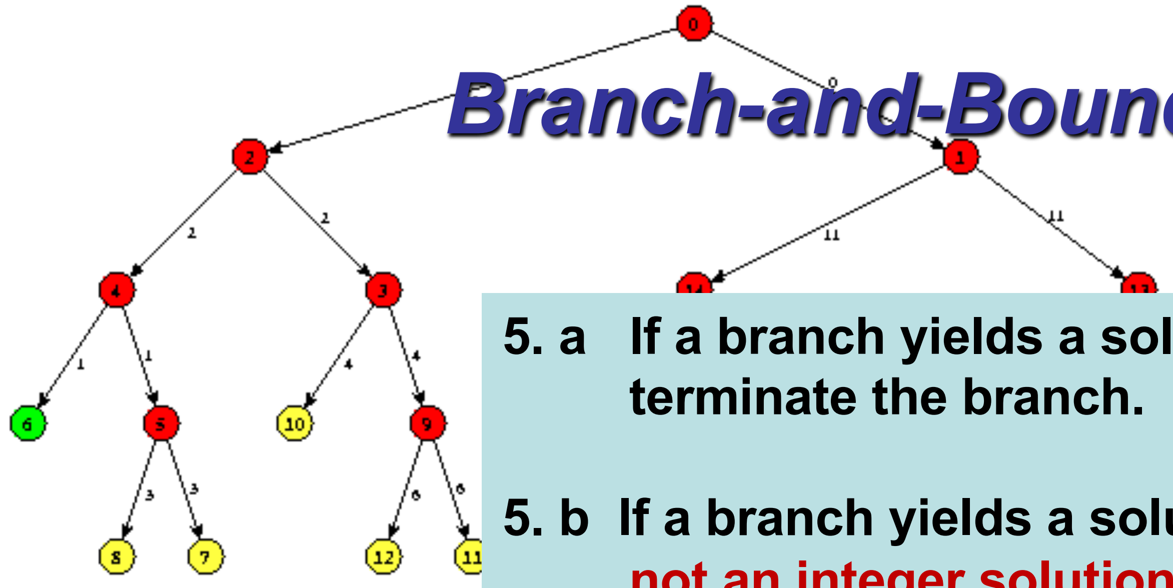
   For example, if $X_2 = 3.75$ was in the optimal linear programming solution, introduce constraint $X_2 => 4$ in the first subproblem, and $X_2 =< 3$ in the second subproblem.

Branch-and-Bound Method



4. Create nodes at the top of these new branches by solving the new problems.

Branch-and-Bound Method



5. a If a branch yields a solution that is not *feasible*, terminate the branch.

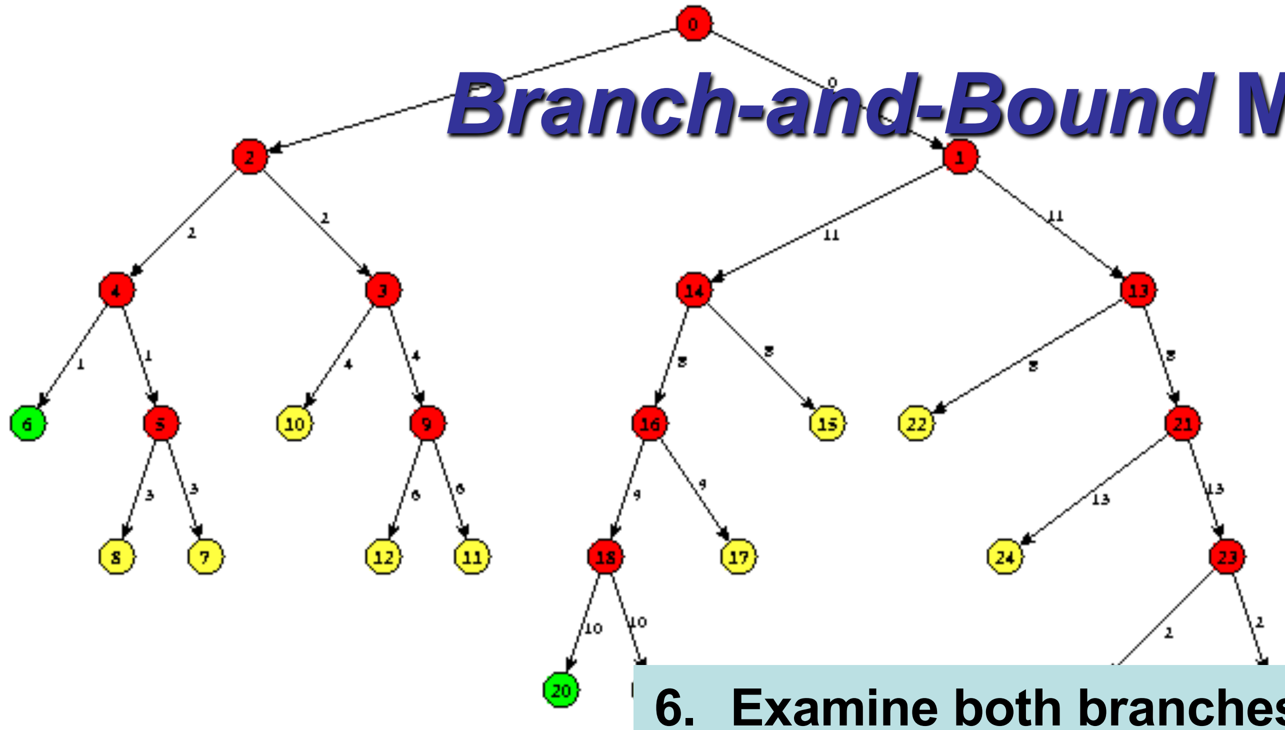
5. b If a branch yields a solution that is feasible, but not an integer solution, go to step 6.

5. c If the branch yields a feasible integer solution, look at the objective function. If its value equals the upper bound, an optimal solution has been reached.

If it is not equal to the upper bound, but exceeds the lower bound, set it as the new lower bound and go to step 6.

Finally, if it is less than the lower bound, terminate this branch.

Branch-and-Bound Method



6. Examine both branches again and set the upper bound equal to the maximum value of the objective function at all final nodes.

   If the upper bound equals the lower bound, stop.

   If not, go back to step 3 .

REVISITED

Harrison Electric Company

Maximize profit = $\$7.00\ X_1 + \$6.00\ X_2$

subject to:

$2X_1 + 3X_2 =< 12$ (wiring hours)

$6X_1 + 5X_2 =< 30$ (assembly hours)

