April 12, 2019

Lectures 6 and 7

1 Retarded time

An observer A sees light coming from the position $\boldsymbol{w}$, see Figure (1). The light that arrived at time t was then sent from $\boldsymbol{w}$ at time

$$t_{\mathrm{r}} = t - \frac{|\boldsymbol{r} - \boldsymbol{w}|}{c}$$

We call t_{r} the **retarded time**.

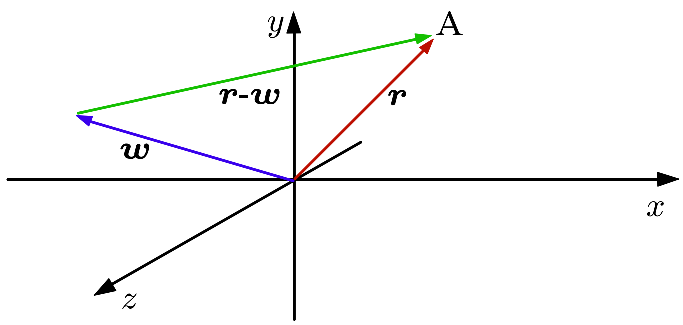

Figure 1: It takes $|\boldsymbol{r} - \boldsymbol{w}|/c$ for light to travel from $\boldsymbol{w}$ to $\boldsymbol{r}$.

Now look at Figure (2). The charge q travels along the curve $\boldsymbol{w}(t)$, where t is time. Assume that observer A at time t receives light sent from q. When was the light sent from q and what was the position of q when the light was sent?

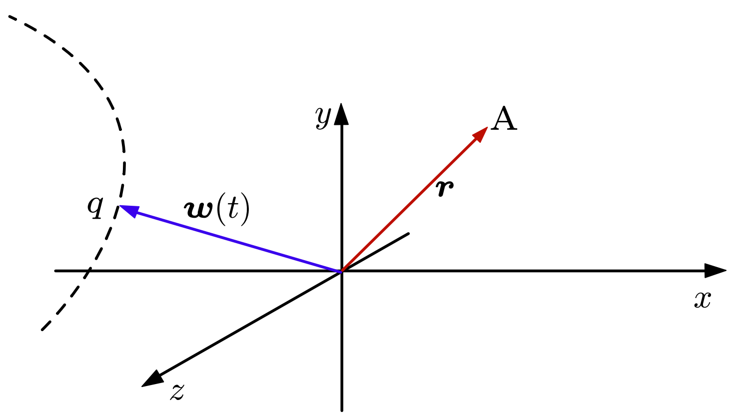

Figure 2: The charge q travels along the curve $\boldsymbol{w}(t)$ and emits light.

The answer is that the light was sent from q at the retarded time

$$t_{\mathrm{r}} = t - \frac{|\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})|}{c} \tag{1.1}$$

Notice that here $\boldsymbol{w}(t_{\mathrm{r}})$ is the position of q at the retarded time. It means that t_{r} is implicitly given by (1.1). To find the position and t_{r} we have to solve equation (1.1) for t_{r}.

2 Sources to electromagnetic fields

We will now derive expressions for the electric and magnetic fields that are radiated Electromagnetic fields are generated by accelerating particles. These moving particles are represented by a current density $\boldsymbol{J}(\boldsymbol{r},t)$ and charge density $\rho(\boldsymbol{r},t)$ in the Maxwell equations. The current and charge density act as sources in the Maxwell equations, such that

$$\nabla\times\boldsymbol{E}(\boldsymbol{r},t) = -\frac{\partial\boldsymbol{B}(\boldsymbol{r},t)}{\partial t} \tag{2.1}$$

$$\nabla\times\boldsymbol{H}(\boldsymbol{r},t) = \boldsymbol{J}(\boldsymbol{r},t)+\varepsilon_0\frac{\partial\boldsymbol{E}(\boldsymbol{r},t)}{\partial t} \tag{2.2}$$

$$\nabla\cdot\boldsymbol{E}(\boldsymbol{r},t) = \frac{\rho(\boldsymbol{r},t)}{\varepsilon_0} \tag{2.3}$$

$$\nabla\cdot\boldsymbol{B}(\boldsymbol{r},t) = 0 \tag{2.4}$$

The divergence of the Ampère law, (2.2), gives, together with (2.3) the relation between the current and charge density

$$\nabla\cdot\boldsymbol{J}(\boldsymbol{r},t) = -\frac{\partial\rho(\boldsymbol{r},t)}{\partial t} \tag{2.5}$$

