Electrodynamics, lecture 8, 2019

Anders Karlsson, anders.karlsson@eit.lth.se

Electrical and information technology

Visits to ESS and MAXIV

- ESS: May 9, 10:00 till 12:15
- MAX IV: May 23, 13.15-14.30

Last lecture

- Liénard-Wiechert potentials
- History of electromagnetism
- $\boldsymbol{E}$ and $\boldsymbol{B}$ from charged particle
- $\boldsymbol{E}$ and $\boldsymbol{B}$ from charged particle with constant velocity

Liénard-Wiechert potentials

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} \frac{q\boldsymbol{v}(t_{\mathrm{r}})}{|\boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}})| - \boldsymbol{\beta}(t_{\mathrm{r}}) \cdot (\boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}}))} \tag{1}$$

$$V(\boldsymbol{r},t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}})| - \boldsymbol{\beta}(t_{\mathrm{r}}) \cdot (\boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}}))} \tag{2}$$

where $\boldsymbol{\beta}(t_{\mathrm{r}}) = \boldsymbol{v}(t_{\mathrm{r}})/c$.

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\boldsymbol{v}(t_{\mathrm{r}})}{c^2} V(\boldsymbol{r},t) \tag{3}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})]$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\boldsymbol{E}(\boldsymbol{r},t)$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})]$$

Here $\boldsymbol{\mathscr{r}}$, $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{a}$ are evaluated at retarded time t_{r}.

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}}) \\
|\boldsymbol{\mathscr{r}}| &= |\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})| = c(t - t_{\mathrm{r}}) \\
\boldsymbol{u} &= c\hat{\boldsymbol{\mathscr{r}}}(t_{\mathrm{r}}) - \boldsymbol{v}(t_{\mathrm{r}}) \\
\boldsymbol{v} &= \frac{d\boldsymbol{w}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} \\
\boldsymbol{a} &= \frac{d\boldsymbol{v}(t_{\mathrm{r}})}{dt_{\mathrm{r}}}
\end{aligned}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge with constant velocity

Let $\boldsymbol{a} = \boldsymbol{0}$. Then

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^3}(c^2 - v^2)\boldsymbol{u}$$

Introduce $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{v}t_\mathrm{r}$.

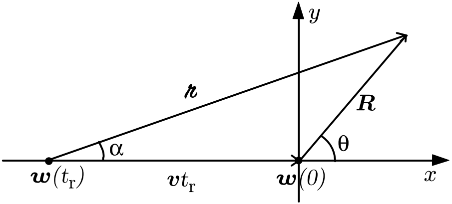


Figure: The particle at $\boldsymbol{w}(t_\mathrm{r})$ and $\boldsymbol{w}(0)$.

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge with constant velocity

Determine $\boldsymbol{E}(\boldsymbol{r},0)$ and $\boldsymbol{B}(\boldsymbol{r},0)$. After some algebra

$$\boldsymbol{E}(\boldsymbol{r},0)=\frac{q}{4\pi\varepsilon_0}\frac{(1-(v/c)^2)\boldsymbol{R}}{R^3(1-v^2\sin^2\theta/c^2)^{3/2}} \tag{4}$$

and

$$\boldsymbol{B}(\boldsymbol{r},0)=\frac{1}{c^2}\boldsymbol{v}\times\boldsymbol{E}(\boldsymbol{r},0) \tag{5}$$

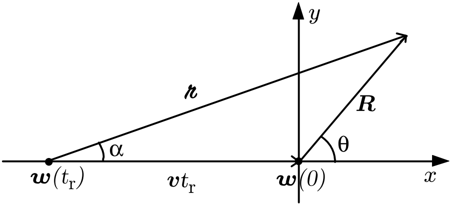


Figure: The particle at $\boldsymbol{w}(t_{\rm r})$ and $\boldsymbol{w}(0)$.

