Electrodynamics

Exercises

Anders Karlsson

Electrical and information technology
Lund University
Box 118
221 00 Lund

Printed in Sweden
Lund, April 2019
First edition

Preface

Exercises for the course Electrodynamics at Lund University

Lund in April 2019

Anders Karlsson

Rules for the ∇-operator

$$(1) \quad \nabla(\varphi + \psi) = \nabla\varphi + \nabla\psi$$
$$(2) \quad \nabla(\varphi\psi) = \psi\nabla\varphi + \varphi\nabla\psi$$
$$(3) \quad \nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = (\boldsymbol{a}\cdot\nabla)\boldsymbol{b} + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a})$$
$$(4) \quad \nabla(\boldsymbol{a}\cdot\boldsymbol{b}) = -\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) + 2(\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}) + \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a})$$

$$(5) \quad \nabla\cdot(\boldsymbol{a}+\boldsymbol{b}) = \nabla\cdot\boldsymbol{a} + \nabla\cdot\boldsymbol{b}$$
$$(6) \quad \nabla\cdot(\varphi\boldsymbol{a}) = \varphi(\nabla\cdot\boldsymbol{a}) + (\nabla\varphi)\cdot\boldsymbol{a}$$
$$(7) \quad \nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b})$$

$$(8) \quad \nabla\times(\boldsymbol{a}+\boldsymbol{b}) = \nabla\times\boldsymbol{a} + \nabla\times\boldsymbol{b}$$
$$(9) \quad \nabla\times(\varphi\boldsymbol{a}) = \varphi(\nabla\times\boldsymbol{a}) + (\nabla\varphi)\times\boldsymbol{a}$$
$$(10) \quad \nabla\times(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a}) + (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}$$
$$(11) \quad \nabla\times(\boldsymbol{a}\times\boldsymbol{b}) = -\nabla(\boldsymbol{a}\cdot\boldsymbol{b}) + 2(\boldsymbol{b}\cdot\nabla)\boldsymbol{a} + \boldsymbol{a}\times(\nabla\times\boldsymbol{b}) + \boldsymbol{b}\times(\nabla\times\boldsymbol{a}) + \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a})$$

$$(12) \quad \nabla\cdot\nabla\varphi = \nabla^2\varphi = \Delta\varphi$$
$$(13) \quad \nabla\times(\nabla\times\boldsymbol{a}) = \nabla(\nabla\cdot\boldsymbol{a}) - \nabla^2\boldsymbol{a}$$
$$(14) \quad \nabla\times(\nabla\varphi) = \boldsymbol{0}$$
$$(15) \quad \nabla\cdot(\nabla\times\boldsymbol{a}) = 0$$
$$(16) \quad \nabla^2(\varphi\psi) = \varphi\nabla^2\psi + \psi\nabla^2\varphi + 2\nabla\varphi\cdot\nabla\psi$$

$$(17) \quad \nabla r = \hat{\boldsymbol{r}}$$
$$(18) \quad \nabla\times\boldsymbol{r} = \boldsymbol{0}$$
$$(19) \quad \nabla\times\hat{\boldsymbol{r}} = \boldsymbol{0}$$
$$(20) \quad \nabla\cdot\boldsymbol{r} = 3$$
$$(21) \quad \nabla\cdot\hat{\boldsymbol{r}} = \frac{2}{r}$$
$$(22) \quad \nabla(\boldsymbol{a}\cdot\boldsymbol{r}) = \boldsymbol{a}, \quad \boldsymbol{a} \text{ constant vector}$$
$$(23) \quad (\boldsymbol{a}\cdot\nabla)\boldsymbol{r} = \boldsymbol{a}$$
$$(24) \quad (\boldsymbol{a}\cdot\nabla)\hat{\boldsymbol{r}} = \frac{1}{r}\left(\boldsymbol{a} - \hat{\boldsymbol{r}}(\boldsymbol{a}\cdot\hat{\boldsymbol{r}})\right) = \frac{\boldsymbol{a}_\perp}{r}$$
$$(25) \quad \nabla^2(\boldsymbol{r}\cdot\boldsymbol{a}) = 2\nabla\cdot\boldsymbol{a} + \boldsymbol{r}\cdot(\nabla^2\boldsymbol{a})$$

$$(26) \quad \nabla u(f) = (\nabla f)\frac{du}{df}$$
$$(27) \quad \nabla\cdot\boldsymbol{F}(f) = (\nabla f)\cdot\frac{d\boldsymbol{F}}{df}$$
$$(28) \quad \nabla\times\boldsymbol{F}(f) = (\nabla f)\times\frac{d\boldsymbol{F}}{df}$$
$$(29) \quad \nabla = \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}}\cdot\nabla) - \hat{\boldsymbol{r}}\times(\hat{\boldsymbol{r}}\times\nabla)$$

Contents

1 The Maxwell equations

1.1

A time-harmonic linearly polarized plane wave propagating in the positive $z-$direction in vacuum reads

$$\boldsymbol{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\boldsymbol{x}}$$

a) Express the angular frequency ω in the wavenumber k and the speed of light c.

b) Determine the magnetic flux density $\boldsymbol{B}(z,t)$.

c) Determine the Poynting vector $\boldsymbol{S}(z,t)$ and its time average.

d) Show that the time average of the electric and magnetic energy densities are the same.

e) Determine the complex electric field $\boldsymbol{E}(z)$ and complex magnetic flux density $\boldsymbol{B}(z)$.

f) Determine the complex Poynting vector $\boldsymbol{S}(z)$ and show that it has the same value as the time average of $\boldsymbol{S}(z,t)$.

1.2

A time-harmonic linearly polarized plane wave propagating in the positive $z-$direction in vacuum reads

$$\boldsymbol{E}(z,t) = E_0 \cos(kz - \omega t)\hat{\boldsymbol{x}}$$

a) Add a wave $\boldsymbol{E}_2(z,t)$ so that the total wave is a circularly polarized wave. There are two ways to do this. Give both of them.

b) Determine the Poynting vector $\boldsymbol{S}(z,t)$ for the circularly polarized wave.

c) Determine the electric and magnetic energy densities, $w_{\mathrm{e}}(z,t)$ and $w_{\mathrm{m}}(z,t)$ for the circularly polarized wave.

1.3

The complex electric far-field from an electric dipole antenna located at the origin and with dipole moment $\boldsymbol{p} = p_0\hat{\boldsymbol{z}}$ is given by

$$\boldsymbol{E}(r,\theta) = E_0 \sin\theta \frac{e^{\mathrm{i}kr}}{kr}\hat{\boldsymbol{\theta}}$$

where $k = \omega/c$ i the wavenumber.

a) Determine $\boldsymbol{B}(r,\theta)$ by using the induction law. Apparently $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the induction law. Do they also satisfy Ampères law?

b) Show that $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the plane wave rule $\boldsymbol{B} = c^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{E}$.

c) Determine the time domain fields $\boldsymbol{E}(r,\theta,t)$ and $\boldsymbol{B}(r,\theta,t)$.

d) Determine the Poynting vector $\boldsymbol{S}(r,\theta,t)$ and its time average.

e) Determine the time average of the radiated power from the dipole.

f) Does $\boldsymbol{E}(r,\theta)$ satisfy $\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = \boldsymbol{0}$? If not why?

g) Neglect all terms that drop off faster than r^{-1} in $\nabla^2\boldsymbol{E}$. Is then $\nabla^2\boldsymbol{E} + k^2\boldsymbol{E} = \boldsymbol{0}$ satisfied?

1.4

The electromagnetic fields from currents and charge distributions that vary slowly in time can often be determined by a quasi static analysis based on a reduced version of the Maxwell equations. There are two quasi static versions. The first is that one skips the term $\frac{\partial \boldsymbol{D}}{\partial t}$ in the Ampère law but keeps the term $\frac{\partial \boldsymbol{B}}{\partial t}$ in the induction law. This is done when the source is a known current distribution. The other version is the other way around, one skips $\frac{\partial \boldsymbol{B}}{\partial t}$ and keeps $\frac{\partial \boldsymbol{D}}{\partial t}$. This is done when the source is a known charge distribution.

a) A circular plate capacitor with area A, distance d between the plates, and with air between the plates, is connected to a time harmonic voltage source. The voltage between the plates is $V_0 \cos \omega t$. Determine the electric field between the plates.

b) Determine the magnetic flux density between the plates.

c) Do $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the full Maxwell equations?

d) Some approximation must have been made since the full Maxwell equations are not satisfied. What is this approximation?

e) A magnetic circuit has an iron core with circular cross section, see figure. The current in the coil is time harmonic which gives a magnetic flux density in the air gap

$$\boldsymbol{B}(t) = B_0 \sin(\omega t)\hat{\boldsymbol{z}}$$

The frequency f is so low that the wavelength $\lambda = c/f$ is much larger than the radius of the air gap. Determine the electric field in the air gap.

f) Do $\boldsymbol{E}$ and $\boldsymbol{B}$ satisfy the full Maxwell equations?

g) Some approximation must have been made since the full Maxwell equations are not satisfied. What is this approximation?

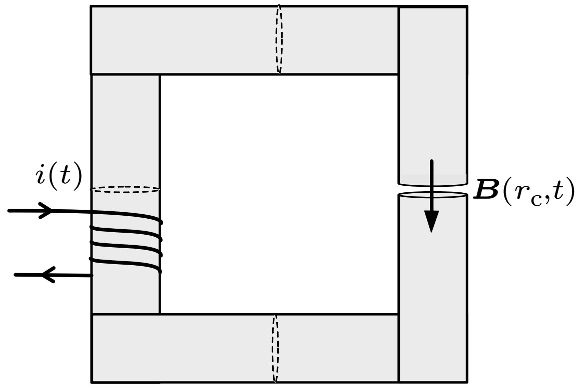


Figure 1: Magnetic circuit

The Maxwell equations: Answers and solutions

S1.1

a) $\omega = kc$

b) The right hand rule gives $\boldsymbol{B}(z,t) = c^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{E}(z,t)$, where $\hat{\boldsymbol{k}} = \hat{\boldsymbol{z}}$. Then $\boldsymbol{B}(z,t) = c^{-1}E_0 \cos(kz - \omega t)\hat{\boldsymbol{y}}$.

c) $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}$. Then $\boldsymbol{S}(z,t) = \dfrac{1}{\eta_0}E_0^2 \cos^2(kz - \omega t)\hat{\boldsymbol{z}}$. The time average is $< \boldsymbol{S}(z,t) >= \dfrac{1}{2\eta_0}E_0^2$.

d) The electric energy density is $w_{\mathrm{E}}(z,t) = \dfrac{1}{2}\varepsilon_0|\boldsymbol{E}(z,t)|^2$. Then $w_{\mathrm{E}}(z,t) = \dfrac{1}{2}\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$. The time average is

$$< w_{\mathrm{E}}(z,t) >= \frac{1}{4}\varepsilon_0 E_0^2$$

The electric energy density is $w_{\mathrm{M}}(z,t) = \dfrac{1}{2}\mu_0|\boldsymbol{H}(z,t)|^2$. Then $w_{\mathrm{H}}(z,t) = \dfrac{1}{2}\varepsilon_0 E_0^2 \cos^2(kz - \omega t)$. The time average is

$$< w_{\mathrm{E}}(z,t) >= \frac{1}{4}\varepsilon_0 E_0^2$$

This is always the case with waves. A wave is a resonance where the energy switch between two energy states.

e) The complex electric field and magnetic flux density are defined by $\boldsymbol{E}(z,t) = \mathrm{Re}\{\boldsymbol{E}(z)e^{-\mathrm{i}\omega t}\}$. Then $\boldsymbol{E}(z) = E_0 e^{\mathrm{i}kz}$ and $\boldsymbol{B}(z) = c^{-1}E_0 e^{\mathrm{i}kz}$.

f) The definition of the complex Poynting vector is $\boldsymbol{S}(z) = \dfrac{1}{2}\boldsymbol{E}(z) \times \boldsymbol{H}^*(z)$, where $*$

denotes complex conjugate. From e) we get $\boldsymbol{S}(z) = \dfrac{1}{2\eta_0} E_0^2$. This is the same value as in c).

S1.2

a) If we add $E_2(z,t) = E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}}$ then the total wave is a circular polarized plane wave. We can also add $E_2(z,t) = -E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}}$.

b) The magnetic field is given by $\boldsymbol{H}(z,t) = \eta_0^{-1}\hat{\boldsymbol{z}} \times (\boldsymbol{E}(z,t) + \boldsymbol{E}_2(z,t))$. With $\boldsymbol{E}_2(z,t) = E_0 \sin(kz - \omega t)\hat{\boldsymbol{y}}$ then $\boldsymbol{H}(z,t) = \eta_0^{-1} E_0(\cos(kz - \omega t)\hat{\boldsymbol{y}} - \sin(kz - \omega t)\hat{\boldsymbol{x}})$. The Poynting vector reads

$$\boldsymbol{S}(z,t) = \frac{1}{\eta_0} E_0^2 \hat{\boldsymbol{z}} \tag{1}$$

c) $w_{\mathrm{E}}(z,t) = w_{\mathrm{M}}(z,t) = \dfrac{1}{2}\varepsilon_0 E_0^2$.

S1.3

a) The induction law gives $\boldsymbol{B} = -\mathrm{i}\dfrac{1}{\omega}\nabla \times \boldsymbol{E}$. In spherical coordinates

$$\nabla \times (E_\theta(r,\theta)\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta(r,\theta))$$

This gives

$$\boldsymbol{B} = \frac{1}{c}E_0 \frac{e^{\mathrm{i}kr}}{kr}\sin\theta\hat{\boldsymbol{\phi}}$$

They do not satisfy the Ampères law since there is a term in the $\hat{\boldsymbol{r}}$-direction that drop off as r^{-2} and does not cancel.

b) The plane wave rule with $\hat{\boldsymbol{k}} = \hat{\boldsymbol{r}}$ also gives $\boldsymbol{B} = \dfrac{1}{c}E_0\dfrac{e^{\mathrm{i}kr}}{kr}\sin\theta\hat{\boldsymbol{\phi}}$.

c) $\boldsymbol{E}(r,\theta,t) = E_0\dfrac{\cos(kr-\omega t)}{kr}\sin\theta\hat{\boldsymbol{\theta}}$ and $\boldsymbol{B}(r,\theta,t) = c^{-1}E_0\dfrac{\cos(kr-\omega t)}{kr}\sin\theta\hat{\boldsymbol{\phi}}$.

d) $\boldsymbol{S}(r,\theta,t) = \dfrac{1}{\eta_0}\dfrac{E_0^2\cos^2(kr-\omega t)}{(kr)^2}\sin^2\theta\hat{\boldsymbol{r}}$, and $<\boldsymbol{S}(r,\theta,t)> = \dfrac{1}{2\eta_0}\dfrac{E_0^2}{(kr)^2}\sin^2\theta\hat{\boldsymbol{r}}$

e) Integrate the Poynting vector over a sphere with radius r. Then the time average of the power is

$$P = \int_0^\pi \int_0^{2\pi} <\boldsymbol{S}(r,\theta,t)> \cdot\hat{\boldsymbol{r}} r^2 \sin\theta\,\mathrm{d}\phi\,\mathrm{d}\theta$$

Since $\int_0^\pi \sin^3\theta\,\mathrm{d}\theta = \int_0^\pi (1-\cos^2\theta)\sin\theta\,\mathrm{d}\theta = \frac{4}{3}$ we get

$$P = \frac{4\pi}{3\eta_0 k^2} E_0^2$$

f) Since $\boldsymbol{E}(r,\theta)$ only depends on r and θ, and is directed in the $\hat{\boldsymbol{\theta}}$ direction $\nabla^2\boldsymbol{E}(r,\theta)$ is reduced to, see formula in appendix 2 in the end of the book,

$$\nabla^2\boldsymbol{E}(r,\theta) = -\hat{\boldsymbol{r}}\left(\frac{2}{r^2}\frac{\partial E_\theta}{\partial\theta} + \frac{2\cot\theta}{r^2}E_\theta\right) + \hat{\boldsymbol{\theta}}\left(\nabla^2 E_\theta - \frac{E_\theta}{r^2\sin^2\theta}\right) \tag{2}$$

where

$$\nabla^2 E_\theta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\theta}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial E_\theta}{\partial\theta}\right) \tag{3}$$

It is straightforward to see that Helmholtz equation is not satisfied since there are terms that drop off as r^{-2} and r^{-3} that do not cancel.

g) Yes. The only term of $\mathcal{O}(r^{-1})$ remaining in $\nabla^2\boldsymbol{E}$ is $\hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial^2}{\partial r^2}(rE_\theta)$ and this term equals $-k^2\boldsymbol{E}$.

S1.4

a) We use the induction law but make the approximation that $\frac{\partial\boldsymbol{B}}{\partial t} \approx \boldsymbol{0}$. Then $\nabla\cdot\boldsymbol{E} = \boldsymbol{0}$ and $\boldsymbol{E} = -\nabla V$. From that we conclude that

$$\boldsymbol{E} = -\frac{V\cos\omega t}{d}\hat{\boldsymbol{z}}$$

b) We use Ampères law for this. $\nabla\times\boldsymbol{H} = \varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}$. Since the capacitor is axially symmetric it must be that $\boldsymbol{H}(r_c.t) = H(r_c,t)\hat{\boldsymbol{\phi}}$. Stoke's theorem then give

$$\boldsymbol{B}(r_c,t) = \varepsilon_0\mu_0\frac{V_0 r_c\omega\sin(\omega t)}{2d}\hat{\boldsymbol{\phi}}$$

c) The induction law is not satisfied since $\nabla\times\boldsymbol{E} = \boldsymbol{0}$ and $\frac{\partial\boldsymbol{B}}{\partial t} \neq \boldsymbol{0}$.

d) The approximation is that we use $\nabla\cdot\boldsymbol{E} = \boldsymbol{0}$ when we calculate $\boldsymbol{E}$. However, when the radius of the capacitor is much smaller than the wavelength $\lambda = \omega/c$, then it is a good approximation.

e) Axially symmetry says that $\boldsymbol{E} = E(r_c,t)\hat{\boldsymbol{\phi}}$. The induction law gives

$$\boldsymbol{E}(r_c,t) = -\frac{\omega}{2}B_0 r_c\cos(\omega t)\hat{\boldsymbol{\phi}}$$

f) No. The Ampères law is not satisfied since $\nabla \times \boldsymbol{H} = \mathbf{0}$ but $\dfrac{\partial \boldsymbol{E}}{\partial t} \neq \mathbf{0}$.

g) We started with a magnetic field that is constant in space. This is an approximation since $\boldsymbol{B}$ should satisfy $\nabla^2 \boldsymbol{B} + k^2 \boldsymbol{B} = \mathbf{0}$. However, when the radius of the core is much smaller than the wavelength $\lambda = \omega/c$, then it is a good approximation.

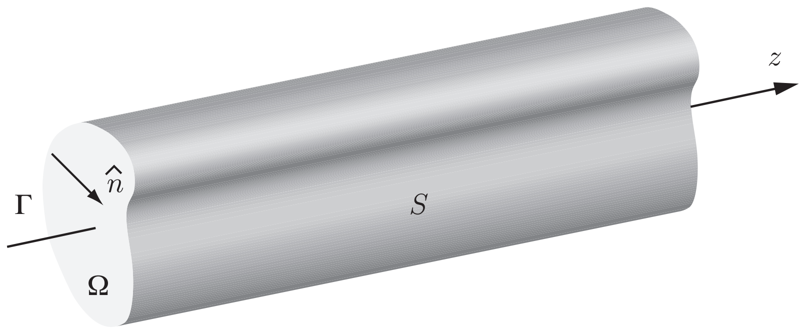


Figure 2: Geometry for waveguide

2 Waveguides and cavities

There are different types of waveguides for electromagnetic waves. Transmission lines, hollow waveguides and dielectric waveguides (e.g. optical fibers), are the most common ones, but here we only consider hollow waveguides. These are metal tubes where the waves propagate by bouncing between the walls. We first present basic results for hollow waveguides with arbitrary cross sections. The derivations of these results are given in the book Microwave theory which can be downloaded from the home page.

Waveguides are structures that guide waves along a given direction. Figure 2 gives an example of the geometry for a waveguide. The surface of the waveguide is denoted S and the normal to the surface $\hat{\boldsymbol{n}}$. The surface is considered to be perfectly conducting and we assume that there is air or vacuum inside the waveguide ($\varepsilon_r = 1$). Note that the normal $\hat{\boldsymbol{n}}$ is a function of the coordinates x and y, but not of the coordinate z. The cross section of the waveguide is denoted Ω and it has the generating curve Γ. The analysis in this chapter is valid for waveguides with general cross section.

Specific z-dependence of the fields

The Maxwell equations in vacuum lead to the vector Helmholtz equations for $\boldsymbol{E}$ and $\boldsymbol{H}$,

$$\nabla^2 \boldsymbol{E}(\boldsymbol{r}) + k^2 \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0} \tag{4}$$
$$\nabla^2 \boldsymbol{H}(\boldsymbol{r}) + k^2 \boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{0}. \tag{5}$$

Decomposition

From now on we let the $z-$axis be parallel to the guiding structures. Then

$$\boldsymbol{r} = \boldsymbol{\rho} + \hat{\boldsymbol{z}} z$$

Also

$$\nabla^2 = \nabla_{\mathrm{T}}^2 + \frac{\partial^2}{\partial z^2}$$

where $\nabla_{\mathrm{T}}^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

In a waveguide the complex electromagnetic fields can be decomposed into a transverse and a longitudinal vector as

$$\boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{E}_{\mathrm{T}}(\boldsymbol{r}) + \hat{\boldsymbol{z}} E_z(\boldsymbol{r}) \tag{6}$$
$$\boldsymbol{H}(\boldsymbol{r}) = \boldsymbol{H}_{\mathrm{T}}(\boldsymbol{r}) + \hat{\boldsymbol{z}} H_z(\boldsymbol{r}) \tag{7}$$

Boundary conditions

The sufficient boundary conditions on a perfectly conducting surface are,

$$\begin{cases} \hat{\boldsymbol{n}} \times \boldsymbol{E}(\boldsymbol{r}) = \boldsymbol{0} \\ \hat{\boldsymbol{n}} \cdot \boldsymbol{H}(\boldsymbol{r}) = 0 \end{cases} \qquad \boldsymbol{r} \text{ on } S \tag{8}$$

since $\boldsymbol{B} = \mu_0 \mu \boldsymbol{H}$ for an isotropic material.

On the surface S these boundary conditions reduce to

$$\begin{cases} E_z(\boldsymbol{r}) = 0 \\ \dfrac{\partial H_z(\boldsymbol{r})}{\partial n} = 0 \end{cases} \qquad \boldsymbol{r} \text{ on } S \tag{9}$$

where $\dfrac{\partial H_z(\boldsymbol{r})}{\partial n} = \hat{\boldsymbol{n}} \cdot \nabla_{\mathrm{T}} H_z(\boldsymbol{r})$. The two boundary conditions are sufficient for determining the waves that can exist in a hollow waveguide.

