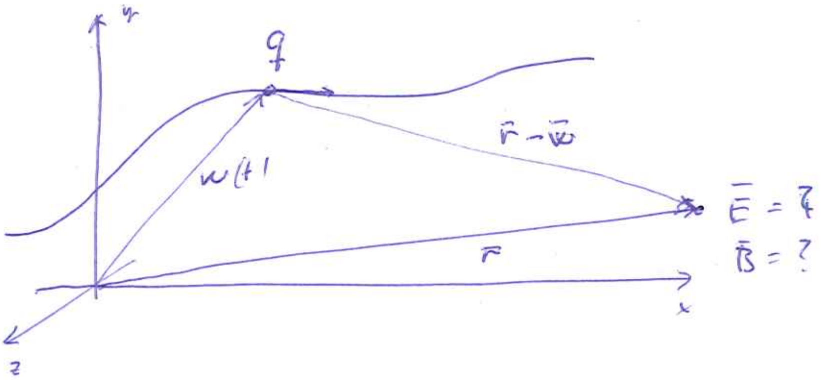
Föreläsning 7

Liénard-Wiechert potentials

Find $\bar{E}(\bar{r})$ and $\bar{B}(\bar{r})$ from q that moves along the curve $\bar{w}(t)$. t = time



Position $\bar{w}(t)$

Velocity $\bar{v}(t) = \dfrac{d\bar{w}(t)}{dt}$

acceleration $\bar{a}(t) = \dfrac{d^2\bar{w}(t)}{dt^2}$

Charge density $= \rho(\bar{r}, t)$

$\rho(\bar{r}', t) = q\, \delta^3(\bar{r}' - \bar{w}(t))$

where $\delta^3(\bar{r} - \bar{r}_0) = \delta(x - x_0)\, \delta(y - y_0)\, \delta(z - z_0)$

$$\int_{\mathcal{V}} \delta^3(\bar{r} - \bar{r}_0)\, dv = \begin{cases} 1 & \text{if } \bar{r}_0 \in \mathcal{V} \\ 0 & \text{otherwise} \end{cases}$$

$\cancel{V(\bar{r}, t) = \frac{1}{4\pi\varepsilon_0}}$

Current density $= \bar{J}(\bar{r}, t) = \rho(\bar{r}, t)\, \bar{v}(\bar{r}, t)$

$$V(\bar{r}, t) = \frac{1}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\rho(\bar{r}', t_r)}{|\bar{r} - \bar{r}'|}\, dv'$$

$$\bar{A}(\bar{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\bar{J}(\bar{r}', t_r)}{|\bar{r} - \bar{r}'|}\, dv'$$

$t_r = t - \dfrac{|\bar{r} - \bar{r}'|}{c}$ = retarded time

$$\Rightarrow \quad V(\bar{r},t) = \frac{q}{4\pi\varepsilon_0} \int_{\mathcal{V}} \frac{\delta^3(\bar{r}' - w(t_r))}{|\bar{r} - \bar{r}'|} \, d\tau'$$

$$\bar{A}(\bar{r},t) = \frac{q\mu_0}{4\pi} \int_{\mathcal{V}} \frac{\bar{v}(t_r)\ \delta^3(\bar{r}' - w(t_r))}{|\bar{r} - \bar{r}'|} \, d\tau'$$

Note $\quad w(t_r) = w\left(t - \dfrac{|\bar{r} - w(t_r)|}{c}\right)$

gives problem

Change of variables in integrals gives

$$V(\bar{r},t) = \frac{q}{4\pi\varepsilon_0\left(|\bar{r} - \bar{w}(t_r)| - \bar{\beta}\cdot(\bar{r} - \bar{w}(t_r))\right)}$$

$$\bar{A}(\bar{r},t) = \frac{\mu_0\, q\, \bar{v}(t_r)}{4\pi\left(|\bar{r} - \bar{w}(t_r)| - \bar{\beta}\cdot(\bar{r} - \bar{w}(t_r))\right)} = \frac{1}{c^2} V(\bar{r},t)\cdot\bar{v}(t_r)$$

$$\bar{\beta} = \frac{\bar{v}(t_r)}{c}$$

$\underline{\bar{E}, \bar{B}}$

$$\bar{E}(\bar{r},t) = -\nabla V(\bar{r},t) - \frac{\partial \bar{A}(\bar{r},t)}{\partial t}$$

$$\bar{B}(\bar{r},t) = \nabla \times \bar{A}(\bar{r},t)$$

$\Rightarrow$ see Griffiths

$$\bar{\mathscr{r}} = \bar{r} - \bar{w}(t_r), \quad \mathscr{r} = |\bar{\mathscr{r}}|, \quad \bar{u} = c\,\hat{\mathscr{r}}(t_r) - \bar{v}(t_r)$$

$$\bar{E}(\bar{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{\mathscr{r}}{(\bar{\mathscr{r}}\cdot\bar{u})^3}\left[(c^2 - v^2)\,\bar{u} + \bar{\mathscr{r}}\times(\bar{u}\times\bar{a})\right]$$