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge with constant velocity

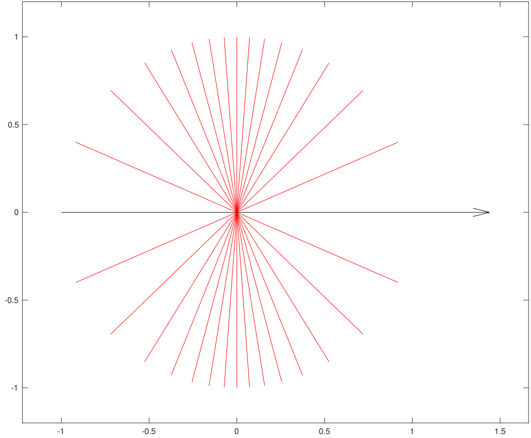


Figure: $\boldsymbol{E}(\boldsymbol{r}, 0)$ for $\beta = 0.9$.

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge with constant velocity

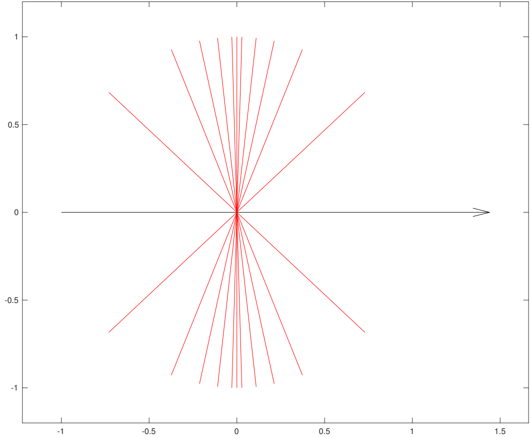


Figure: $\boldsymbol{E}(\boldsymbol{r}, 0)$ for $\beta = 0.95$.

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge with constant velocity

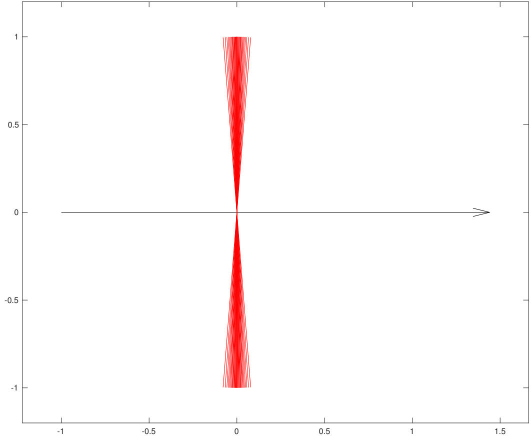


Figure: $\boldsymbol{E}(\boldsymbol{r}, 0)$ for $\beta = 0.999$.

Today

- Radiation from currents and charges
- Larmor formula
- Liénard's generalization of the Larmor formula

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

Question: Can we identify what part of $\boldsymbol{E}$ that contains the radiated field?

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u}+\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})]$$

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}})\\
|\boldsymbol{\mathscr{r}}| &= |\boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}})| = c(t-t_{\mathrm{r}})\\
\boldsymbol{u} &= c\hat{\boldsymbol{\mathscr{r}}}(t_{\mathrm{r}})-\boldsymbol{v}(t_{\mathrm{r}})\\
\boldsymbol{v} &= \frac{d\boldsymbol{w}(t_{\mathrm{r}})}{dt_{\mathrm{r}}}\\
\boldsymbol{a} &= \frac{d\boldsymbol{v}(t_{\mathrm{r}})}{dt_{\mathrm{r}}}
\end{aligned}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

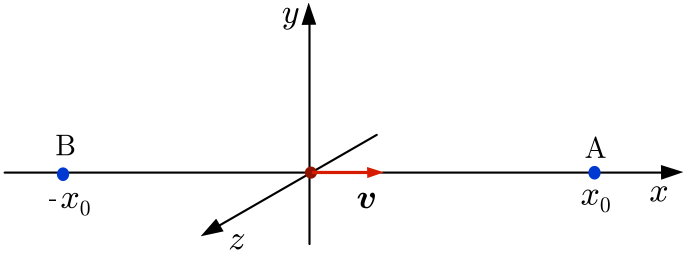
Answer: Yes!

