Simulation

Lecture O1 Optimization: Linear Programming

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Outline of the course

- Linear Programming (1 lecture)
- Integer Programming (1 lecture)
- Heuristics and Metaheursitics (3 lectures)

Lab Exercises

- Two Lab Exercises
  - ✓Solving linear and integer programs using MATLAB and Excel
  - ✓Implementing and simulation of heuristics and metaheuristics
  - ✓Compulsory

Outline of this lecture

- Introduction
- Classification of Optimization Problems
- Linear Programming (LP)
- Examples of LP
- Development of LP
- Graphical Solution to LP Problems
- The Simplex Method and Simplex Tableau
- Duality
- Marginal Values and Sensitivity Analysis
- LP Solvers

Introduction (1)

- **Optimization:**
  - the act of obtaining the best result under given circumstances.
  - also, defined as the process of finding the conditions that lead to optimal solution(s)
- **Mathematical programming**:
  - methods to seek the optimum solution(s) to a problem
- **Steps involved in mathematical programming**
  - Conversion of stated problem into a mathematical model that abstracts all the essential elements of the problem.
  - Exploration of different solutions of the problem.
  - Finding out the most suitable or optimum solution.
- **Operations research:**
  - a branch of mathematics concerned with the application of scientific methods and techniques to decision making problems and with establishing the best or optimal solutions.

Introduction (2)

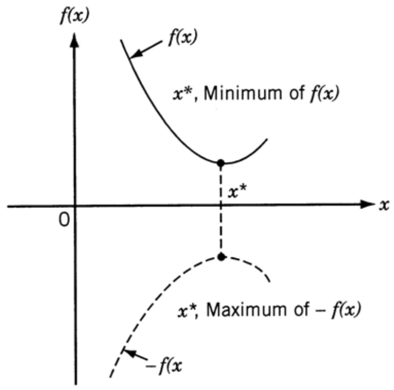
Methods of Operational Research

Mathematical Programming Techniques	Stochastic Process Techniques	Statistical Methods
Calculus methods	Statistical decision theory	Regression analysis
Calculus of variations	Markov processes	Cluster analysis, pattern recognition
Nonlinear programming	Queueing theory	Design of experiments
Geometric programming	Renewal theory	Discriminate analysis (factor analysis)
Quadratic programming	Simulation methods	
Linear programming	Reliability theory	
Dynamic programming		
Integer programming		
Stochastic programming		
Separable programming		
Multiobjective programming		
Network methods: CPM and PERT		
Game theory		
Simulated annealing		
Genetic algorithms		
Neural networks		

Introduction (3)

- **Mathematical optimization problem:**

$$\text{minimize } f(x)$$
$$\text{subject to } g_i(x) \le b_i, \quad i = 1,\ldots,m$$



- $f: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$: objective function
- $x=(x_1,\ldots,x_n)$: design variables (unknowns of the problem, they must be linearly independent)
- $g_i: \boldsymbol{R}^n \rightarrow \boldsymbol{R}$: $(i=1,\ldots,m)$: inequality constraints
- There might be equality constraints as well

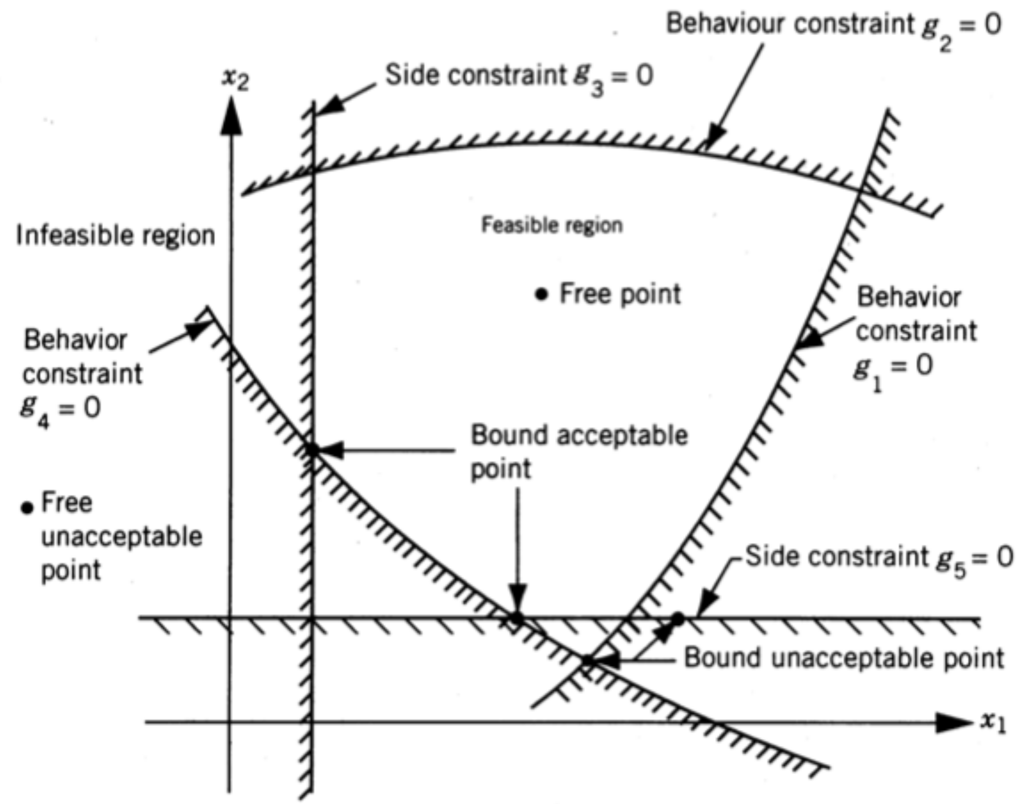
Introduction (4)
Constraint Types

- **Behaviour constraints:** represent limitations on the behaviour or performance of the system