Notice! $\bar{\mathscr{r}} = \bar{\mathscr{r}}(t_r)$, $v = v(t_r)$, $\bar{a} = \bar{a}(t_r)$ in this expression

$$\bar{B}(\bar{r},t) = \frac{1}{c}\,\hat{n} \times \bar{E}(\bar{r},t)$$

Notice: $\bar{E} \perp \bar{B}$

Example Assume $\bar{w}(t_r) = \bar{r}_0$ (constant)

$$\Rightarrow \bar{v} = \frac{d\bar{w}}{dt} = \bar{0} \quad \text{and} \quad \bar{a} = \frac{\partial^2 w}{\partial t^2} = \bar{0}$$

$$\Rightarrow \bar{u} = c\,\hat{n}$$

$$\Rightarrow \bar{E} = \frac{q}{4\pi\varepsilon_0}\frac{n}{(cn)^3}c^3\hat{n} = \frac{q}{4\pi\varepsilon_0}\frac{\hat{n}}{n^2}$$

$$\bar{B} = \bar{0}$$

$\Rightarrow$ Coulomb's field

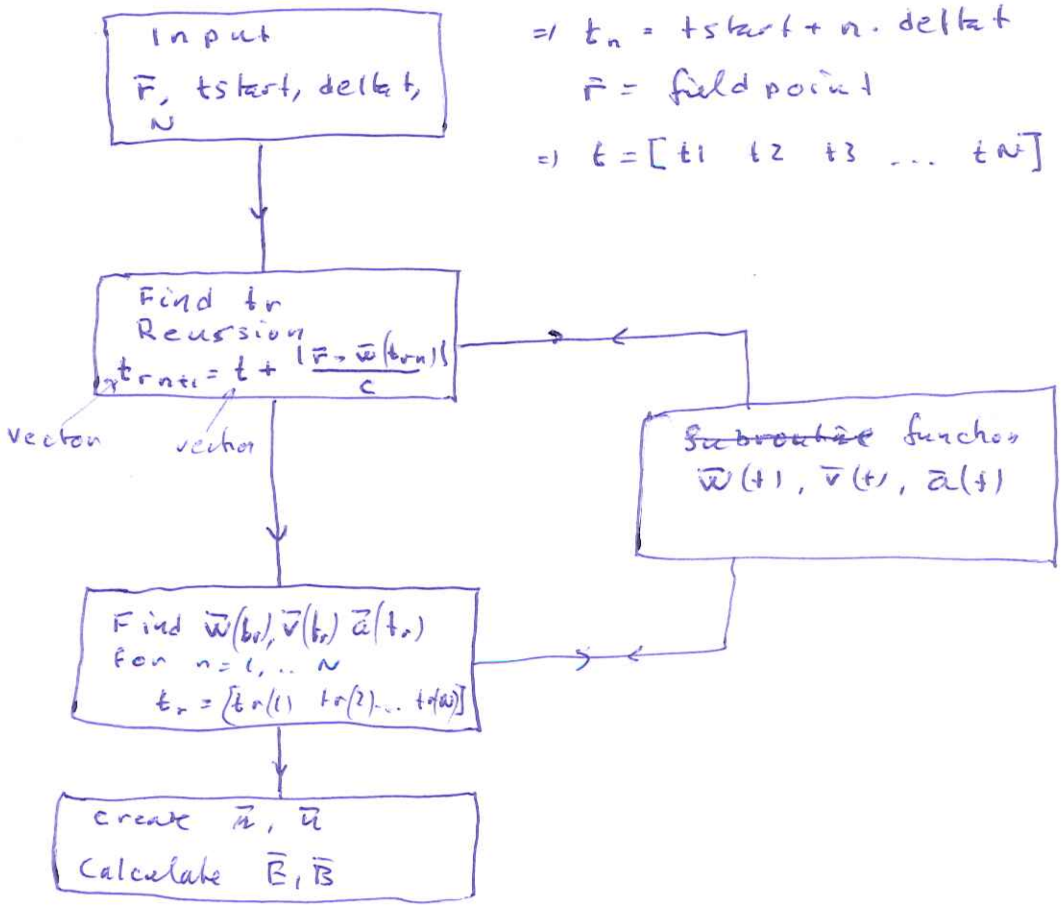
Matlab script

Use function on script

$\Rightarrow t_n = tstart + n \cdot deltat$

$\bar{r}$ = field point

$\Rightarrow t = [t1 \;\; t2 \;\; t3 \;\; \dots \;\; tN]$



Dot and cross in Matlab

Ex

Let
```
A = [Ax  Ay  Az];
B = [Bx  By  Bz];
```
row vectors $\bar{A} = (A_x, A_y, A_z)$
$\bar{B} = (B_x, B_y, B_z)$

$\Rightarrow$
```
C = dot(A, B);
```
$C = \bar{A} \cdot \bar{B}$ scalar product

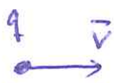
```
D = cross(A, B);
```
$\bar{D} = \bar{A} \times \bar{B}$ vector product.

Notice `dot(A, B) = A.*B`

Recursion

```
tr = t - r/c;
tol = 1e-4;   (for instance)
while diff > tol
     :
   trn = t - (r-w(tr))/c;
   diff = max(abs(tr-trn))/..
   tr = trn
end
```

