April 16, 2019

Lectures 8

$\boldsymbol{E}$ and $\boldsymbol{B}$ for point charge

The electric and magnetic fields are given by

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u}+\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})] \tag{0.1}$$

and

$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\boldsymbol{E}(\boldsymbol{r},t) \tag{0.2}$$

where

$$\boldsymbol{w}(t_\mathrm{r}) = \text{position of particle at retarded time} \tag{0.3}$$
$$\boldsymbol{r} = \text{field point} \tag{0.4}$$
$$\boldsymbol{\mathscr{r}}=\boldsymbol{r}-\boldsymbol{w}(t_\mathrm{r}) \tag{0.5}$$
$$t-t_\mathrm{r}=\frac{|\boldsymbol{r}-\boldsymbol{w}(t_\mathrm{r})|}{c} \tag{0.6}$$
$$t-t_\mathrm{r}=\frac{|\boldsymbol{\mathscr{r}}|}{c} \tag{0.7}$$
$$\boldsymbol{u}=c\hat{\boldsymbol{\mathscr{r}}}-\boldsymbol{v} \tag{0.8}$$

Example: Charge with constant velocity

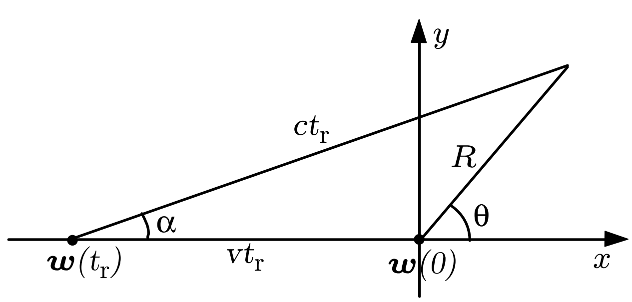

Figure 1: The particle at $\boldsymbol{w}(t_\mathrm{r})$ and $\boldsymbol{w}(0)$.

When the velocity is constant, $\boldsymbol{v}=v\hat{\boldsymbol{x}}$ then $\boldsymbol{a}=\boldsymbol{0}$. Then from (0.19)

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q}{4\pi\varepsilon_0}\frac{|\boldsymbol{\mathscr{r}}|(c^2-v^2)\boldsymbol{u}}{(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3} \tag{0.9}$$

Assume that the particle passes $\boldsymbol{r} = \boldsymbol{0}$ at time $t = 0$. Let us determine the field when $t = 0$. Introduce $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{v}t$. Fom figure (1)

$$(ct_{\mathrm{r}})^2 = (vt_{\mathrm{r}} + R\cos\theta)^2 + (R\sin\theta)^2 \tag{0.10}$$
$$|\boldsymbol{\mathscr{r}}| = ct_{\mathrm{r}} \tag{0.11}$$
$$|\boldsymbol{\mathscr{r}}|\boldsymbol{u} = c\boldsymbol{\mathscr{r}} - |\boldsymbol{\mathscr{r}}|\boldsymbol{v} = c(\boldsymbol{\mathscr{r}} - \boldsymbol{v}t_{\mathrm{r}}) = c\boldsymbol{R} \tag{0.12}$$

This gives

$$t_{\mathrm{r}} = \frac{vR\cos\theta}{c^2 - v^2} - Rc\frac{\sqrt{1 - (v/c)^2\sin^2\theta}}{c^2 - v^2} \tag{0.13}$$

Now

$$\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = c|\boldsymbol{\mathscr{r}}| - v|\boldsymbol{\mathscr{r}}|\cos\alpha, \tag{0.14}$$

but

$$|\boldsymbol{\mathscr{r}}|\cos\alpha = vt_{\mathrm{r}} + R\cos\theta \tag{0.15}$$

and then

$$\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = (c^2 - v^2)t_{\mathrm{r}} - vR\cos\theta \tag{0.16}$$

Plug in (0.13) to get

$$\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = Rc\sqrt{1 - (v/c)^2\sin^2\theta} \tag{0.17}$$

Since

$$|\boldsymbol{\mathscr{r}}|\boldsymbol{u} = c\boldsymbol{R} \tag{0.18}$$

we get

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\frac{(1 - (v/c)^2)\boldsymbol{R}}{R^3(1 - (v/c)^2\sin^2\theta)^{3/2}} \tag{0.19}$$

It is now easy to transfer this to an arbitrary t. We just let $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{v}t$, i.e. the vector from the position of the particle at time t to the field point $\boldsymbol{r}$ and let θ be the angle between $\boldsymbol{R}$ and $\boldsymbol{v}$. For the magnetic field we use

$$\hat{\boldsymbol{\mathscr{r}}} = \frac{\boldsymbol{r} - \boldsymbol{v}t}{ct_{\mathrm{r}}} = \frac{\boldsymbol{r} - \boldsymbol{v}t + (t - t_{\mathrm{r}})\boldsymbol{v}}{ct_{\mathrm{r}}} = \frac{\boldsymbol{R}}{ct_{\mathrm{r}}} + \frac{\boldsymbol{v}}{c} \tag{0.20}$$

Then (0.2) gives

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c^2}\boldsymbol{v} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{0.21}$$

Radiation

Far-field

We now find the radiated electromagnetic waves from an accelerating point charge. When $r \gg r'$ and $r \gg \lambda$, where λ is the wavelength of the waves generated by the accelerating particle, then we can make a number of approximations that lead to quite nice expressions for the electromagnetic waves. This is very important since we intend to describe the light produced in synchrotrons. Even though we will find expressions valid when $r \gg r'$ we can argue that the expressions are valid also for smaller r.