TM- and TE-modes

In this section we solve the Maxwell equations in a waveguide with general cross-section Ω and perfectly conducting walls S. The conditions in (9) separate the z-component of the electric field, E_z, from the $z-$component of the magnetic field, H_z. We look for solutions where either E_z *or* H_z is zero, *ie.*

$$\begin{cases} H_z(\boldsymbol{r}) = 0 \qquad \text{(TM-case)} \\ E_z(\boldsymbol{r}) = 0 \qquad \text{(TE-case).} \end{cases}$$

The first case is the transverse magnetic case (TM-case), where the magnetic field lacks $z-$component. The other case is the transverse electric case (TE-case). The

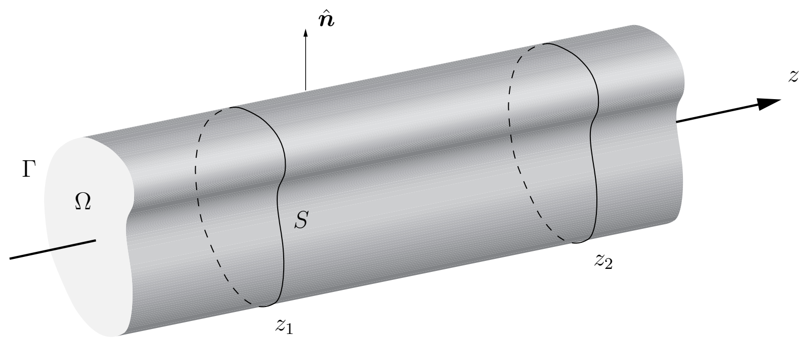


Figure 3: The source free region in the waveguide.

solutions to the two cases do not couple since there is no coupling via the differential equations or the boundary conditions. We will later also discuss the conditions that have to be satisfied in order to obtain waves with *both* E_z and H_z zero.

We let the region $z_1 < z < z_2$ be source free, *ie.* $\boldsymbol{J} = \mathbf{0}$, see figure 3 and determine the waves that can exist in this region.

We first describe our strategy for finding general solutions. The waveguide is assumed to be filled with an isotropic, homogeneous material with material parameters ϵ and μ. The $z-$components of the equations (4) and (5), and the boundary conditions for $E_z(\boldsymbol{r})$ and $H_z(\boldsymbol{r})$ are summarized as

$$\begin{cases} \nabla^2 E_z(\boldsymbol{r}) + k^2 E_z(\boldsymbol{r}) = 0 \\ E_z(\boldsymbol{r}) = 0 \qquad \boldsymbol{r} \text{ on } S \end{cases} \qquad z \in [z_1, z_2], \boldsymbol{\rho} \in \Omega \text{ (TM-case)}$$
$$\begin{cases} \nabla^2 H_z(\boldsymbol{r}) + k^2 H_z(\boldsymbol{r}) = 0 \\ \dfrac{\partial H_z}{\partial n}(\boldsymbol{r}) = 0 \qquad \boldsymbol{r} \text{ on } S \end{cases} \qquad z \in [z_1, z_2], \boldsymbol{\rho} \in \Omega \text{ (TE-case)} \tag{10}$$

where the wave number is

$$k^2 = \frac{\omega^2}{c^2}$$

On page 11 it shown that we can determine the entire vector field $\boldsymbol{E}$ and $\boldsymbol{H}$ from (15) if we know E_z and H_z.

We use the method of separation of variables to solve the two boundary value problems in (10). The method is frequently used in mathematical physics and in our case it leads to a complete set of functions in the transverse coordinates x and y. The $z-$component of the electric (TM-case) or magnetic field (TE-case) is expanded in this system. The other components follow from the relations between the transverse and longitudinal components.

TEM-modes

If Ω is not simply connected TEM-modes can exist. These are modes with both $E_z = 0$ and $H_z = 0$. A waveguide with N surfaces can have $N-1$ TEM-modes. The most common type of a waveguide with a TEM-mode is the coaxial cable. One can treat TEM-modes by solving the Maxwell equations, but it is easier to use transmission line theory, where wave propagation is expressed in terms of currents and voltages, rather than electric and magnetic fields. The transmission line theory can be found in the book *Microwave theory*. We do not treat TEM-modes here.

The longitudinal components of the fields

We make the following ansatz

$$\begin{cases} E_z(\boldsymbol{r}) = v(\boldsymbol{\rho})e^{\mathrm{i}k_z z}, & \text{(TM-case)} \\ H_z(\boldsymbol{r}) = w(\boldsymbol{\rho})e^{\mathrm{i}k_z z}, & \text{(TE-case)} \end{cases}$$

where $\boldsymbol{\rho} = \hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}$.

We identify the following two eigenvalue problems for the hollow waveguide

$$\begin{cases} \nabla_{\mathrm{T}}^2 v(\boldsymbol{\rho}) + k_t^2 v(\boldsymbol{\rho}) = 0 \\ v(\boldsymbol{\rho}) = 0 \qquad \boldsymbol{\rho} \text{ on } \Gamma \end{cases} \qquad \text{(TM-case)} \tag{11}$$

and

$$\begin{cases} \nabla_{\mathrm{T}}^2 w(\boldsymbol{\rho}) + k_t^2 w(\boldsymbol{\rho}) = 0 \\ \dfrac{\partial w}{\partial n}(\boldsymbol{\rho}) = 0 \qquad \boldsymbol{\rho} \text{ on } \Gamma \end{cases} \qquad \text{(TE-case)} \tag{12}$$

where

$$k_t^2 = k^2 - k_z^2. \tag{13}$$

There are only non-trivial solutions v, or w, for discrete values of k_t^2. We call these values eigenvalues and the corresponding solutions eigenfunctions.

Here are some properties of the eigenvalues and eigenfunctions:

- All eigenvalues are positive and can be numbered such that $0 < k_{t1}^2 \leq k_{t2}^2 \leq k_{t3}^2 \leq \ldots$, where $k_{tn}^2 \to \infty$ as $n \to \infty$.
- There is only a finite number of eigenvalues that have the same values.
- We always let the eigenfunctions $v_n(\boldsymbol{\rho})$ and $w_n(\boldsymbol{\rho})$ be *real valued*

- The eigenfunctions v_n and v_m, or w_n and w_m, that belong to different eigenvalues k_{tn}^2 and k_{tm}^2 in the TM- and TE-cases are orthogonal on Ω.
- Each of the sets of eigenfunctions, $\{v_n(\boldsymbol{\rho})\}_{n=1}^{\infty}$ and $\{w_n(\boldsymbol{\rho})\}_{n=1}^{\infty}$, constitutes a complete set of functions in the plane.

The set of functions $v_n(\boldsymbol{\rho})$ is orthogonal and normalized (also called orthonormalized) if

$$\int_{\Omega} v_n(\boldsymbol{\rho}) v_m(\boldsymbol{\rho})\, \mathrm{d}S = \delta_{nm} \tag{14}$$

where δ_{nm} is the Kronecker delta.[1]

The transverse components $\boldsymbol{E}_\mathrm{T}$ and $\boldsymbol{H}_\mathrm{T}$

In Chapter 4 of the book *Microwave theory* it is shown that the transverse components of $\boldsymbol{E}$ and $\boldsymbol{H}$ can be expressed in E_z and H_z:

$$\begin{cases} \boldsymbol{E}_\mathrm{T}(\boldsymbol{r}) = \dfrac{\mathrm{i}}{k_t^2}\left\{k_z \nabla_T E_z(\boldsymbol{r}) - \omega\mu_0 \hat{\boldsymbol{z}} \times \nabla_\mathrm{T} H_z(\boldsymbol{r})\right\} \\ \boldsymbol{H}_\mathrm{T}(\boldsymbol{r}) = \dfrac{\mathrm{i}}{k_t^2}\left\{k_z \nabla_\mathrm{T} H_z(\boldsymbol{r}) + \omega\varepsilon_0 \hat{\boldsymbol{z}} \times \nabla_\mathrm{T} E_z(\boldsymbol{r})\right\} \end{cases} \tag{15}$$

This for a mode propagating in the positive $z-$direction. A mode that propagates in the negative $z-$direction has $z-$dependence $e^{-\mathrm{i}k_z z}$ and then all k_z in (15) change sign.

Waveguide modes

The electromagnetic field that corresponds to a certain eigenwavenumber k_{tn}^2 is called a waveguide mode. The modes for the TE-case are called TE-modes and the ones for TM-modes are called TM-modes.

Cut-off frequency, phase speed and group speed

We define the cut-off frequency as

$$f_c = \frac{c}{2\pi} k_t \tag{16}$$

Consider a mode with eigenwavenumber k_{tn}^2. Since $k_{zn}^2 = k^2 - k_{tn}^2$ we have three cases:

[1] $\delta_{nm} = 1$ if $n = m$ and 0 otherwise

- $k < k_t$ For these frequencies k_z is imaginary. The mode attenuates (decays exponentially) and we say that it is a non-propagating mode. The frequency is below the cut-off frequency $f_c = \dfrac{c}{2\pi}k_t$.
- $k = k_t$ Then $k_z = 0$. The frequency is equal to the cut-off frequency $f_c = \dfrac{c}{2\pi}k_t$.
- $k > k_t$ For these frequencies k_z is real. The frequency is above the cut-off frequency $f_c = \dfrac{c}{2\pi}k_t$. The mode propagates as a wave without attenuation.

For a propagating TM-mode we have $E_z(\boldsymbol{r}) = v_n(\boldsymbol{\rho})e^{\mathrm{i}k_{zn}z}$ and then the corresponding time domain component is $E_z(\boldsymbol{r},t) = \mathrm{Re}\{E_z(\boldsymbol{r})e^{\mathrm{i}\omega t}\}$. this gives

$$E_z(\boldsymbol{r},t) = v_n(\rho)\cos(\omega t - k_z z) \tag{17}$$

The phase $\omega t - k_z z$ is constant when $z = \dfrac{\omega}{k_z}t$. It means that the phase travels with the phase speed

$$v_p = \frac{\omega}{k_z}. \tag{18}$$

Notice that this speed is always larger than the speed of light and goes to infinity as $f \to f_c$. There is no contradiction with special relativity since one can show that the power travels with the group speed, which is given by

$$v_g = \frac{k_z}{k}c. \tag{19}$$

This speed is always lower than the speed of light and goes to zero when $f \to f_c$.

We now give examples of important cross-sections for which we can derive explicit expressions of the vector basis functions.

Waveguide with rectangular cross-section

We start with the eigenfunctions for the rectangular waveguide. This is the most common type of hollow waveguide. The geometry is depicted in figure 4. The surface is simply connected and hence no TEM-mode exists. The convention is to let the longest side of the rectangle be along the x-axis.

The eigenvalues that are to be solved are

$$\begin{cases} \dfrac{\partial^2 v(\boldsymbol{\rho})}{\partial x^2} + \dfrac{\partial^2 v(\boldsymbol{\rho})}{\partial y^2} + k_t^2 v(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \text{ in } \Omega \\ v(\boldsymbol{\rho}) = 0, & \boldsymbol{\rho} \text{ on } \Gamma \end{cases} \qquad \text{(TM-case)}$$

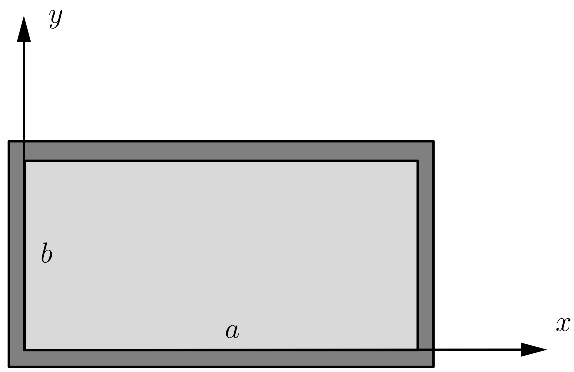


Figure 4: The geometry for a waveguide with rectangular cross-section.

and

$$\begin{cases} \dfrac{\partial^2 w(\boldsymbol{\rho})}{\partial x^2} + \dfrac{\partial^2 w(\boldsymbol{\rho})}{\partial y^2} + k_t^2 w(\boldsymbol{\rho}) = 0 \qquad \boldsymbol{\rho}, \text{ in } \Omega \\ \dfrac{\partial w}{\partial n}(\boldsymbol{\rho}) = 0 \qquad \boldsymbol{\rho}, \text{ on } \Gamma \end{cases} \qquad \text{(TE-case)}.$$

The solution is based on the following one-dimensional eigenvalue problems:

$$\begin{cases} \dfrac{\partial^2 X(x)}{\partial x^2} + \gamma X(x) = 0, \quad 0 \le x \le a \\ X(x) = 0, \quad x = 0, a \end{cases}$$

and

$$\begin{cases} \dfrac{\partial^2 \tilde{X}(x)}{\partial x^2} + \gamma \tilde{X}(x) = 0, \quad 0 \le x \le a \\ \dfrac{d\tilde{X}}{dx}(x) = 0, \quad x = 0, a. \end{cases}$$

The solutions to these two problems are

$$X_m(x) = \sin\left(\frac{m\pi x}{a}\right), \quad m = 1, 2, 3, \dots$$

and

$$\tilde{X}_m(x) = \cos\left(\frac{m\pi x}{a}\right), \quad m = 0, 1, 2, 3, \dots,$$

respectively. These sets of functions are orthogonal and complete on the interval $x \in [0, a]$. The solution to the two-dimensional eigenvalue problems for the rectangular waveguide are obtained as a product of these sets of one-dimensional eigenfunctions[2],

[2] A common method to create complete sets of functions in two dimensions is to take the product of one-dimensional systems, *ie.* if $\{f_m(x)\}_{m=1}^{\infty}$ and $\{g_n(y)\}_{n=1}^{\infty}$ are complete systems on the intervals $x \in [a, b]$ and $y \in [c, d]$, respectively, then

$$\{f_m(x)g_n(y)\}_{m,n=1}^{\infty}$$

is a complete set of functions in the rectangle $[a, b] \times [c, d]$.

	Eigenfunctions v_{mn}, w_{mn}	Eigenvalues $k_{t\,mn}^2$
TM_{mn}	$v_{mn} = \frac{2}{\sqrt{ab}} \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi y}{b}\right)$	$\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$
TE_{mn}	$w_{mn} = \sqrt{\frac{\varepsilon_m \varepsilon_n}{ab}} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$	$\pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)$

Table 1: Table of normalized eigenfunctions to equations (11) and (12) for rectangular waveguides, see figure 4. The integers m and n can have values $m, n = 0, 1, 2, 3, \ldots$, with the exception that m and n are not zero for TM-modes, and m and n cannot **both** be zero for the TE-modes ($\varepsilon_m = 2 - \delta_{m,0}$). The convention in this book is always to have the long side of the rectangle along the x-axis, *ie.* $a > b$. The mode with the lowest cut-off frequency is then the TE_{10} mode. This mode is called the fundamental mode and is very important.

ie.

$$\begin{cases} \sin\left(\dfrac{m\pi x}{a}\right) \sin\left(\dfrac{n\pi y}{b}\right), & \text{TM-case} \\[2ex] \cos\left(\dfrac{m\pi x}{a}\right) \cos\left(\dfrac{n\pi y}{b}\right), & \text{TE-case.} \end{cases}$$

The eigenvalues in the two cases are the same $k_t^2 = \pi^2 \left(m^2/a^2 + n^2/b^2\right)$. The normalized functions are

$$\begin{cases} v_{mn} = \dfrac{2}{\sqrt{ab}} \sin\left(\dfrac{m\pi x}{a}\right) \sin\left(\dfrac{n\pi y}{b}\right), & \text{TM-case} \\[2ex] w_{mn} = \sqrt{\dfrac{\varepsilon_m \varepsilon_n}{ab}} \cos\left(\dfrac{m\pi x}{a}\right) \cos\left(\dfrac{n\pi y}{b}\right), & \text{TE-case} \end{cases}$$

where the Neumann-factor is $\varepsilon_m = 2 - \delta_{m,0}$. The results are collected in table 1.

Example

The fundamental mode of a rectangular waveguide with $a > b$ is the TE_{10} mode. It has the cut-off frequency $f_{c_{10}} = \dfrac{c}{2a}$ and $w_{10} = \sqrt{\dfrac{2}{ab}} \cos\left(\dfrac{\pi x}{a}\right)$. The normalized electric field is

$$\boldsymbol{E}_{10\text{TE}}(x, \omega) = \hat{\boldsymbol{y}} \frac{\mathrm{i}\omega\mu_0}{\pi} \sqrt{\frac{2a}{b}} \sin\left(\frac{\pi x}{a}\right). \tag{20}$$

If $a > 2b$ then the second mode is TE_{20} that has cut-off frequency $f_{c_{20}} = \dfrac{c}{a}$. If $b < a < 2b$ then TE_{01} is the second mode with cut-off frequency $f_{c_{01}} = \dfrac{c}{2b}$. In

m	1	2	0	1	2	3	3	4	0	1
n	0	0	1	1	1	0	1	0	2	2
$f_{c_{mn}}$ (GHz)	3.19	6.38	6.81	7.52	9.33	9.57	11.7	12.8	13.6	14.0
$k_{z_{mn}}$ (m^{-1})[a]	43.3	$107i$	$119i$	$136i$	$179i$	$184i$	$233i$	$255i$	$274i$	$282i$
$k_{z_{mn}}$ (m^{-1})[b]	144.6	86.6	70.6	22.6	$114i$	$122i$	$188i$	$215i$	$237i$	$246i$

[a]The frequency is $f = 3.8$ GHz.
[b]The frequency is $f = 7.6$ GHz.

Table 2: Table of the lowest cut-off frequencies $f_{c_{mn}}$ and the longitudinal wavenumber $k_{z_{mn}}$ for a rectangular waveguide with dimensions 4.7 cm $\times$ 2.2 cm. Only TE-modes can have m- or n-values that are zero. For frequencies below the cut-off frequency the longitudinal wavenumber $k_{z_{mn}}$ is imaginary and the corresponding mode is non-propagating. The attenuation of that mode is $\exp(-\mathrm{Im}\{k_{z_{mn}}\}z)$.

order to maximize the bandwidth it is common to have rectangular waveguides with $a > 2b$. Then the bandwidth is $\mathrm{BW} = \dfrac{c}{2a}$ and the fractional bandwidth is $b_{\mathrm{f}} = 2(c/2a)/(3c/2a) = 2/3 = 0.67$.

Example

A rectangular waveguide has dimensions 4.7 cm $\times$ 2.2 cm. The cut-off frequencies $f_{c_{mn}}$ for the different modes are easy to calculate from (16) and table 1. The longitudinal wavenumbers $k_{z_{mn}}$, given by (13), are related to the frequency f and the cut-off frequency $f_{c_{mn}}$ in the following way

$$k_{z_{mn}} = \frac{2\pi}{c}\sqrt{f^2 - f_{c_{mn}}^2}.$$

The results are given in table 2. The bandwidth is BW = 3.19 Ghz and the fractional bandwidth is $b_{\mathrm{f}} = 1$.

Waveguide with circular cross-section

The geometry of the circular waveguide with radius a is depicted in figure 5. The geometry has only one simply connected surface and hence there is no TEM-mode. It is best to solve the eigenvalue problem in cylindrical-(polar)coordinates. The eigenvalue problems are given by

$$\begin{cases} \nabla_{\mathrm{T}}^2 v(\boldsymbol{\rho}) + k_t^2 v(\boldsymbol{\rho}) = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial v(\boldsymbol{\rho})}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 v(\boldsymbol{\rho})}{\partial\phi^2} + k_t^2 v(\boldsymbol{\rho}) = 0 \\ v(a,\phi) = 0 \end{cases} \qquad \text{(TM-case)}$$

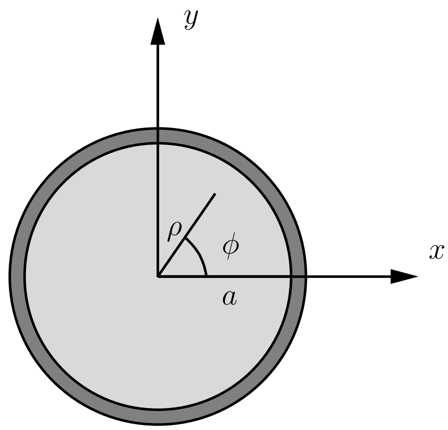


Figure 5: Geometry for waveguide with circular cross-section.

and

$$\begin{cases} \nabla_{\mathrm{T}}^2 w(\boldsymbol{\rho}) + k_t^2 w(\boldsymbol{\rho}) = \dfrac{1}{\rho}\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial w(\boldsymbol{\rho})}{\partial\rho}\right) + \dfrac{1}{\rho^2}\dfrac{\partial^2 w(\boldsymbol{\rho})}{\partial\phi^2} + k_t^2 w(\boldsymbol{\rho}) = 0 \\ \dfrac{\partial w}{\partial n}(a,\phi) = 0 \end{cases} \qquad \text{(TE-case)}.$$

We solve these eigenvalue problems by the method of separation of variables. We make the ansatz $v(\rho,\phi) = f(\rho)g(\phi)$ and insert this into the differential equation. After division with $f(\rho)g(\phi)/\rho^2$ we get

$$\frac{\rho}{f(\rho)}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f(\rho)}{\partial\rho}\right) + k_t^2\rho^2 = -\frac{1}{g(\phi)}\frac{\partial^2 g(\partial\phi^2)}{\phi}.$$

The right hand side depends only on ϕ and the left hand side depends only on ρ. That means that they both have to be equal to a constant. We denote this constant γ and get

$$\begin{cases} \rho\dfrac{\partial}{\partial\rho}\left(\rho\dfrac{\partial f(\rho)}{\partial\rho}\right) + \left(k_t^2\rho^2 - \gamma\right) f(\rho) = 0 \\ \dfrac{\partial^2 g(\phi)}{\partial\phi^2} + \gamma g(\phi) = 0. \end{cases}.$$

The solution to the eigenvalue problem in the variable ϕ is

$$g(\phi) = \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}, \qquad m = 0, 1, 2, 3, \ldots$$

Only integer values of m are allowed since the function must be periodic in ϕ with period 2π, *ie.* only $\gamma = m^2$, $m = 0, 1, 2, 3, \ldots$ are possible values. The corresponding set of functions is complete on the interval $\phi \in [0, 2\pi)$. The solution to the equation

	Eigenfunctions v_{mn}, w_{mn}	Eigenvalues $k_{t\,mn}^2$
TM_{mn}	$v_{mn} = \dfrac{\sqrt{\varepsilon_m} J_m(\xi_{mn}\rho/a)}{\sqrt{\pi} a J_m'(\xi_{mn})} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$	$\dfrac{\xi_{mn}^2}{a^2}$
TE_{mn}	$w_{mn} = \dfrac{\sqrt{\varepsilon_m}\eta_{mn} J_m(\eta_{mn}\rho/a)}{\sqrt{\pi\left(\eta_{mn}^2 - m^2\right)} a J_m(\eta_{mn})} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}$	$\dfrac{\eta_{mn}^2}{a^2}$

Table 3: Table of the normalized eigenfunctions for waveguides with circular cross-section, see figure 5 for definition of geometry. ($\varepsilon_m = 2 - \delta_{m,0}$). The first values of the positive zeros ξ_{mn} to $J_m(x)$ and the positive zeros η_{mn} to $J_m'(x)$, *ie.* $J_m(\xi_{mn}) = 0$ and $J_m'(\eta_{mn}) = 0$, $m = 0, 1, 2, 3, \ldots$, $n = 1, 2, 3, \ldots$ are listed in tables 4 and 5 on page . The mode with the lowest cut-off frequency is the TE_{11} mode.

in the ρ-variable is a Bessel function, see section 8. Only solutions that are regular in $\rho = 0$ are valid, *ie.*

$$f(\rho) = J_m(k_t \rho).$$

The boundary conditions $v_m(a,\phi) = 0$ and $\dfrac{dw_m}{d\rho}(a,\phi) = 0$ for the TM- and TE-cases, respectively, add extra conditions. For these boundary conditions to be satisfied, the transverse wavenumber has to satisfy

$$k_t a = \begin{cases} \xi_{mn}, & \text{(TM-case)} \\ \eta_{mn}, & \text{(TE-case)}, \end{cases}$$

where ξ_{mn} and η_{mn}, $n = 1, 2, 3, \ldots$, are zeros to the Bessel function $J_m(x)$ and to the derivative of the Bessel function, respectively, *ie.* $J_m(\xi_{mn}) = 0$ and $J_m'(\eta_{mn}) = 0$. Numerical values of the first of these zeros are given in appendix 8.

