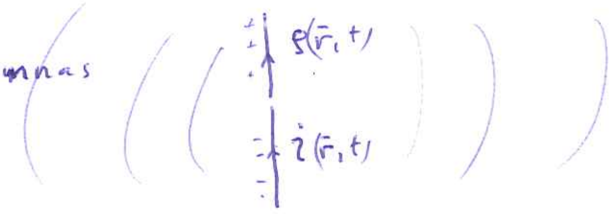
Fields from sources

Antennas



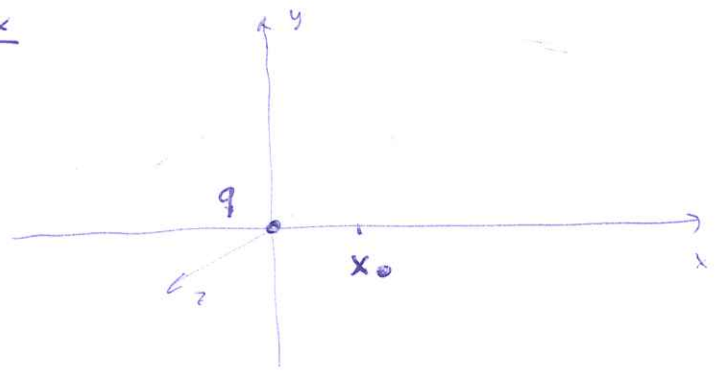
$\bar{E}, \bar{B} = ?$

Accelerating particles

$\bar{F} = q(\bar{E} + \bar{v} \times \bar{B})$

If: $\bar{F} \neq \bar{0} \Rightarrow$ acceleration $\Rightarrow$ radiated fields

Ex

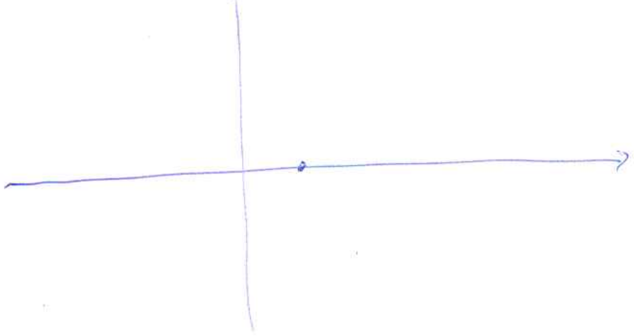


$t < 0 \Rightarrow$ q at rest

$0 < t < \frac{T}{2}$ q accelerates to the right $\bar{a} = a\hat{x} \Rightarrow \bar{v}(t) = a t \hat{x}$

$\frac{T}{2} < t < T$ q decelerates $\bar{a} = -a\hat{x}$.

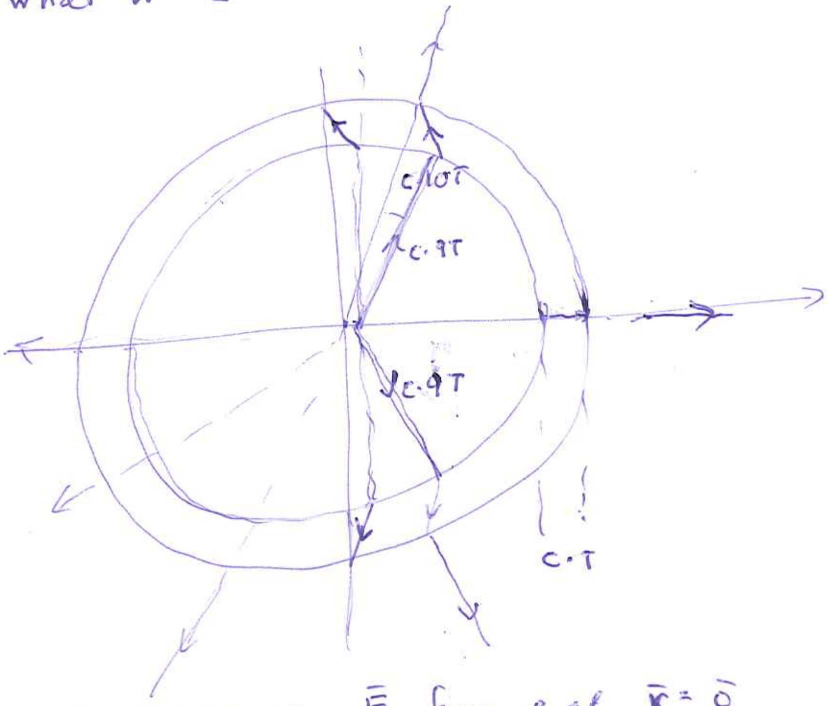
$t > T$ q at rest at x_0



Causality

EM-signals travel with speed of light

What is $\bar{E}$ at $t = 10T$?