$\bar{E}$ and $\bar{B}$ from q with constant $\bar{v}$ $\Rightarrow$ $\bar{w}(t) = \bar{w}(0) + \bar{v} \cdot t$



$\bar{a} = 0 \Rightarrow$

$$\bar{E}(\bar{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{(c^2 - v^2)\,\kappa}{(\bar{\kappa} \cdot \bar{u})^3}\,\bar{u}$$

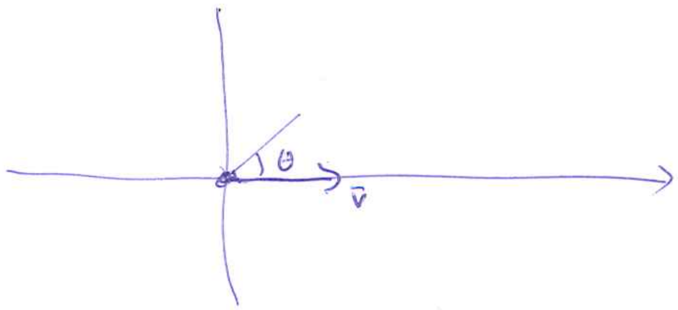
<u>Remember!</u> $\quad \vec{\kappa} = \bar{r} - \bar{w}(t_r)$

$$\bar{u} = c\,\hat{\kappa}(t_r) - \bar{v}$$

we get, see Griffiths

$$\bar{E}(\bar{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{1 - (v/c)^2}{\left(1 - v^2 \sin^2\theta / c^2\right)^{3/2}} \frac{\hat{R}}{R^2}$$

where $\bar{R} = \bar{r} - \bar{v} \cdot t$ $\qquad$ (Notice! t, not t_r)



Forward $\Rightarrow$ $\theta = 0, \pi$ $\Rightarrow$ $\quad \bar{E}(\bar{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{\hat{R}}{R^2} \cdot \left(1 - (v/c)^2\right)$

$\Rightarrow$ reduced by factor $1 - (v/c)^2$

Perpendicular $\Rightarrow$ $\theta = \frac{\pi}{2}$ $\Rightarrow$ $\quad \bar{E}(\bar{r},t) = \frac{q}{4\pi\varepsilon_0} \frac{\hat{R}}{R^2} \frac{1}{\sqrt{1 - (v/c)^2}}$

$\Rightarrow$ enhanced by $\frac{1}{\sqrt{1 - (v/c)^2}}$

Mobile

$$\vec{B}(r,t) = \frac{1}{c}\,\hat{n} \times \vec{E}(r,t)$$

$$\hat{n} = \frac{\vec{r} - \vec{v}\cdot t_r}{n} = \frac{(\vec{r} - \vec{v}t) + (t - t_r)\vec{v}}{n} = \frac{\vec{R}}{n} + \frac{\vec{v}}{c}$$

$$\left(t - t_r = \frac{n}{c} \right)$$

$$\Rightarrow \quad \vec{B}(\vec{r},t) = \frac{1}{c^2}\,\vec{v} \times \vec{E}(\vec{r},t)$$