The sets of functions $\{J_m(\xi_{mn}\rho/a)\}_{n=1}^\infty$, $\{J_m'(\eta_{mn}\rho/a)\}_{n=1}^\infty$ are both complete on the interval $\rho \in [0, a]$ for every value of m. The complete set of functions in the circle is, in analogy with the rectangular waveguide, given by the product of the sets of basis functions. The normalized eigenfunctions (the normalization integrals are given in appendix 8) are

$$\begin{cases} v_{mn} = \dfrac{\sqrt{\varepsilon_m} J_m(\xi_{mn}\rho/a)}{\sqrt{\pi} a J_m'(\xi_{mn})} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}, & \text{TM-case} \\ w_{mn} = \dfrac{\sqrt{\varepsilon_m}\eta_{mn} J_m(\eta_{mn}\rho/a)}{\sqrt{\pi\left(\eta_{mn}^2 - m^2\right)} a J_m(\eta_{mn})} \begin{pmatrix} \cos m\phi \\ \sin m\phi \end{pmatrix}, & \text{TE-case,} \end{cases} \tag{21}$$

where $\varepsilon_m = 2 - \delta_{m,0}$. The results are collected in table 3.

Example

The fundamental mode is the TE_{11} mode. The cut-off frequency is given by $f_{c_{11}} = \frac{c\eta_{11}}{2\pi a}$ where a is the radius of the cylinder and $\eta_{11} = 1.841$ is the first zero of $J_1'(x)$. The second mode is the TM_{01} mode with cut-off frequency $f_{c_{01}} = \frac{c\xi_{01}}{2\pi a}$ where $\xi_{01} = 2.405$ is the first zero of $J_0(x)$. The bandwidth is $\text{BW} = f_{c_{01}} - f_{c_{11}} = \frac{c}{2\pi a}(2.405 - 1.841) = 0.564\dfrac{c}{2\pi a}$. The fractional bandwidth is $b_\text{f} = 2\frac{0.564}{1.841+2.405} = 0.265$.

Analyzing waveguides with Comsol Multiphysics

Waveguides with arbitrary cross-sections can be analyzed with numerical methods and in this book we use the finite element method. The specific calculations are done with the commercial software package Comsol Multiphysics. We use Comsol to find the cut-off frequencies for the TE- and TM-modes in a hollow waveguide filled with a homogenous non-conducting material with permittivity ϵ. We also let Comsol determine the electric and magnetic fields and the power flow density for the lowest modes. In Comsol we do the following steps:

- We choose **2D**> **Radio frequency**> **Electromagnetic waves**> **Eigenfrequency study**.
- We draw the cross section of the waveguide.
- In **Study**>**Eigenfrequency** we define how many modes that are to be determined and the cut-off frequency where Comsol starts to look for eigenfrequencies.
- We let Comsol solve the eigenvalue problem. It then shows the electric field in the cross section of the waveguide for the different modes. It also gives the cut-off frequencies f_c for the modes. From the cut-off frequencies we get the corresponding k_t from $k_t = \omega/c = 2\pi f_c/c$. There are spurious solutions with very low frequencies, or complex frequencies that Comsol presents. These have a fuzzy field plot.
- To distinguish TE- from TM-modes we plot the $z-$component of the electric field. If the plot is fuzzy with very small field values then the mode is a TE-mode, otherwise it is a TM-mode.
- The fields that Comsol presents are not normalized. We use a normalization such that $\iint_\Omega |E_z(\boldsymbol{\rho})|^2\,\mathrm{d}x\mathrm{d}y = 1$ for the TM-modes and $\iint_\Omega |H_z(\boldsymbol{\rho})|^2\,\mathrm{d}x\mathrm{d}y = 1$ for the TE-modes. To obtain this normalization we divide all field values with $\iint_\Omega |E_z(\boldsymbol{\rho})|^2\,\mathrm{d}x\mathrm{d}y$ for the TM-modes and $\iint_\Omega |H_z(\boldsymbol{\rho})|^2\,\mathrm{d}x\mathrm{d}y$ for the TE-modes. To integrate we right click on **Derived values** and choose integration and surface integral.
- Notice that there are many options of surface graphs to choose from.

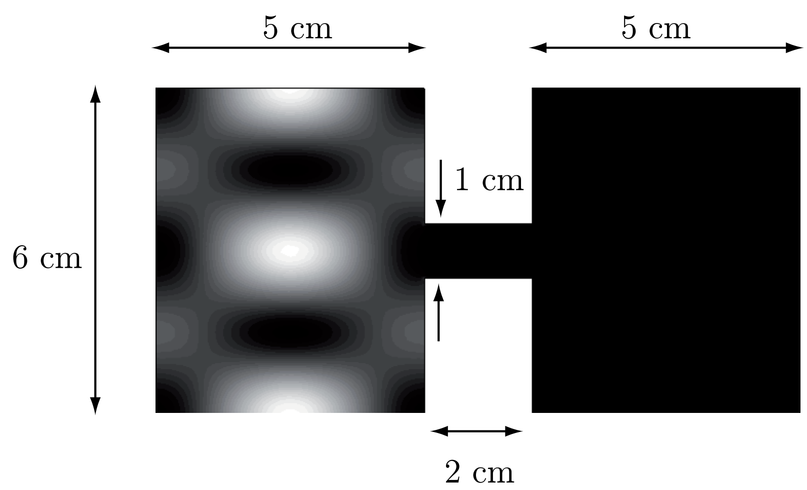


Figure 6: Geometry for the ridge waveguide and the power flow density for the TM-mode with cut-off frequency 5.83 GHz.

Example

We analyze the ridge waveguide in figure 6. This is a waveguide with a large bandwidth since the fundamental mode has a very low cut-off frequency. We use the scheme for Comsol to obtain the modes. The TM modes have cut-off frequencies $f_c = 3.88$ GHz, 3.88 GHz, 5.83 GHz, 5.83 GHz, 6.43 GHz, 6.43 GHz and the TE modes have cut-off frequencies $f_c = 0.663$ GHz, 2.51 GHz, 2.51 GHz, 2.86 GHz, 3.14 GHz. In figure 7 we see the active power flow density for one of the TM-mode with the cut-off frequency 5.83 GHz. The other mode with the same cut-off frequency has its power flow in the right part of the waveguide. This mode is very close to the TM_{12} mode in a 5 cm $\times$ 6 cm rectangular waveguide. The TM_{12} mode has cut-off frequency $f_c = 5.831$ GHz which is very close to the cut-off frequency 5.827 GHz obtained for the mode in figure 6. The fractional bandwidth is defined by $b_f = 2(f_{\text{upper}} - f_{\text{lower}})/(f_{\text{upper}} + f_{\text{lower}})$, and the ridge waveguide has $b_f = 1.16$. This is to be compared with $b_f = 0.67$ for the rectangular waveguide with $a > 2b$.

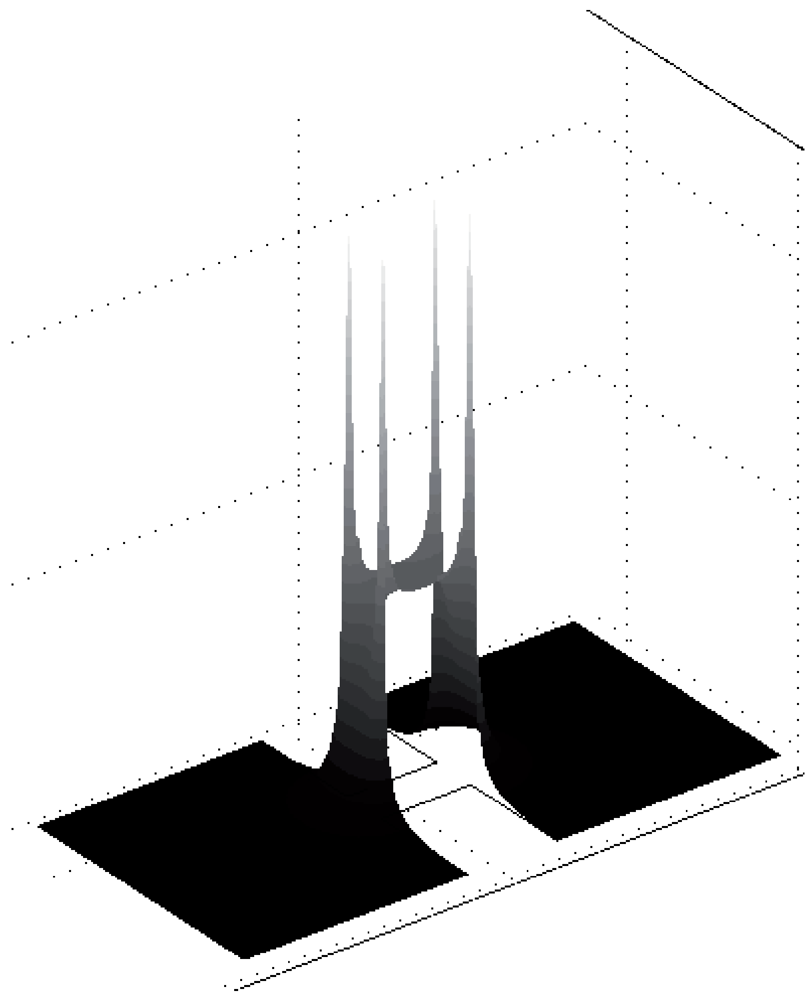


Figure 7: The power flow density for the fundamental mode of the ridge waveguide, *ie.* the TE-mode with cut-off frequency 0.663 GHz. The power flow is concentrated to the narrow section. Notice that the electric field is very strong at edges.

2.1

Assume a rectangular air filled waveguide with dimension $a \times b$, where $b < a$ and $a = 0.3$ m. Determine the largest b such that the fundamental mode is the only propagating mode in the interval $(f_0, 2f_0)$, where f_0 is the cut-off frequency for the fundamental mode.

2.2

Sketch the electric field in the $xy-$plane for the fundamental mode of a rectangular waveguide. Use vectors where the length of the vector indicates the field strength.

2.3

The electric field of a TE_{m0}-mode is $\boldsymbol{E}(x,z) = E_m \sin\left(\frac{m\pi x}{a}\right) e^{\mathrm{i}k_z z}\hat{\boldsymbol{y}}$. Determine $\boldsymbol{H}$ by using the induction law.

2.4

Sketch the magnetic field in the $xy-$plane for the fundamental mode of a rectangular waveguide.

2.5

Determine the surface current density on the surfaces $x = 0$ and $x = a$ for the fundamental mode of a rectangular waveguide.
Hint: The surface current density of a perfect conductor is given by $\boldsymbol{J}_S = \hat{\boldsymbol{n}} \times \boldsymbol{H}$, where $\hat{\boldsymbol{n}}$ is the outward directed unit normal vector to the surface.

2.6

Assume a rectangular airfilled waveguide with dimension $a \times b$, $a = 0.3$ m, $b = 0.15$ m. The wavelength in the z-direction is defined by $\lambda_z = \dfrac{2\pi}{k_z}$. Use $c = 3 \cdot 10^8$ m/s.

a) Determine k_z for $m = 1, 2, 3$ for the TE_{m0} modes when $f = 704$ MHz.

b) Determine λ_z for the TE_{10} mode at $f = 704$ MHz.

c) Determine λ_z for the TE_{10} mode at $f = 500$ MHz.

d) Let $f = 704$ MHz. Assume that the amplitude of the TE_{20} mode is 10 V/m at $z = 0$. Determine z such that the amplitude is 5 V/m.

e) How many of the TE_{mn} and TM_{mn} modes are propagating at 2 GHz?

2.7

The phase speed in the z-direction is defined by $v_f = \dfrac{\omega}{k_z}$ and the group speed by the derivative $v_g = \left(\dfrac{dk_z}{d\omega}\right)^{-1}$. Assume that the TE_{10}-mode propagates in a rectangular airfilled waveguide with dimension $a \times b$, $a = 0.3$ m, $b = 0.15$ m. Use $c = 3 \cdot 10^8$ m/s.

a) Determine v_f and v_g for $f = 704$ MHz.

b) Determine v_f and v_g for $f = 500$ MHz.

c) Determine v_f and v_g when $f \to \infty$ MHz.

d) Use Matlab to plot v_g as a function of frequency in the interval [500 MHz, 5 GHz].

2.8

A TE_{10} mode with $\boldsymbol{E} = E_0 \sin(\pi x/a) e^{ik_z z} \hat{\boldsymbol{y}}$ is propagating in the positive $z-$direction for $z < 0$. At $z = 0$ the waveguide is terminated by a perfectly conducting plate. Determine the total electric field in the waveguide.

2.9

Determine the three lowest cut-off frequencies for the waveguides described below by using the analytic formulas and confirm your solutions by determining the cut-off frequencies with COMSOL. Give the ten lowest cut-off frequencies obtained by COMSOL for each waveguide. Check the accuracy in the COMSOL solutions and give a rough estimate of the error.

a) A rectangular waveguide with $a = 8$ cm and $b = 3$ cm.

b) A circular waveguide with radius $R = 5$ cm.

c) A waveguide with a cross section as a half circle with radius $R = 5$ cm.

2.10

A waveguide has a cross section in the shape of a quarter circle with radius R. Determine expressions for the cut-off frequencies and E_z and H_z for all TE- and TM-modes for the waveguide. You don't have to normalize E_z and H_z.

2.11

Use COMSOL Multiphysics to determine the five lowest modes of the waveguide in the previous problem when the radius is 2 cm. Check the cut-off frequencies with the analytic expressions.

2.12

Use COMSOL Multiphysics to determine the five lowest modes of a waveguide with elliptic cross-section and where the ellipse has a semi-axis of 5 cm and 3 cm. Is there any of the modes that resembles the fundamental mode of a circular waveguide?

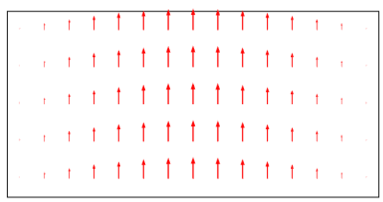
Waveguides and cavities: Answers and solutions

S2.1

$b = 0.15$ m.

S2.2

See figure



S2.3

$$\boldsymbol{H} = \frac{\mathrm{i}E_0}{\omega\mu_0}\left(\mathrm{i}k_z \sin\left(\frac{m\pi x}{a}\right), 0, -\frac{m\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\right) e^{\mathrm{i}k_z z}$$

S2.4

Same figure as in problem 2 but with all arrows rotated 90 degrees.

S2.5

$$\boldsymbol{J}_S \frac{\mathrm{i}E_0\pi}{\omega\mu_0 a} e^{\mathrm{i}k_z z}\boldsymbol{e}_y$$

S2.6

a) $m = 1$ gives $k_z = 10.3798$ m^{-1}, $m = 2$ gives $k_z = \mathrm{i}14.8744$ m^{-1} (for a wave in positive $z-$direction) and $m = 3$ gives $k_z = \mathrm{i}27.7409$ m^{-1}.

b) 0.605 m

c) ∞

d) $z = (-\mathrm{Im}(k_z))^{-1} \ln 2 = 4.66$ cm.

e) 7 TE-modes (10, 01, 20, 11, 30, 21, 31) and 3 TM-modes (11, 21, 31)

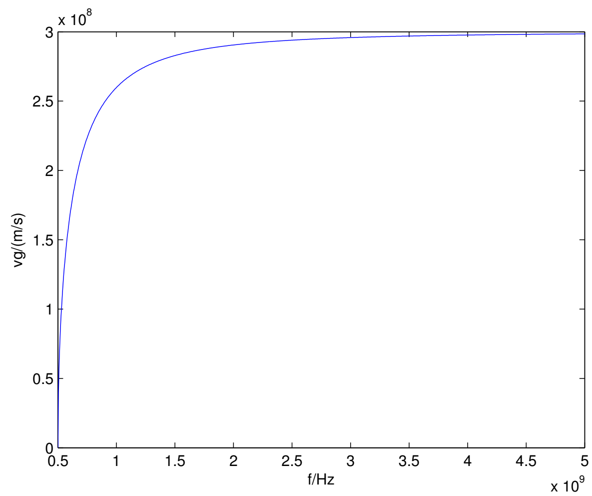
S2.7

a) $v_f = \dfrac{\omega}{k_z}$ and $v_g = \dfrac{k_z c}{k}$ gives $v_f = 4.26 \cdot 10^8$ m/s and $v_g = 2.11 \cdot 10^8$ m/s.

b) $v_f = \infty$ and $v_g = 0$.

c) $v_f = v_g = c = 3 \cdot 10^8$ m/s when $f \to \infty$.

d)



S2.8

$\boldsymbol{E}(x, z) = 2\mathrm{i}E_0 \sin(\pi x/a) \sin(k_z z)\boldsymbol{e}_y$

S2.9

a) The three lowest cut-off frequencies are $f_{TE_{10}} = 1.87370286$ GHZ, $f_{TE_{20}} = 3.74740573$ GHZ and $f_{TE_{01}} = 4.99654099$ GHz.

b) The three lowest cut-off frequencies are $f_{TE_{11}} = 1.75680928$ GHz, $f_{TM_{01}} = 2.29501702$ GHZ and $f_{TE_{21}} = 2.91433762$ GHz.

c) For the half-circle the modes are the same as for the full circle except that the TM_{0n} modes cannot exist. The boundary condition at the flat surfaces $\phi = 0$ and $\phi = \pi$ require that the E_z is zero there and that cannot be fullfilled by the TM_{0n} mode since E_z is then proportional to $\cos\phi$. Then the modes with the lowest cut-off frequencies are $f_{TE_{11}} = 1.75680928$ GHz, $f_{TE_{21}} = 2.91433762$ GHz and $f_{TE_{01}} = f_{TM_{11}} = 3.656478349513779$ GHz.

S2.10

TE-modes:

$$H_z = A_{nj} J_{2n}(\beta_{2nj}\rho)\cos(2n\phi)\exp(\mathrm{i}k_{z_{nj}}z)$$

where $k_{z_{nj}}^2 = (\omega/c_0)^2 - (k_{t_{2nj}})^2$ and $J'_{2n}(k_{t_{2nj}}R) = 0$

TM-modes:

$$E_z = B_{n,j} J_{2n}(k_{t_{2nj}}\rho)\sin(2n\phi)\exp(\mathrm{i}k_{z_{nj}}z)$$

where $k_{z_{nj}}^2 = (\omega/c_0)^2 - (k_{t_{2nj}})^2$ and $J_{2n}(k_{t_{2nj}}R) = 0$.

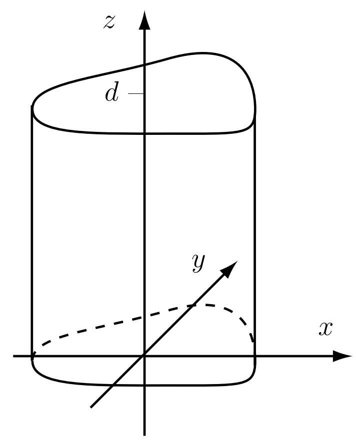


Figure 8: Geometry for cylindric resonance cavity.

3 Microwave cavities

A finite volume of air or vacuum, enclosed by a metallic surface, constitutes a resonance cavity. Only electromagnetic fields with certain frequencies can exist in the cavity. These fields are called cavity modes, or eigenmodes, and the corresponding frequencies are called eigenfrequencies, or resonance frequencies. In this chapter we describe how cavity modes and resonance frequencies can be obtained by analytical and numerical methods. Resonance cavities are frequently used as bandpass and bandstop filters in microwave systems. The losses are much smaller than in traditional bandpass filters based on circuit components and that makes the filters based on cavities very narrow banded. In modern particle accelerators the particles are accelerated by the electric fields in microwave cavities. Another important application is klystrons and magnetrons, which are generators for time-harmonic electromagnetic waves. Magnetrons are used in radars and also in microwave ovens. Klystrons are used as sources in, eg., particle accelerators and high power communication systems.

Cylindrical cavities

We analyze a common type of resonance cavity that consists of a hollow waveguide terminated by metallic plane surfaces at $z = 0$ and $z = d$, see figure 8. In order to determine the fields that can exist in such a cavity we need boundary conditions for the z-component of the electric and magnetic fields at $z = 0$ and $z = d$. Since $\boldsymbol{E}_T(\boldsymbol{\rho}, 0) = \boldsymbol{E}_T(\boldsymbol{\rho}, d) = \boldsymbol{0}$ for all $\boldsymbol{\rho}$, it follows that $\nabla_T \cdot \boldsymbol{E}_T(\boldsymbol{\rho}, 0) = \nabla_T \cdot \boldsymbol{E}_T(\boldsymbol{\rho}, d) = 0$. There are no charges inside the cavity and then $\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = 0$, *ie.* $0 = \nabla_T \boldsymbol{E}_T(\boldsymbol{r}) + \partial E_z(\boldsymbol{r})/\partial z$. It follows that the z-derivative of E_z is zero at the end surfaces. The magnetic field $\boldsymbol{H}$ is zero in the metal and $\boldsymbol{B}$ is always continuous and then it follows that H_z is zero at the end surfaces. We conclude that the boundary conditions at

$z = 0$ and $z = d$ are

$$\begin{cases} \dfrac{\partial E_z(x,y,0)}{\partial z} = \dfrac{\partial E_z(x,y,d)}{\partial z} = 0 \\ H_z(x,y,0) = H_z(x,y,d) = 0. \end{cases} \tag{22}$$

A cavity mode in a cylindrical cavity is a superposition of a waveguide mode propagating in the positive and negative z-directions. The z-component of the fields of mode n is expressed as

$$\begin{cases} E_z(\boldsymbol{r}) = (a^+_{n\nu} e^{\mathrm{i}k_z z} - a^-_{n\nu} e^{-\mathrm{i}k_z z}) v_n(\boldsymbol{\rho}) & \nu = \mathrm{TM} \\ H_z(\boldsymbol{r}) = (a^+_{n\nu} e^{\mathrm{i}k_z z} + a^-_{n\nu} e^{-\mathrm{i}k_z z}) w_n(\boldsymbol{\rho}) & \nu = \mathrm{TE}. \end{cases}$$

The boundary conditions give $a^+_{n\nu} = -a^-_{n\nu}$ and $\sin k_z d = 0$. Hence k_z can only take the discrete values

$$k_{z_\ell} = \frac{\ell\pi}{d} \quad \begin{cases} \ell = 0,\, 1,\, 2\ldots & \nu = \mathrm{TM} \\ \ell = 1,\, 2\ldots & \nu = \mathrm{TE}. \end{cases}$$

There exists no TE-mode with value $\ell = 0$ since then $H_z = 0$. The frequencies that can exist in the cavity are determined by $k^2 = k^2_{t_n} + k^2_{z_\ell}$ and thus

$$f_{n\ell} = \frac{c}{2\pi}\sqrt{k^2_{t_n} + \left(\frac{\ell\pi}{d}\right)^2}. \tag{23}$$

The transverse fields for the corresponding resonances follow from (15). For the wave traveling in the negative $z-$direction we need to change sign on k_z in these formulas.

The pill-box cavity

The pill-box cavity is a circular cylindric cavity. Almost all resonance cavities that are used for accelerating particles in acceerators are related to this cavity.