$X_1, X_2 => 0$

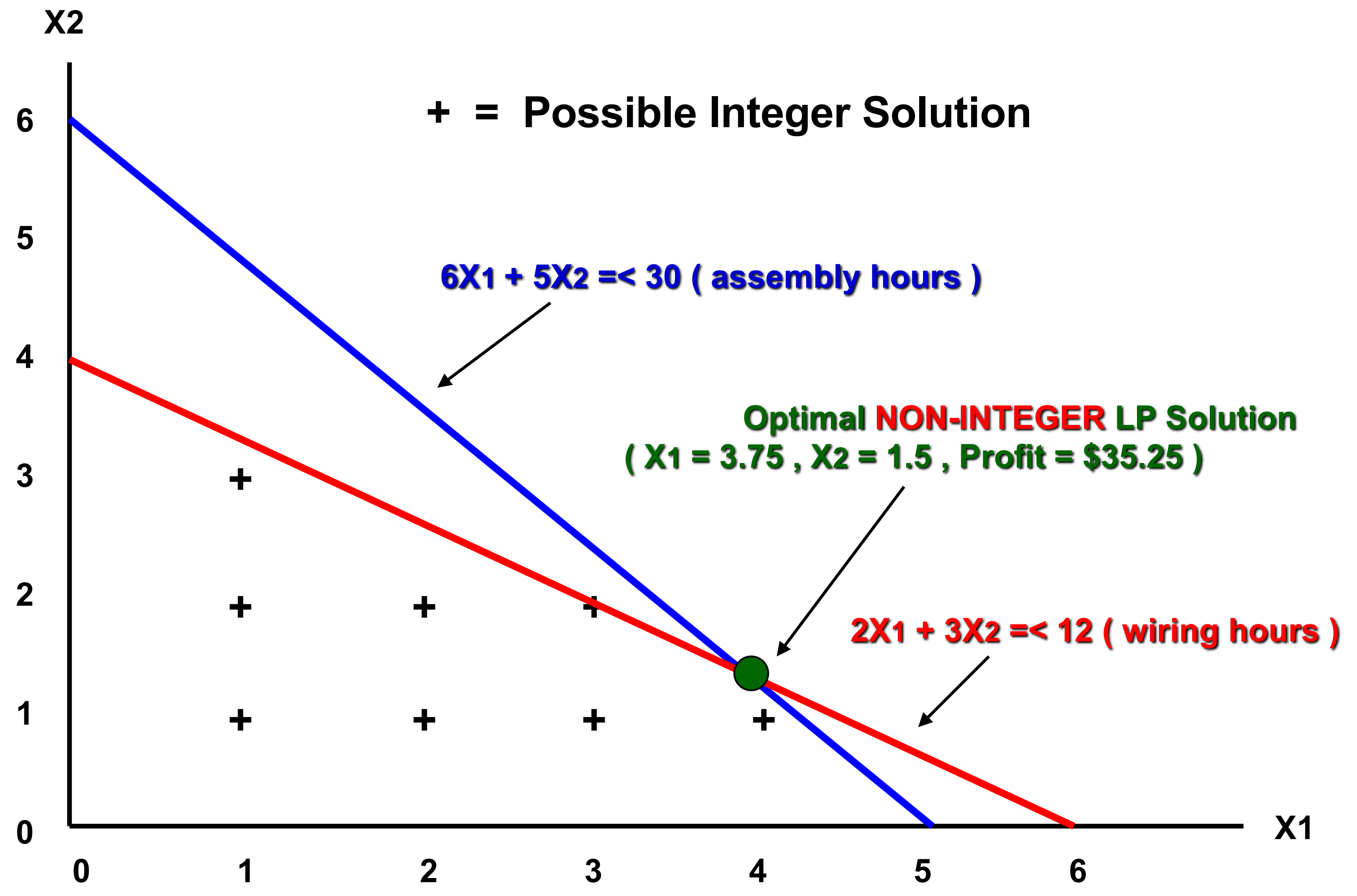
The Model

where:

X_1 = number of chandeliers produced

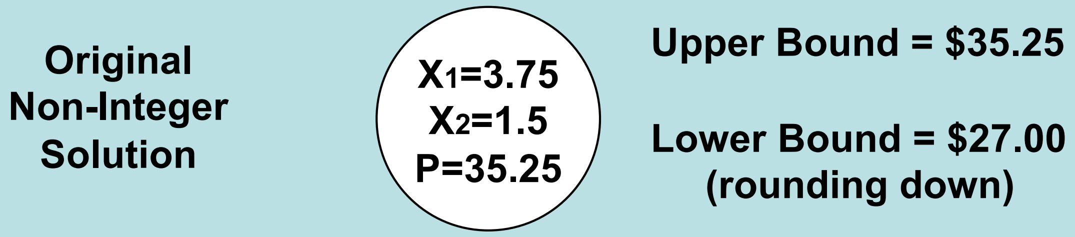
X_2 = number of ceiling fans produced

REVISITED Harrison Electric Company



Branch-and-Bound Method

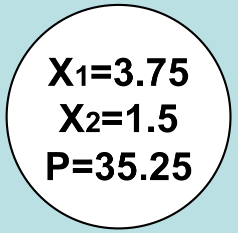
- Since X_1 and X_2 are not integers, the solution is not valid.
- The profit of $35.25 will be the initial *upper bound*.
- Rounding down gives $X_1 = 3$, $X_2 = 1$, profit = $27.00 , which is feasible and can be used as a *lower bound*.


Original Non-Integer Solution
X_1=3.75
X_2=1.5
P=35.25
Upper Bound = $35.25
Lower Bound = $27.00 (rounding down)

Branch-and-Bound Method

- We divide the problem into two subproblems, A and B
- We can branch on either the non-integer X_1 or X_2
- We choose X_1 this time

Original
Non-Integer
Solution



Upper Bound = $35.25

Lower Bound = $27.00
(rounding down)

Branch-and-Bound Method

Subproblem A

$$\begin{aligned} \text{Max } Z &= \$7X_1 + \$6X_2 \\ \text{s.t.} \quad & 2X_1 + 3X_2 =< 12 \\ & 6X_1 + 5X_2 =< 30 \\ & X_1 => 4 \end{aligned}$$

Subproblem B

$$\begin{aligned} \text{Max } Z &= \$7X_1 + \$6X_2 \\ \text{s.t.} \quad & 2X_1 + 3X_2 =< 12 \\ & 6X_1 + 5X_2 =< 30 \\ & X_1 =< 3 \end{aligned}$$

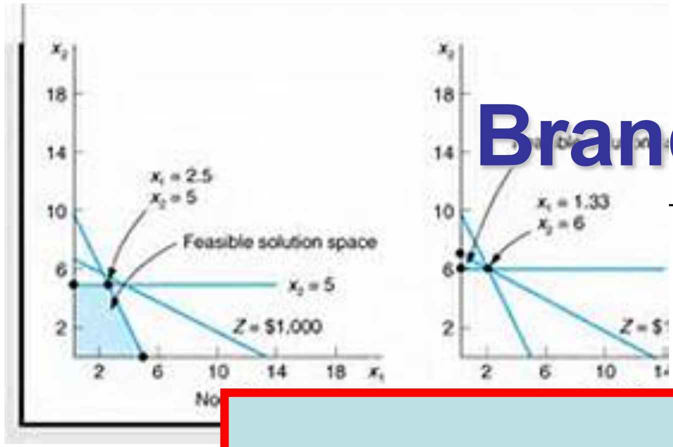
Original Non-Integer Solution

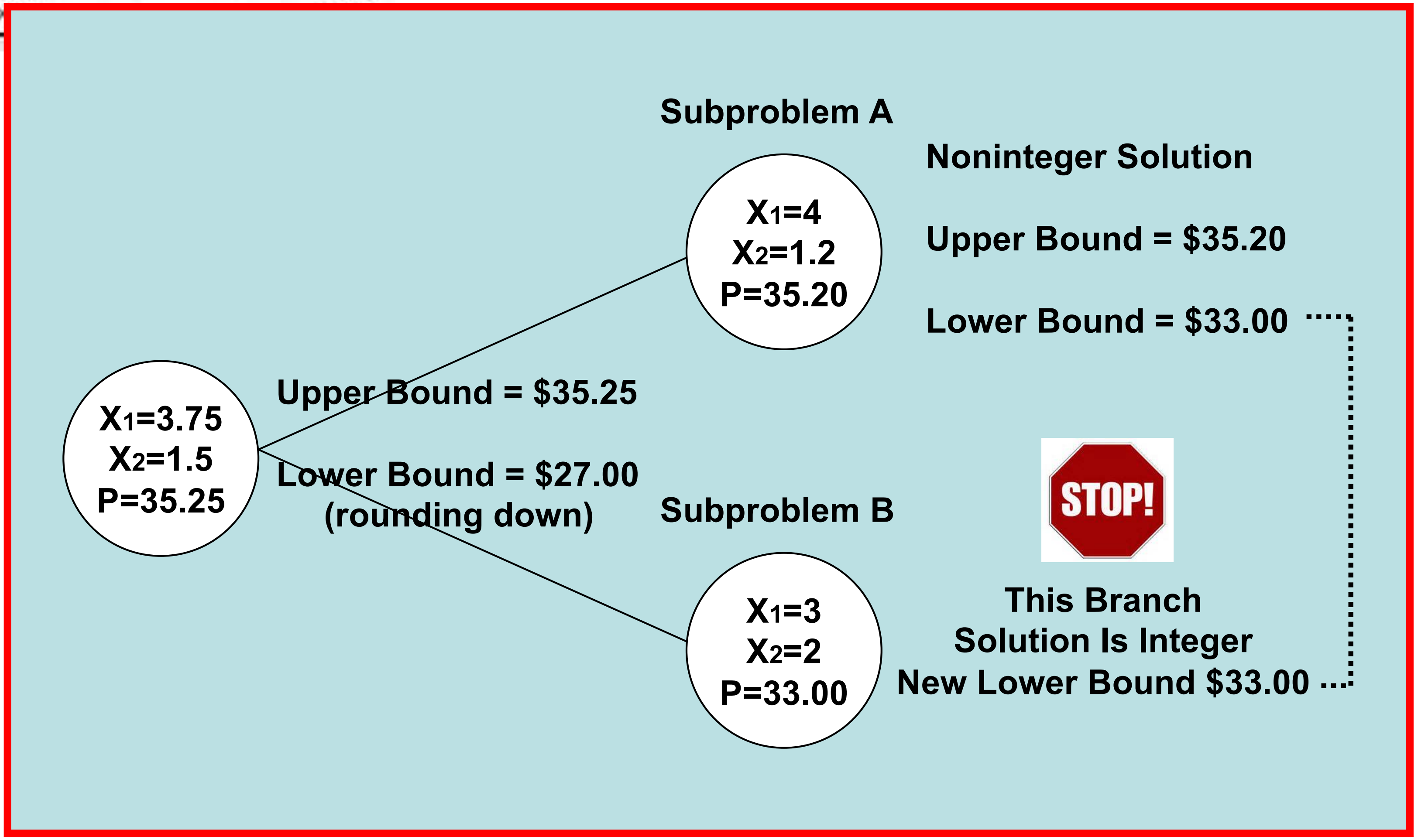
$X_1=3.75$
$X_2=1.5$
$P=35.25$

Upper Bound = \$35.25

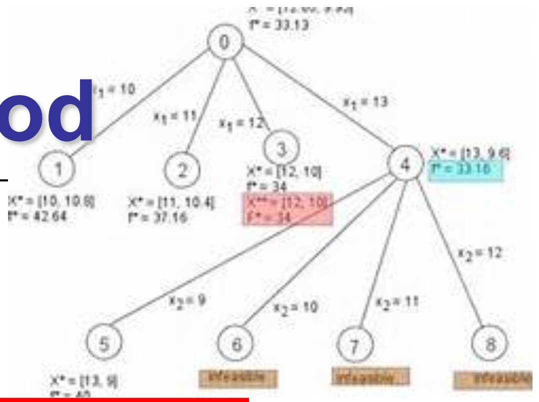
Lower Bound = \$27.00 (rounding down)