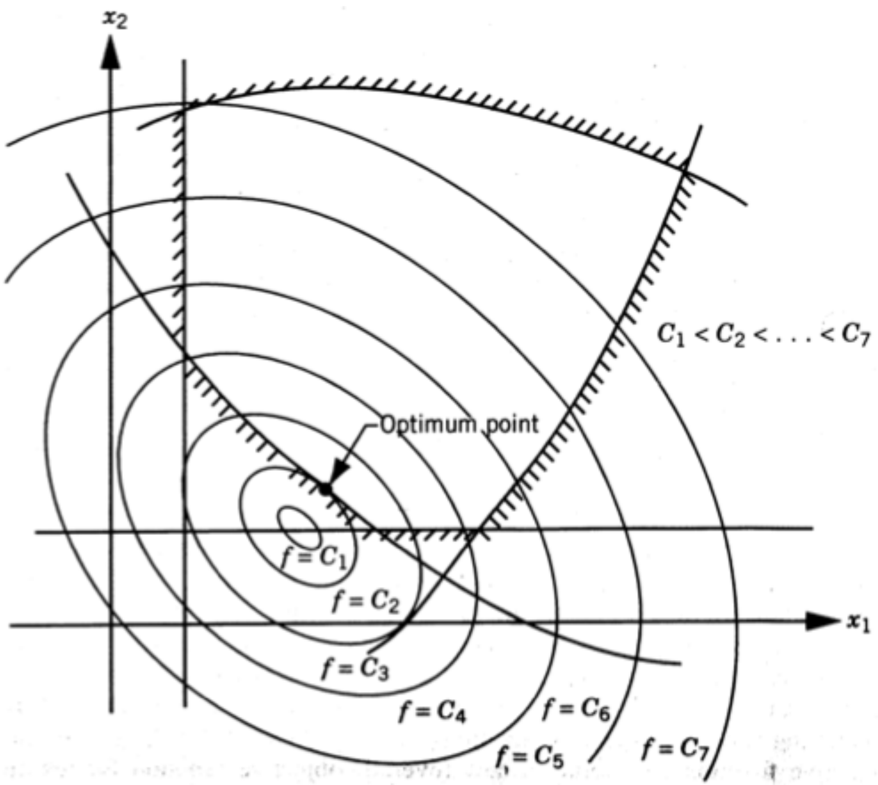
- **Side constraints:** represent physical limitations on design variables such as manufacturing limitations

Introduction (5)
Constraints Surface

- Free and acceptable point
- Free and unacceptable point
- Bound and acceptable point
- Bound and unacceptable point



Introduction (6)
Objective Function Surfaces



- In simple problems, the optimum point can be determined using graphical method.
- In large problems, the constraint and objective function surfaces become complex even for visualization
  - the problem has to be solved using mathematical techniques.

Classification of Optimization Problems (1)

- Constraints → constrained vs. non-constrained
- Design variables → static vs. dynamic and deterministic vs. stochastic
- Equations (objective function and constraints) → linear, non-linear, geometric, quadratic
- Permissible values of design variables → integer vs. real-valued
- Number of objectives → single vs. multi-objective
- …

Classification of Optimization Problems (2)

Types of Problems

Linear

constant

$\sum c_i x_i$

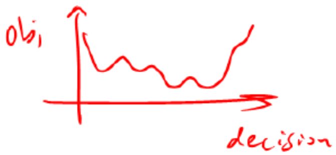
decision

Non-Linear

Convex

one local minima

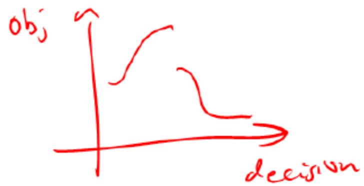
Multiple Minima (Multi-Modal)



Combinatorial

Discrete

Discontinuous



Decision : x

Objective : $F(x)$ ← Cost Function

Linear Programming Model (1)

Let: $X_1, X_2, X_3, \ldots\ldots\ldots, X_n$ = decision variables

Z = Objective function or linear function

Requirement: Maximization of the linear function Z.

$$Z = c_1X_1 + c_2X_2 + c_3X_3 + \ldots\ldots\ldots + c_nX_n \qquad (1)$$

subject to the following linear constraints:

$$\begin{aligned}
&a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\
&a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (2) \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m
\end{aligned}$$

where a_{ij}, b_i, and c_j are given constants.

Linear Programming Model (2)

- A more efficient notation:

$$\text{maximize } Z = \sum_{j=1}^{n} c_j x_j$$

$$\text{subject to} \sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1, \dots, m \qquad (3)$$

$$x_j \ge 0, \ i = 1, \dots, n$$

- A compact matrix form:

$$\text{maximize } Z = \mathbf{c}^T \mathbf{x}$$

$$\text{subject to } \mathbf{Ax} \le \mathbf{b}$$

$$x_j \ge 0, \ i = 1, \dots, n$$

Why we talk about Linear Programming?