Let the cylinder have radius a and height d. We use the expressions for v_{mn} and W_{mn} given in (21). Then the TM-modes have the E_z field

$$E_{zmn\ell}(\rho,\phi,z) = A_{mn\ell} J_m(\xi_{mn}\rho/a)\cos(m\phi)\cos\left(\frac{\ell z}{d}\right),\ m = 0,1,\ldots,\ n = 1,2\ldots,\ \ell = 0,1\ldots \tag{24}$$

and the TE-mode has the H_z field

$$H_{zmn\ell}(\rho,\phi,z) = B_{mn\ell} J_m(\eta_{mn}\rho/a)\cos(m\phi)\sin\left(\frac{\ell z}{d}\right),\ m = 0,1,\ldots,\ n = 1,2\ldots,\ \ell = 1,2\ldots,$$

(25)

where $A_{mn\ell}$ and $B_{mn\ell}$ are amplitudes. Here ξ_{mn} is the n:th zero of $J_m(x)$ and η_{mn} the n:th zero of $J'_m(x)$. The first zeros are given in tables in appendix 1. The eigenfrequencies are given by

$$f_{mn\ell} = \begin{cases} \frac{c}{2\pi}\sqrt{\left(\frac{\xi_{mn}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2} & \text{TM} \\ \frac{c}{2\pi}\sqrt{\left(\frac{\eta_{mn}}{a}\right)^2 + \left(\frac{\ell\pi}{d}\right)^2} & \text{TE.} \end{cases} \tag{26}$$

The fundamental mode is the mode with lowest eigenfrequency. The lowest TM-mode is TM_{010}, with frequency

$$f_{010} = \frac{c}{2\pi}\frac{2.405}{a} \tag{27}$$

and the lowest TE-mode is TE_{111}, with frequency

$$f_{111} = \frac{c}{2\pi}\sqrt{\left(\frac{1.841}{a}\right)^2 + \left(\frac{\pi}{d}\right)^2}. \tag{28}$$

When $d < \frac{\pi}{\sqrt{2.405^2 - 1.841^2}}a \approx 2.03a$ then the fundamental mode is TM_{010} otherwise it is TE_{111}. It is the TM_{010} that is used in the cavities of accelerators. We can see why when we know the total electric field. Equation (21) says that $\boldsymbol{E}_T = \boldsymbol{0}$ for TM_{010}, since $k_z = 0$. This means that the electric field of TM_{010} is

$$\boldsymbol{E}(\boldsymbol{r}) = AJ_0(\xi_{01}\rho/a)\hat{\boldsymbol{z}}. \tag{29}$$

This is shown in figure (9).

The TM_{010} cavity is perfect for accelerating particles. In figure (10) the beam pipe and the pillbox cavity is shown. The particles travel along the beam pipe. As they enter the pillbox cavity the electric field is strong and accelerates that particle in the forward direction. The particles must come in bunches separated in time with $T = 1/f$, where T is the period of the frequency of the TM_{010} mode.

In practice the pilbox cavities used in accelerators are deformed in order to optimize their performance. Figure (11) shows an elliptic cavity of the same type that is to be used in ESS. It consists of five cells[3] and each cell can be viewed as a deformed pillbox cavity. The electric field and the bunches are depicted in figure (12). Notice that there is a phase difference of 180 degrees between two adjacent cells. In that way the particles get accelerated in all of the cells.

[3]In ESS the elliptic cavities will have six cells.

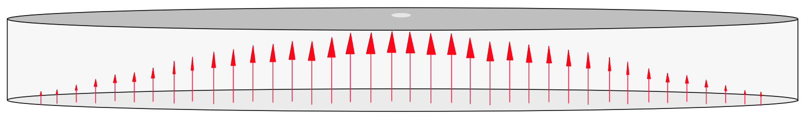


Figure 9: The electric field of the TM_{010} mode in a pill-box cavity.

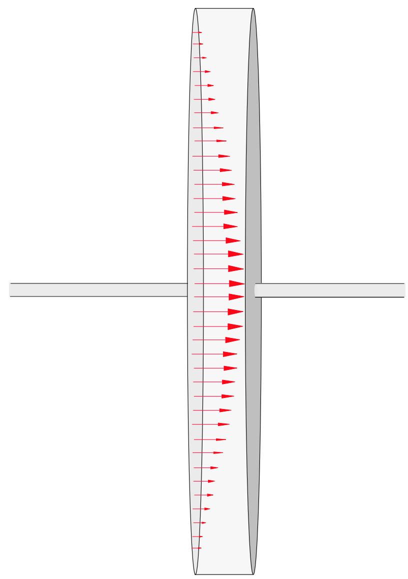


Figure 10: The pillbox cavity and beam pipe in an accelerator.

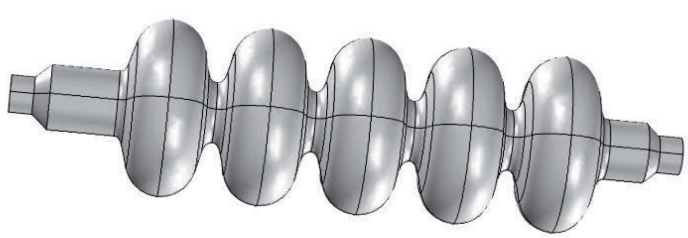


Figure 11: The elliptic cavity to be used in the LINAC of ESS.

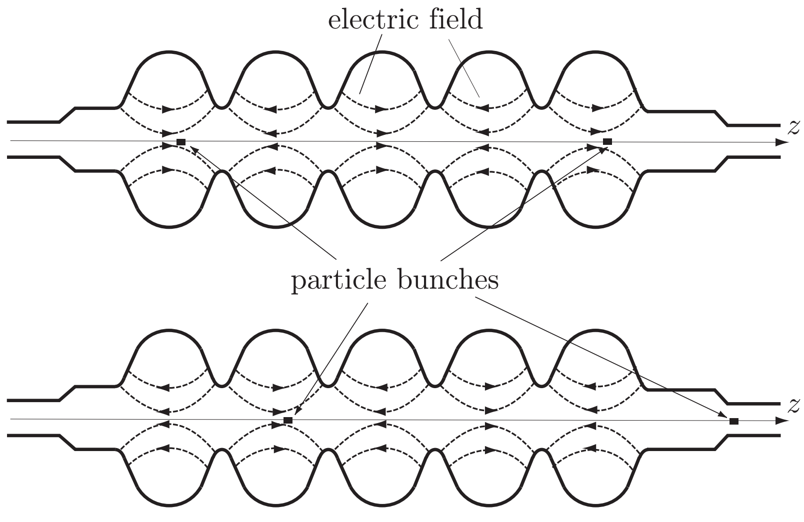


Figure 12: The cross section of the elliptic cavity and the electric field. The bunches are traveling to the right. The frequency of the fundamental mode is 704 MHz and the bunches come with a frequency of 352MHz. The lower figure is at half a period later than the upper one.

Traveling wave cavities

We have seen that the pill-box cavity is very good for accelerating particles. There is another type of cavity, called traveling wave cavity, that is very common for accelerating electrons. It is used in the LINAC of MAX IV and also in accelerators that are used for radio therapy at hospitals.

To understand the idea behind the traveling wave cavity we can go back to the circular waveguide. Let the radius of the waveguide be a. According to (21) and (15) the TM_{01} mode has the electric field

$$\boldsymbol{E}(\rho, z) = E_0 \left(-\frac{\mathrm{i}k_z}{k_t} J_1(k_t\rho)\hat{\boldsymbol{\rho}} + J_0(k_t\rho)\hat{\boldsymbol{z}} \right) e^{\mathrm{i}k_z z},$$

where $k_t = \dfrac{2.405}{a}$ and $k_z = \sqrt{k^2 - k_t^2}$. On the symmetry axis the electric field is directed in the $z-$direction. The idea is now to let the electrons surf on the electric wave. The problem is that the wave is traveling with the phase speed $v_p = \dfrac{\omega}{k_z}$ and that is always larger than the speed of light. We need to slow down the wave so that the wave is traveling with the same speed as the electrons. Since electrons are very light particles they travel with almost the speed of light already at quite small energies. The goal is then to let the phase speed equal the speed of light. We do this by adding irises, which are walls with apertures. This is seen in figure (13). This results in the wave in figure (14). The electric field is at this moment directed to the left in the red cells. The electron bunches are located in these cells. There is

Figure 13: The traveling wave cavity. The cells are the same as the one used in M the MAX IV LINAC.

a phase difference of $2\pi/3$ between two adjacent cells. At a time $T/3$ later the red region has moved one cell to the right. Also the bunch must have moved to this cell. This sets the length of each cell to $L_{\text{cell}} = cT/3$. The wave continues to travel to the right and the bunches of electrons are traveling with the same speed. They are then accelerated in each cell and, by that, along the entire structure. To prevent that the wave is reflected at the end of the cavity, the cavity is terminated by an absorber.

Analyzing resonance cavities with Comsol

The resonance cavities can be analyzed by FEM. There are three different cases that are of interest:

a) If the cavity is axisymmetric we use **2D axisymmetric** in Comsol. The axially symmetric geometry makes it possible to expand the electric and magnetic fields in a Fourier series in the system $e^{\mathrm{i}n\phi}$. Then the problem is reduced to a two-dimensional problem in the cylindrical coordinates ρ and z. Each n value is treated separately. The reduction from 3D to 2D makes the solver very fast and accurate.

b) If the resonance cavity consists of a hollow waveguide with plane metallic walls at $z = 0$ and $z = h$ it is easy to analyze it with FEM. First the cut-off frequencies for the different modes are determined using the scheme on page 32. Then the resonance frequencies of the cavity are obtained from (23).

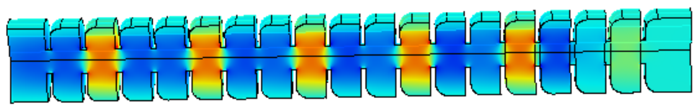


Figure 14: A Comsol simulation of the electric field in the traveling wave cavity.

c) If neither of the two previous cases are relevant then we have to use the three-dimensional solver.

We now give an example of the first case.

Example

We determine some of the resonances of a hollow sphere with radius $a = 1$ m. We use the solver **2D axisymmetric** since the sphere is axially symmetric. All of the field components can be expanded in a Fourier series

$$f(\rho, \phi, z) = \sum_{m=-\infty}^{\infty} f_m(\rho, z) e^{\mathrm{i}m\phi}.$$

Fields with different m-values do not couple to each other and then each m value can be treated separately.

a) First choose **2D axisymmetric**>**Radio Frequency**>**Electromagnetic waves** >**Eigenfrequency**.

b) Draw a circle with radius $a = 0.1$ m and put its center at $(0, 0)$.

c) Choose **Sector angle** 180 degrees and **Rotation** -90 degrees. By that you have a half circle in the right half-plane.

d) Choose **Air** as material.

e) Go to **Electromagnetic waves** and choose **perfect conductor** as boundary condition for the circular line. The symmetry axis has the condition **Axial Symmetry** by default.

f) Choose **Electromagnetic waves** and the azimuthal index m.

g) In **Study>Eigenfrequency** we set the frequency to e.g. 1 GHz. This is the frequency where Comsol starts to look for an eigenfrequency. We can also choose the number of resonances that it will determine.

h) The mesh size is **Normal** by default. If we need a better accuracy then we choose a finer mesh.

i) We now let Comsol solve the problem.

j) Comsol calculates the lowest resonant frequencies and their electric fields. There might be spurious solutions that are unphysical. The resonance frequency for spurious solutions are usually very far from 1 GHz, or even complex, and the corresponding field plots are fuzzy.

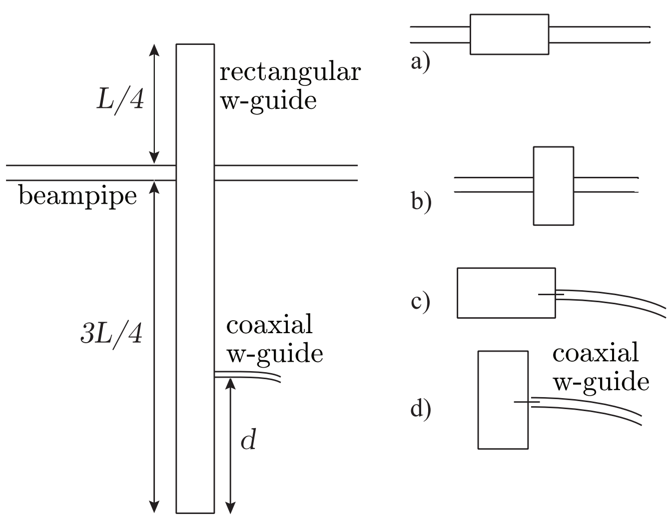
3.1

Determine the ratio between radius a and length d for a circular cylinder such that the lowest resonance frequency for the TE and TM modes are the same. All walls are perfectly conducting and there is vacuum inside the cavity. Confirm your result with COMSOL.

3.2

The fundamental mode TE_{10} is used for accelerating particles in a rectangular waveguide, see figure. The waveguide has dimension $0.2\,\text{m} \times 0.1\,\text{m}$ and is terminated in both ends by perfect conducting plates. There will then be a standing wave in the waveguide. The waveguide is fed by a coaxial cable with an inner conductor that extends into the waveguide.

The frequency and length is adapted such that the electric field has a maximum where the beam pipe is attached to the waveguide. The particles come in bunches with frequency $f = 500$ MHz, but in order to reduce the dimensions of the waveguide it is fed with the double frequency $f_v = 1$ GHz.



a) Is the pipe attached at the waveguide as in figure a) or b)? Motivate why.

b) Is the coaxial cable attached as in figure c) or d)? Motivate your answer.

c) Determine the length, L, of the waveguide expressed in $a = 0.2$ m, f_v and speed of light c_0. Choose the shortest possible length for which the pipe is attached where the electric field has a maximum.

d) Determine the distance d between the end of the waveguide and the feed point, expressed in L. Motivate

e) Assume that that you like to feed the waveguide with 500 MHz. Is it possible to do this by just changing the length of the waveguide?

3.3

The fundamental mode TM_{010} is used for accelerating particles that travel along the symmetry axis of a circular cylindric cavity with radius a and length h. The particles come in short bunches with time interval T. All bunches should get the same acceleration.

a) Determine the radius a of the cavity, expressed in T and c_0. The radius should be as large as possible.

b) You want to avoid higher order modes with frequencies that are a multiple of the frequency of the fundamental mode, f_0. The reason is that the beam acts as a radiating antenna that excites cavity modes. The modes with a frequency close to or at a multiple of the frequency of the beam will grow strong and affect the trajectories of the particles.

Determine the length h that should be avoided in order for the TM_{011} mode to get the frequency $2f_0$.

Hint: The three lowest zeros of the Bessel function $J_0(x)$ are $\xi_{01} = 2.405$, $\xi_{02} = 5.520$, $\xi_{03} = 8.654$. The TM_{mnl}-mode has $E_z(r_c, \phi, z) = E_{mnl} J_m(\xi_{mn} r_c/a) \cos(m\phi) \cos\left(\frac{l\pi z}{h}\right)$.

3.4

Assume a cylindric cavity with radius a and length h.

a) Sketch the electric field for the TM_{010} mode in a cross section of the cavity. The cross section should be in the xz-plane, where the $z-$axis is the symmetry axis. The electric field should be shown as arrows where the length of the arrow indicates the field strength.

b) Sketch the magnetic field in a cross section in the xy-plane of the TM_{010} mode.

c) Determine the resonant frequency of the TM_{010} mode when $a = 0.1$ m and $h = 0.3$ m.

d) The cylindric cavity is fed by a source with frequency equal to the resonant frequency of the TM_{010} mode. The phase of the source is such that the electric field in the cavity is proportional to $\sin(\omega t)$. The maximum amplitude of the electric field in the cavity is E_0. Assume that a proton with speed v enters the cavity at time $t = 0$. Determine the kinetic energy that the proton will gain in the cavity as a function of the length h. The gain in energy is small enough compared to the initial kinetic energy of the proton, such that the speed of the proton can be considered to be constant in the cavity. The radius a, the length h, E_0, ω and v are assumed to be known.

3.5

Consider the TM_{010}-mode in a cylindric cavity with radius a and length L.

a) At what radius r_c is the magnetic field maximal.

b) Where is the surface current density maximal?

c) Where is the surface charge density maximal?

d) Confirm you results from a COMSOL 2D axisymmetric eigenfrequency calculation. You can plot the quantities along lines by using line graph. You can find line graph under Results→1D Plot Group. If you don't find surface charge density then you can plot the absolute value of the normal component of the D-field, or equivalently, the norm of the the D-field along the surface.

3.6

A resonance cavity is a cylinder with elliptic cross section. The ellipse has major half-axis $a = 3$ cm and minor half-axis $b = 2$ cm. The length of the cylinder is 3 cm.

Determine the three lowest resonance frequencies of the cavity by using COMSOL. All walls are perfectly conducting and there is vacuum inside the cavity.

Help: One can solve problem a) in COMSOL using either a 2D or a 3D calculation. Do both.

Microwave cavities: Answers and solutions

S3.1

$$\frac{d}{a} = 2.03$$

S3.2

a) The pipe is attached as in b) since the electric field is directed parallell with the end plates.

b) The coaxial cable is attached as in d) since the inner conductor then couples to the electric field.

c) The length L is given by $L = \lambda_z = \frac{2\pi}{k_z}$ where $k_z = \sqrt{k^2 - (\pi/a)^2}$. Since the frequency is f_v the wavenumber is $k = 2\pi f_v/c_0$. Then

$$L = \frac{2\pi}{\sqrt{k^2 - (\pi/a)^2}} = \frac{2c_0 a}{\sqrt{(2f_v a)^2 - c_0^2}} \tag{30}$$

Numerically $L = 45.3$ cm when $a = 20.0$ cm.

d) $d = L/4$ since the feed point should be where the electric field is maximal.

e) No, it will not work. The cut-off frequency for the TE_{10} mode is $f_c = \frac{c_0}{2a} \approx$750 MHz.

S3.3

a) The resonance frequency is given by $f = \frac{c_0}{2\pi}\frac{\xi_{01}}{a}$, where $\xi_{01} = 2.405$. Thern $a = \frac{2.405 T c_0}{2\pi}$

b) For TM_{011}

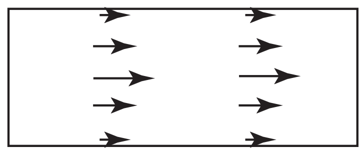
$$k^2 = \left(\frac{\xi_{01}}{a}\right)^2 + \left(\frac{\pi}{h}\right)^2$$

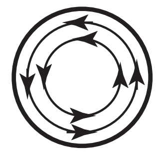
where $k = \frac{4\pi}{T c_0}$.

The following length should be avoided:

$$h = \frac{\pi}{\sqrt{\left(\frac{4\pi}{c_0 T}\right)^2 - \left(\frac{2.405}{a}\right)^2}} = \frac{c_0 T}{2\sqrt{3}}$$

S3.4





a) See left figure.

b) See right figure.

c) $f = \dfrac{c}{2\pi}\dfrac{2.405}{.1} = 1.15$ GHz

d) Since $z = vt$ the proton will experience the force $\boldsymbol{F} = eE_0 \sin(\omega t)\hat{\boldsymbol{z}} = eE_0 \sin(\omega z/v)\hat{\boldsymbol{z}}$. The gain in kinetic energy is $\Delta W_k = \int_0^h eE_0 \sin(\omega z/v)dz = \dfrac{eE_0 v}{\omega}(1 - \cos(\omega h/v))$

S3.5

a) The r_c at which the magnetic field has its largest amplitude is when $\xi_{01} r_c = \eta_{11} a$, where $\xi_{01} = 2.505$ and $\eta_{11} = 1.841$. Then $r_c = 0.7656a$.

b) $r_c = 0.7656a$

c) In the center of the flat surfaces since $\rho_{\mathrm{S}} = \varepsilon_0 \hat{\boldsymbol{n}} \cdot \boldsymbol{E}$ and $\boldsymbol{E} = E_0 J_0(\xi_{01} r_c)\hat{\boldsymbol{z}}$, where $J_0(x)$ is maximal at $x = 0$.

S3.6

The three lowest resonance frequencies are

$f_1 = 4.86$ GHz, $f_2 = 5.81$ GHz and $f_3 = 6.59$ GHz.

4 Potentials and fields

4.1

Show that the differential equations for V and $\boldsymbol{A}$ (10.4 and 10.5) can be written in the more symmetric form

$$\Box^2 V + \frac{\partial L}{\partial t} = -\frac{1}{\varepsilon_0}\rho$$
$$\Box^2 \boldsymbol{A} - \nabla L = -\mu_0 \boldsymbol{J}$$

where $\Box^2 \equiv \nabla^2 - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}$ and $L \equiv \nabla \cdot \boldsymbol{A} + \mu_0\varepsilon_0 \frac{\partial V}{\partial t}$.

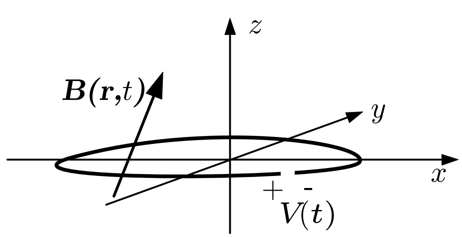
Problem 10.1 in Griffiths 5th ed..

4.2

Suppose $V = 0$ and $\boldsymbol{A} = A_0 \sin(kx - \omega t)\hat{\boldsymbol{y}}$, where A_0, ω, and k are constants. Find $\boldsymbol{E}$ and $\boldsymbol{B}$ and check that they satisfy Maxwell's equations in vacuum. What condition must you impose on ω and k?

Problem 10.4 in Griffiths 5th ed..

4.3



Consider a closed curve $\mathcal{C}$ in the plane $z = 0$. Along the curve is a thin metal wire (perfectly conducting) that is almost closed, see figure. A magnetic flux density $\boldsymbol{B}(\boldsymbol{r}, t)$ will induce a voltage $V(t)$ in the gap of the loop. Show that, if the self-inductance of the loop can be neglected,

$$V(t) = -\frac{d\Phi(t)}{dt}$$

where $\Phi(t) = \int_S \boldsymbol{B} \cdot \hat{\boldsymbol{z}}\, \mathrm{d}S$ and S is the planar surface enclosed by $\mathcal{C}$. One can show this in many ways, but you should do it by going through the following steps:

a) Show that $\Phi(t) = \oint_C \boldsymbol{A}(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{\ell}$.

b) What is the electric field inside the metal of the wire?

c) Is $\boldsymbol{A}$ affected by the presence of the wire?

d) Use $\int_{C_\mathrm{m}} \boldsymbol{A}(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{\ell} \approx \oint_C \boldsymbol{A}(\boldsymbol{r},t)\cdot \mathrm{d}\boldsymbol{\ell}$, $\boldsymbol{E} = -\nabla V - \dfrac{\partial \boldsymbol{A}}{\partial t}$ and show that

$$V(t) = -\frac{d\Phi(t)}{dt}$$

here C_m is the almost closed curve along the metal wire.

e) Can we use the same formula for V if the gap is large?

4.4

A time-dependent point charge $q(t)$ at the origin, $\rho(\boldsymbol{r},t) = q(t)\delta^3(\boldsymbol{r})$, is fed by a current $\boldsymbol{J}(\boldsymbol{r},t) = -\dfrac{\dot{q}(t)}{4\pi r^2}\hat{\boldsymbol{r}}$.

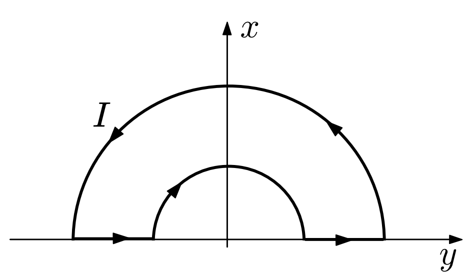
a) Check that charge is conserved, by confirming that the continuity equation is obeyed.

b) Find the scalar and vector potentials in the Coulomb gauge.

c) Find the fields, and check that they satisfy all of Maxwell's equations.

Problem 10.7 in Griffiths 5th ed..

