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1 Lecture 1

The Maxwell equations in a source free region with vacuum are

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} \tag{1.1}$$

$$\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \frac{\partial \boldsymbol{D}(\boldsymbol{r}, t)}{\partial t} \tag{1.2}$$

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}, t) = 0 \tag{1.3}$$

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}, t) = 0. \tag{1.4}$$

Here

$$\boldsymbol{E}(\boldsymbol{r}, t) = \text{electric field} \tag{1.5}$$

$$\boldsymbol{D}(\boldsymbol{r}, t) = \text{electric flux density} \tag{1.6}$$

$$\boldsymbol{H}(\boldsymbol{r}, t) = \text{magnetic field} \tag{1.7}$$

$$\boldsymbol{B}(\boldsymbol{r}, t) = \text{magnetic flux density.} \tag{1.8}$$

In vacuum $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$ and $\boldsymbol{B} = \mu_0 \boldsymbol{H}$, where ε_0 is the permittivity of vacuum and μ_0 the permeability of vacuum, with values

$$\varepsilon_0 = 8.854 \cdot 10^{-12}\,\text{As/Vm} \tag{1.9}$$

$$\mu_0 = 4\pi \cdot 10^{-7}\,\text{Vs/Am} \tag{1.10}$$

Only regions with vacuum are considered in the course.

1.1 The wave equation

The wave equations for $\boldsymbol{E}$ and $\boldsymbol{H}$ follow from the Maxwell equations

$$\nabla^2 \boldsymbol{E} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} = \boldsymbol{0} \tag{1.11}$$

$$\nabla^2 \boldsymbol{H} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{H}}{\partial t^2} = \boldsymbol{0}. \tag{1.12}$$

The conditions $\nabla \cdot \boldsymbol{E} = 0$ and $\nabla \cdot \boldsymbol{H} = 0$ also have to hold.

1.1.1 Plane wave solutions

A linearly polarized plane wave propagating in the positive $z-$direction is given by

$$\boldsymbol{E}(z, t) = E(z - ct)\hat{\boldsymbol{x}}. \tag{1.13}$$

The magnetic field is given by the right hand rule

$$\boldsymbol{H}(z, t) = \eta_0^{-1}\hat{\boldsymbol{z}} \times \boldsymbol{E}(z, t) \tag{1.14}$$

where

$$\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} \tag{1.15}$$

is the wave impedance of vacuum.

A more general plane wave is propagating in the $\hat{\boldsymbol{k}}$-direction is $\boldsymbol{E}(\hat{\boldsymbol{k}} \cdot r - ct)$, where $\hat{\boldsymbol{k}} \perp \boldsymbol{E}$. Then

$$\boldsymbol{H}(\boldsymbol{r}, t) = \eta_0^{-1} \hat{\boldsymbol{k}} \times \boldsymbol{E}(\hat{\boldsymbol{k}} \cdot r - ct). \tag{1.16}$$

Here $\hat{\boldsymbol{k}} = \dfrac{\boldsymbol{k}}{k}$ is the unit vector pointing in the direction the wave propagates.

1.2 The Poynting theorem

Consider a volume V with vacuum and enclosed by a surface S. The surface S has an outward directed unit normal $\hat{\boldsymbol{n}}$. In V there can be a current density $\boldsymbol{J}$. The Poynting theorem is:

$$\oint_S \boldsymbol{S} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = -\int_V \left(\mu_0 \boldsymbol{H} \cdot \frac{\partial \boldsymbol{H}}{\partial t} + \varepsilon_0 \boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \right) \mathrm{d}V - \int_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}S, \tag{1.17}$$

where

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H} \tag{1.18}$$

is the Poynting vector.