We can start with the expression for the electric and magnetic fields in (0.1) and (0.2). In the far zone we keep all terms that drop off as r^{-1} and skip those who drop off faster than this. Everywhere, except in the expression for t_r, we use

$$\boldsymbol{\mathscr{r}} \approx r\hat{\boldsymbol{r}} \tag{0.22}$$
$$\boldsymbol{u} \approx c\hat{\boldsymbol{r}} - \boldsymbol{v}(t_\mathrm{r}) \tag{0.23}$$
$$(c^2 - v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{a}) \approx r\hat{\boldsymbol{r}} \times ((c\hat{\boldsymbol{r}} - \boldsymbol{v}) \times \boldsymbol{a}) \tag{0.24}$$
$$\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} = r(c - \hat{\boldsymbol{r}} \cdot \boldsymbol{v}) \tag{0.25}$$

Then from (0.1)

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q\mu_0}{4\pi r} \frac{\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} - \boldsymbol{\beta}) \times \boldsymbol{a})}{(1 - \hat{\boldsymbol{r}} \cdot \boldsymbol{\beta})^3} \tag{0.26}$$
$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c}\hat{\boldsymbol{r}} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{0.27}$$

We have to remember that $\hat{\boldsymbol{r}}$, $\boldsymbol{a}$, and $\boldsymbol{\beta}$ are functions of the retarded time time t_r.

Example: Charge in circular motion

Let the particle travel along a circle with radius b. The speed v is constant so that

$$\boldsymbol{w} = b(\cos(\omega t_\mathrm{r}), \sin(\omega t_\mathrm{r})) \tag{0.28}$$
$$\boldsymbol{v} = v(-\sin(\omega t_\mathrm{r}), \cos(\omega t_\mathrm{r})) \tag{0.29}$$
$$\boldsymbol{a} = \omega v(\cos(\omega t_\mathrm{r}), \sin(\omega t_\mathrm{r})) \tag{0.30}$$

where $\omega = \dfrac{v}{b}$ is the angular frequency. Let $\boldsymbol{r} = x\hat{\boldsymbol{x}}$, then

$$\hat{\boldsymbol{r}} \times ((\hat{\boldsymbol{r}} - \boldsymbol{\beta}) \times \boldsymbol{a}) = \omega v(\sin(\omega t_\mathrm{r}) + \beta)\hat{\boldsymbol{y}} \tag{0.31}$$

and

$$\boldsymbol{E}(x, 0, 0, t) = \frac{q\mu_0\omega v}{4\pi x} \frac{(\sin(\omega t_\mathrm{r}) + \beta)}{(1 + \beta\sin(\omega t_\mathrm{r}))^3}\hat{\boldsymbol{y}} \tag{0.32}$$
$$\boldsymbol{B}(x, 0, 0, t) = \frac{q\mu_0\omega\beta}{4\pi x} \frac{(\sin(\omega t_\mathrm{r}) + \beta)}{(1 + \beta\sin(\omega t_\mathrm{r}))^3}\hat{\boldsymbol{z}} \tag{0.33}$$

For $x > 0$ it is clear that $|\boldsymbol{E}(x, 0, 0, t)|$ is maximum when $\omega t_\mathrm{r} = \dfrac{3\pi}{2} + n2\pi$. It means that the light is sent when q travels towards the observer.

Radiated power by a point charge

Assume a point charge that travels along a path $\boldsymbol{w}(t_\mathrm{r})$. Let say that we are at a point on this path and see the particle pass by at time $\bar{t}_\mathrm{r}$. What is the power radiated at that time? To find this we make a sphere with radius R, with its center at $\boldsymbol{w}(\bar{t}_\mathrm{r})$. We wait a time R/c and then we measure the power transmitted through the sphere for the fields that were generated at time t_r. From our position we see that $\hat{\boldsymbol{r}}(t_\mathrm{r})$ is directed in the radial direction, which a make its easy to find the radiation pattern and the radiated power. The field reaches the sphere at time $t = t_\mathrm{r} + R/c$.