4.5



A piece of a wire bent into a loop, as shown in the figure, carries a current that increases linearly with time:

$$I(t) = kt \qquad (-\infty < t < \infty)$$

Calculate the retarded vector potential $\boldsymbol{A}$ at the center. Find the electric field at the center. Why does the (neutral) wire produce an electric field? (Why can't you determine the magnetic field from this expression for $\boldsymbol{A}$)

Problem 10.12 in Griffiths 5th ed..

4.6

A particle of charge q moves in a circle of radius a at a constant angular velocity ω. Assume that the circle lies in the $xy-$plane, centered at the origin, and at time $t = 0$ the charge is at $(a, 0)$, on the positive $x-$axis. Find the Liénard-Wiechert potentials for points on the $z-$axis.

Problem 10.15 in Griffiths 5th ed..

4.7

It was shown in Griffiths that at most one point on the particle trajectory communicates with $\boldsymbol{r}$ at any given time. In some cases there might be no such point (an observer at $\boldsymbol{r}$ would not see the particle). As an example, consider a particle in hyperbolic motion along the $x-$axis:

$$\boldsymbol{w}(t) = \sqrt{b^2 + (ct)^2}\hat{\boldsymbol{x}} \qquad (-\infty < t < \infty) \tag{31}$$

(In special relativity this is the trajectory of a particle subject to a constant force $F = mc^2/b$.) Sketch the graph of w versus t. At four or five different representative points on the curve, draw the trajectory of a light signal emitted by the particle at that point–both in the plus x direction and the minus x direction. What region on your graph corresponds to points and times (x, t) from which the particle cannot be seen? At what time does someone at point x first see the particle? (Prior to this the potential is zero.) Is it possible for a particle, once seen, to disappear from view?

Problem 10.17 in Griffiths 5th ed..

4.8

Suppose a point charge q that is constrained to move along the $x-$axis. Show that the fields at points x on the axis to the right of the charge are given by

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0(x - w_x)^2}\left(\frac{c + v}{c - v}\right)\hat{\boldsymbol{x}}$$
$$\boldsymbol{B} = \boldsymbol{0}$$

where $w_x = w_x(t_{\mathrm{r}})$ is the position of the charge at the retarded time. (Do not assume v is constant!) What are the fields on the axis to the left of the charge?

Problem 10.20 in Griffiths 5th ed..

Potentials and fields: Answers and solutions

S4.1

S4.2

$$\begin{aligned}\boldsymbol{E} &= -\nabla V - \frac{\partial \boldsymbol{A}}{\partial t} = A_0\omega\cos(kx-\omega t)\hat{\boldsymbol{y}}\\ \boldsymbol{B} &= \nabla\times\boldsymbol{A} = A_0 k\cos(kx-\omega t)\hat{\boldsymbol{z}}\end{aligned}$$

It follows that $\nabla\cdot\boldsymbol{E}=0$ and $\nabla\cdot\boldsymbol{B}=0$.

It is also straightforward to see that $\nabla\times\boldsymbol{E} = -\dfrac{\partial\,\boldsymbol{B}}{\partial t}$.

If $k^2=\omega^2\varepsilon_0\mu_0$ then $\nabla\times\boldsymbol{B}=\mu_0\varepsilon_0\dfrac{\partial\,\boldsymbol{E}}{\partial t}$.

S4.3

a) Use $\boldsymbol{B}=\nabla\times\boldsymbol{A}$ and Stokes theorem.

b) The electric field is zero.

c) No, not in the quasi statics. The magnetic flux density $\boldsymbol{B}$ is, by assumption, not affected and then $\boldsymbol{A}$ is not affected either.

d)

e) No. The voltage between two points that are not close to each other is not unique. It depends on the integration path between the points.

S4.4

S4.5

$$\boldsymbol{A} = \frac{\mu_0}{4\pi}\oint\frac{I(t_\mathrm{r})}{r'}\,\mathrm{d}\boldsymbol{\ell} = \frac{\mu_0 k}{4\pi}\oint\frac{t-r'/c}{r'}\,\mathrm{d}\boldsymbol{\ell} = \frac{\mu_0 k}{4\pi}\left(t\oint\frac{1}{r'}\,\mathrm{d}\boldsymbol{\ell} - \frac{1}{c}\oint\mathrm{d}\boldsymbol{\ell}\right) \tag{32}$$

But for the complete loop $\oint\mathrm{d}\boldsymbol{\ell}=\boldsymbol{0}$, so

$$\boldsymbol{A} = \frac{\mu_0 kt}{4\pi}\left(\frac{1}{a}\int_1\mathrm{d}\boldsymbol{\ell} + \frac{1}{b}\int_2\mathrm{d}\boldsymbol{\ell} + 2\hat{\boldsymbol{x}}\int_a^b\frac{\mathrm{d}x}{x}\right) \tag{33}$$

Here $\int_1\mathrm{d}\boldsymbol{\ell}=2a\hat{\boldsymbol{x}}$ (inner circle) and $\int_2\mathrm{d}\boldsymbol{\ell}=-2b\hat{\boldsymbol{x}}$ (outer circle), so

$$\boldsymbol{A} = \frac{\mu_0 kt}{4\pi}\left(\frac{1}{a}(2a) - \frac{1}{b}(2b) + 2\ln(b/a)\right)\hat{\boldsymbol{x}}$$

Thus

$$\begin{aligned} \boldsymbol{A} &= \frac{\mu_0 k t}{2\pi} \ln(b/a)\hat{\boldsymbol{x}} \\ \boldsymbol{E} &= -\frac{\partial \boldsymbol{A}}{\partial t} = -\frac{\mu_0 k}{2\pi} \ln(b/a)\hat{\boldsymbol{x}} \end{aligned}$$

The changing magnetic field induces the electric field. Since we only know $\boldsymbol{A}$ at one point (the center), we can't compute $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

S4.6

At time t the particle is at $\boldsymbol{w}(t) = a(\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}})$, so $\boldsymbol{v}(t) = a\omega(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}})$. Therefore

$$\boldsymbol{r} - \boldsymbol{w} = z\hat{\boldsymbol{z}} - a(\cos(\omega t_{\mathrm{r}})\hat{\boldsymbol{x}} + \sin(\omega t_{\mathrm{r}})\hat{\boldsymbol{y}})$$

and, as expected, $|\boldsymbol{r} - \boldsymbol{w}| = \sqrt{a^2 + z^2}$. It must also be that $(\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})) \cdot \boldsymbol{v}(t_{\mathrm{r}}) = 0$ (check this if you don't believe it). So $\left(1 - \dfrac{(\boldsymbol{r} - \boldsymbol{w}) \cdot \boldsymbol{v}}{|\boldsymbol{r} - \boldsymbol{w}|c}\right) = 1.$

Then

$$\begin{aligned} V(z, t) &= \frac{q}{4\pi\varepsilon_0\sqrt{a^2 + z^2}} \\ \boldsymbol{A}(z, t) &= \frac{q\omega a}{4\pi\varepsilon_0 c^2\sqrt{a^2 + z^2}} \left(-\sin(\omega t_{\mathrm{r}})\hat{\boldsymbol{x}} + \cos(\omega t_{\mathrm{r}})\hat{\boldsymbol{y}}\right) \end{aligned}$$

where $t_{\mathrm{r}} = t - \dfrac{\sqrt{a^2 + z^2}}{c}$.

S4.7

The particle is invisible for times $t < -\dfrac{x}{c}$ for a person at x. the light reaches the person at $t = -\dfrac{x}{c}$ and after that it will be visible for all times $t > -\dfrac{x}{c}$.

S4.8

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{|\boldsymbol{\mathscr{r}}|}{(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})} \left[(c^2 - v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{\mathscr{r}})\right]$$

here $\boldsymbol{v} = v\hat{\boldsymbol{x}}$, $\boldsymbol{a} = a\hat{\boldsymbol{x}}$, and, for points to the right, $\hat{\boldsymbol{\mathscr{r}}} = \hat{\boldsymbol{x}}$. So $\boldsymbol{u} = (c - v)\hat{\boldsymbol{x}}$, $\boldsymbol{u} \times \boldsymbol{a} = \boldsymbol{0}$, and $\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = |\boldsymbol{\mathscr{r}}|(c - v)$.

$$\boldsymbol{E} = \frac{q}{4\pi\varepsilon_0} \frac{|\boldsymbol{\mathscr{r}}|}{|\boldsymbol{\mathscr{r}}|^3 (c - v)^3} (c^2 - v^2)(c - v)\hat{\boldsymbol{x}} = \frac{q}{4\pi\varepsilon_0 |\boldsymbol{\mathscr{r}}|^2} \left(\frac{c + v}{c - v}\right) \hat{\boldsymbol{x}} \tag{34}$$

$$\boldsymbol{B} = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E} = \boldsymbol{0} \tag{35}$$

where $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{w}(t_r)$.

For field points to the left, $\hat{\boldsymbol{\mathscr{r}}} = -\hat{\boldsymbol{x}}$ and $\boldsymbol{u} = -(c+v)\hat{\boldsymbol{x}}$, so $\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = |\boldsymbol{\mathscr{r}}|(c+v)$, and

$$\boldsymbol{E} = -\frac{q}{4\pi\varepsilon_0}\frac{|\boldsymbol{\mathscr{r}}|}{|\boldsymbol{\mathscr{r}}|^3(c-v)^3}(c^2-v^2)(c-v)\hat{\boldsymbol{x}} = -\frac{q}{4\pi\varepsilon_0|\boldsymbol{\mathscr{r}}|^2}\left(\frac{c+v}{c-v}\right)\hat{\boldsymbol{x}} \tag{36}$$

$$\boldsymbol{B} = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E} = \boldsymbol{0} \tag{37}$$

5 Radiation

5.1

Write a Matlab script that determines the retarded time t_r from the time t given the position vector $\boldsymbol{w}(t)$ as a function of time and the field point $\boldsymbol{r}$. Check your program by calculating t_r for the following case of circular motion of the particle:

$$\boldsymbol{w}(t) = A(\cos(\omega t), \sin(\omega t), 0) \tag{38}$$

where A is the the radius of the circle. The speed of the particle is $v = \beta c$ and the field point is $\boldsymbol{r}$. Let

$$A = 10\,\mathrm{m} \tag{39}$$
$$\beta = 0.95 \tag{40}$$
$$\boldsymbol{r} = (100, 0, 0)\,\mathrm{m} \tag{41}$$
$$0 < t < 250\,\mathrm{ns} \tag{42}$$

Notice that $\omega = v/A$. Plot $t_\mathrm{r}(t)$ as a function of t in the interval $0 < t < 250\,\mathrm{ns}$ and compare with the curve in the answer.

5.2

Write a Matlab script that evaluates the electric and magnetic fields from a point charge that moves along a prescribed curve $\boldsymbol{w}(t)$.

a) Check your program by determining the electric field from a particle in circular motion with the values given in 5.1.

b) Check your program by determining the electric field from a particle traveling through an undulator.

5.3

Write a Matlab script that evaluates the frequency spectrum of the electric field from a a point charge that moves along a prescribed curve $\boldsymbol{w}(t)$.

a) Check your program by determining the frequency spectrum of E_y from a particle in circular motion with the values given in 5.1.

b) Check your program by determining the Fourier transform of the electric field from a particle traveling through an undulator.

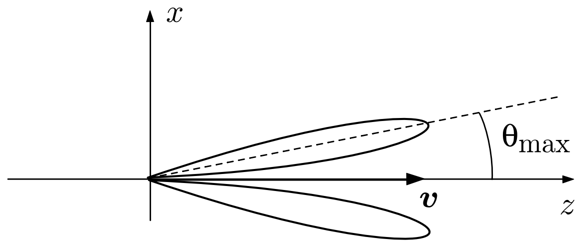


Figure 15: Angle θ_{max} at maximum radiation.

5.4

In Bohr's theory of hydrogen, the electron in its ground state was supposed to travel in a circle of radius 5×10^{-11} m, held in orbit by the Coulomb attraction of the proton. According to classical electrodynamics, this electron should radiate, and hence spiral in to the nucleus. Show that $v \ll c$ for most of the trip (so you can use the Larmor formula), and calculate the lifespan of the Bohr's atom. (Assume each revolution is essentially circular.)

Problem 11.14 in Griffiths 5th ed..

5.5

When $\boldsymbol{v}$ and $\boldsymbol{a}$ of a particle is instantaneously collinear (at time t_{r}), as, for example in a straight-line motion. Then the angular distribution of the radiation is, according to Griffiths, given by

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c} \frac{\sin^2\theta}{(1-\beta\cos\theta)^5}$$

Find the angle θ_{max}, see figure (15), at which the maximum radiation is emitted. Show that for ultrarelativistic speeds (v close to c), $\theta_{\text{max}} \approx \sqrt{0.5 - 0.5\beta}$. What is the intensity of the radiation in this maximal direction (in the ultrarelativistic case), in proportion to the same quantity for a particle instantaneously at rest? Give your answer in terms of γ.

Problem 11.15 in Griffiths 5th ed..

5.6

In the previous problem we assumed that the velocity and acceleration were (instantaneously, at least) collinear. Carry out the same analysis for the case where they are perpendicular. Choose your axes so that $\boldsymbol{v}$ lies along the $z-$axis and $\boldsymbol{a}$ along the

$x$axis, so that $\boldsymbol{v} = v\hat{\boldsymbol{z}}$, $\boldsymbol{a} = a\hat{\boldsymbol{x}}$, and $\hat{\boldsymbol{r}} = \sin\theta\cos\phi\hat{\boldsymbol{x}} + \sin\theta\sin\phi\hat{\boldsymbol{y}} + \cos\theta\hat{\boldsymbol{z}}$. Check that P is consistent with the Liénard formula.

(*Comment:* For relativistic velocities ($\beta \approx 1$) the radiation is again sharply peaked in the forward direction. The most important application of these formulas is to circular motion– in this case the radiation is called synchrotron radiation. For a relativistic electron, the radiation sweeps around like a locomotive's headlight as the particle moves.)

Problem 11.16 in Griffiths 5th ed..

5.7

a) A particle of charge q moves in a circle of radius R at a constant speed v. To sustain the motion, you must, of course, provide a centripetal force mv^2/R. What additional force ($\boldsymbol{F}_{\mathrm{e}}$) must you exert, in order to counteract the radiation reaction? [It's easiest to express the answer in terms of the instantaneous velocity $\boldsymbol{v}$.] What power (P_{e}) does this extra force deliver? Compare P_{e} with the power radiated (use Larmor formula).

b) Repeat part (a) for a particle in simple harmonic motion with amplitude A and angular frequency ω ($\boldsymbol{w}(t) = A\cos(\omega t)\hat{\boldsymbol{z}}$). Explain the discrepancy.

c) Consider the case of a particle in free fall (constant acceleration g). What is the radiation reaction force? What is the power radiated? Comment on the results.

Problem 11.17 in Griffiths 5th ed..

Radiation: Answers and solutions

S5.1

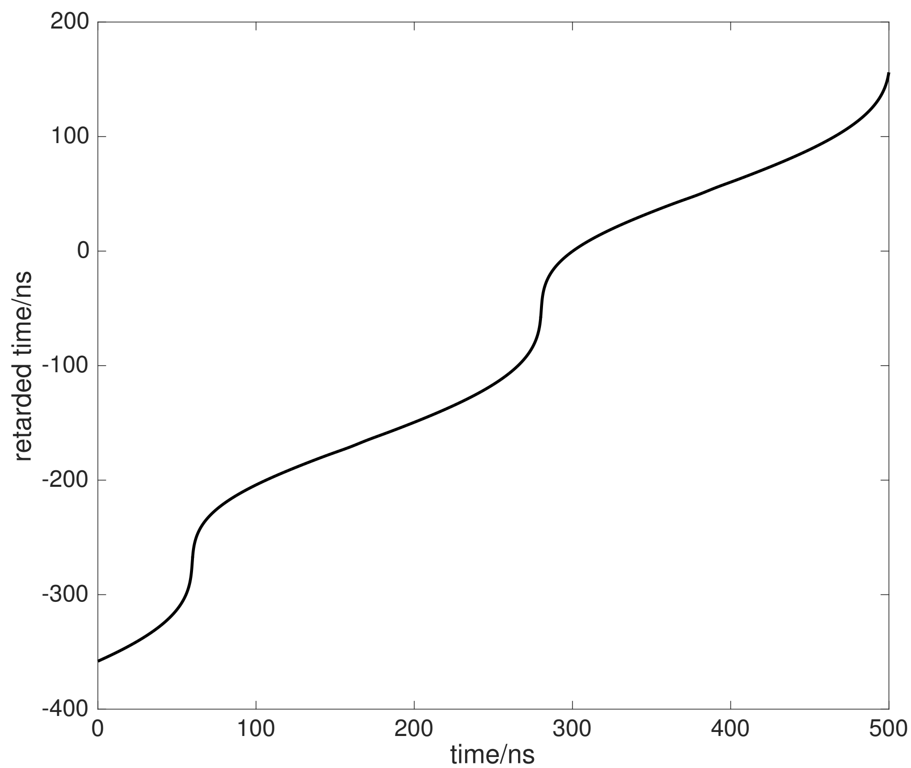


Figure 16: The retarded time t_{r} as a function of time t for a particle in circular motion.

One may use the function **fsolve** in Matlab. Do **help fsolve** in Matlab to find out more.

Another option is to solve t_{r} from t by iteration. Start with $t_{\mathrm{r}0} = t$ and use the scheme

$$t_{\mathrm{r}n+1} = t - \sqrt{r^2 + |\boldsymbol{w}(t_{\mathrm{r}n})|^2 - 2\boldsymbol{r} \cdot \boldsymbol{w}(t_{\mathrm{r}n})}/c$$

S5.2

It is convenient to use the Matlab functions dot and cross in order to take the scalar and vector products of three-dimensional vectors.

a) The circular motion is $\boldsymbol{w}(t) = 10(\cos\omega t, \sin\omega t, 0)$ m. The speed is $0.95c$ and the field point is $(100, 0, 0)$ m. The $y-$component of the electric field is given in Figure (17).

b) The motion is given by $\boldsymbol{w}(t) = (vt, A\sin\omega t, 0)$, where $A = 0.001$ m, $\omega = 1.6953 \cdot 10^{10}$ rad/s, and speed $v = 0.9c$. The period of the magnetic field is 0.1 m, the length of the undulator is 0.2 m, and the field point is $(20, 0, 0)$ m. The $y-$component of the electric field is given in Figure (18). The time in the figure is translated.

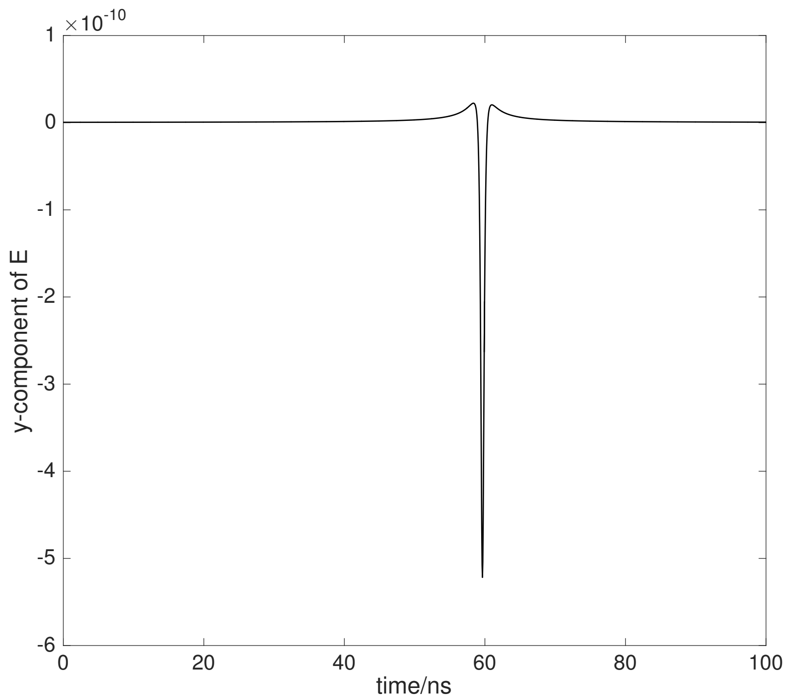


Figure 17: $E_y(t)$ as a function of time t for a particle in circular motion.

S5.3

One can here use the function fft in Matlab. In the reference page for fft it is explained how the fourier transform, as a function of frequency, is obtained from the fft.

S5.4

$$F = \frac{q}{4\pi\varepsilon_0 r^2} = ma = m\frac{v^2}{r}$$

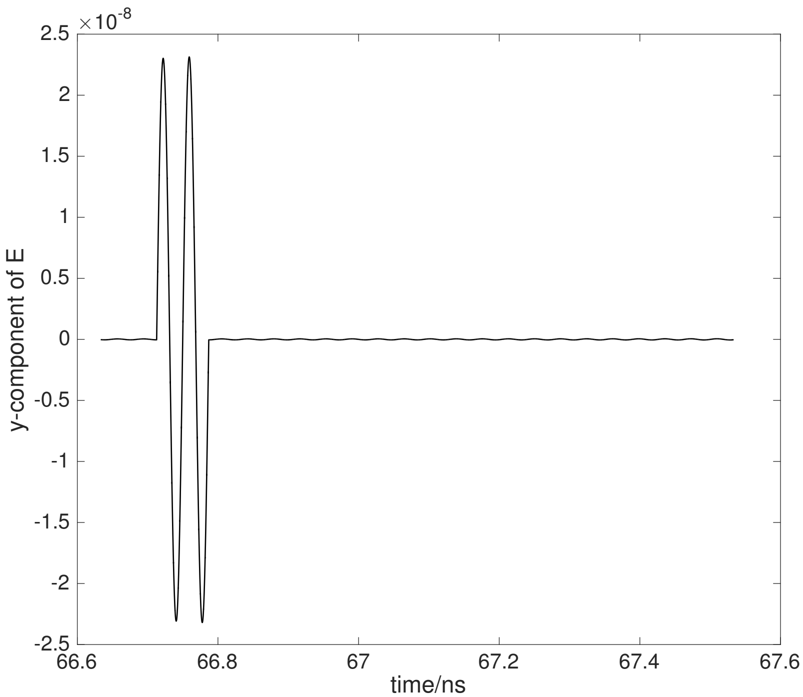


Figure 18: $E_y(t)$ as a function of time t for a particle running through an undulator.

then $v = \sqrt{\dfrac{q^2}{4\pi\varepsilon_0 m r}}$. At the beginning ($r_0 = 0.5$ Å) Numerically

$$\frac{v}{c} = 0.0075$$

and when the radius is one hundreth of this , v/c is only 10 times greater (0.075), so for most of the trip velocity is safely non-relativistic.