- LP is simpler than NLP, hence, good for a foundation
- Linearity has some unique features for optimization
- A lot of problems are or can be converted to a LP formulation
- Some NLP algorithms are based upon LP simplex method

LP Problems
Example 1: Blending Problem

- An oil company makes two blends of fuel by mixing three oils. The costs and daily availability of the oils are given below:

Oil	Cost ($/litre)	Amount available (litres)
A	0.30	6,000
B	0.40	10,000
C	0.48	12,000

- The requirements of the blends of fuel:

Blend 1	at least 30% of *A* at most 50% of *B* at least 30% of *C*
Blend 2	at most 40% of *A* at least 35% of *B* at most 40% of *C*

- Each litre of blend 1 can be sold for $1.10 and each litre of blend 2 can be sold for $1.20. Long-term contracts require at least 10,000 litres of each blend to be produced. Formulate this blending problem as a linear programming problem

- Objective: the problem wants to maximize the profit obtained from making and selling the two types of blends of fuel

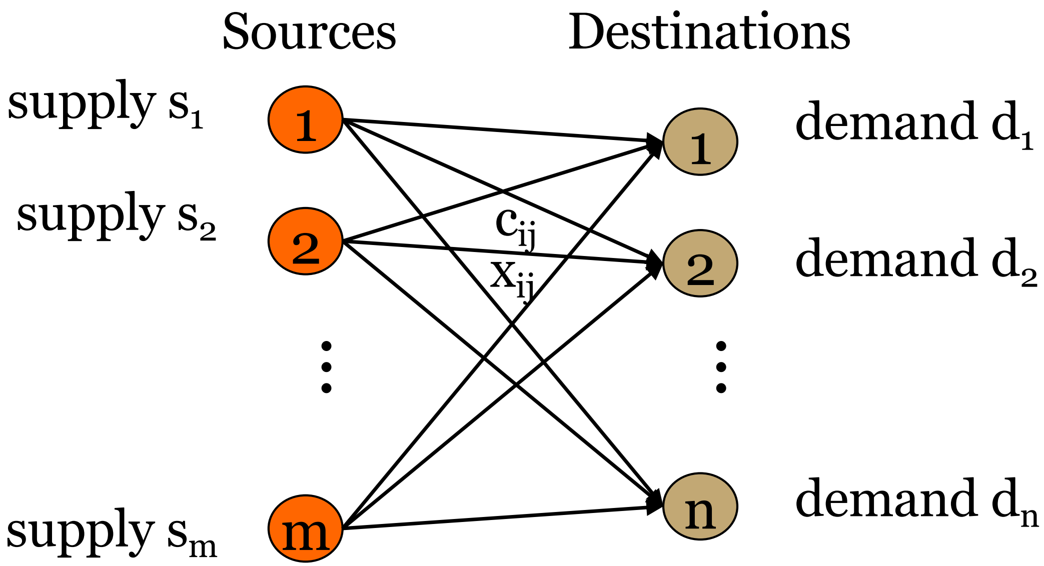
LP Problems
Example 2: Product Scheduling

- A manufacturer knows that it must supply a given number of items of a certain product each month for the next n months.
- They can be produced either in regular time, subject to a maximum each month, or in overtime. The cost of producing an item during overtime is greater than during regular time. A storage cost is associated with each item not sold at the end of the month.
- The problem is to determine the production schedule that minimizes the sum of production and storage costs.

LP Problems
Example 3: Transportation Problem

- The problem of finding the **minimum-cost** distribution of a given commodity from a group of supply centers (**sources**) $i=1,\ldots,m$ to a group of demanding centers (**destinations**) $j=1,\ldots,n$
- Each source i has a certain supply (s_i)
- Each destination j has a certain demand (d_j)
- The cost of shipping from a source to a destination is directly proportional to the number of units shipped

LP Problems
Example 3: Transportation Problem



x_{ij} : The amount of commodity to be shipped from i^{th} source to j^{th} destination

c_{ij} : The cost of shipment from i^{th} source to j^{th} destination

Developing LP Model (1)

- Steps:
  - Determine the objective and formulate it as a function of decision variables.
  - Specify and formulate the constraints.
  - Use/develop a method to solve the problem
    - Graphically
    - Simplex, revised simplex, Ellipsoid, etc.
    - Interior point
  - Do further analysis (shadow price, sensitivity, etc.)

Developing LP Model (2)
Example: Product mix

- The N. Dustrious Company produces two products: I and II. The raw material requirements, space needed for storage, production rates, and selling prices for these products are given as below:

	Products		
	I	II	Capacity
Storage Space (ft^2/unit)	4	5	1500 ft^2
Raw Material (lb/unit)	5	3	1575 lb
Production rate (units/hr)	60	30	7 hours
Selling price ($/unit)	13	11	

- Manufactured products are shipped out of the storage area at the end of the day → the two products must share the total raw material, storage space, and production time

Developing LP Model (3)
Example: Product mix

- **Objective:**
  - the company wants to determine how many units of each product to produce per day to maximize its total income.
- **Solution**
  - The company has decided that it wants to maximize its sale income, which depends on the number of units of product I and II that it produces.
  - The decision variables, x_1 and x_2 represent the number of units of products I and II, respectively, produced per day.

Developing LP Model (4)
Example: Product mix

- The object is to maximize:

$$Z = 13x_1 + 11x_2$$

subject to the constraints on storage space, raw materials, and production time.

- Each unit of product I requires 4 ft^2 of storage space and each unit of product II requires 5 ft^2. The total space (?) must be less than or equal to the available storage space, which is 1500 ft^2. Therefore,

$$4X_1 + 5X_2 \leq 1500$$

- Similarly, each unit of product I and II produced requires 5 and 3 lbs, respectively, of raw material. Hence a total of $5x_l + 3x_2$ lb of raw material is used.

