Electrodynamics 2019: Lecture 3

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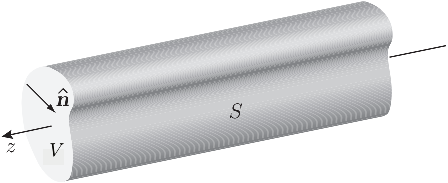
Electrical and information technology

Last lecture

- Waveguide modes.
- TE-and TM-modes.
- Rectangular waveguide.
- Cut-off frequency.
- Short introduction to Comsol.

Hollow waveguides



Two type of waves:

- TE-waves$\Rightarrow$ $E_z = 0,\ H_z = w(\boldsymbol{\rho})e^{\mathrm{i}k_z z}$
- TM-waves$\Rightarrow$ $H_z = 0,\ E_z = v(\boldsymbol{\rho})e^{\mathrm{i}k_z z}$

TE-waves: $E_z = 0$
$\boldsymbol{E} = (E_x, E_y, 0)$ and $\boldsymbol{H} = (H_x.H_y, H_z)$.
Find H_z! Then:

$$\begin{cases} \boldsymbol{E}_\mathrm{T}(\boldsymbol{r}) = -\mathrm{i}\dfrac{\omega\mu_0}{k_t^2}\hat{\boldsymbol{z}} \times \nabla_\mathrm{T} H_z(\boldsymbol{r}) \\ \boldsymbol{H}_\mathrm{T}(\boldsymbol{r}) = \mathrm{i}\dfrac{k_z}{k_t^2}\nabla_\mathrm{T} H_z(\boldsymbol{r}) \end{cases} \tag{1}$$

$H_z = w(\boldsymbol{\rho})e^{\mathrm{i}k_z z}$

Eigenvalue problem

$$\begin{aligned} \nabla^2 w(\boldsymbol{\rho}) + k_t^2 w(\boldsymbol{\rho}) &= 0,\ \boldsymbol{\rho} \in \Omega \\ \hat{\boldsymbol{n}} \cdot \nabla w(\boldsymbol{\rho}) &= 0,\ \boldsymbol{\rho} \in \Gamma \end{aligned}$$

Eigenvalues k_{tn}^2 and eigenfunctions $w_n(\boldsymbol{\rho})$

TM-waves: $H_z = 0$
$\boldsymbol{E} = (E_x, E_y, E_z)$ and $\boldsymbol{H} = (H_x.H_y, 0)$.
Find E_z! Then:

$$\begin{cases} \boldsymbol{E}_{\mathrm{T}}(\boldsymbol{r}) = \mathrm{i}\dfrac{k_z}{k_t^2}\nabla_T E_z(\boldsymbol{r}) \\ \boldsymbol{H}_{\mathrm{T}}(\boldsymbol{r}) = \mathrm{i}\dfrac{\omega\varepsilon_0}{k_t^2}\hat{\boldsymbol{z}} \times \nabla_{\mathrm{T}} E_z(\boldsymbol{r}) \end{cases} \tag{2}$$

$E_z = v(\boldsymbol{\rho})e^{\mathrm{i}k_z z}$

Eigenvalue problem

$$\nabla^2 v(\boldsymbol{\rho}) + k_t^2 v(\boldsymbol{\rho}) = 0,\ \boldsymbol{\rho} \in \Omega$$
$$v(\boldsymbol{\rho}) = 0,\ \boldsymbol{\rho} \in \Gamma$$

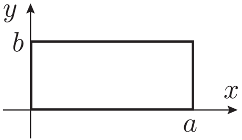
Eigenvalues k_{tn}^2 and eigenfunctions $v_n(\boldsymbol{\rho})$

Rectangular waveguide TE-modes

TE-waves $\Rightarrow H_z = w(\boldsymbol{\rho})e^{\mathrm{i}k_z z}$
Eigenvalue problem

$$\nabla^2 w(\boldsymbol{\rho}) + k_t^2 w(\boldsymbol{\rho}) = 0,\ \boldsymbol{\rho} \in \Omega$$
$$\hat{\boldsymbol{n}} \cdot \nabla w(\boldsymbol{\rho}) = 0, \text{ on all four sides}$$

Rectangular waveguide TE-modes



Eigenvalues: $k_{tmn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$

Eigenfunctions: $w_{mn}(\boldsymbol{\rho}) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$

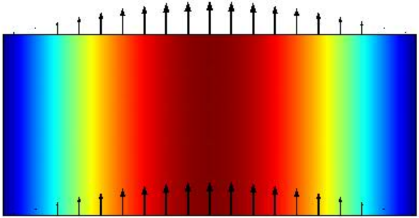
$m = 0, 1, 2 \ldots$, $n = 0, 1, 2 \ldots$

<u>but</u> $(m, n) \neq (0, 0)$

Rectangular waveguide

The fundamental mode is TE_{10}.