From the Larmor formula, $P = \dfrac{\mu_0 q^2}{6\pi c}\left(\dfrac{v^2}{r}\right)^2 = \dfrac{\mu_0 q^2}{6\pi c}\left(\dfrac{q^2}{4\pi\varepsilon_0 m r^2}\right)^2$. (since $a = v^2/r$), and $P = \dfrac{dU}{dt}$, where U is the (total) energy of the electron:

$$U = U_{\text{kin}} + U_{\text{pot}} = \frac{1}{2}mv^2 - \frac{q^2}{4\pi\varepsilon_0 r} = -\frac{q^2}{8\pi\varepsilon_0 r}$$

So

$$-\frac{dU}{dt} = -\frac{q^2}{8\pi\varepsilon_0 r^2}\frac{dr}{dt} = P = \frac{q^2}{6\pi\varepsilon_0 c^3}\left(\frac{q^2}{4\pi\varepsilon_0 m r^2}\right)^2$$

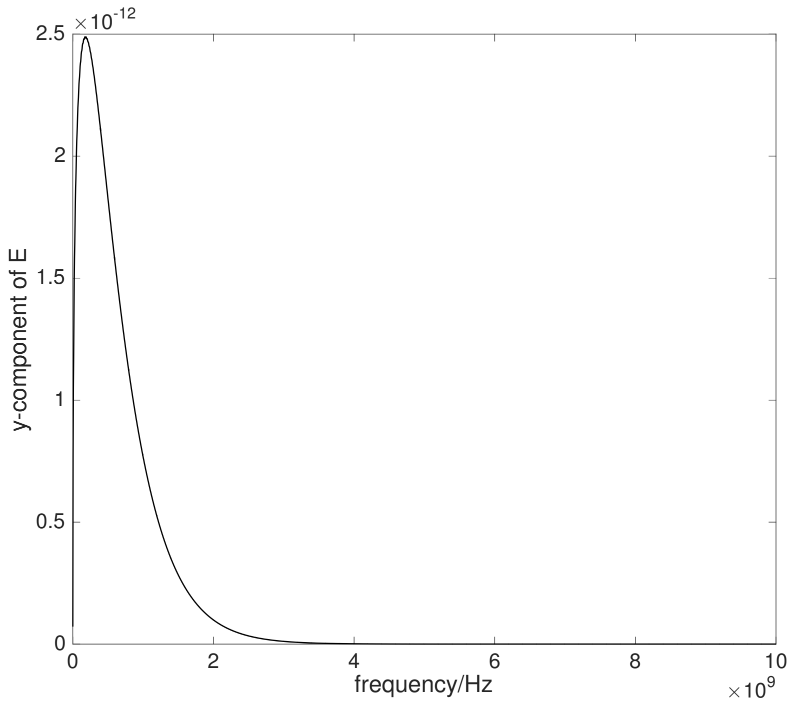


Figure 19: The absolute value of the Fourier transform of $E_y(t)$ in figure (17).

and hence $\dfrac{dr}{dt} = -\dfrac{1}{3c}\left(\dfrac{q^2}{2\pi\varepsilon_0 mc}\right)^2 \dfrac{1}{r^2}$, or

$$\mathrm{d}t = -3c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 r^2 \mathrm{d}r$$

Integration from r_0 to 0 gives

$$t = c\left(\frac{2\pi\varepsilon_0 mc}{q^2}\right)^2 r_0^3$$

Numerically $t = 1.3 \cdot 10^{-11}$ s.

S5.5

The maximum occurs when

$$\frac{d}{d\theta}\left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right) = 0$$

This gives

$$\cos\theta = \frac{1}{3\beta}(\pm\sqrt{1+15\beta^2} - 1)$$

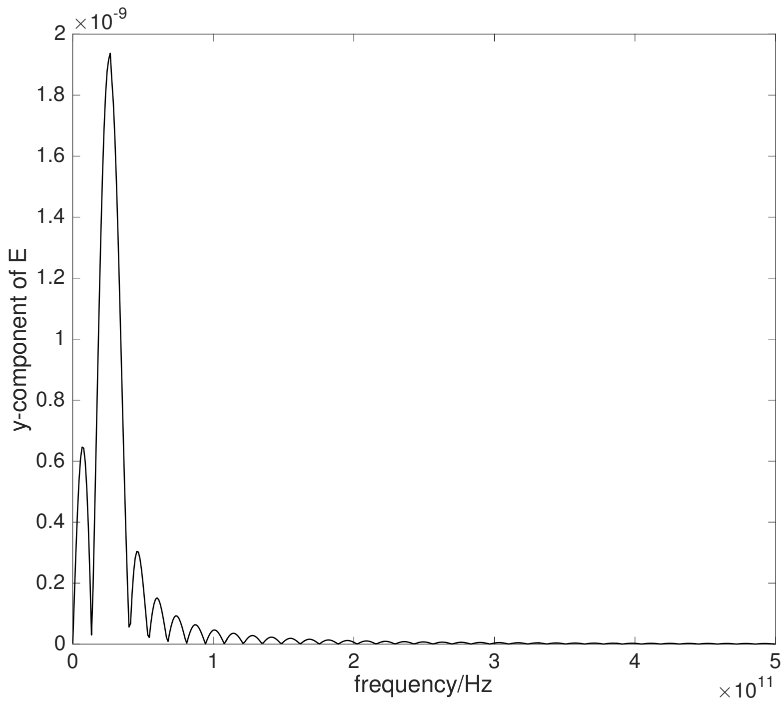


Figure 20: The absolute value of the Fourier transform of $E_y(t)$ in figure (18).

Since $\theta_{\max} \to 90°$ ($\cos\theta_{\max} = 0$) when$\beta \to 0$ we use the + sign. Then

$$\theta_{\max} = \cos^{-1}\left(\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right)$$

For $v \approx c$, $\beta \approx 1$ write $\beta = 1-\epsilon$, where $\epsilon \ll 1$, and expand to first order in ϵ:

$$\left(\frac{\sqrt{1+15\beta^2}-1}{3\beta}\right) = \ldots = 1 - \frac{1}{4}\epsilon$$

Evidently $\theta_{\max} \approx 0$, so $\cos\theta_{\max} \approx 1 - \frac{1}{2}\theta_{\max}^2 = 1 - \frac{1}{4}\epsilon$, or

$$\theta_{\max} \approx \sqrt{\frac{\epsilon}{2}} = \sqrt{0.5(1-\beta)}$$

Let

$$f = \frac{(dP/d\Omega|_{\theta_m})_{\text{ur}}}{(dP/d\Omega|_{\theta_m})_{\text{rest}}} = \left(\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}\right)_{\text{ur}}$$

Now since $\sin^2\theta_{\max} \approx \epsilon/2$, and

$$(1-\beta\cos\theta_{\max}) \approx 1-(1-\epsilon)(1-0.25\epsilon) \approx 1-(1-\epsilon-0.25\epsilon) = 1.25\epsilon$$

So $f = \left(\frac{4}{5}\right)^5 \frac{1}{2\epsilon^4}$. But

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(1-\epsilon)^2}} \approx \frac{1}{\sqrt{2\epsilon}}$$

Therefore $\epsilon = \frac{1}{2\gamma^2}$ and

$$f = \left(\frac{4}{5}\right)^5 \frac{1}{2}(2\gamma^2)^4 = 2.62\gamma^8$$

S5.6

Use

$$\frac{dP}{d\Omega} = \frac{q^2}{16\pi^2\varepsilon_0}\frac{|\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \boldsymbol{a})|^2}{(\hat{\boldsymbol{\imath}} \cdot \boldsymbol{u})^5}$$

Also $\hat{\boldsymbol{\imath}} \cdot \boldsymbol{u} = c(1-\beta\cos\theta)$, $\boldsymbol{a} \cdot \boldsymbol{u} = ac\sin\theta\cos\phi$ and $u^2 = c^2 + v^2 - 2cv\cos\theta$. Since

$$\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \hat{\boldsymbol{a}}) = (\hat{\boldsymbol{\imath}} \cdot \boldsymbol{a})\boldsymbol{u} - (\hat{\boldsymbol{\imath}} \cdot \boldsymbol{u})\boldsymbol{a}$$

then

$$|\hat{\boldsymbol{\imath}} \times (\boldsymbol{u} \times \hat{\boldsymbol{\imath}})|^2 = a^2c^2((1-\beta\cos\theta)^2 - (1-\beta^2)(\sin\theta\cos\phi)^2)$$

and

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2 a^2}{16\pi^2 c}\frac{((1-\beta\cos\theta)^2 - (1-\beta^2)(\sin\theta\cos\phi)^2)}{(1-\beta\cos\theta)^5}$$

Integrate over θ and ϕ. The integrals over ϕ are easy. The remaining integral is

$$P = \frac{\mu_0 q^2 a^2}{6\pi^2 c}\int_0^\pi \frac{(2(1-\beta\cos\theta)^2 - (1-\beta^2)\sin^\theta)}{(1-\beta\cos\theta)^5}\sin\theta\,\mathrm{d}\theta$$

Use change of variables to $y = (1-\beta\cos\theta)$. The integral can then be solved by Wolphram Alpha. The result is

$$P = \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c}$$

where $\gamma = \frac{1}{\sqrt{1-\beta^2}}$.

Is this consistent with the Liénard generalization of the Larmor formula?

$$P = \frac{\mu_0 q^2 \gamma^6}{6\pi^2 c}\pi\left(a^2 - \frac{|\boldsymbol{v} \times \boldsymbol{a}|^2}{c^2}\right)$$

Now $\boldsymbol{v} \times \boldsymbol{a} = va(\hat{\boldsymbol{z}} \times \hat{\boldsymbol{x}}) = va\hat{\boldsymbol{y}}$, so

$$a^2 - \frac{|\boldsymbol{v} \times \boldsymbol{a}|^2}{c^2} = a^2\gamma^{-2}$$

Then $P = \frac{\mu_0 q^2 a^2 \gamma^4}{6\pi c}$ which is the same as above.

S5.7

a) To counteract the radiation reaction you must exert a force $\boldsymbol{F}_{\rm e} = -\dfrac{\mu_0 q^2}{6\pi c}\dfrac{d\boldsymbol{a}}{dt}$. For circular motion , $\boldsymbol{r}(t) = R(\cos(\omega t)\hat{\boldsymbol{x}} + \sin(\omega t)\hat{\boldsymbol{y}})$, $\boldsymbol{v}(t) = R\omega(-\sin(\omega t)\hat{\boldsymbol{x}} + \cos(\omega t)\hat{\boldsymbol{y}})$, and $\dfrac{d\boldsymbol{a}}{dt} = -\omega^2\boldsymbol{v}$. Then

$$\boldsymbol{F}_{\rm e} = \frac{\mu_0 q^2}{6\pi c}\omega^2\boldsymbol{v}$$

The power one has to supply is

$$P = \boldsymbol{F}_{\rm e}\cdot\boldsymbol{v} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2$$

This is the same as the radiated power given by the Larmor formula.

b) For simple harmonic motion, $\boldsymbol{r}(t) = A\cos(\omega t)\hat{\boldsymbol{z}}$, $\boldsymbol{v} = -A\omega\sin(\omega t)\hat{\boldsymbol{z}}$ and $\boldsymbol{a} = -\omega^2\boldsymbol{r}$. Then

$$\boldsymbol{F}_{\rm e} = \frac{\mu_0 q^2}{6\pi c}\omega^2\boldsymbol{v} \qquad (43)$$

$$P_{\rm e} = \frac{\mu_0 q^2}{6\pi c}\omega^2 v^2 \qquad (44)$$

Now $v^2 = A^2\omega^2\sin^2(\omega t)$ but $P_{\rm rad} = \dfrac{\mu_0 q^2}{6\pi c}\omega^4 A^2\cos^2(\omega t) \neq P_{\rm e}$. However, the time averages are the same.

c) In free fall, $\boldsymbol{r} = \dfrac{1}{2}gt^2\hat{\boldsymbol{y}}$, $\boldsymbol{v} = gt\hat{\boldsymbol{y}}$, $\boldsymbol{a} = g\hat{\boldsymbol{y}}$ and $d\boldsymbol{a}/dt = \boldsymbol{0}$. So $\boldsymbol{F}_{\rm e} = \boldsymbol{0}$. The radiation action is zero and hence $P_{\rm e} = 0$. But there is radiation $P_{\rm rad} = \dfrac{\mu_0 q^2}{6\pi c}g^2$. Evidently energy is being continuously extracted from the nearby fields. This paradox persists even in the exact solution.

6 Relativistic electrodynamics

6.1

a) What's the percent error introduced when you use Galileo's rule, instead of Einstein's, with $v_{\text{AB}} = 8$ km/h and 96 km/h?

b) Suppose that you could run at half the speed of light down the corridor of a train going three quarters the speed of light. What would your speed be relative to the ground?

c) Prove, using Einstein's formula for velocity addition that if $v_{\text{AB}} < c$ and $v_{\text{BC}} < c$, then $v_{\text{AC}} < c$

Problem 12.3 in Griffiths 5th ed..

6.2

As the outlaws escape in their getaway car, which goes $\frac{3}{4}c$, the police officer fires a bullet from the pursuit car, which only goes $\frac{1}{2}c$. The muzzle velocity of the bullet (relative to the gun) is $\frac{1}{3}c$. Does the bullet reach its target (a) according to Galileo, (b) according to Einstein?

Problem 12.4 in Griffiths 5th ed..

6.3

In a laboratory experiment a muon is observed to travel 800 m before disintegrating. A graduate student looks up the lifetime of a muon (2×10^{-6} s) and concludes that its speed was

$$v = \frac{800\,\text{m}}{2\times10^{-6}\text{s}} = 4\times10^{8}\,\text{m}$$

Faster than light! Identify the student's error, and find the actual speed of the muon.

Problem 12.7 in Griffiths 5th ed..

6.4

A Lincoln Continental is twice as long as a VW Beetle, when they are at rest. As the Continental overtakes the VW, going to a speed trap, a (stationary) policeman observes that they both have the same length. The VW is going at half the speed of light. How fast is the Lincoln going? (Leave your answer as a multiple of c)

Problem 12.9 in Griffiths 5th ed..

6.5

a) Write out the matrix that describes a Galilean transformation.

b) Write out the matrix describing a Lorentz transformation along the $y-$axis.

c) Find the matrix describing a Lorentz transformation with velocity v along the $x-$axis, followed by a Lorentz transformation with velocity $\bar{v}$ along the y-axis. Does it matter in what order the transformations are carried out?

6.6

If a particle's kinetic energy is n times its rest energy, what is its speed?

Problem 12.30 in Griffiths 5th ed..

6.7

A particle of mass m whose total energy is twice its rest energy collides with an identical particle at rest. If they stick together, what is the mass of the resulting composite particle? What is its velocity?

Problem 12.33 in Griffiths 5th ed..

6.8

In the past, most experiments in particle physics involved stationary targets: one particle (usually a proton or an electron) was accelerated to a high energy E, and collided with a target particle at rest. Far higher relative energies are obtainable (with the same accelerator) if you accelerate both particles to energy E, and fire them at each other. Classically the energy $\bar{E}$ on one particle, relative to the other, is just $4E$ (why?)- not much of a gain (only a factor of 4). But relativistically the gain can be enormous. Assuming the two particles have the same mass, m, show that

$$\bar{E} = \frac{2E^2}{mc^2} - mc^2$$

Suppose that you use protons ($mc^2 = 1$ GeV) with $E = 30$ GeV. What $\bar{E}$ do you get? What multiple of E does this amount to?

Problem 12.35 in Griffiths 5th ed..

6.9

a) The electrons in the large storage ring of MAX IV has 3 GeV energy. Determine $\beta = v/c$ for the electrons.

b) In the large hadron collider at Cern protons are accelerated to high energies in two directions such that they collide. The protons have 7 TeV energy. Determine $\beta = v/c$ for the protons and compare that with β for the electrons in MAX IV. How many m/s difference is there between the electrons of MAX IV and the protons of LHC? How many m/s difference is there between the speed of light and the 7 TeV protoms?

c) Assume that MAX IV would like to beat LHC in particle speed. What energy does that require?

d) Determine the ratio $\bar{E}/E$, where $\bar{E}$ is the energy when two 7 TeV protons collide and E the energy when a 7 TeV proton collides with a proton at rest.

6.10

a) Charge q_A is at rest at the origin in system S; charge q_B flies by at speed v on a trajectory parallel to the $x-$axis, but at $y = d$. What is the electromagnetic force on q_B as it crosses the $y-$axis?

b) Now study the same problem from system $\bar{S}$, which moves to the right with speed v. What is the force on q_B when q_A passes the $\bar{y}$-axis? [Do it two ways: (i) by using your answer to (a) and transforming the force; (ii) by computing the fields in $\bar{S}$ and using the Lorentz force law.]

Problem 12.45 in Griffiths 5th ed..

6.11

Consider a beam of protons traveling in an accelerator. We assume that along a quite long distance the beam can be approximated by a circular cylinder with radius a and constant charge density ρ, seen from the stationary system S_0. The $z-$axis is the symmetry axis of the cylinder and the cylinder travels with velocity $\boldsymbol{v}_0\hat{\boldsymbol{z}}$.

a) Determine the electric field $\boldsymbol{E}_0$ and magnetic flux density $\boldsymbol{B}_0$ for $r_\text{c} < a$ seen from S_0.

b) Show that the Lorentz force $q(\boldsymbol{E}_0 + \boldsymbol{v} \times \boldsymbol{B}_0)$ goes to zero as $v \to c$. This explains why bunches of particles in an accelerator can stay confined for a long time.

c) What is the electric field $\bar{\boldsymbol{E}}$ and magnetic flux density for $r_c < a$ in a system $\bar{S}$ that travels with the protons? What is the force on an electron in the system $\bar{S}$

compared to the force in a system S at rest relative the accelerator?

Relativistic electrodynamics: Answers and solutions

S6.1

a) $6.7 \cdot 10^{-14}\%$ error.

b) $\dfrac{10}{11}c$.

c) Let $\beta = v_{\mathrm{AC}}/c$, $\beta_1 = v_{\mathrm{AB}}/c$ and $\beta_2 = v_{\mathrm{BC}}/c$. then

$$\beta^2 = \frac{\beta_1^2 + 2\beta - 1\beta_2 + \beta_2^2}{(1 + 2\beta_1\beta_2 + \beta_1^2\beta_2^2)} = 1 - \Delta$$

where $\Delta = (1 - \beta_1^2)(1 - \beta_2^2)/(1 + \beta_1\beta_2)^2$ is clearly a positive number. So $\beta_2 < 1$, and hence $|v_{\mathrm{AC}}| < c$

S6.2

a) Velocity of bullet realtive ground: $\frac{1}{2}c + \frac{1}{3}c = \frac{10}{12}c$.
Velocity of getaway car: $\frac{3}{4}c = \frac{9}{12}c$ so buller does reach target.

b) Velocity of bullet relative ground: $\frac{c/2+c/3}{1+1/6} = \frac{20}{28}c$.
Velocity of getaway car: $\frac{3}{4}c = \frac{21}{28}c$ so bullet does not reach target.

S6.3

The student has not taken into account time dilatation of the muon's internal clock. In the laboratory, the muon lasts $\gamma\tau$, where τ is the proper lifetime $2 \cdot 10^{-6}$ s. Thus, after some analysis, $v = 0.8c$.

S6.4

$$v = \frac{\sqrt{13}}{4}c$$

S6.5

a)

$$\begin{pmatrix} ct \\ \bar{x} \\ \bar{y} \\ \bar{z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

b)

$$\Lambda = \begin{pmatrix} \gamma & 0 & -\gamma\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\beta & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

1.5 **c)** Multiply the matrices:

$$\Lambda = \begin{pmatrix} \bar\gamma & 0 & -\bar\gamma\bar\beta & 0 \\ 0 & 1 & 0 & 0 \\ -\bar\gamma\bar\beta & 0 & \bar\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma\bar\gamma & -\gamma\bar\gamma\beta & -\bar\gamma\bar\beta & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ -\bar\gamma\gamma\bar\beta & \bar\gamma\gamma\bar\beta\beta & \bar\gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Yes, the order does matter. In the other order bars and non-bars would switch.

S6.6

$$\gamma mc^2 - mc^2 = nmc^2$$

This gives $\gamma = n + 1$ and from that

$$u = \frac{\sqrt{n(n+2)}}{n+1} c$$

S6.7

Initial momentum is obtained from: $E^2 - p^2c^2 = m^2c^4$, then $p = \sqrt{3}mc$.

Initial energy: $2mc^2 + mc^2 = 3mc^2$.

Each is conserved, so final energy is $3mc^2$ and final momentum is $\sqrt{3}mc$. The mass, M, of the two particles that are stuck is obtained from

$$E^2 - p^2c^2 = (3mc^2)^n 2 - (\sqrt{3}mc)^2c^2 = 6m^2c^4 = M^2c^4$$

Then $M = \sqrt{6}m \approx 2.45m$. Some of the kinetic energy is transformed into rest energy.

the speed of the two particles is

$$v = \frac{pc^2}{E} = \frac{\sqrt{3}mcc^2}{3mc^2} = \frac{c}{\sqrt{3}}$$

S6.8

Classically, $E = \frac{1}{2}mv^2$. In a colliding beam experiment, the relativity velocity is twice the velocity of either one, so the relative energy is $4E$.

For relativistic particles we use the fact that $(E/c, p_x, p_y, p_z)$ is a four vector and hence transform according to the Lorentz transformationen. Thus we know how to transform from the system where both particles have energy W and are moving with velocities $v\hat{\boldsymbol{x}}$ and $-v\hat{\boldsymbol{x}}$, respectively, to the system where one of the particles is at rest. The energy $\bar{E}$ of the moving particle in the new system is, according to the Lorentz transformation, given by

$$\frac{\bar{E}}{c} = \gamma\frac{E}{c} - \gamma\beta p$$

where $\beta = -v/c$, $\gamma = 1/\sqrt{1-\beta^2}$ and $p = m\gamma v$. Since

$$\beta^2 = \frac{\gamma^2 - 1}{\gamma^2}$$
$$\bar{E} = m\bar{\gamma}c^2$$
$$E = m\gamma c^2$$

we get

$$\bar{\gamma} = \gamma^2(1+\beta^2) = 2\gamma^2 - 1$$

and finally

$$\bar{E} = \frac{\bar{\gamma}}{\gamma}E = 2\gamma E - E/\gamma = 2\frac{E^2}{mc^2} - mc^2$$

For $E = 30$ GeV and $mc^2 = 1$ GeV, we have $\bar{E} = 1799$ GeV$= 60E$.

S6.9

a) Kinetic energy is $E = m\gamma c^2 - mc^2$. This gives $\beta = \sqrt{1 - \left(\frac{mc^2}{E + mc^2}\right)^2}$. With $m = 9.11 \cdot 10^{-31}$ kg one get $mc^2 = 0.51$ MeV and $\beta = 0.999999986$.

b) With $m = 1.673 \cdot 10^{-27}$ kg, $mc^2 = 0.939$ GeV and $\beta = 0.999999991$. The protons of LHC are just 1.5 m/s faster than the the electrons in MAX IV. They are on the other hand only 2.7 m/s slower than the speed of light.

c) They have to go up to 3.8 GeV, so the 1.5 m/s cost 0.8 GeV.

d) $\frac{\bar{E}}{E} = \frac{(2E^2/mc^2 - mc^2)}{E} = 14900$

S6.10

a) Fields of A and B: $\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_A}{d^2}\hat{\boldsymbol{y}}$ and $\boldsymbol{B} = \boldsymbol{0}$. So force on q_B is $\boldsymbol{F} = \frac{1}{4\pi\varepsilon_0}\frac{q_A q_B}{d^2}\hat{\boldsymbol{y}}$.

b) (i) The particle is at rest in $\bar{S}$ and then the force is $\bar{\boldsymbol{F}} = \frac{\gamma}{4\pi\varepsilon_0}\frac{q_A q_B}{d^2}\hat{\boldsymbol{y}}$

(ii) With $\theta = 90°$ we get $\boldsymbol{E} = \frac{1}{4\pi\varepsilon_0}\frac{q_A(1-\beta^2)}{(1-\beta^2)^{3/2}}\frac{1}{d^2}\hat{\boldsymbol{y}} = \frac{\gamma}{4\pi\varepsilon_0}\frac{q_A}{d^2}\hat{\boldsymbol{y}}$

$\boldsymbol{B} \neq \boldsymbol{0}$ but since $v_B = 0$ in $\bar{S}$, there is no magnetic force anyway, and $\boldsymbol{F} = \frac{\gamma}{4\pi\varepsilon_0}\frac{q_A q_B}{d^2}\hat{\boldsymbol{y}}$

S6.11

a) Gauss law gives the electric field for $r_c < a$:

$$\boldsymbol{E}(r_c) = \frac{\rho\pi r_c^2}{\varepsilon_0 2\pi r_c}\hat{\boldsymbol{r}}_c = \frac{\rho r_c}{2\varepsilon_0}\hat{\boldsymbol{r}}_c$$

The current density is $\boldsymbol{J}(r_c) = \rho\boldsymbol{v}$ for $r_c < a$ and zero for $r_c > a$. Amperes law gives

$$\boldsymbol{B}(r_c) = \frac{\mu_0 v\rho\pi r_c^2}{2\pi r_c}\hat{\boldsymbol{\phi}} = \frac{\mu_0 v\rho r_c}{2}\hat{\boldsymbol{\phi}}$$

b) The force on a particle traveling with the beam and at a distance r_c from the symmetry axis is:

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) = \frac{q\rho r_c}{2\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)\hat{\boldsymbol{r}}_c$$

We see that this force goes to zero as $v \to c$.

c) The electric and magnetic fields in system $\bar{S}$ are

$$\bar{\boldsymbol{E}} = \gamma(\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B}) = \gamma\frac{\rho r_c}{2\varepsilon_0}\left(1 - \frac{v^2}{c^2}\right)\hat{\boldsymbol{r}}_c \tag{45}$$

$$\bar{\boldsymbol{B}} = \gamma\left(\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}\right) = \boldsymbol{0} \tag{46}$$

No surprise that $\bar{\boldsymbol{B}} = \boldsymbol{0}$.