Developing LP Model (5)
Example: Product mix

- This must be less than or equal to the total amount of raw material available, which is 1575 lb. Therefore,

$$5x_1 + 3x_2 \leq 1575$$

- Product I can be produced at the rate of 60 units per hour. Therefore, it must take 1 minute or 1/60 of an hour to produce 1 unit. Similarly, it requires 1/30 of an hour to produce 1 unit of product II. Hence a total of $x_1/60 + x_2/30$ hours is required for the daily production. This quantity must be less than or equal to the total production time available each day. Therefore,

$$x_1 / 60 + x_2 / 30 \leq 7$$

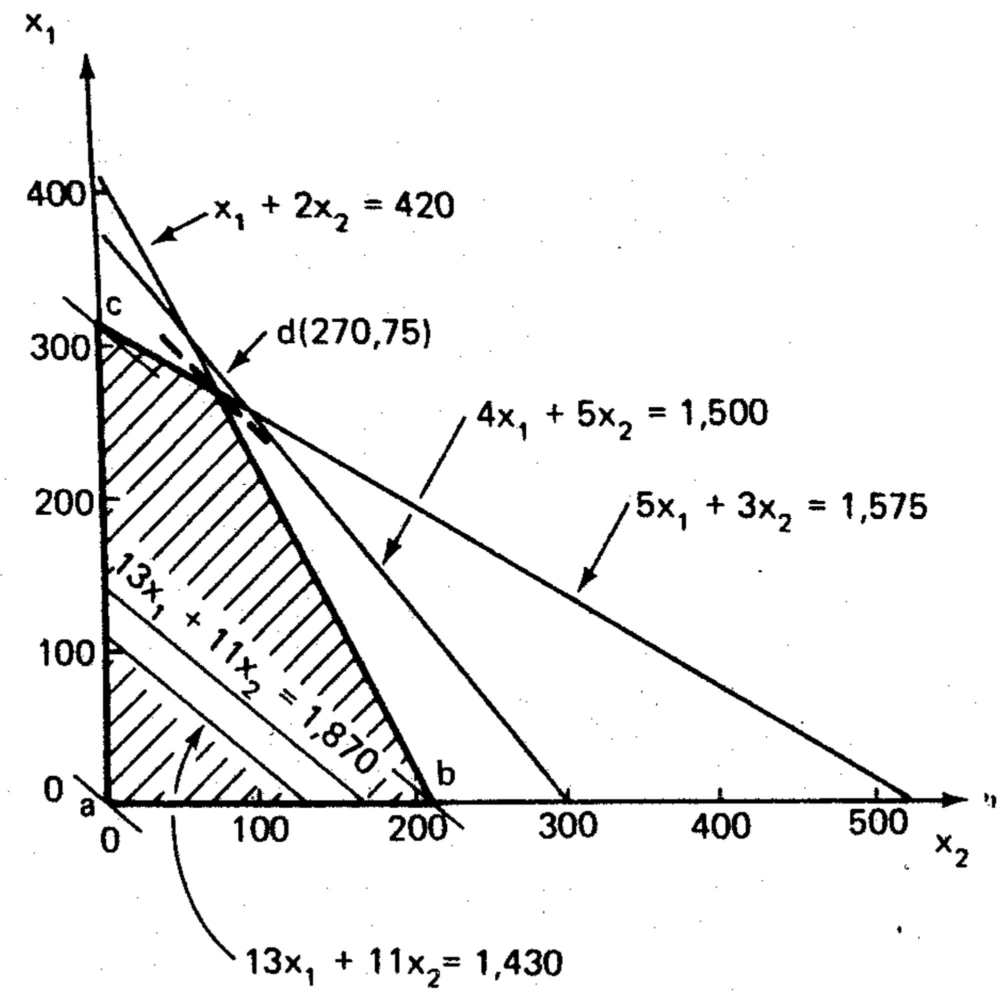
or $x_1 + 2x_2 \leq 420$

- Finally, the company cannot produce a negative quantity of any product, therefore x_1 and x_2 must each be greater than or equal to zero

Developing LP Model (6)
Example: Product mix

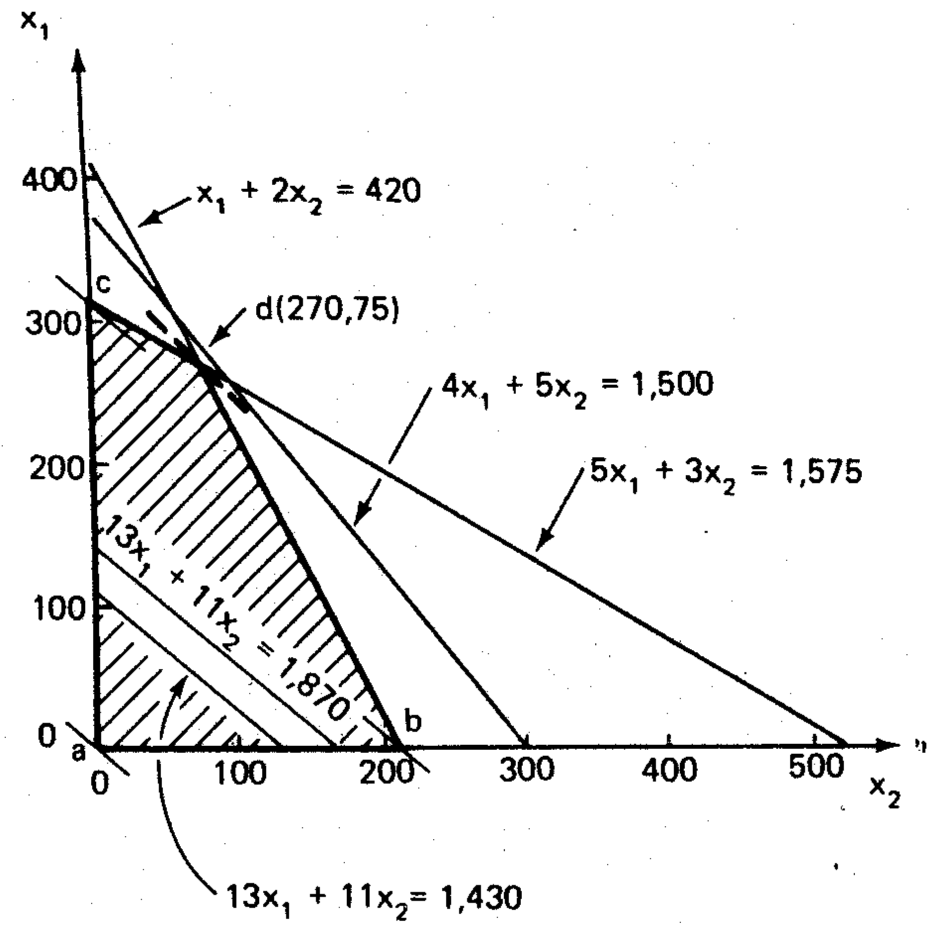
$$
\begin{aligned}
&\textit{Maximize} \quad 13x_1 + 11x_2 \\
&\text{Subject to:} \\
&\qquad 4x_1 + 5x_2 \quad \le 1500 \quad \text{(Storage constraint))} \\
&\qquad 5x_1 + 3x_2 \quad \le 1575 \quad \text{(Raw material constraint)} \\
&\qquad x_1 + 2x_2 \quad \le 420 \quad \text{(Prodcution time constraint)} \\
&\qquad\qquad x_1 \ge 0 \\
&\qquad\qquad x_2 \ge 0
\end{aligned}
\qquad (4)
$$