Branch-and-Bound Method





Branch-and-Bound Method



- Subproblem A is now branched into two new subproblems, C and D
- Subproblem C has the *additional* constraint of X_2 => 2
- Subproblem D has the *additional* constraint of X_2 =< 1
- The logic here is that since A's optimal solution of X_1 = 1.2 is not feasible, the integer feasible answer must lie at X_2 => 2 or X_2 =< 1

Branch-and-Bound Method

Subproblem C

$$\text{Max } Z = \$7X_1 + \$6X_2$$
$$\text{s.t.} \quad 2X_1 + 3X_2 =< 12$$
$$6X_1 + 5X_2 =< 30$$
$$X_1 => 4$$
$$X_2 => 2$$

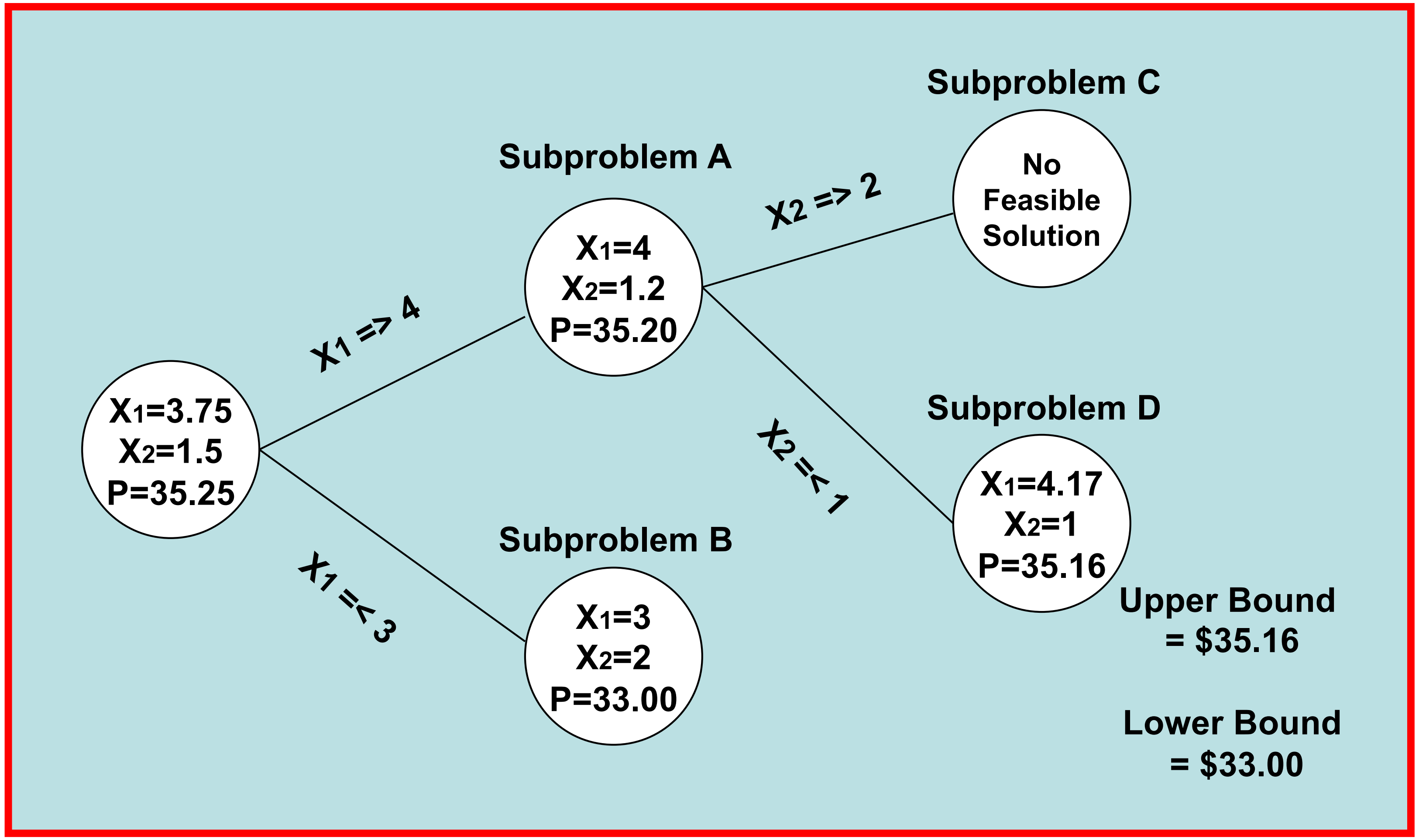
Subproblem D

$$\text{Max } Z = \$7X_1 + \$6X_2$$
$$\text{s.t.} \quad 2X_1 + 3X_2 =< 12$$
$$6X_1 + 5X_2 =< 30$$
$$X_1 => 4$$
$$X_2 =< 1$$

Subproblem C has no feasible solution whatsoever because the first two constraints are violated if $X_1 => 4$ and $X_2 => 2$ constraints are observed. We terminate this branch and do not consider its solution.

Subproblem D's solution is $X_1 = 4.17$, $X_2 = 1$, profit = \$35.16. This non-integer solution yields a new upper bound of \$35.16.

Branch-and-Bound Method



Branch-and-Bound Method

Finally, we create subproblems E and F and solve for X_1 and X_2 with the additional constraints $X_1 =< 4$ and $X_1 => 5$.

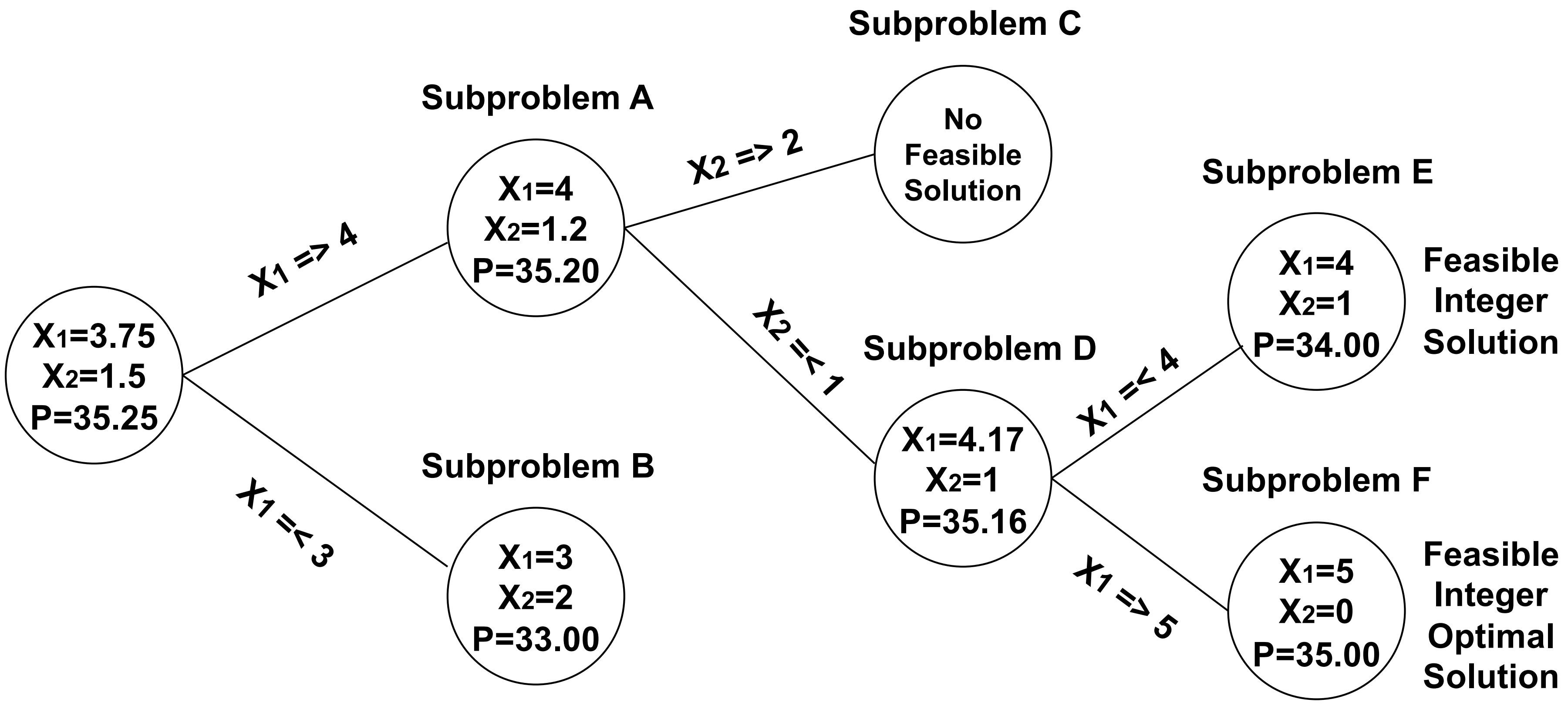
Subproblem E

$$\begin{aligned} \text{Max } Z &= \$7X_1 + \$6X_2 \\ \text{s.t.} \quad & 2X_1 + 3X_2 =< 12 \\ & 6X_1 + 5X_2 =< 30 \\ & X_1 => 4 \\ & X_1 =< 4 \\ & X_2 =< 1 \end{aligned}$$