The Poynting vector is

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{E} \times \boldsymbol{B} \tag{0.34}$$

According to (0.2) $\boldsymbol{B}(\boldsymbol{r}, t) = \dfrac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}(\boldsymbol{r}, t)$ and we get

$$\boldsymbol{S} = \frac{1}{\mu_0 c}(\boldsymbol{E} \times (\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E})) = \frac{1}{\mu_0 c}(E^2 \hat{\boldsymbol{\mathscr{r}}} - (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{E})\boldsymbol{E}) \tag{0.35}$$

The far-field, or acceleration field, as Griffiths calls it, is given by (1.5), or,

$$\boldsymbol{E}_\mathrm{rad} = \frac{q}{4\pi\varepsilon_0} \frac{|\boldsymbol{\mathscr{r}}|}{(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^3} (\boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{a})) \tag{0.36}$$

$$\boldsymbol{B}_\mathrm{rad}(\boldsymbol{r}, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{0.37}$$

This is the field that falls off as $|\boldsymbol{\mathscr{r}}|^{-1}$ for large $|\boldsymbol{\mathscr{r}}|$. We see that $\boldsymbol{E}_\mathrm{rad}$ is perpendicular to $\boldsymbol{\mathscr{r}}$ and then

$$\boldsymbol{S}_\mathrm{rad} = \frac{1}{\mu_0 c} \left(\hat{\boldsymbol{\mathscr{r}}} |\boldsymbol{E}(\boldsymbol{r}, t)|^2 - \boldsymbol{E}(\boldsymbol{r}, t) \hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{E}(\boldsymbol{r}, t) \right) \tag{0.38}$$

The second term is zero since $\boldsymbol{E} \perp \hat{\boldsymbol{\mathscr{r}}}$ and then

$$\boldsymbol{S}_\mathrm{rad}(R, \theta, \phi, t) = \frac{1}{\mu_0 c} \hat{\boldsymbol{\mathscr{r}}} |\boldsymbol{E}_\mathrm{rad}(R, \theta, \phi, t)|^2 \tag{0.39}$$

For particles with speed much less than the speed of light it is quite straightforward to carry out the integral. Then $\boldsymbol{u} \approx c \hat{\boldsymbol{\mathscr{r}}}$ and

$$\boldsymbol{E}_\mathrm{rad}(R, \theta, \phi, t) = \frac{q}{4\pi\varepsilon_0 c^2} \frac{1}{R} (\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{a})) \tag{0.40}$$

$$\boldsymbol{B}_\mathrm{rad}(R, \theta, \phi, t) = \frac{1}{c} \hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{0.41}$$

where $\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{a}) = \hat{\boldsymbol{\mathscr{r}}}(\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a}) - \boldsymbol{a}$ and $|\hat{\boldsymbol{\mathscr{r}}} \times (\hat{\boldsymbol{\mathscr{r}}} \times \boldsymbol{a})|^2 = a^2 - (\hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{a})^2 = a^2 \sin^2 \theta$, where θ is the angle between $\hat{\boldsymbol{\mathscr{r}}}$ and $\boldsymbol{a}$. We recognise the the radiation pattern from an electric elementary dipole with its dipole moment directed parallell to $\boldsymbol{a}$. The radiated power is

$$P(t) = \int_0^{2\pi} \int_0^{\pi} \hat{\boldsymbol{\mathscr{r}}} \cdot \boldsymbol{S}_\mathrm{rad}(R, \theta, \phi, t) R^2 \sin\theta \, \mathrm{d}\theta \mathrm{d}\phi \tag{0.42}$$

This gives

$$P(t) = \frac{\mu_0 q^2 a^2}{6\pi c} \tag{0.43}$$

This formula is called the Larmor formula.

There is a generalization of Poynting vector and the Larmor formula for the case when v is not small compared to c. The generalization is called the Liénard's generalization and reads

$$P(t) = \frac{\mu_0 q^2 \gamma^6}{6\pi c} \left(a^2 - \left| \frac{\boldsymbol{v} \times \boldsymbol{a}}{c} \right|^2 \right) \tag{0.44}$$

Example: If the particle is traveling in a magnetic field then $\boldsymbol{v} \perp \boldsymbol{a}$ and the expression reduces to

$$P(t) = \frac{\mu_0 q^2 \gamma^4 a^2}{6\pi c} \tag{0.45}$$

Thus in LHC and storage rings for electrons the radiation loss scales as the particle energy to the fourth power. This becomes a problem. For the protons in LHC this is not the limiting factor. It is instead the magnetic field that sets the limit.

Example: Just before collision, the energy of a proton in LHC is roughly 2000 times the energy of an electron in the MAX IV storage ring. On the other hand the proper mass of the proton is about 2000 times the proper mass of the electron. By that γ is almost the same for LHC proton and the MAX IV electron. The circumference of the large ring in LHC is 27 km and the storage ring in MAX IV is 528 m. It means that a is $27000/528 \approx 50$ times larger for the MAX IV electrons than for the LHC protons. Then the radiation loss per particle is 2500 larger in MAXIV than in LHC.