Graphical Solution
Example: Product mix



Key Observations

- Optimum must be at the intersection of constraints
- Intersections are easy to find
- Intersections are called basic solutions
- Some intersections are outside the feasible region and so need not be considered
- The others (at the corners of the feasible region) are called basic feasible solution



Simplex Method (1)

- When decision variables are more than 2, it is always advisable to use Simplex Method to avoid lengthy graphical procedure.
- The simplex method is not used to examine all the feasible solutions.
- It deals only with a finite and usually small set of feasible solutions, the set of corner points (i.e., extreme points) of the convex feasible space that contains the optimal solution.

Simplex Method (2)

- **Steps:**

1. Find an initial corner point in the feasible region.
2. Examine each boundary edge intersecting at this point to see whether movement along any edge increases the value of the objective function.
3. If the value of the objective function increases along an edge, move along this edge to the adjacent corner point. Test optimality.
4. Repeat steps 2 and 3 until movement along any edge no longer increases the value of the objective function.

Simplex Method (3)
Example: Product mix

- **Step 1: find an initial feasible corner point**
  - Convert all the inequality constraints into equalities using slack variables:

$$S_1 = \text{unused storage space}$$
$$S_2 = \text{unused raw materials}$$
$$S_3 = \text{unused production time}$$

  - Our original LP model was:

$$\begin{aligned}
&\textit{Maximize} \quad 13x_1 + 11x_2 \\
&\text{Subject to:} \\
&\qquad 4x_1 + 5x_2 \le 1500 \quad \text{(Storage constraint))} \\
&\qquad 5x_1 + 3x_2 \le 1575 \quad \text{(Raw material constraint)} \\
&\qquad x_1 + 2x_2 \le 420 \quad \text{(Prodcution time constraint)} \\
&\qquad x_1 \ge 0 \\
&\qquad x_2 \ge 0
\end{aligned} \qquad (4)$$

Simplex Method (4)
Example: Product mix

- Apply the slack variables to (4) to get this augmented form of LP as follows:

$$Z - 13x_1 - 11x_2 = 0 \qquad (A1)$$

Subject to:

Storage constraint: $4x_1 + 5x_2 + S_1 = 1500 \qquad (B1)$

Material constraint: $5x_1 + 3x_2 + S_2 = 1575 \qquad (C1)$

Prod. time constraint: $x_1 + 2x_2 + S_3 = 420 \qquad (D1)$

$$x_1 \geq 0, x_2 \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0$$

(5)

- A feasible solution to (5) is: (how?)

$(x_1 , x_2 , S_1 , S_2 , S_3) = (0 , 0 , 1500 , 1575 , 420)$

Simplex Method (5)
Example: Product mix

- **Step 2: check if improvement is possible**
  - The coefficients of x_1 and x_2 in Eq. (A1) are negative→ Z can be increased by increasing either x_1 or x_2. Choose x_1 (?)
  - In Eq. (B1), if $x_2 = S_1 = 0$, then $x_1 = 1500/4 = 375$→ there is only sufficient storage space to produce 375 units of product I.
  - From Eq. (C1)→there is only sufficient raw materials to produce 1575/5 = 315 units of product I.
  - From Eq. (D1)→ there is only sufficient time to produce 420/1 = 420 units of product I.
  - From all three constraints, there is sufficient resource to produce only 315 units of x_1 → thus the maximum value of x_1 is limited by Eq. (C1).