Subproblem F

$$\begin{aligned} \text{Max } Z &= \$7X_1 + \$6X_2 \\ \text{s.t.} \quad & 2X_1 + 3X_2 =< 12 \\ & 6X_1 + 5X_2 =< 30 \\ & X_1 => 4 \\ & X_1 => 5 \\ & X_2 =< 1 \end{aligned}$$

Full Branch-and-Bound Solution



$X_1=3.75$ $X_2=1.5$ $P=35.25$
$X_1 => 4$
Subproblem A
$X_1=4$ $X_2=1.2$ $P=35.20$
$X_1 =< 3$
Subproblem B
$X_1=3$ $X_2=2$ $P=33.00$
$X_2 => 2$
Subproblem C
No Feasible Solution
$X_2 =< 1$
Subproblem D
$X_1=4.17$ $X_2=1$ $P=35.16$
$X_1 =< 4$
Subproblem E
$X_1=4$ $X_2=1$ $P=34.00$
Feasible Integer Solution
$X_1 => 5$
Subproblem F
$X_1=5$ $X_2=0$ $P=35.00$
Feasible Integer Optimal Solution

The stopping rule for the branching process is that we continue until the new upper bound is less than or equal to the lower bound or no further branching is possible. The latter is the case here since both branches yielded feasible integer solutions.

Branch & Bound (for Minimization IP)

- Branch and Bound Algorithm
  1. Solve LP relaxation to get a lower bound on cost for current branch
     - If solution exceeds upper bound, branch is terminated
     - If solution is integer, replace upper bound on cost
  2. Create two branched problems by adding constraints to original problem
     - Select integer variable with fractional LP solution
     - Add integer constraints to the original LP
  3. Repeat until no branches remain, return optimal solution.