We see that the force is γ times larger in $\bar{S}$ than in S. This is accordance with the transformation formula for forces.

7 Motion of relativistic particles in electromagnetic fields

Solving the equation of motion for a particle with MATLAB

Consider a particle with charge q and mass m that moves in a vacuum region with a static electric field $\boldsymbol{E}(x,y,z)$ and a static magnetic flow density $\boldsymbol{B}(x,y,z)$. The motion of the particle is governed by Newton's second law

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

where $\boldsymbol{F}$ is the force and $\boldsymbol{p} = m\boldsymbol{\eta} = m\gamma\boldsymbol{u} = m\dfrac{\boldsymbol{u}}{\sqrt{1-u^2/c^2}}$ is the relativistic momentum. Here $\boldsymbol{\eta}$ is the proper velocity and m is the proper mass (sometimes called rest mass). The force is the Lorentz' force

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}).$$

Thus

$$m\frac{d\gamma\boldsymbol{u}}{dt} = q(\boldsymbol{E}(\boldsymbol{r}) + \boldsymbol{u}\times\boldsymbol{B}(\boldsymbol{r})). \tag{47}$$

There are some special cases when this equation can be solved analytically. In other cases one has to solve the equations by numerical methods. To use the MATLAB routines we need to rewrite the equation as a system of coupled first order ordinary differential equations. To do this we first use the relation

$$\frac{d(m\gamma\boldsymbol{u})}{dt} = m\gamma\left(\boldsymbol{a} + \frac{\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{a})}{c^2-u^2}\right) \tag{48}$$

where $\boldsymbol{a}$ is the acceleration $\boldsymbol{a} = \dfrac{d\boldsymbol{u}}{dt}$. The derivation of this expression is left as an exercise. From this relation and equation (47) one can show that

$$\boldsymbol{a} = \frac{q}{m\gamma}\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B} - \frac{1}{c^2}\boldsymbol{u}(\boldsymbol{u}\cdot\boldsymbol{E})\right) \tag{49}$$

Also the derivation of this relation is left as an exercise.

The non-relativistic case

We show in detail how the non-relativistic version of equation (50) can be solved by MATLAB. The relativistic case is left to the reader. The non-relativistic version of equation (49) is obtained by letting $\dfrac{u}{c} = 0$ and thus $\gamma = 1$

$$\boldsymbol{a} = \frac{q}{m}\left(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B}\right) \tag{50}$$

This equation is a system of three equations:

$$\frac{d}{dt}\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} a_1(u_x, u_y, u_z) \\ a_2(u_x, u_y, u_z) \\ a_3(u_x, u_y, u_z) \end{pmatrix} \tag{51}$$

where

$$\begin{aligned} a_1(u_x, u_y, u_z) &= \frac{q}{m}\left(E_x + u_y B_z - u_z B_y\right) \\ a_2(u_x, u_y, u_z) &= \frac{q}{m}\left(E_y + u_z B_x - u_x B_z\right) \\ a_3(u_x, u_y, u_z) &= \frac{q}{m}\left(E_z + u_x B_y - u_y B_x\right) \end{aligned} \tag{52}$$

The system (51) is not sufficient, even if we know the velocity at a certain time. The reason is that $\boldsymbol{E}$ and $\boldsymbol{B}$ are space dependent and we then need to know also the position of the particle. For this reason we add three equations to the system (51) as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \\ u_z \\ a_1(u_x, u_y, u_z) \\ a_2(u_x, u_y, u_z) \\ a_3(u_x, u_y, u_z) \end{pmatrix}$$

Initial conditions are needed for the velocity and position. For convenience we solve the equation for $t > 0$ and then we need the initial conditions $(x(0), y(0), z(0))$ and $(u_x(0), u_y(0), u_z(0))$.

An analytic solution for the case when $\boldsymbol{E} = E_z\hat{\boldsymbol{z}}$ and $\boldsymbol{B} = B_z\hat{\boldsymbol{z}}$

Assume that the particle starts at the origin $(x, y, z) = (0, 0, 0)$ with velocity $(u_x, u_y, u_z) = (10^4, 0, 0)$ m/s. When both $\boldsymbol{E}$ and $\boldsymbol{B}$ are directed in the $z-$direction and they are constant in space then the equation of motion can be solved analytically. From (52) we get the three equations

$$\begin{aligned} \frac{du_x}{dt} &= \frac{q}{m}\left(u_y B_z\right) \\ \frac{du_y}{dt} &= -\frac{q}{m}\left(u_x B_z\right) \\ \frac{du_z}{dt} &= \frac{q}{m}\left(E_z\right) \end{aligned} \tag{53}$$

That means that the motion in the $xy-$plane is decoupled from the motion in the z.plane. In the $xy-$plane the particle moves along a circular path with radius

$R = \dfrac{mu}{qB}$, where $u = \sqrt{u_x^2 + u_y^2} = 10^4$ m/s is a constant speed, and in the $z-$direction it has a constant acceleration such that $z(t) = \dfrac{q}{2m} E_z t^2$.

We can use this analytic solution to check the accuracy of the different ode-solvers in MATLAB.solution.

The MATLAB script

There are at least eight different solvers for ordinary differential equations in MATLAB. It is hard to tell which of these solvers that is the best one for a specific problem. Sometimes one has to try different solvers and pick the one that gives the best result. When we applied the different solvers to the problem above the solver ode23t gave the best results. Here is an example of a script that solves the equation

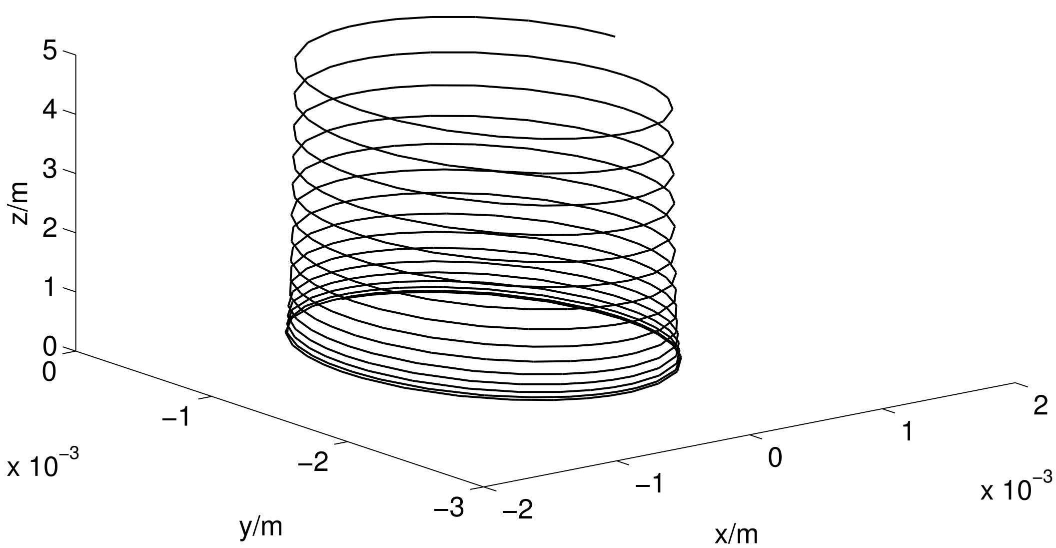
```
clear
% This script determines and plots the trajectory of a
% charged particle in a static electric field
% and a static magnetic flux density
% y(1:3) is the position vector and y(4:6) the velocity vector
[t,y]=ode23t(@uppgift1NonR,[0 1e-3],[0 0 0 100 0 0]);
figure(1)
% Plot the trajectory
plot3(y(:,1),y(:,2),y(:,3));
figure(2)
% Plot z(t)
plot(t,y(:,3))
```

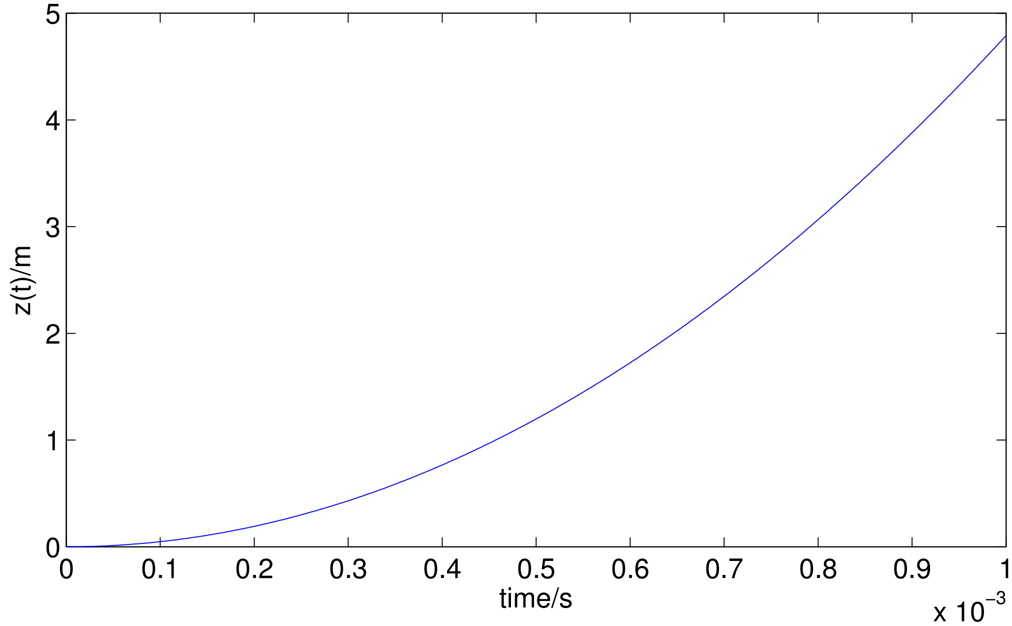
Here [00.01] says that it solves from $t = 0$ to $t = 0.01$ s and
$[00010000] = [x(0)y(0)z(0)u_x(0)u_y(0)u_z(0)]$ in m/s. The routine ode23 calls the function uppgift1NonR that can be written as

```
function yout=uppgift1NonR(t,yin)
% This function is used by the script uppgift1NonRel.m
% q= charge, m0=mass, E=electric field, B=magnetic flux
% density;
% yin is a vector for which yin(1:3) is the position and
%  yin(4:6) is the velocity of the particle.
q=1.6e-19;
m0=1.67e-27;
E=[0 0 0.1];
B=[0 0 0.001];
% Notice that one can use the function dot to do a scalar
% product in MATLAB
v=sqrt(dot(yin(4:6),yin(4:6)));
```

```
% Notice that one can use the function cross to do a
% vector product in MATLAB
vcrossB=cross(yin(4:6),B);
yout=[yin(4:6);q/m0*(E'+vcrossB')];
```

The program determines the trajectory of a proton that at $t = 0$ has velocity $(0, 0, 100)$ m/s and position $(0, 0, 0)$ and moves in a region with the electric field $\boldsymbol{E} = (0, 0, 0.1)$ V/m and magnetic flux density $\boldsymbol{B} = (0, 0, 0.001)$ T. When we run the script we get the trajectory of the particle and the graph of $z(t)$ as a function of t as seen by the figures





7.1

Write a Matlab program that solves the equation of motion for a relativistic particle in an arbitrary space dependent electric field $\boldsymbol{E}(\boldsymbol{r})$ and magnetic flux density $\boldsymbol{B}(\boldsymbol{r})$. The program should be able to plot the trajectory $(x(t), y(t), z(t))$ for a given time interval and given initial conditions $(x(0), y(0), z(0))$ and $(u_x(0), u_y(0), u_z(0))$.
Hint: First write the system of equations as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \\ z \\ u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} u_x \\ u_y \\ u_z \\ \alpha_1(u_x, u_y, u_z) \\ \alpha_2(u_x, u_y, u_z) \\ \alpha_3(u_x, u_y, u_z) \end{pmatrix}$$

and use the routine ode23t. The plot subroutine plot3 is suitable for trajectories in three dimensions.

7.2

Check your Matlab program by first considering $\boldsymbol{E} = (0, 0, 0)$ and $\boldsymbol{B} = B_0(0, 0, 1)$. Determine $(x(t), y(t), z(t))$ when $(x(0), y(0), z(0)) = 0$ and $(u_x(0), u_y(0), u_z(0)) = (10^8, 0, 0)$ m/s. Let $B_0 = 1$ mT. The time should be large enough so that the particle comes back to where it started from. Consider first an electron and then a proton. You can compare the radius of the circle that the charge moves along with the analytic result.

7.3

a) Consider a particle with mass m and charge q. Show that when $\boldsymbol{E} = (0, 0, E_0)$, $\boldsymbol{B} = (0, 0, 0)$, $(x(0), y(0), z(0)) = \mathbf{0}$ and $(u_x(0), u_y(0), u_z(0)) = \mathbf{0}$ then the velocity of the particle is given by $u_x(t) = u_y(t) = 0$ and

$$u_z(t) = \frac{qE_0ct}{\sqrt{(mc)^2 + (qE_0t)^2}}$$
$$z(t) = \frac{c}{qE_0}\sqrt{(mc)^2 + (qE_0t)^2} - \frac{mc^2}{qE_0}$$

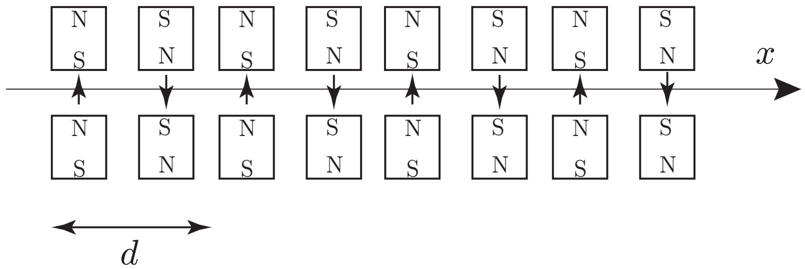
b) Check your Matlab program by considering an electron that is accelerated by the field $\boldsymbol{E} = (0, 0, -10^6)$ V/m and $\boldsymbol{B} = B_0(0, 0, 0)$. Determine $(x(t), y(t), z(t))$ when $(x(0), y(0), z(0)) = 0$ and $(u_x(0), u_y(0), u_z(0)) = (0, 0, 0)$ m/s. Plot $u_z(t)$ and $z(t)$ as a function of time in the interval $0 < t < 20$ ns. Compare with the analytic result.

7.4

Consider $\boldsymbol{E} = (0, 0, E_0)$ and $\boldsymbol{B} = (0, 0, B_0)$. Use your Matlab program to plot the trajectory $(x(t), y(t), z(t))$ when $(x(0), y(0), z(0)) = \mathbf{0}$ and $(u_x(0), u_y(0), u_z(0)) = (10^8, 0, 0)$ m/s. Consider first an electron in the time interval $0 < t < 5$ ns and then a proton in the interval $0 < t < 5\,\mu$s. Let $B_0 = 0.1$ T and $E_0 = 10^4$ V/m.

7.5

In the synchrotrons of Max IV much of the light is created by letting the electrons pass through undulators. An undulator usually consists of a number of permanent magnets arranged as in the figure. They create an x-dependent magnetic field that can be approximated by $\boldsymbol{B}(x) = \left(0, 0, B_0 \cos\left(\frac{2\pi x}{d}\right)\right)$ where d is the period of the undulator, given by the figure. Let $B_0 = 0.01$ T, $0 < x < 15d$, $d = 2$ cm, $(x(0), y(0), z(0)) = (0, 0, 0)$ and $(u_x(0), u_y(0), u_z(0)) = (0.999 \cdot c, 0, 0)$, where c is the speed of light. Plot the trajectory $(x(t), y(t))$ for the interval $0 < t < 1$ ns.



N
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N
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x
d

Motion of relativistic particles in electromagnetic fields: Answers and solutions

S7.1

The main program can look like this

```
clear
% start with initial conditions
%[t,y]=ode23t(@uppgift1rhs,[0 1/3e8],[0 0 0 2.999e8 0 0]);
[t,y]=ode23t(@uppgift1rhs,[0 5e-8],[0 0 0 1.e8 10 0]);
figure
% Plot of the trajectory in 3-dimensions
plot3(y(:,1),y(:,2),y(:,3));
% Plot of trajectory in the xy-plane
%plot(y(:,1),y(:,2))
axis equal
% plot of speed in the z-dirction as a function of time
%plot(t,y(:,6))
```

Here is the function that is called by ode23t

```
function yout=uppgift1rhs(t,yin)
% This function is used by the script uppgift1Matlab.m
% q= charge, m0=mass, c= speed of light,
% E=electric field, B=magnetic flux
% density,d=distance between pairs of mgnets in an undulator.
% yin is a vector for which yin(1:3) is the position and yin(4:6) is the
% velocity of the particle.
q=1.6e-19;
m0=1.67e-27;%9.1e-31;
c=3e8;
d=0.05;
%E=[0 0 1e6];
E=[0 0 0];
%B=[0 0 0];
%B=[0 0 .001];
B=[0 0 .1*cos(2*pi*yin(1)/d)];
% Notice that one can use the function dot to do
% a scalar product in Matlab
v=sqrt(dot(yin(4:6),yin(4:6)));
gamma=1/sqrt(1-(v/c)^2);
vdotE=dot(yin(4:6),E);
% Notice that one can use the function cross to do
% a vector product in Matlab
```

```
vcrossB=cross(yin(4:6),B);
yout=[yin(4:6);q/m0/gamma*(E'+vcrossB'-1/c^2*yin(4:6)*vdotE)];}
```

S7.2

The analytic result is that $\omega = \frac{qB_0}{m\gamma} = \frac{qB_0\sqrt{1-(u/c)^2}}{m}$. The radius is given by $R = u/\omega = \frac{um}{qB_0\sqrt{1-(u/c)^2}}$. When $u = 10^8$ m/s, $B_0 = 1$ mT, $q = 1.6 \cdot 10^{-19}$ m/s then we get the radius R and period $T = 2\pi/\omega$ to be

$R = 0.60$ m and $T = 37.9$ ns for the electron.

$R = 1107$ m and $T = 69.56\,\mu$s for the proton.

The program gives these values with high accuracy.

S7.3

a) Integrate $\frac{dm\gamma u_z(t)}{dt} = qE_0$ from 0 to t. This gives $\frac{u_z(t)}{\sqrt{1-(u_z(t)/c)^2}} = \frac{qE_0}{m}t$. The solution is

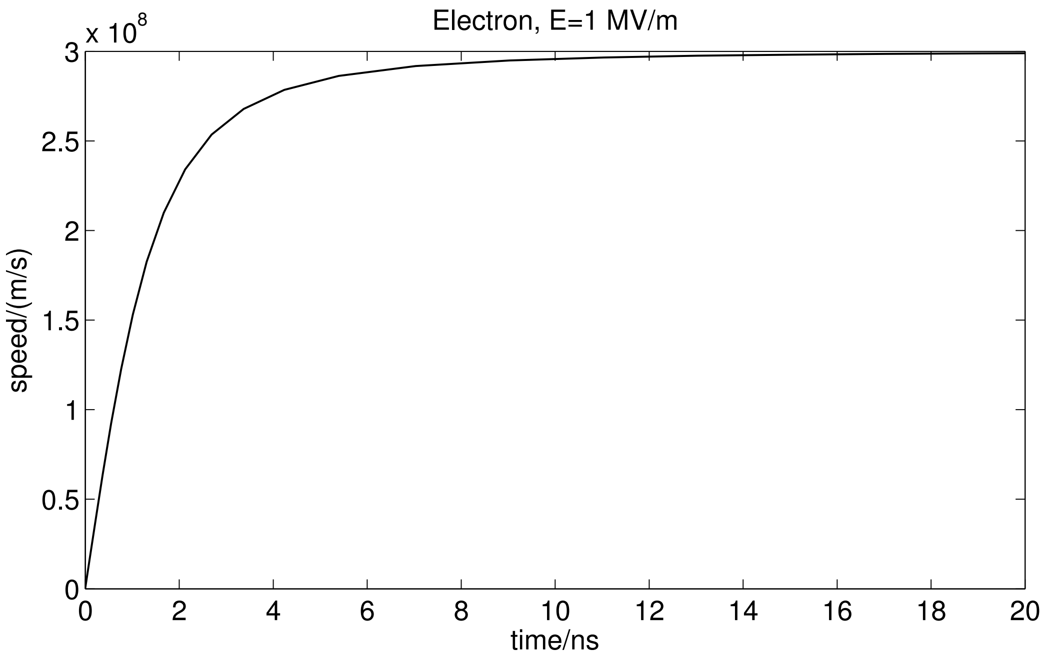
$$u_z(t) = \frac{qE_0ct}{\sqrt{(mc)^2 + (qE_0t)^2}} \tag{54}$$

Notice that $u_z(t)$ is a function of qE_0t/m. This means that the electric field required to obtain a speed of a particle with charge q in a certain time is proportional to the mass of the particle.