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})$$

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}}) \\
|\boldsymbol{\mathscr{r}}| &= |\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})| = c(t - t_{\mathrm{r}}) \\
\boldsymbol{u} &= c\hat{\boldsymbol{\mathscr{r}}}(t_{\mathrm{r}}) - \boldsymbol{v}(t_{\mathrm{r}}) \\
\boldsymbol{v} &= \frac{d\boldsymbol{w}(t_{\mathrm{r}})}{dt_{\mathrm{r}}} \\
\boldsymbol{a} &= \frac{d\boldsymbol{v}(t_{\mathrm{r}})}{dt_{\mathrm{r}}}
\end{aligned}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

Assume that $\boldsymbol{w}(t_{\mathrm{r}}) = \boldsymbol{0}$ and $\boldsymbol{v}(t_{\mathrm{r}}) = v\hat{\boldsymbol{x}}$.



Observer A is at $\boldsymbol{r}_A = x_0\hat{\boldsymbol{x}}$, where $x_0 > 0$ and observer B is at $\boldsymbol{r}_B = -x_0\hat{\boldsymbol{x}}$.
Who will receive the strongest radiated $\boldsymbol{E}$-field?

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})$$

For A

$$\begin{aligned}\boldsymbol{\mathscr{r}}&=x_0\hat{\boldsymbol{x}}\\|\boldsymbol{\mathscr{r}}|&=x_0\\\boldsymbol{u}&=(c-v)\hat{\boldsymbol{x}}\end{aligned}$$

For B

$$\begin{aligned}\boldsymbol{\mathscr{r}}&=-x_0\hat{\boldsymbol{x}}\\|\boldsymbol{\mathscr{r}}|&=x_0\\\boldsymbol{u}&=-(c+v)\hat{\boldsymbol{x}}\end{aligned}$$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})$$

For A

$$|\boldsymbol{E}(\boldsymbol{r},t)|\sim\frac{1}{x_0(c-v)^2}$$

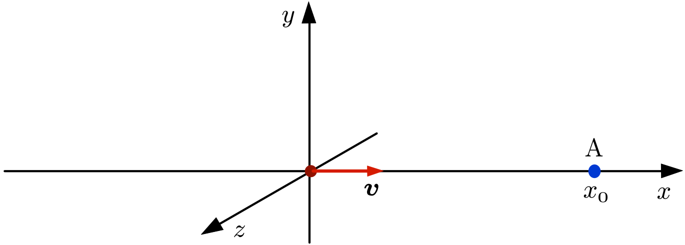
For B

$$|\boldsymbol{E}(\boldsymbol{r},t)|\sim\frac{1}{x_0(c+v)^2}$$

At MAX IV: 3GeV electrons $\Rightarrow \dfrac{v}{c}=0.999999986$. Then $\boldsymbol{E}$ is 10^{16} times larger at A than at B. The power flux is 10^{32} times larger at A than at B.

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

Assume that $\boldsymbol{w}(t_{\rm r}) = \boldsymbol{0}$ and $\boldsymbol{v}(t_{\rm r}) = v\hat{\boldsymbol{x}}$.



Observer A is at $\boldsymbol{r}_A = x_0\hat{\boldsymbol{x}}$, where $x_0 > 0$.
In what direction should $\boldsymbol{a}$ be in order to maximize $\boldsymbol{E}$ at A?

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

Assume that $\boldsymbol{w}(t_\mathrm{r}) = \boldsymbol{0}$ and $\boldsymbol{v}(t_\mathrm{r}) = v\hat{\boldsymbol{x}}$.
Observer A is at $\boldsymbol{r}_A = x_0\hat{\boldsymbol{x}}$, where $x_0 > 0$.
In what direction should $\boldsymbol{a}$ be in order to maximize $\boldsymbol{E}$ at A?

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u})^3}\boldsymbol{\mathscr{r}} \times (\boldsymbol{u} \times \boldsymbol{a})$$

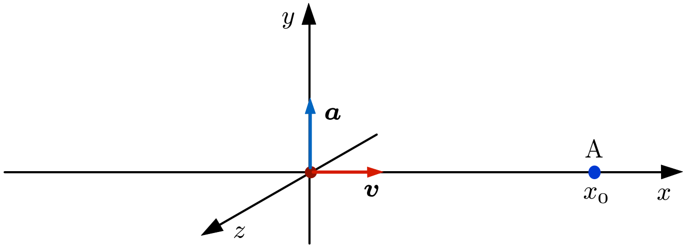
Here

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= x_0\hat{\boldsymbol{x}} \\
|\boldsymbol{\mathscr{r}}| &= x_0 \\
\boldsymbol{u} &= (c - v)\hat{\boldsymbol{x}} \\
\boldsymbol{\mathscr{r}} \cdot \boldsymbol{u} &= x_0(c - v)
\end{aligned}$$

Answer: $\boldsymbol{a}$ should be perpendicular to $\hat{\boldsymbol{x}}$.