An Example Minimization Problem

Branch & Bound

- Example: a problem with 4 variables, all required to be integer

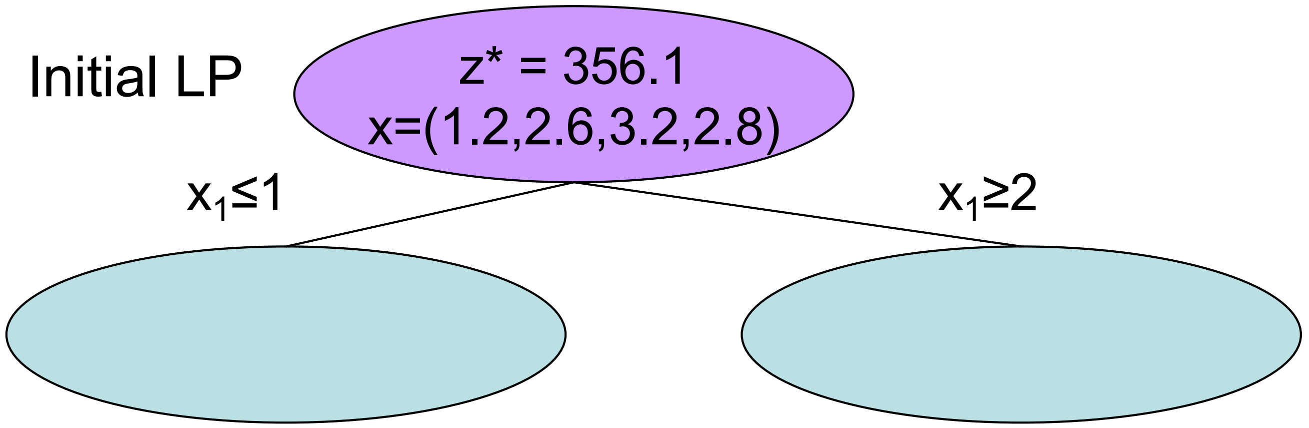
Branch & Bound

Initial LP

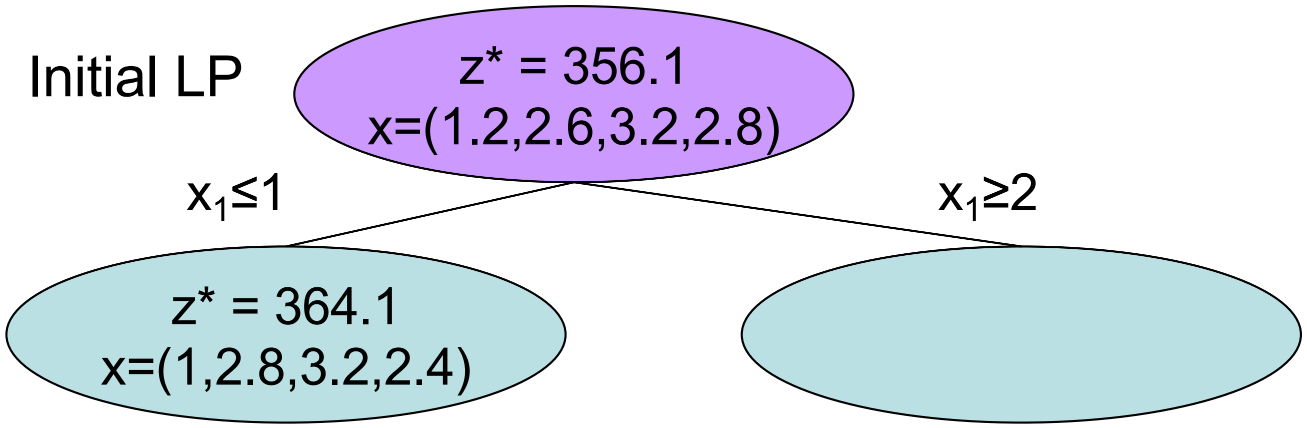
$z^* = 356.1$

$x=(1.2,2.6,3.2,2.8)$

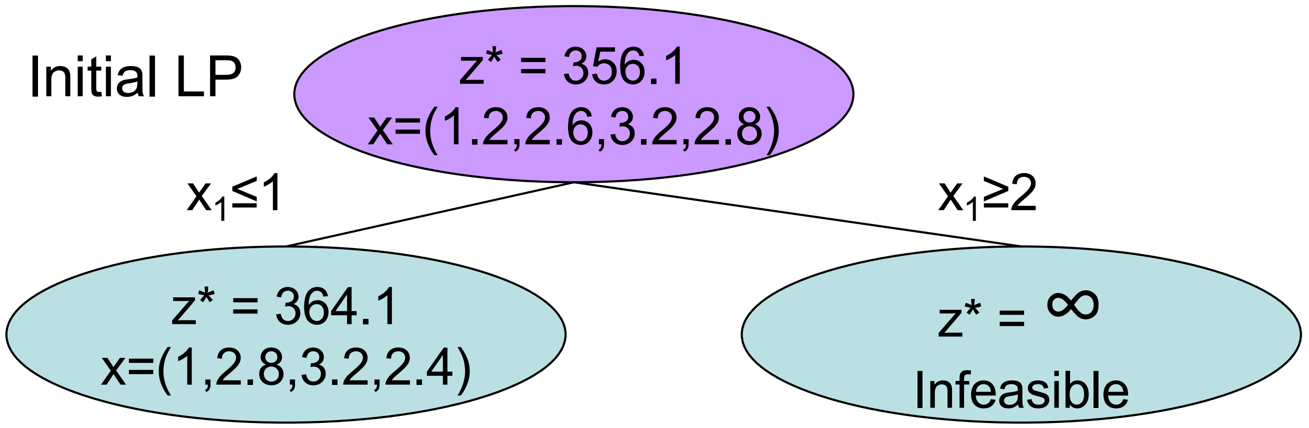
Branch & Bound



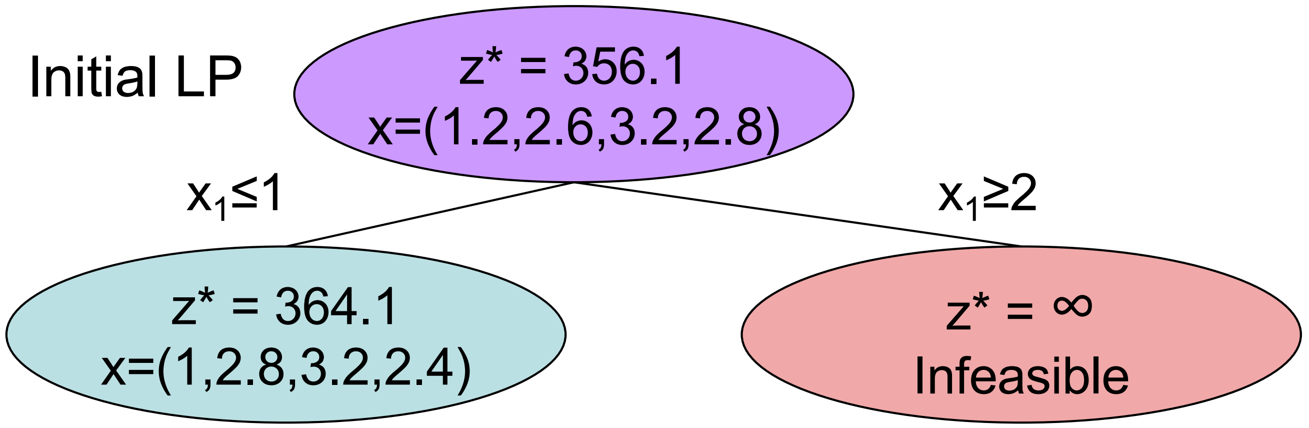
Branch & Bound



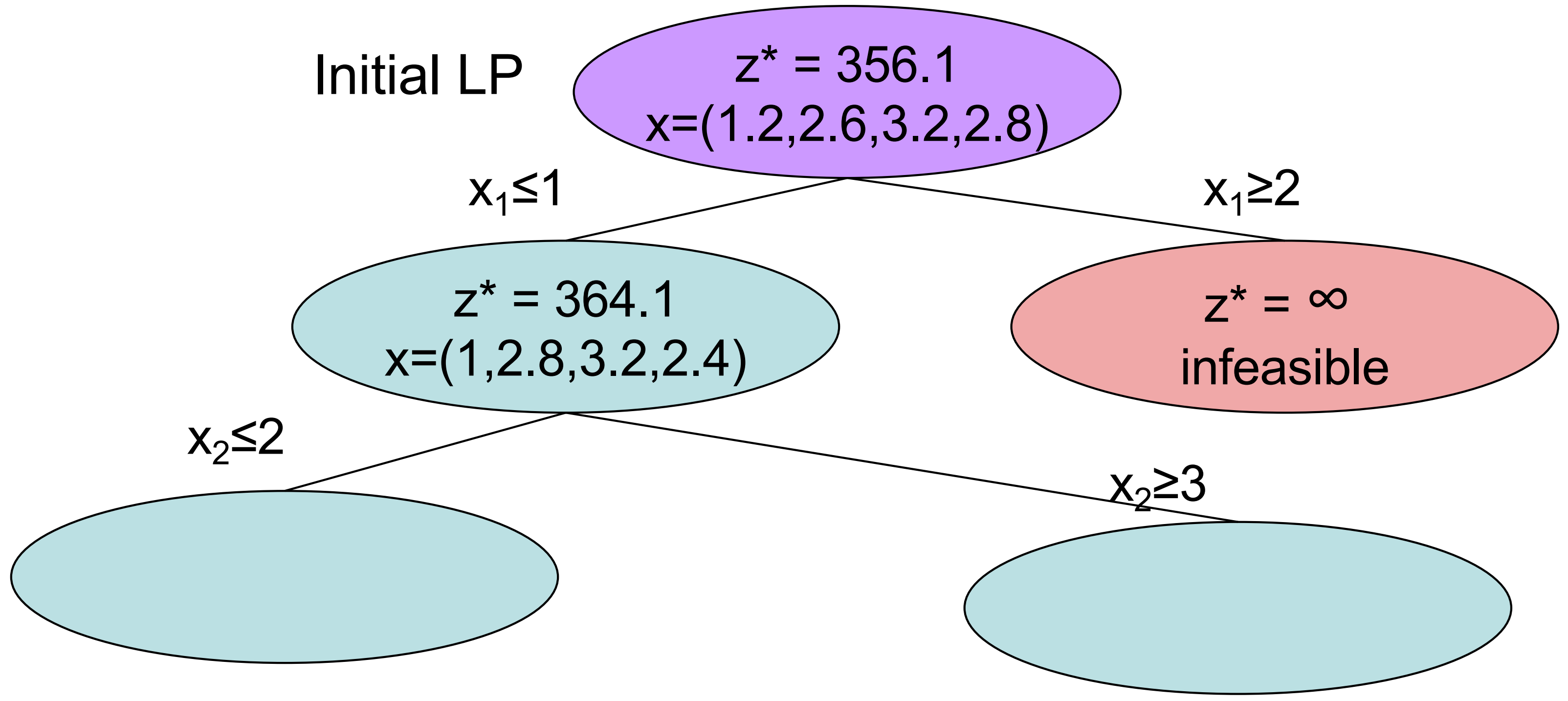