To get $z(t)$we integrate Eq. 54 from 0 to t and use $z(0) = 0$. This gives

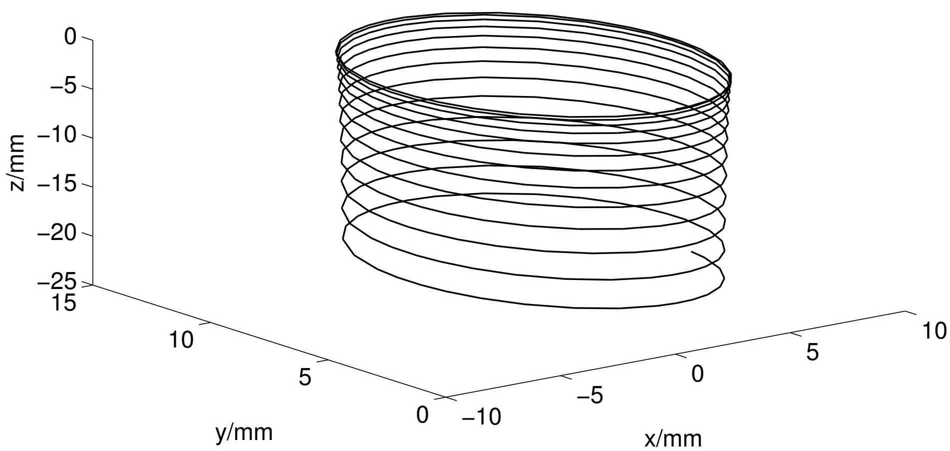
$$z(t) = \frac{c}{qE_0}\sqrt{(mc)^2 + (qE_0t)^2} - \frac{mc^2}{qE_0}$$

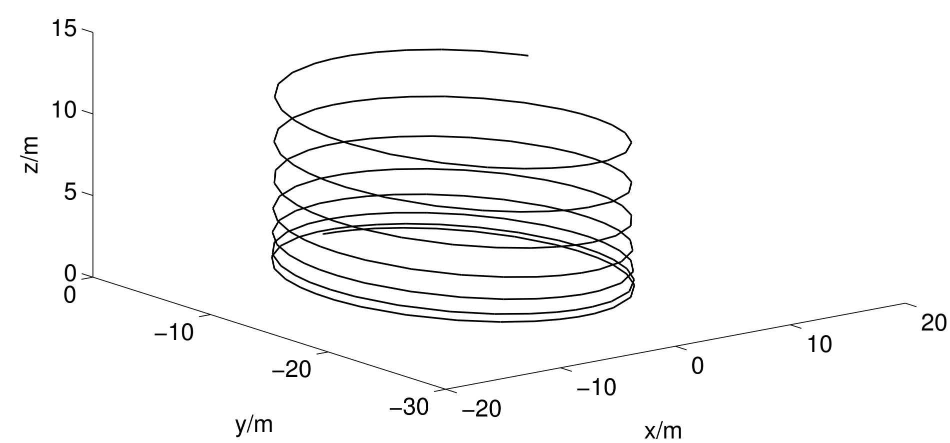
b)


Electron, E=1 MV/m
x 10^8
3
2.5
2
1.5
1
0.5
0
speed/(m/s)
0
2
4
6
8
10
12
14
16
18
20
time/ns

S7.4

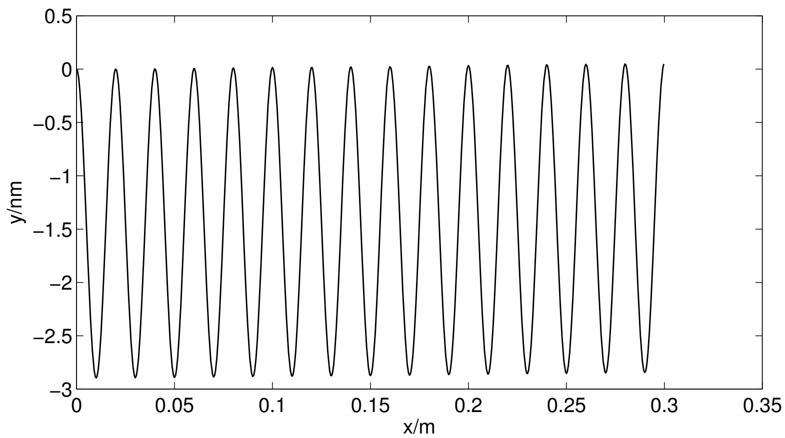
The initial conditions are $(x(0), y(0), z(0)) = \mathbf{0}$ and $(u_x(0), u_y(0), u_z(0)) = (10^8, 0, 0)$ m/s. The first graph shows the electron trajectory for the time interval $0 < t < 5$ ns when $\boldsymbol{E} = (0, 0, 10^4)$ V/m and $\boldsymbol{B} = (0, 0, 0.1)$ T. The second graph shows the proton trajectory for the time interval $0 < t < 5\,\mu$s when $\boldsymbol{E} = (0, 0, 10^4)$ V/m and $\boldsymbol{B} = (0, 0, 0.1)$ T.





S7.5

The figure shows the electron trajectory in an undulator for the time interval $0 < t < 1$ ns. $\boldsymbol{E} = (0, 0, 0)$ V/m and $\boldsymbol{B} = (0, 0, 0.01 \cos(2\pi x/d)$ T, where $d = 2$ cm, and $(x(0), y(0), z(0)) = \mathbf{0}$, $(u_x(0), u_y(0), u_z(0)) = (0.999c, 0, 0)$.



8 Appendix 1: Bessel functions

In wave propagation problems the Bessel differential equation often appears, especially in problems showing axial or spherical symmetries. This appendix collects some useful and important results for the solution of the Bessel differential equation.

Bessel and Hankel functions

The Bessel differential equation is

$$z^2 \frac{d^2}{dz^2} Z_n(z) + z \frac{d}{dz} Z_n(z) + (z^2 - n^2) Z_n(z) = 0 \tag{55}$$

where n is assumed integer[4].

There exist two linearly independent solutions to this differential equation. One is regular at the origin, $z = 0$, and this solution is the Bessel function $J_n(z)$ of order n. The argument z is a complex number. These solutions are often called cylindrical Bessel function of order n, which stresses the affinity to problems with the axial symmetry. The Bessel functions $J_n(z)$ are defined real-valued for a real argument z. An everywhere in the complex z-plane convergent power series is

$$J_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{z}{2}\right)^{n+2k} \tag{56}$$

We notice immediately that $J_n(z)$ is an even function for even n and odd for odd n, *ie.*

$$J_n(-z) = (-1)^n J_n(z)$$

A commonly used integral representation of the Bessel functions is

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos\left(z \sin t - nt\right)\, \mathrm{d}t = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\cos t} \mathrm{e}^{\mathrm{i}n\left(t - \frac{1}{2}\pi\right)}\, \mathrm{d}t \tag{57}$$

From this integral representation, we see that the Bessel functions for positive and negative *integer* orders, n, are related to each other.

$$J_{-n}(z) = (-1)^n J_n(z)$$

The power series representation in (56) implies that for small arguments we have

$$J_n(z) = \frac{1}{n!} \left(\frac{z}{2}\right)^n + O(z^{n+2})$$

[4] A more general definition with eg. complex-valued n is also possible, but the expressions and the results often differ.

	Root j			
Order n	1	2	3	4
0	2.4048	5.5201	8.6537	11.7915
1	3.8317	7.0156	10.1735	13.3237
2	5.1356	8.4172	11.6198	14.7960
3	6.3802	9.7610	13.015	16.2235
4	7.5883	11.0647	14.3725	17.6160

Table 4: Table of the roots ξ_{nj} to $J_n(z)$.

	Root j			
Order n	1	2	3	4
0	3.8317	7.0156	10.1735	13.3237
1	1.8412	5.3314	8.5363	11.7060
2	3.0542	6.7061	9.9695	13.1704
3	4.2012	8.0152	11.3459	14.5858
4	5.3175	9.2824	12.6819	15.9641

Table 5: Table of the roots η_{nj} to $J'_n(z)$.

For large arguments hold ($-\pi < \arg z < \pi$)

$$J_n(z) = \left(\frac{2}{\pi z}\right)^{1/2} \left\{P_n(z)\cos\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right) - Q_n(z)\sin\left(z - \frac{n\pi}{2} - \frac{\pi}{4}\right)\right\}$$

where the functions $P_n(z)$ and $Q_n(z)$ have the following asymptotic expansions ($\nu = 4n^2$)

$$\begin{cases} P_n(z) \sim 1 - \dfrac{(\nu-1)(\nu-9)}{2!(8z)^2} + \dfrac{(\nu-1)(\nu-9)(\nu-25)(\nu-49)}{4!(8z)^4} - \dots \\ Q_n(z) \sim \dfrac{\nu-1}{8z} - \dfrac{(\nu-1)(\nu-9)(\nu-25)}{3!(8z)^3} + \dots \end{cases} \tag{58}$$

The roots of the Bessel function $J_n(z)$ are all real, and the first roots, ξ_{nj}, are listed in Table 4. The derivative of the Bessel function $J_n(z)$ has also only real roots, η_{nj}, and the first ones are listed in Table 5. Larger roots (larger j values) are asymptotically given by

$$\xi_{nj} = j\pi + \left(n - \frac{1}{2}\right)\frac{\pi}{2}, \qquad \eta_{nj} = j\pi + \left(n - \frac{3}{2}\right)\frac{\pi}{2}$$

Another, linearly independent solution to the Bessel differential equation, which is real-valued for real arguments, is the Neumann function[5] $N_n(z)$. The power series

[5]These solutions are also called Bessel functions of the second kind.

expansion is

$$\begin{aligned}N_n(z) &= \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right)+\gamma-\frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\right)J_n(z)\\ &-\frac{1}{\pi}\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{z}{2}\right)^{n+2k}}{k!(n+k)!}\sum_{l=1}^{k}\left(\frac{1}{l}+\frac{1}{l+n}\right)\\ &-\frac{1}{\pi}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{-n+2k}\end{aligned}$$

where the Euler constant $\gamma = 0.577\,215\,66\ldots$, and where all sums are dined as zero if the summation index exceeds the upper summation limit. This solution is singular at the origin $z = 0$. For small arguments the dominant contribution is

$$\begin{aligned}N_0(z) &= \frac{2}{\pi}\left(\ln\left(\frac{z}{2}\right)+\gamma\right)+O(z^2)\\ N_n(z) &= -\frac{(n-1)!}{\pi}\left(\frac{z}{2}\right)^{-n}+\ldots\end{aligned}$$

For large arguments the Neumann function has an asymptotic expansion ($-\pi < \arg z < \pi$)

$$N_n(z) = \left(\frac{2}{\pi z}\right)^{1/2}\left(P_n(z)\sin\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)+Q_n(z)\cos\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)\right)$$

where the functions $P_n(z)$ and $Q_n(z)$ are given by (58).

In the solution of scattering problems, linear combinations of Bessel and Neumann functions, *ie.* the Hankel functions, $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ of the first and the second kind, respectively, are natural[6]. These are defined as

$$\begin{aligned}H_n^{(1)}(z) &= J_n(z)+\mathrm{i}N_n(z)\\ H_n^{(2)}(z) &= J_n(z)-\mathrm{i}N_n(z)\end{aligned}$$

The Hankel functions of the first and second kind have integral representations

$$\begin{aligned}H_n^{(1)}(z) &= \frac{2}{\mathrm{i}\pi}\mathrm{e}^{-\mathrm{i}n\frac{\pi}{2}}\int_0^\infty \mathrm{e}^{\mathrm{i}z\cosh s}\cosh ns\,ds, \qquad 0<\arg z<\pi\\ H_n^{(2)}(z) &= \frac{2\mathrm{i}}{\pi}\mathrm{e}^{\mathrm{i}n\frac{\pi}{2}}\int_0^\infty \mathrm{e}^{-\mathrm{i}z\cosh s}\cosh ns\,ds, \qquad -\pi<\arg z<0\end{aligned}$$

For large argumens, the Hankel functions have asymptotic expansions

$$\begin{aligned}H_n^{(1)}(z) &= \left(\frac{2}{\pi z}\right)^{1/2}\mathrm{e}^{\mathrm{i}\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)}\left(P_n(z)+\mathrm{i}Q_n(z)\right), \qquad -\pi<\arg z<2\pi\\ H_n^{(2)}(z) &= \left(\frac{2}{\pi z}\right)^{1/2}\mathrm{e}^{-\mathrm{i}\left(z-\frac{n\pi}{2}-\frac{\pi}{4}\right)}\left(P_n(z)-\mathrm{i}Q_n(z)\right), \qquad -2\pi<\arg z<\pi\end{aligned} \tag{59}$$

[6]These also called Bessel functions of the third kind.

where the functions $P_n(z)$ and $Q_n(z)$ are given by (58).

Solutions to the Bessel differential equation of different order are related to each other by recursion relations. Some of the more important ones are ($n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots$)[7]

$$\begin{aligned}
&Z_{n-1}(z) - Z_{n+1}(z) = 2Z_n'(z) \\
&Z_{n-1}(z) + Z_{n+1}(z) = \frac{2n}{z} Z_n(z) \\
&Z_{n+1}(z) = \frac{n}{z} Z_n(z) - Z_n'(z) \\
&Z_n'(z) = Z_{n-1}(z) - \frac{n}{z} Z_n(z) \\
&\left(\frac{d}{z\,dz}\right)^m [z^n Z_n(z)] = z^{n-m} Z_{n-m}(z) \\
&\left(\frac{d}{z\,dz}\right)^m \left[z^{-n} Z_n(z)\right] = (-1)^m z^{-n-m} Z_{n+m}(z)
\end{aligned}$$

Here $Z_n(z)$ is a fixed arbitrary linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ or $H_n^{(2)}(x)$. Specifically, we have

$$J_1(z) = -J_0'(z)$$

which is frequently used in the analysis in this textbook.

Some useful indefinite integrals with solutions to the Bessel differential equation, which are often used in the text, are ($n = 0, 1, 2, \ldots$)

$$\begin{aligned}
&\int x^{n+1} Z_n(x)\,dx = x^{n+1} Z_{n+1}(x) = -x^{n+1}\left(Z_n'(x) - \frac{n}{x} Z_n(x)\right) \\
&\int x^{-n+1} Z_n(x)\,dx = -x^{-n+1} Z_{n-1}(x) = -x^{-n+1}\left(Z_n'(x) + \frac{n}{x} Z_n(x)\right) \\
&\int x \left(Z_n(x)\right)^2\,dx = \frac{x^2}{2}\left[(Z_n(x))^2 - Z_{n-1}(x) Z_{n+1}(x)\right] \\
&\qquad\qquad\qquad\quad = \frac{x^2}{2}\left(Z_n'(x)\right)^2 + \frac{1}{2}(x^2 - n^2)\left(Z_n(x)\right)^2
\end{aligned}$$

As above, $Z_n(x)$ is an arbitrary linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ or $H_n^{(2)}(x)$. Some additional — more complex —but useful determined integrals are

[7] These recursion relations hold for non-integer values of n, eg. $n = 1/2$. The index m, however, must be an integer.

$(n = 0, 1, 2, \ldots, m = 0, 1, 2, \ldots)$

$$\int \left[(\alpha^2 - \beta^2)x - \frac{m^2 - n^2}{x}\right] Z_m(\alpha x) Y_n(\beta x)\, dx = \beta x Z_m(\alpha x) Y_{n-1}(\beta x) - \alpha x Z_{m-1}(\alpha x) Y_n(\beta x) + (m-n) Z_m(\alpha x) Y_m(\beta x)$$

$$\int x Z_m(\alpha x) Y_m(\beta x)\, dx = \frac{\beta x Z_m(\alpha x) Y_{m-1}(\beta x) - \alpha x Z_{m-1}(\alpha x) Y_m(\beta x)}{\alpha^2 - \beta^2}$$

$$\int \frac{Z_m(\alpha x) Y_n(\alpha x)}{x}\, dx = \alpha x \frac{Z_{m-1}(\alpha x) Y_n(\alpha x) - Z_m(\alpha x) Y_{n-1}(\alpha x)}{m^2 - n^2} - \frac{Z_m(\alpha x) Y_n(\alpha x)}{m+n}$$

Here, $Z_n(\alpha x)$ and $Y_n(\beta x)$ is an arbitrary linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ or $H_n^{(2)}(x)$.

For Bessel functions, $J_n(z)$, Neumann functions, $N_n(z)$, and Hankel functions, $H_n^{(1)}(z)$ or $H_n^{(2)}(z)$, we have for a complex argument z

$$\begin{cases} J_n(z^*) = (J_n(z))^* \\ N_n(z^*) = (N_n(z))^* \end{cases} \qquad \begin{cases} H_n^{(1)}(z^*) = \left(H_n^{(2)}(z)\right)^* \\ H_n^{(2)}(z^*) = \left(H_n^{(1)}(z)\right)^* \end{cases}$$

The Graf addition theorem for Bessel functions is useful. Let $Z_n(x)$ be any linear combination of $J_n(x)$, $N_n(x)$, $H_n^{(1)}(x)$ and $H_n^{(2)}(x)$. The Graf addition theorem is

$$Z_n(w) \begin{pmatrix} \cos n\phi \\ \sin n\phi \end{pmatrix} = \sum_{k=-\infty}^{\infty} Z_{n+k}(u) J_k(v) \begin{pmatrix} \cos k\alpha \\ \sin k\alpha \end{pmatrix}, \qquad \left|v\mathrm{e}^{\pm \mathrm{i}\alpha}\right| < |u|$$

where w is

$$w = \sqrt{u^2 + v^2 - 2uv\cos\alpha}$$

Useful integrals

Some integrals related to Bessel functions used in this book are derived in this subsection. We start with the integral representation for integer order n, (57)

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\cos\phi} \mathrm{e}^{\mathrm{i}n\phi}\ \mathrm{d}\phi = 2\pi \mathrm{i}^n J_n(z)$$

From this we easily conclude by a simple change of variables that

$$\int_0^{2\pi} \mathrm{e}^{\mathrm{i}z\cos(\phi-\alpha)} \mathrm{e}^{\mathrm{i}n\phi}\ \mathrm{d}\phi = \mathrm{e}^{\mathrm{i}n\alpha} \int_{-\alpha}^{2\pi-\alpha} \mathrm{e}^{\mathrm{i}z\cos\psi} \mathrm{e}^{\mathrm{i}n\psi}\ \mathrm{d}\psi = 2\pi \mathrm{i}^n J_n(z) \mathrm{e}^{\mathrm{i}n\alpha}$$

This integral is a function of the variables z and α.

9 Appendix 2: ∇ in curvlinear coordinate systems

In this appendix some important expressions with the ∇-operator in two curvilinear coordinate systems, cylindrical and spherical, are collected. For completeness we start with the Cartesian coordinate system.

Cartesian coordinate system

The Cartesian coordinates (x, y, z) is the most basic coordinate system. The gradient and the Laplace-operator of a scalar field $\psi(x, y, z)$ in this coordinate system are

$$\begin{aligned}
\nabla\psi &= \hat{\boldsymbol{x}}\frac{\partial\psi}{\partial x} + \hat{\boldsymbol{y}}\frac{\partial\psi}{\partial y} + \hat{\boldsymbol{z}}\frac{\partial\psi}{\partial z} \\
\nabla^2\psi &= \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}
\end{aligned}$$

The divergence, the curl, and the Laplace-operator of a vector field $\boldsymbol{A}(x, y, z) = \hat{\boldsymbol{x}}A_x(x, y, z) + \hat{\boldsymbol{y}}A_y(x, y, z) + \hat{\boldsymbol{z}}A_z(x, y, z)$ are

$$\begin{aligned}
\nabla\cdot\boldsymbol{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
\nabla\times\boldsymbol{A} &= \hat{\boldsymbol{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\boldsymbol{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\boldsymbol{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
\nabla^2\boldsymbol{A} &= \hat{\boldsymbol{x}}\nabla^2 A_x + \hat{\boldsymbol{y}}\nabla^2 A_y + \hat{\boldsymbol{z}}\nabla^2 A_z
\end{aligned}$$

Circular cylindrical (polar) coordinate system

We now treat the first curvilinear coordinate system, and start with the circular cylindrical coordinate system (ρ, ϕ, z) defined by

$$\begin{cases}
\rho = \sqrt{x^2 + y^2} \\
\phi = \begin{cases} \arccos\frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos\frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases} \\
z = z
\end{cases}$$

The gradient and the Laplace-operator of a scalar field $\psi(\rho, \phi, z)$ in this coordinate system are

$$\begin{aligned}
\nabla\psi &= \hat{\boldsymbol{\rho}}\frac{\partial\psi}{\partial\rho} + \hat{\boldsymbol{\phi}}\frac{1}{\rho}\frac{\partial\psi}{\partial\phi} + \hat{\boldsymbol{z}}\frac{\partial\psi}{\partial z} \\
\nabla^2\psi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}
\end{aligned}$$

The divergence, the curl, and the Laplace-operator of a vector field $\boldsymbol{A}(\rho, \phi, z) = \hat{\boldsymbol{\rho}} A_\rho(\rho, \phi, z) + \hat{\boldsymbol{\phi}} A_\phi(\rho, \phi, z) + \hat{\boldsymbol{z}} A_z(\rho, \phi, z)$ are

$$
\begin{aligned}
\nabla \cdot \boldsymbol{A} &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho A_\rho\right) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} \\
\nabla \times \boldsymbol{A} &= \hat{\boldsymbol{\rho}}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\boldsymbol{\phi}}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{\boldsymbol{z}}\frac{1}{\rho}\left(\frac{\partial}{\partial \rho}\left(\rho A_\phi\right) - \frac{\partial A_\rho}{\partial \phi}\right) \\
\nabla^2 \boldsymbol{A} &= \hat{\boldsymbol{\rho}}\left(\nabla^2 A_\rho - \frac{A_\rho}{\rho^2} - \frac{2}{\rho^2}\frac{\partial A_\phi}{\partial \phi}\right) + \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi - \frac{A_\phi}{\rho^2} + \frac{2}{\rho^2}\frac{\partial A_\rho}{\partial \phi}\right) + \hat{\boldsymbol{z}}\nabla^2 A_z
\end{aligned}
$$

Spherical coordinates system

The spherical coordinate system (r, θ, ϕ) (polar angle θ and the azimuth angle ϕ) is defined by

$$
\begin{cases}
r = \sqrt{x^2 + y^2 + z^2} \\
\theta = \arccos \frac{z}{\sqrt{x^2+y^2+z^2}} \\
\phi = \begin{cases} \arccos \frac{x}{\sqrt{x^2+y^2}} & y \geq 0 \\ 2\pi - \arccos \frac{x}{\sqrt{x^2+y^2}} & y < 0 \end{cases}
\end{cases}
$$

The gradient and the Laplace-operator of a scalar field $\psi(r, \theta, \phi)$ in this coordinate system are

$$
\begin{aligned}
\nabla \psi &= \hat{\boldsymbol{r}}\frac{\partial \psi}{\partial r} + \hat{\boldsymbol{\theta}}\frac{1}{r}\frac{\partial \psi}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{r \sin \theta}\frac{\partial \psi}{\partial \phi} \\
\nabla^2 \psi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \psi}{\partial r}\right) + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 \psi}{\partial \phi^2} \\
&= \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(r \psi\right) + \frac{1}{r^2 \sin \theta}\frac{\partial}{\partial \theta}\left(\sin \theta \frac{\partial \psi}{\partial \theta}\right) + \frac{1}{r^2 \sin^2 \theta}\frac{\partial^2 \psi}{\partial \phi^2}
\end{aligned}
$$

and the divergence, the curl, and the Laplace-operator of a vector field $\boldsymbol{A}(r,\theta,\phi) = \hat{\boldsymbol{r}}A_r(r,\theta,\phi) + \hat{\boldsymbol{\theta}}A_\theta(r,\theta,\phi) + \hat{\boldsymbol{\phi}}A_\phi(r,\theta,\phi)$ are

$$\begin{aligned}
\nabla \cdot \boldsymbol{A} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \\
\nabla \times \boldsymbol{A} &= \hat{\boldsymbol{r}}\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta A_\phi\right) - \frac{\partial A_\theta}{\partial\phi}\right) \\
&\quad + \hat{\boldsymbol{\theta}}\frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}\left(rA_\phi\right)\right) + \hat{\boldsymbol{\phi}}\frac{1}{r}\left(\frac{\partial}{\partial r}\left(rA_\theta\right) - \frac{\partial A_r}{\partial\theta}\right) \\
\nabla^2 \boldsymbol{A} &= \hat{\boldsymbol{r}}\left(\nabla^2 A_r - \frac{2A_r}{r^2} - \frac{2}{r^2}\frac{\partial A_\theta}{\partial\theta} - \frac{2\cot\theta}{r^2}A_\theta - \frac{2}{r^2\sin\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
&\quad + \hat{\boldsymbol{\theta}}\left(\nabla^2 A_\theta - \frac{A_\theta}{r^2\sin^2\theta} + \frac{2}{r^2}\frac{\partial A_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\phi}{\partial\phi}\right) \\
&\quad + \hat{\boldsymbol{\phi}}\left(\nabla^2 A_\phi - \frac{A_\phi}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\frac{\partial A_r}{\partial\phi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial A_\theta}{\partial\phi}\right)
\end{aligned}$$