Simplex Method (6)
Example: Product mix

- **Step 3: move to the next corner point**
  - From Equation C1, which limits the maximum value of x_1.

$$x_1 = -\frac{3}{5}x_2 - \frac{1}{5}S_2 + 315 \tag{6}$$

  - Substituting Eq.(6) in Eq.(5), we get this new formulation:

$$\begin{aligned}
Z - \frac{16}{5}x_2 \qquad\quad + \frac{13}{5}S_2 \qquad\quad &= 4095 \quad \text{(A2)} \\
+\frac{13}{5}x_2 + S_1 - \frac{4}{5}S_2 \qquad\quad &= 240 \quad \text{(B2)} \\
x_1 + \frac{3}{5}x_2 \qquad\quad + \frac{1}{5}S_2 \qquad\quad &= 315 \quad \text{(C2)} \\
\frac{7}{5}x_2 \qquad\quad - \frac{1}{5}S_2 + S_3 &= 105 \quad \text{(D2)}
\end{aligned} \tag{7}$$

Simplex Method (7)
Example: Product mix

- From equations (7), the new feasible solution is:

$x_1 = 315$, $x_2 = 0$, $S_1 = 240$, $S_2 = 0$, $S_3 = 105$, and $Z = 4095$

- Optimality Test: this is not the optimum solution (from Eq.(A2)).
  - The coefficient of x_2 in the objective function represented by A2 is negative (-16/5), which means that the value of Z can be further increased by giving x_2 some positive value.

Simplex Method (8)
Example: Product mix

- **Repeat steps 2 to 3:**
  - Following the same analysis procedure used in step 1, it is clear that::
    - In Eq. (B2), if $S_1 = S_2 = 0$, then $x_2 = (5/13)(240) = 92.3$.
    - From Eq. (C2), x_2 can take on the value $(5/3\ )(315) = 525$ if $x_1 = S_2 = 0$
    - From Eq. (D2), x_2 can take on the value $(5/7)(105) = 75$ if $S_2 = S_3 = 0$
  - Therefore, constraint D2 limits the maximum value of x_2 to 75. Thus a new feasible solution includes $x_2 = 75$, $S_2 = S_3 = 0$.

Simplex Method (9)
Example: Product mix

- From Equation D2:

$$x_2 = \frac{1}{7}S_2 - \frac{5}{7}S_3 + 75 \qquad (8)$$

- Substituting this equation into Eq. (7) yield:

$$\begin{aligned}
Z \qquad\qquad\quad + \frac{15}{7}S_2 + \frac{16}{7}S_3 &= 4335 \quad (\text{A}3)\\
S_1 - \frac{3}{7}S_2 - \frac{13}{7}S_3 &= 45 \quad (\text{B}3)\\
x_1 \qquad\quad + \frac{2}{7}S_2 - \frac{3}{7}S_3 &= 270 \quad (\text{C}3)\\
x_2 \quad - \frac{1}{7}S_2 + \frac{5}{7}S_3 &= 75 \quad (\text{D}3)
\end{aligned} \qquad (9)$$

From these equations, the new feasible solution is readily found to be: $x_1 = 270$, $x_2 = 75$, $S_1 = 45$, $S_2 = 0$, $S_3 = 0$, $Z = 4335$.

The Simplex Method (10)

- Optimality Test: the coefficients in the objective function represented by Eq. (A3) are all positive, this new solution is the optimum solution.
  - Optimal solution: $x_1 = 270$, $x_2 = 75$, $S_1 = 45$, $S_2 = 0$, $S_3 = 0$, $Z = 4335$.

Simplex Tableau for Maximization (1)

- **Step 1:** Set up the initial tableau using Eq. (5).

$$
\begin{aligned}
&Z - 13x_1 - 11x_2 = 0 \quad \text{(A1)}\\
&\text{Subject to:}\\
&4x_1 + 5x_2 + S_1 = 1500 \quad \text{(B1)}\\
&5x_1 + 3x_2 + S_2 = 1575 \quad \text{(C1)}\\
&x_1 + 2x_2 + S_3 = 420 \quad \text{(D1)}\\
&x_1 \geq 0, x_2 \geq 0, S_1 \geq 0, S_2 \geq 0, S_3 \geq 0
\end{aligned}
\tag{5}
$$

In any iteration, a variable that has a nonzero value in the solution is called a **basic variable**.

Row Number	Basic Vari-able	Coefficients of: Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
Initial tableau									
A1	Z	1	−13	−11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420

Simplex Tableau for Maximization (2)

- **Step 2:** Identify the variable that will be assigned a nonzero value in the next iteration so as to increase the value of the objective function. This variable is called the entering variable.
  - It is the non-basic variable which is associated with the smallest negative coefficient in the objective function.
  - If two or more non-basic variables are tied with the smallest coefficients, select one of these arbitrarily and continue.

- **Step 3:** Identify the variable, called the leaving variable, which will be changed from a nonzero to a zero value in the next solution.

Simplex Tableau for Maximization (3)

➢ **Step 4:** Enter the basic variables for the second tableau. The row sequence of the previous tableau should be maintained, with the leaving variable being replaced by the entering variable.

Row Number	Basic Variable	Coefficients of: Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
Initial tableau									
A1	Z	1	-13	-11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420
Second tableau at end of first iteration									
A2	Z	1	0	$-\frac{16}{5}$	0	$+\frac{13}{5}$	0	4095	
B2	S_1	0	0	$\frac{13}{5}$	1	$-\frac{4}{5}$	0	240	92.3
C2	x_1	0	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	315	525
D2	S_3	0	0	$\frac{7}{5}$	0	$-\frac{1}{5}$	1	105	75

Simplex Tableau for Maximization (4)

- **Step 5:** Compute the coefficients for the second tableau. A sequence of operations will be performed so that at the end the x_1 column in the second tableau will have the following coefficients:

	x_1
Z	0
S_1	0
x_1	1
S_3	0

The second tableau yields the following feasible solution:

$x_1 = 315$, $x_2 = 0$, $S_1 = 240$, $S_2 = 0$, $S_3 = 105$, and $Z = 4095$

Simplex Tableau for Maximization (5)

- The row operations proceed as follows:
  - The coefficients in row C2 are obtained by dividing the corresponding coefficients in row C1 by 5.
  - The coefficients in row A2 are obtained by multiplying the coefficients of row C2 by 13 and adding the products to the corresponding coefficients in row AI.
  - The coefficients in row B2 are obtained by multiplying the coefficients of row C2 by -4 and adding the products to the corresponding coefficients in row BI.
  - The coefficients in row D2 are obtained by multiplying the coefficients of row C2 by -1 and adding the products to the corresponding coefficients in row DI.