Branch & Bound



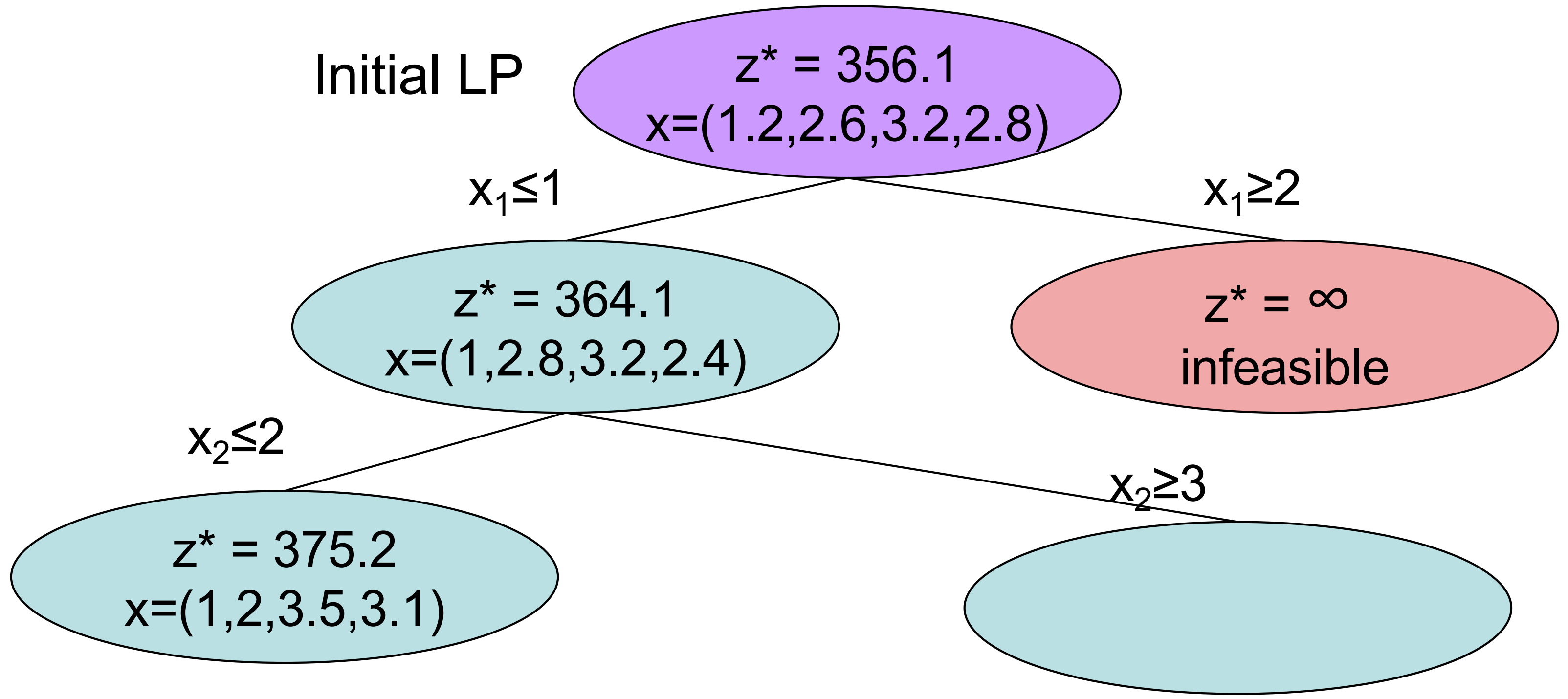
Branch & Bound



Branch & Bound



Branch & Bound



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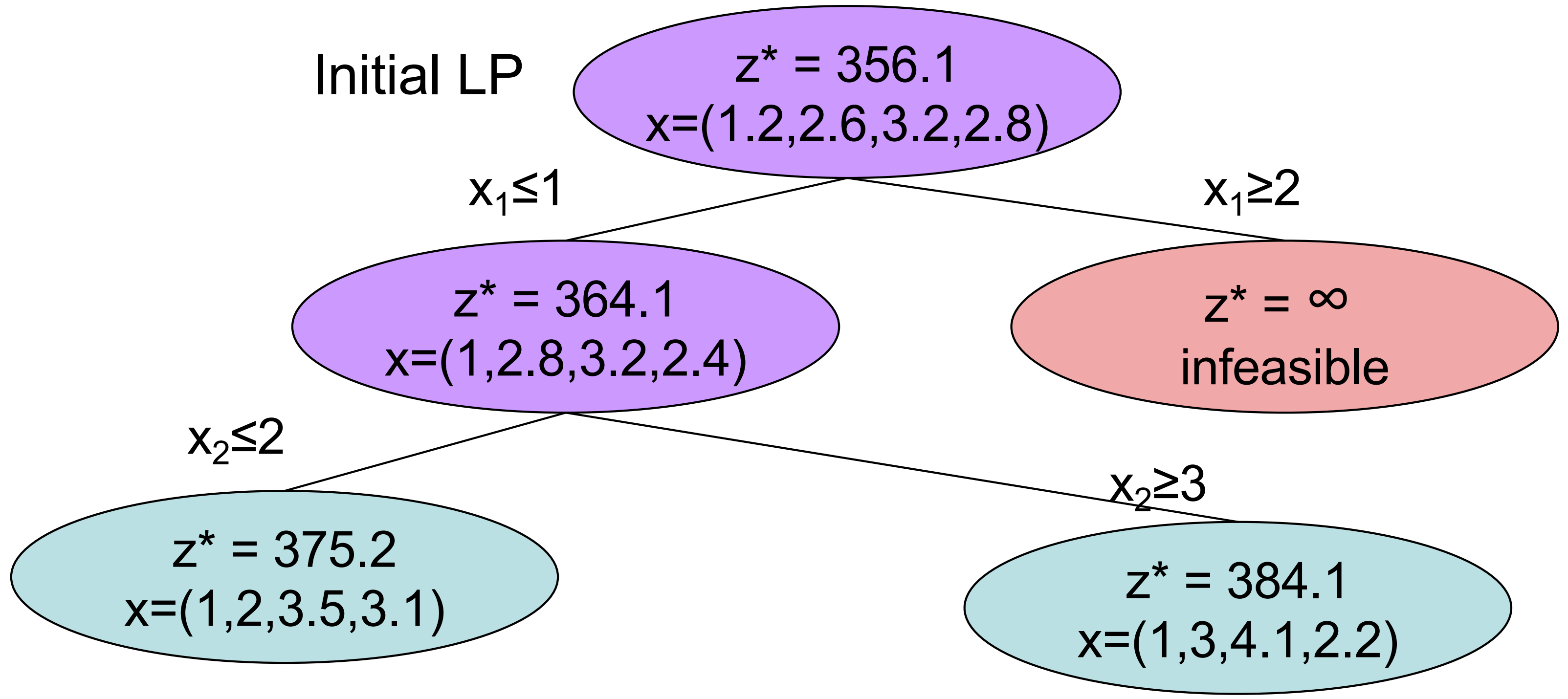

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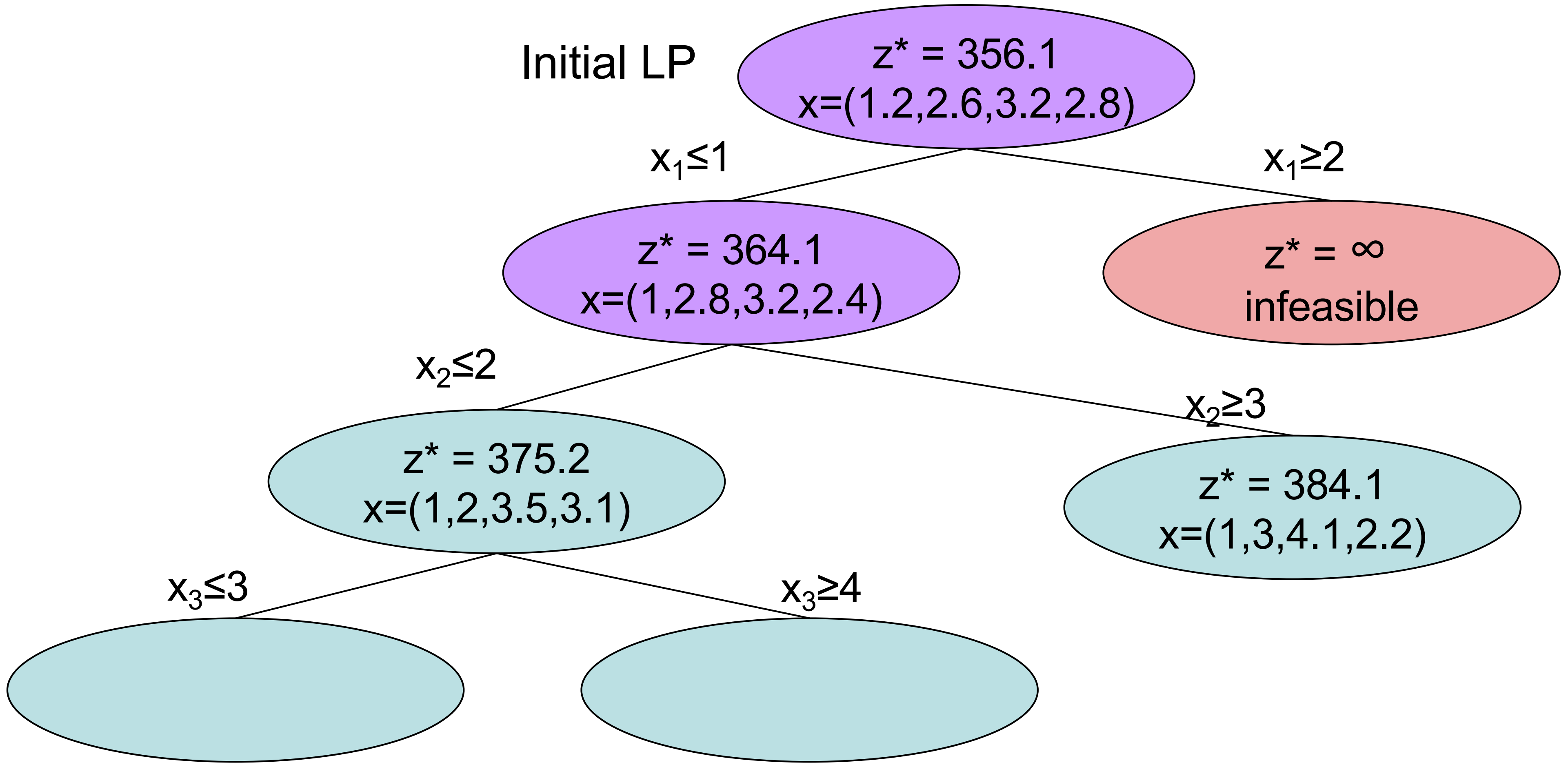

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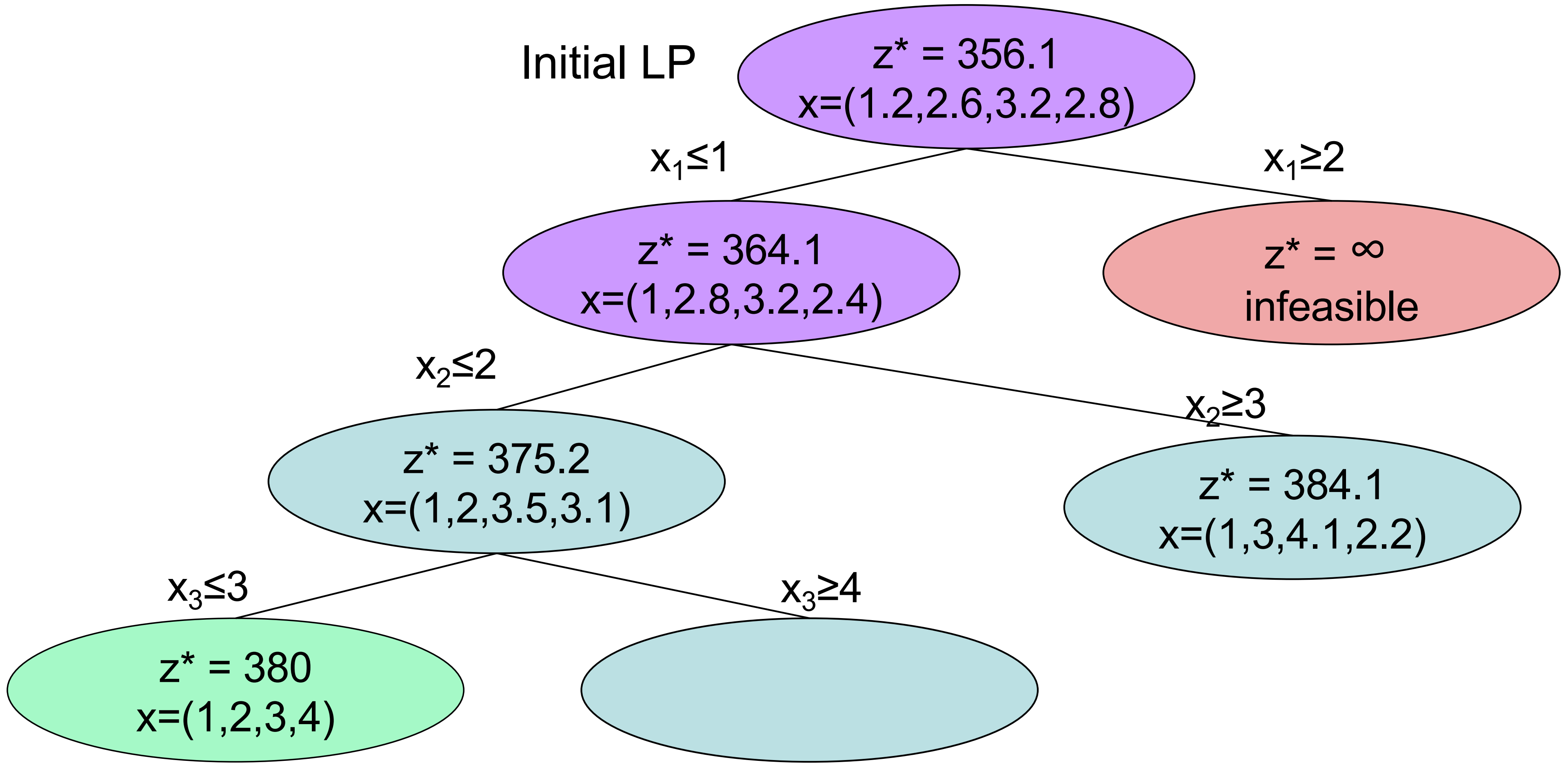