$r > c \cdot 10T \Rightarrow \bar{E}$ from q at $\bar{r} = \bar{0}$

$r' < c \cdot 9T \Rightarrow \bar{E}$ from q at $\bar{r} = (x_0, 0, 0)$

Between circles a wave propagating ~~outw~~ in the radial direction.

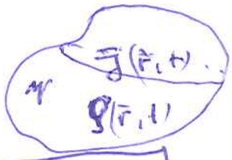
Tools to find $\bar{E}$ and $\bar{B}$

$\bar{A}$ = vector potential

V = scalar potential

$\bar{A}$ and $\bar{V}$

Consider a volume $\mathcal{V}$ with currents $\bar{J}(\bar{r},t)$ and charge densities $\rho(\bar{r},t)$, in an otherwise empty space.



vacuum

$$\nabla \cdot \bar{B} \equiv 0 \Rightarrow \boxed{\bar{B} = \nabla \times \bar{A}}$$

$\bar{A}$ = vector potential

Equation for $\bar{A}$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t} = -\frac{\partial\, \nabla \times \bar{A}}{\partial t}$$

$$\Rightarrow \quad \nabla \times \left(\bar{E} + \frac{\partial \bar{A}}{\partial t}\right) = \bar{0}$$

$$\Rightarrow \quad \boxed{\bar{E} + \frac{\partial \bar{A}}{\partial t} = -\nabla V}$$

V = scalar electric potential

Ampères law $\quad \nabla \times \bar{B} = \mu_0 \bar{J} + \frac{1}{c^2}\frac{\partial \bar{E}}{\partial t}$

$$\Rightarrow \quad \nabla \times (\nabla \times \bar{A}) = \mu_0 \bar{J} - \frac{1}{c^2}\left(\frac{\partial^2 \bar{A}}{\partial t^2} + \nabla \frac{\partial V}{\partial t}\right)$$

But $\quad \nabla \times (\nabla \times \bar{A}) = \nabla\, \nabla \cdot \bar{A} - \nabla^2 \bar{A}$

$$\Rightarrow \quad \nabla^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = -\mu_0 \bar{J} + \nabla\left(\nabla \cdot \bar{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right)$$

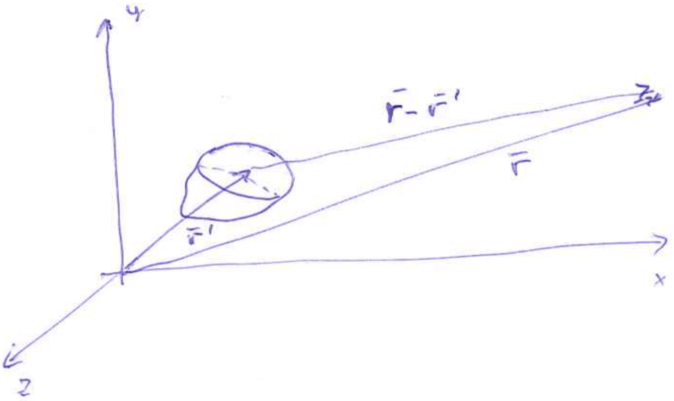
Lorentz gauge

Choose $\bar{A}$ such that $\boxed{\nabla \cdot \bar{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0}$

$$\Rightarrow \quad \boxed{\nabla^2 \bar{A} - \frac{1}{c^2}\frac{\partial^2 \bar{A}}{\partial t^2} = -\mu_0 \bar{J}} \qquad (1)$$