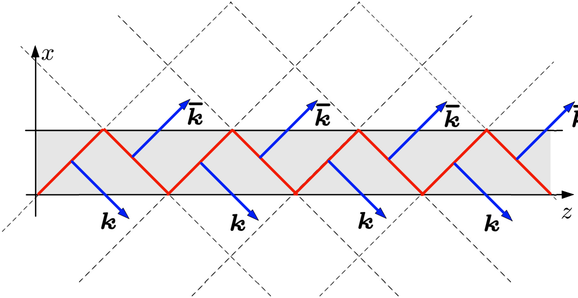
The electric field in the plane $z = 0$ from Comsol:



Eq. (15) $\Rightarrow \boldsymbol{E}(\boldsymbol{r}) = E_0 \sin\left(\frac{\pi x}{a}\right) e^{\mathrm{i}k_z z}\hat{\boldsymbol{y}}$

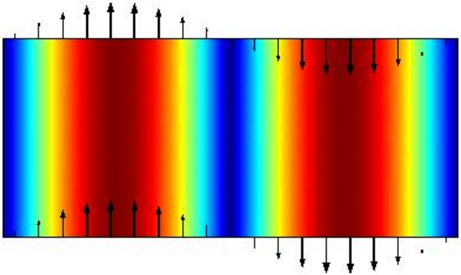
Rectangular waveguide

The fundamental mode TE_{10} is a superposition of two linearly polarized plane waves:



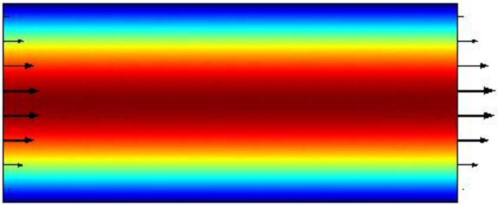
$$\boldsymbol{k} = \left(-\frac{\pi}{a}, 0, k_z\right) \text{ and } \bar{\boldsymbol{k}} = \left(\frac{\pi}{a}, 0, k_z\right)$$

The TE_{20} mode



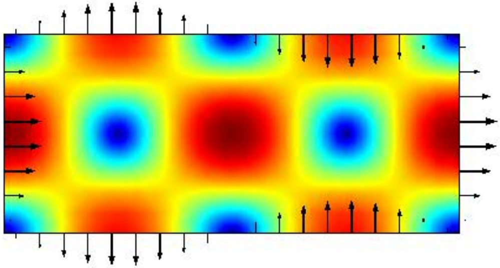
Eq. (15) $\Rightarrow \boldsymbol{E}(\boldsymbol{r}) = E_0 \sin\left(\frac{2\pi x}{a}\right) e^{\mathrm{i}k_z z}\hat{\boldsymbol{y}}$

The TE_{01} mode



Eq. (15) $\Rightarrow \boldsymbol{E}(\boldsymbol{r}) = E_0 \sin\left(\frac{\pi y}{b}\right) e^{\mathrm{i}k_z z}\hat{\boldsymbol{x}}$

The TE_{21} mode



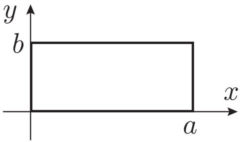
Cut-off frequencies

$e^{\mathrm{i}k_z z}$

$k_z = \sqrt{k^2 - k_{tmn}^2}$.

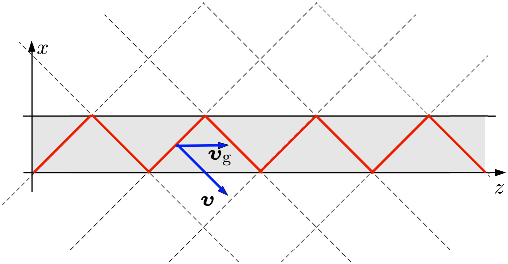
1. $k > k_{tmn} \implies k_z$ real $\implies$ propagating mode
2. $k = k_{tmn} \implies k_z = 0 \implies$ standing wave, cut-off frequency f_c
3. $k < k_{tmn} \implies k_z$ imaginary $\implies$ non-propagating mode

Cut-off frequencies: example



Rectangular waveguide $a = 5$ cm, $b = 2$cm.
TE$_{10}$: $f_c = 3$ GHz
TE$_{20}$: $f_c = 6$ GHz
TE$_{01}$: $f_c = 7.5$ GHz
$f < 3$ GHz No waves can propagate
3 GHz$< f <$ 6 GHz Only TE$_{10}$ can propagate
6 GHz$< f <$ 7.5 GHz TE$_{10}$ and TE$_{20}$ can propagate

Phase- and group speed



Group speed$=v_{\mathrm{g}} = \dfrac{k_z}{k}c=$ speed that power travels with.

Notice: $v_{\mathrm{g}} < c$

Phase- and group speed

Phase speed$=v_\mathrm{p} = \dfrac{k}{k_z}c=$ speed that pattern travels with.

Notice $v_\mathrm{p} > c$ since $k_z < k$

Outline for today

- Circular cylindrical waveguides
- Bessel functions
- The fundamental mode
- Comsol example