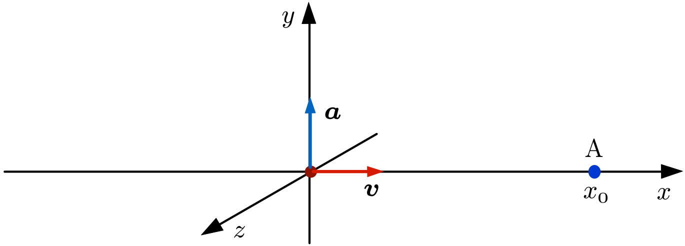
$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

$\boldsymbol{w}(t_{\mathrm{r}}) = \mathbf{0}$, $\boldsymbol{v}(t_{\mathrm{r}}) = v\hat{\boldsymbol{x}}$ and $\boldsymbol{a}(t_{\mathrm{r}}) = a\hat{\boldsymbol{y}}$



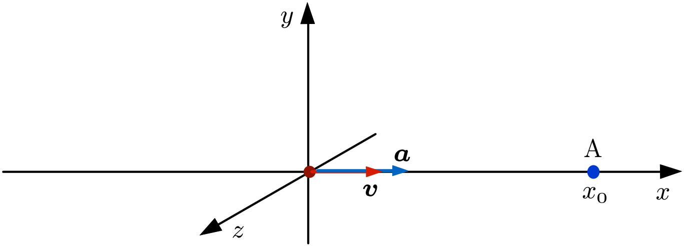
$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

The intensity of the light from q is maximized when q travels towards you and its acceleration is perpendicular to the velocity!



$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

Question: What is the radiated $\boldsymbol{E}$ at A if $\boldsymbol{a} = a\hat{\boldsymbol{x}}$?



Answer: Zero

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

Question: What is the radiated $\boldsymbol{E}$ at A if $\boldsymbol{a} = a\hat{\boldsymbol{x}}$?

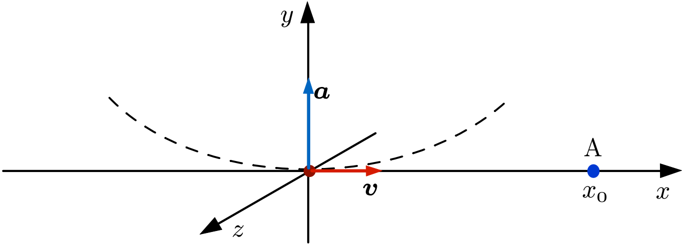
$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}\boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})$$

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= x_0\hat{\boldsymbol{x}} \\
|\boldsymbol{\mathscr{r}}| &= x_0 \\
\boldsymbol{u} &= (c-v)\hat{\boldsymbol{x}} \\
\boldsymbol{v} &= v\hat{\boldsymbol{x}} \\
\boldsymbol{a} &= a\hat{\boldsymbol{x}}
\end{aligned}$$

Answer: Zero since $\boldsymbol{u}\times\boldsymbol{a} = \boldsymbol{0}$

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

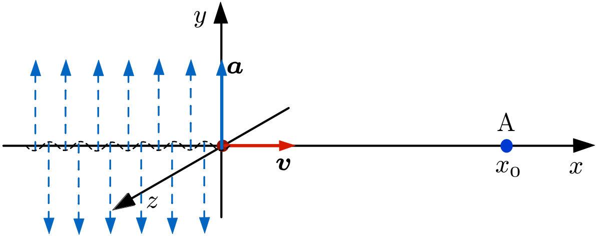
The intensity of the light from q is obtained when q travels towards you and the acceleration is perpendicular to the velocity!



Circular motion in dipole magnet

$\boldsymbol{E}$ and $\boldsymbol{H}$ from point charge

The intensity of the light from q is obtained when q travels towards you and the acceleration is perpendicular to the velocity!



Sinusoidal motion in undulator