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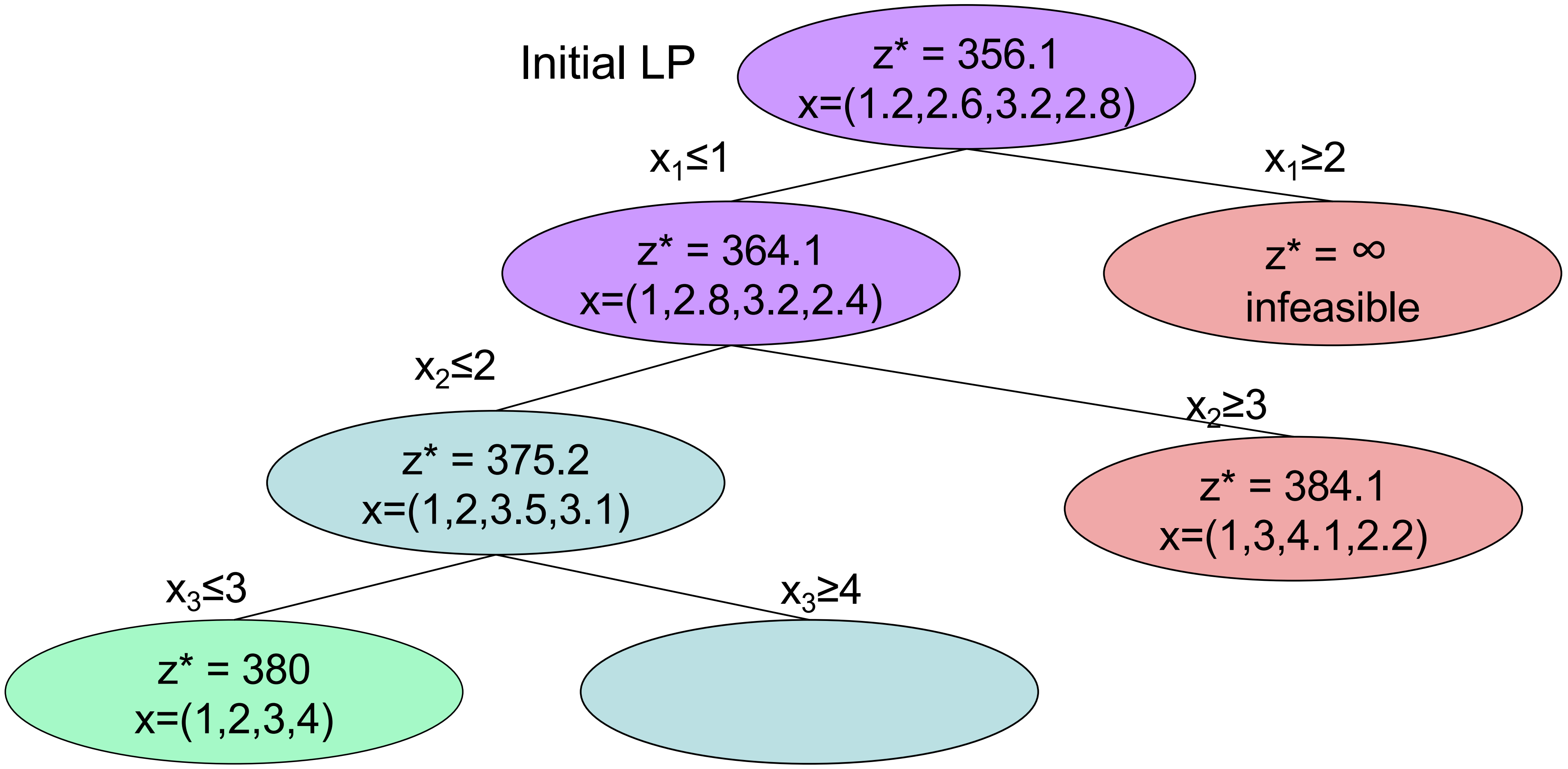

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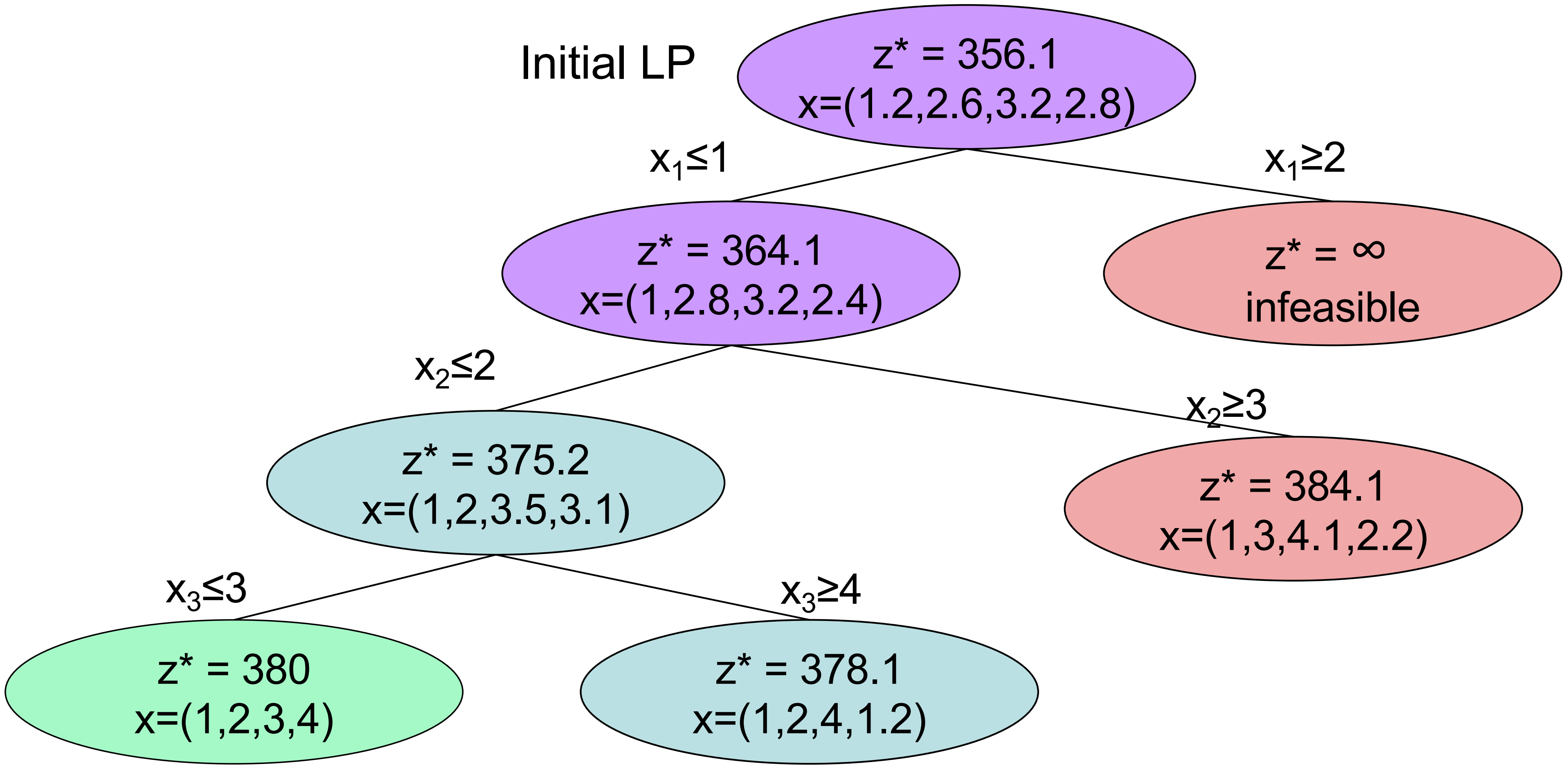