Simplex Tableau for Maximization (6)

- **Step 6:** Check for optimality. The second feasible solution is also not optimal, because the objective function (row A2) contains a negative coefficient. <u>Another iteration beginning with step 2 is necessary</u>.

- In the third tableau (next slide), all the coefficients in the objective function (row A3) are positive. Thus an optimal solution has been reached and it is as follows:

$$x_1 = 270,\ x_2 = 75,\ S_1 = 45,\ S_2 = 0,\ S_3 = 0, \text{ and } Z = 4335$$

Row Number	Basic Variable	Z	x_1	x_2	S_1	S_2	S_3	Right-Hand Side	Upper Bound on Entering Variable
		Coefficients of:							
Initial tableau									
A1	Z	1	-13	-11	0	0	0	0	
B1	S_1	0	4	5	1	0	0	1500	375
C1	S_2	0	5	3	0	1	0	1575	315
D1	S_3	0	1	2	0	0	1	420	420
Second tableau at end of first iteration									
A2	Z	1	0	$-\frac{16}{5}$	0	$+\frac{13}{5}$	0	4095	
B2	S_1	0	0	$\frac{13}{5}$	1	$-\frac{4}{5}$	0	240	92.3
C2	x_1	0	1	$\frac{3}{5}$	0	$\frac{1}{5}$	0	315	525
D2	S_3	0	0	$\frac{7}{5}$	0	$-\frac{1}{5}$	1	105	75
Third tableau at end of second and final iteration									
A3	Z	1	0	0	0	$+\frac{15}{7}$	$+\frac{16}{7}$	4335	
B3	S_1	0	0	0	1	$-\frac{3}{7}$	$-\frac{13}{7}$	45	
C3	x_1	0	1	0	0	$\frac{2}{7}$	$-\frac{3}{7}$	270	
D3	x_2	0	0	1	0	$-\frac{1}{7}$	$\frac{5}{7}$	75	

Duality (1)

- With every linear programming problem, there is associated another linear programming problem which is called the dual of the original (or the primal) problem.

Interpretation and Formulation of the Dual Problem

- Consider again the production mix problem of N. Dustrious Company.
  - Suppose that the company is considering leasing out the entire production facility to another company, and it must decide on the <u>minimum daily rental price</u> that will be acceptable.
  - This decision problem can also be formulated as a linear programming problem.

Duality (2)

- Let y_1 y_2 and y_3 represent the unit price of each unit of storage space, raw materials, and production time
- The unit prices are in fact the income values of each unit of resource to the N. Dustrious Company.
- There are available 1500 ft^2 of storage space, 1575 lb of raw materials, and 420 minutes of production time per day
- Thus the total income value (**P**) of all the available resources may be expressed as follows :

$$P = 1500y_1 + 1575y_2 + 420y_3$$

- The objective of the problem is to minimize P subject to the condition that the N. Dustrious Company will earn at least as much income as when it operates the production facility itself

Duality (3)

- Since the market value (or selling price) of 1 unit of product I is \$13 and it requires 4 ft^2 of storage space, 5 lbs of raw materials, and 1 minute of production time, the following constraint must be satisfied:

$$4y_1 + 5y_2 + y_3 \geq 13$$

- Similarly, for Product II:

$$5y_1 + 3y_2 + 2y_3 \geq 11$$

- In addition, the unit prices y_1, y_2 and y_3 must all be greater than or equal to zero.

Duality (4)

- The new linear programming problem may now be summarized as follows :

$$\begin{aligned}
&\textit{Minimize} \quad 1500\mathrm{y}_1 + 1575\mathrm{y}_2 + 420 y_3 && \text{(Dual Problem)} \\
&\text{Subject to:} \\
&\qquad 4\mathrm{y}_1 + 5\mathrm{y}_2 + 1 y_3 \quad \geq 13 \quad \text{(C1)} \\
&\qquad 5\mathrm{y}_1 + 3\mathrm{y}_2 + 2 y_3 \quad \geq 11 \quad \text{(C2)} && (1) \\
&\qquad \mathrm{y}_1 \geq 0,\ \mathrm{y}_2 \geq,\ \mathrm{y}_3 \geq 0
\end{aligned}$$

- The following interesting observations can now be made
  - o P = Z = \$4335
  - o y_1 = \$0; y_2 = \$15/7 and y_3 = \$16/7
- This problem is the same as maximization problem in the previous example and can now be solved accordingly.

Duality (5)

The primal-dual relationship

Primal Problem

Maximize

$$Z = c_1 x_1 + c_2 x_2$$

subject to:

$$k_{11}x_1 + k_{12}x_2 \leq b_1$$

$$k_{21}x_1 + k_{22}x_2 \leq b_2$$

$$k_{31}x_1 + k_{32}x_2 \leq b_3$$

$$\text{all } x_i \geq 0$$

Dual Problem

Minimize

$$P = b_1 y_1 + b_2 y_2 + b_3 y_3$$

subject to:

$$k_{11}y_1 + k_{21}y_2 + k_{31}y_3 \geq c_1$$

$$k_{12}y_2 + k_{22}y_2 + k_{32}y_3 \geq c_2$$

$$\text{all } y_i \geq 0$$

Duality (6)

Complete Regularization of the Primal Problem

❖ Consider the following primal problem:

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$4x_1 + 7x_2 \leq 56$$

$$2x_1 + 5x_2 \geq 20$$

$$5x_1 + 4x_2 = 40$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

❖ The first inequality requires no modification.