Solution

$$\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\bar{J}(\bar{r}, t_r)}{|\bar{r} - \bar{r}'|} \, dv'$$



$$t_r = t - \frac{|\bar{r} - \bar{r}'|}{c} = \text{retarded time}$$

Eqution for V

Gauss law + Lorenz gauge $\Rightarrow$

$$\nabla^2 V - \frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0}$$

$$\boxed{V(\bar{r}, t) = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{V}} \frac{\rho(\bar{r}', t - t_r)}{|\bar{r} - \bar{r}'|} \, dv'} \quad (2)$$

From (1), (2), $B = \nabla \times \bar{A}$ and $\bar{E} = -\nabla V - \frac{\partial \bar{A}}{\partial t}$ we get E and $\bar{B}$

Frequency domain

Assume time harmonic $\bar{J}(\bar{r},t)$ and $\rho(\bar{r},t) \Rightarrow$

Complex $\bar{J}(\bar{r})$, $\rho(\bar{r})$, $\bar{A}$, where $\bar{J}(\bar{r},t) = \mathrm{Re}\{\bar{J}(\bar{r})\, e^{-i\omega t}\}$

$$\Rightarrow \nabla^2 \bar{A}(\bar{r}) + k^2 \bar{A}(\bar{r}) = -\mu_0 \bar{J}(\bar{r})$$

$$\Rightarrow \bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\bar{J}(\bar{r}')\, e^{ik|\bar{r}-\bar{r}'|}}{|\bar{r}-\bar{r}'|}\, dv'$$

Example Far-fields $\bar{E}$, $\bar{B}$ from antenna

Far zone $\Rightarrow r \gg r'$, $r \gg \lambda$

$$\Rightarrow |\bar{r}-\bar{r}'| = \sqrt{(\bar{r}-\bar{r}')\cdot(\bar{r}-\bar{r}')} = \sqrt{r^2 + \underbrace{r'^2}_{\simeq 0} - 2\bar{r}\cdot\bar{r}'} \simeq \cancel{r - \hat{r}\cdot\bar{r}'}$$

$$\simeq r\sqrt{1 - 2\frac{\bar{r}\cdot\bar{r}'}{r^2}} \simeq r - \hat{r}\cdot\bar{r}'$$

$$\Rightarrow \quad e^{ik|\bar{r}-\bar{r}'|} \simeq e^{ikr} \cdot e^{-ik\hat{r}\cdot\bar{r}'}$$

$$\frac{1}{|\bar{r}-\bar{r}'|} \simeq \frac{1}{r}$$

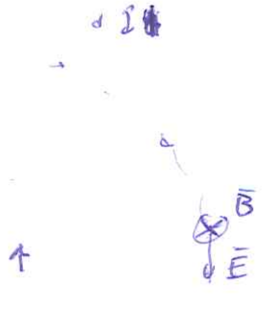
$$\Rightarrow \bar{A}(\bar{r}) = \frac{\mu_0}{4\pi} \int_V \bar{J}(\bar{r}')\, e^{-ik\hat{r}\cdot\bar{r}'}\, dv' \; \frac{e^{ikr}}{r}$$

Far zone $\Rightarrow \nabla \rightarrow ik\hat{r}$

$$\Rightarrow \bar{B} = \nabla \times \bar{A} \simeq ik\, \hat{r} \times \bar{A}$$

Ampere $\Rightarrow$
$$\bar{E} = i\frac{c^2}{\omega} \nabla \times (\nabla \times \bar{A}) = -i\omega\, \hat{r} \times (\hat{r} \times \bar{A})$$
$$= i\frac{c^2}{\omega} \nabla \times \bar{B} = -\frac{c^2}{\omega} k\, \hat{r} \times \bar{B} = -c\, \hat{r} \times \bar{B}$$

Example Electric dipole $\Rightarrow \bar{J}(\bar{r}) = dI_0\, \delta(\bar{r})\, \hat{z}$



$$\Rightarrow \bar{A} = \frac{\mu_0}{4\pi} dI_0 \frac{e^{ikr}}{r}\hat{z}$$

$$\bar{B} = ik\frac{\mu_0}{4\pi} dI_0 \frac{e^{ikr}}{r} \sin\theta\, \hat{\varphi}$$

$$\bar{E} = +ikc\frac{\mu_0}{4\pi} dI_0 \frac{e^{ikr}}{r} \sin\theta\, \hat{\theta}$$