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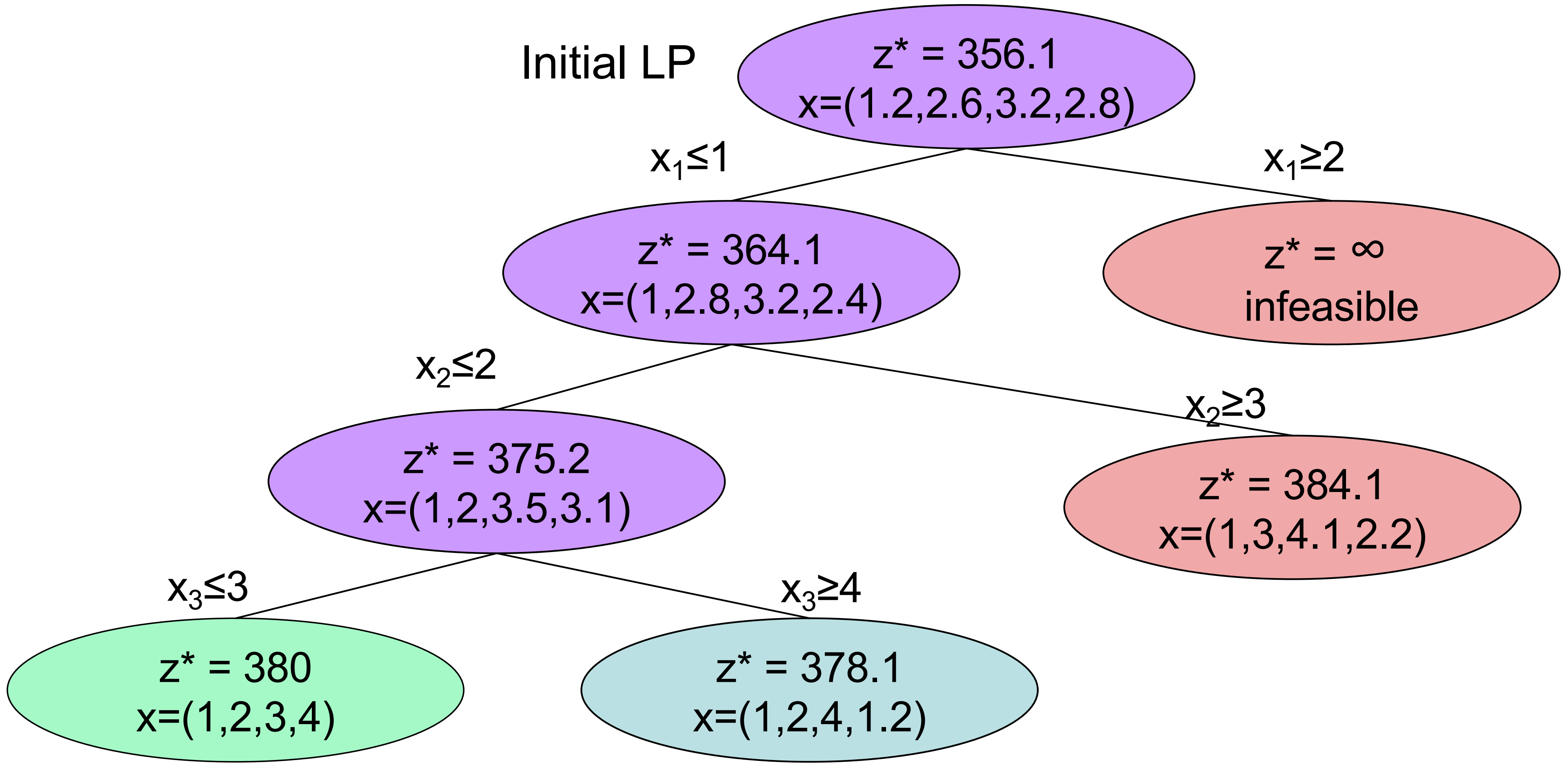

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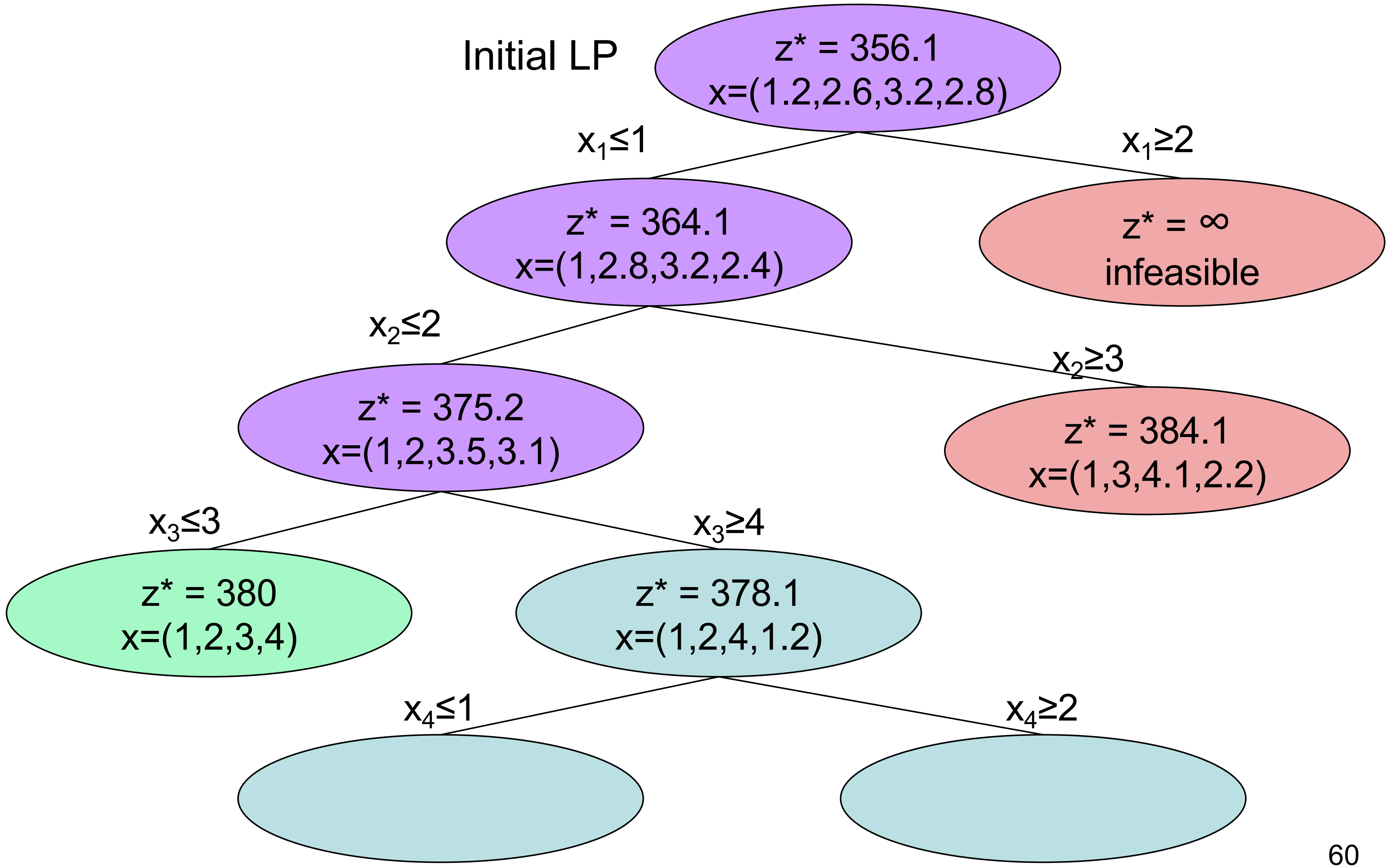
Initial LP
$z^* = 356.1$
$x=(1.2,2.6,3.2,2.8)$
$x_1 \leq 1$
$x_1 \geq 2$
$z^* = 364.1$
$x=(1,2.8,3.2,2.4)$
$z^* = \infty$
infeasible
$x_2 \leq 2$
$x_2 \geq 3$
$z^* = 375.2$
$x=(1,2,3.5,3.1)$
$z^* = 384.1$
$x=(1,3,4.1,2.2)$
$x_3 \leq 3$
$x_3 \geq 4$
$z^* = 380$
$x=(1,2,3,4)$
$z^* = 378.1$
$x=(1,2,4,1.2)$
$x_4 \leq 1$
$x_4 \geq 2$