Interpretations:

- $\oint\limits_S \boldsymbol{S} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S$ is the power radiated out from V.
- $\mu_0 \int\limits_V \boldsymbol{H} \cdot \dfrac{\partial \boldsymbol{H}}{\partial t} \, \mathrm{d}V = \dfrac{\mu_0}{2} \dfrac{\partial}{\partial t} \int\limits_V |\boldsymbol{H}|^2 \, \mathrm{d}V$ is the change per unit time of the magnetic energy in V.
- $\dfrac{\varepsilon_0}{2} \int\limits_V \boldsymbol{E} \cdot \dfrac{\partial \boldsymbol{E}}{\partial t} \, \mathrm{d}V = \dfrac{\varepsilon_0}{2} \dfrac{\partial}{\partial t} \int\limits_V |\boldsymbol{E}|^2 \, \mathrm{d}V$ is the change per unit time of the electric energy in V.
- $P_d(t) = \int\limits_V \boldsymbol{E} \cdot \boldsymbol{J} \, \mathrm{d}S$ is the power related to $\boldsymbol{J}$. If $P_d(t) > 0$ then it is the power consumed by $\boldsymbol{J}$. This is the case if there is a conductive region in V. If $P_d(t) < 0$ then $\boldsymbol{J}$ adds the power $-P_d(t)$ to the system. This is for instance the case if there is an antenna in V, or, as in an accelerator, a beam of particles traveling through V.
- The Poynting vector $\boldsymbol{S}$ is interpreted as the power flow density (W/m^2) in the direction $\hat{\boldsymbol{k}} = \boldsymbol{S}/|\boldsymbol{S}|$.
- $\dfrac{\mu_0}{2} |\boldsymbol{H}|^2$ is interpreted as the magnetic energy density (J/m^3).
- $\dfrac{\varepsilon_0}{2} |\boldsymbol{E}|^2$ is interpreted as the electric energy density (J/m^3).

1.2.1 Energies for propagating waves

Propagation of waves requires that two types of energies interact. For water waves potential interacts with kinetic energy. For waves along a string the stress energy interacts with the kinetic energy. For an electromagnetic wave the two interacting energies are the electric and magnetic energy. The electric and magnetic energy densities are equal for all propagating electromagnetic waves.

Example: The electric energy density of a plane wave is

$$w_{\mathrm{e}} = \frac{1}{2}\varepsilon_0 |\boldsymbol{E}(\hat{\boldsymbol{k}} \cdot r - ct)|^2 \tag{1.19}$$

and the magnetic energy density is

$$w_{\mathrm{m}} = \frac{1}{2}\mu_0 |\boldsymbol{H}(\hat{\boldsymbol{k}} \cdot r - ct)|^2 \tag{1.20}$$

Since $\boldsymbol{H} = \eta_0^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{E}(\hat{\boldsymbol{k}} \cdot r - ct)$ it follows that $w_{\mathrm{e}}(\boldsymbol{r}, t) = w_{\mathrm{m}}(\boldsymbol{r}, t)$.

1.3 The Maxwell equations in frequency domain

We may transform the time domain Maxwell equations into frequency domain. This can be done by Fourier transformationor by using phasors. In both cases time derivative $\frac{\partial}{\partial t}$ is transformed to multiplication with $-\mathrm{i}\omega$. The frequency domain Maxwell equations in a source free region with vacuum are

$$\nabla \times \boldsymbol{E}(\boldsymbol{r}) = \mathrm{i}\omega \boldsymbol{B}(\boldsymbol{r}) \tag{1.21}$$
$$\nabla \times \boldsymbol{H}(\boldsymbol{r}) = -\mathrm{i}\omega D(\boldsymbol{r}) \tag{1.22}$$
$$\nabla \cdot \boldsymbol{E}(\boldsymbol{r}) = 0 \tag{1.23}$$
$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = 0. \tag{1.24}$$

where the frequency domain fields have argument $(\boldsymbol{r})$ instead of $(\boldsymbol{r}, t)$. The wave equations (1.11) and (1.12) are transformed to the Helmholtz equations

$$\nabla^2 \boldsymbol{E} + k^2 \boldsymbol{E} = \boldsymbol{0} \tag{1.25}$$
$$\nabla^2 \boldsymbol{H} + k^2 \boldsymbol{H} = \boldsymbol{0}. \tag{1.26}$$

where $k = \frac{\omega}{c}$ is the wavenumber of vacuum.

1.4 Waveguides

See pages 7–12 in the exercise book.