Duality (7)

- The second inequality can be changed to the less-than-or-equal-to type by multiplying both sides of the inequality by -1 and reversing the direction of the inequality; that is,

$$-2x_1 - 5x_2 \le -20$$

- The equality constraint can be replaced by the following two inequality constraints:

$$5x_1 + 4x_2 \le 40$$
$$5x_1 + 4x_2 \ge 40$$

- If both of these inequality constraints are satisfied, the original equality constraint is also satisfied.

Duality (8)

- Multiplying both sides of the inequality by –1 and reversing the direction of the inequality yields:

$$-5x_1 - 4x_2 \leq -40$$

- The primal problem can now take the following standard form:

Maximize

$$Z = 12x_1 + 4x_2$$

subject to:

$$\begin{aligned} 4x_1 + 7x_2 &\leq 56 \\ -2x_1 - 5x_2 &\leq -20 \\ 5x_1 + 4x_2 &\leq 40 \\ -5x_1 - 4x_2 &\leq -40 \\ x_1 &\geq 0 \\ x_2 &\geq 0 \end{aligned}$$

Duality (9)

- The dual of this problem can now be obtained as follows:

Minimize

$$P = 56y_1 - 20y_2 + 40y_3 - 40y_4$$

subject to:

$$\begin{aligned} 4y_1 - 2y_2 + 5y_3 - 5y_4 &\geq 12 \\ 7y_1 - 5y_2 + 4y_3 - 4y_4 &\geq 4 \\ \text{all } y_i &\geq 0 \end{aligned}$$

Marginal Values of Additional Resources (1)

- The simplex solution yields the optimum production program for N. Dustrious Company.
  - The company can maximize its sale income to \$4335 by producing 270 units of product I and 75 units of product II.
  - There will be no surplus of raw materials and production time
  - But there will be 45 units of unused storage space
- **We want to know if it is worthwhile to increase the production by procuring additional units of raw materials or by working overtime.**

Marginal Values of Additional Resources (2)

- **Some insightful questions:**
  - What is the income value (or marginal value) of each additional unit of each type of resources?
  - What is the maximum cost (or marginal cost) worthwhile to pay for each additional unit of resources?
- Answers to these questions can be obtained from the objective function in the last tableau of the simplex solution:

$$Z + \frac{15}{7}S_2 + \frac{16}{7}S_3 = \$4335$$

that is,

$$Z = \$4335 - \frac{15}{7}S_2 - \frac{16}{7}S_3$$

Marginal Values of Additional Resources (3)

- In the final Tableau, S_1, S_2 and S_3 represent surplus resources
  - Thus, $-S_1$, $-S_2$, $-S_3$ represent additional units of the resources that can be made available.
- The marginal value (in Excel solver, it is called shadow price) of one additional unit of each resource:

$$\text{Storage space} = \frac{\partial Z}{\partial(-S_1)} = \$0$$

$$\text{Raw materials} = \frac{\partial Z}{\partial(-S_2)} = \$\frac{15}{7}$$

$$\text{Production time} = \frac{\partial Z}{\partial(-S_3)} = \$\frac{16}{7}$$

Sensitivity Analysis

- Sensitivity analysis helps to test the sensitivity of the optimum solution with respect to changes of the coefficients in the objective function, coefficients in the constraints inequalities, or the constant terms in the constraints.

- **In the Product Mix example:**

  - The original selling prices (i.e. \$13 and \$11) of the two products may vary from time to time (i.e. $\$13 \pm \Delta$ and $\$11 \pm \Delta$). How much these prices can change without affecting the optimality of the present solution?

  - The amount of each type of resources needed to produce one unit of each type of product can be either increased or decreased slightly. Will such changes affect the optimal solution ?

  - Will the present solution remain the optimum solution if the amount of raw materials, production time, or storage space is suddenly changed because of shortages, machine failures, or other events?

LP Solvers

- Excel Solver → good for small problems, it comes with a sensitivity analysis tool
- ILOG CPLEX → is the world's leader
- Xpress-MP from Dash Optimization → good for medium scale problems
- lp_solve open source project → very useful
- MATLAB solvers → efficient and simple, a lot of optimization methods are available. Sensitivity analysis tool (ssatool) is only available in the latest version.

MATLAB LP Solver

Linprog() for medium scale problems uses the Simplex algorithm

Example

Find x that minimizes

$f(x) = -5x_1 - 4x_2 - 6x_3,$

subject to

$x_1 - x_2 + x_3 \leq 20$

$3x_1 + 2x_2 + 4x_3 \leq 42$

$3x_1 + 2x_2 \leq 30$

$0 \leq x_1, 0 \leq x_2, 0 \leq x_3.$

```
>> f = [-5; -4; -6];
>>  A = [1 -1 1
         3 2 4
         3 2 0];
>> b = [20; 42; 30];
>> lb = zeros(3,1);
>> x = linprog(f,A,b,[],[],lb);
>> Optimization terminated.
>> x
x =
    0.0000
   15.0000
    3.0000
```

References

- Book:
  - Linear Programming: Foundations and Extensions / by Robert J Vanderbei
- Presentation slides:
  - Linear Programming / by Syed M. Ahmed
  - Optimization /by Pelin Gündeş