This is the continuity equation. If we integrate this over a volume V, enclosed by the surface S, we get

$$\frac{\partial Q(t)}{\partial t} = -i(t) \tag{2.6}$$

where

$$Q(t) = \int_V \rho(\boldsymbol{r},t)\,\mathrm{d}v \tag{2.7}$$

is the charge in V and

$$i(t) = \oint_S \boldsymbol{J}(\boldsymbol{r},t)\cdot\hat{\boldsymbol{n}}\,\mathrm{d}S \tag{2.8}$$

is the current running out from V. Here $\hat{\boldsymbol{n}}$ is the outward directed unit normal vector to S.

To obtain the electromagnetic fields from a current density we use the magnetic vector potential $\boldsymbol{A}$. It is related to the magnetic flux density by

$$\boldsymbol{B} = \nabla\times\boldsymbol{A} \tag{2.9}$$

The induction law implies

$$\nabla\times\left(\boldsymbol{E}+\frac{\partial\boldsymbol{A}}{\partial t}\right) = \boldsymbol{0} \tag{2.10}$$

All irrotational fields can be expressed in terms of a scalar field. We then use

$$\boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} = -\nabla V \tag{2.11}$$

where V is the electric scalar potential[1]. Notice that for static fields $\boldsymbol{E} = -\nabla V$, which is the same relation as in electrostatics. Now insert (2.9) and (2.11) into Amperes law

$$\nabla \times (\nabla \times \boldsymbol{A}) + \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} + \frac{1}{c^2}\nabla\frac{\partial V}{\partial t} = \mu_0 \boldsymbol{J} \tag{2.12}$$

Since $\nabla \times (\nabla \times \boldsymbol{A}) = \nabla(\nabla \cdot \boldsymbol{A}) - \nabla^2 \boldsymbol{A}$ we get

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} - \nabla\left(\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) = -\mu_0 \boldsymbol{J} \tag{2.13}$$

We are free to choose $\nabla \cdot \boldsymbol{A}$ as we like and we let

$$\nabla \cdot \boldsymbol{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0 \tag{2.14}$$

This condition is called the Lorenz condition. The equation for the vector potential is, by that,

$$\nabla^2 \boldsymbol{A} - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{A}}{\partial t^2} = -\mu_0 \boldsymbol{J} \tag{2.15}$$

The solution is

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int_V \frac{J(\boldsymbol{r}', t_\mathrm{r})}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}v' \tag{2.16}$$

where

$$t_\mathrm{r} = t - \frac{|\boldsymbol{r} - \boldsymbol{r}'|}{c} \tag{2.17}$$

is the retarded time. The equation for the scalar potential V is obtained from Gauss law

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\varepsilon_0} \tag{2.18}$$

and the Lorenz condition (2.14):

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\varepsilon_0} \tag{2.19}$$

The solution is

$$V(\boldsymbol{r}, t) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\boldsymbol{r}', t_\mathrm{r})}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}v' \tag{2.20}$$

[1] We use the same notation V for potential as Griffiths, even though we also use V for volume

3 Liénard-Wiechert Potentials

Assume a point charge that moves along a curve with parametrization $\boldsymbol{w}(t)$, where t is time and $\boldsymbol{w}(t)$ the position of the particle at time t. To obtain the electric and magnetic fields we need to calculate the current density $\boldsymbol{J}(\boldsymbol{r}, t)$ and the charge density $\rho(\boldsymbol{r}, t)$ and plug them into the vector and scalar potentials, (2.16) and (2.20). The charge and current distribution for a point charge moving along $\boldsymbol{w}(t_\mathrm{r})$ is

$$\rho(\boldsymbol{r}', t_\mathrm{r}) = q\delta^3(\boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r})) \tag{3.1}$$

$$\boldsymbol{J}(\boldsymbol{r}', t_\mathrm{r}) = q\frac{\partial \boldsymbol{w}(t_\mathrm{r})}{\partial t_\mathrm{r}}\delta^3(\boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r})) \tag{3.2}$$

The three-dimensional delta-function is defined by $\delta^3(\boldsymbol{r} - \boldsymbol{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$. From (2.16) and (2.20) we get

$$V(\boldsymbol{r}, t) = \frac{q}{4\pi\varepsilon_0}\int_V \frac{\delta^3(\boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r}))}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}v' \tag{3.3}$$

$$\boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi}\int_V \frac{q\boldsymbol{v}(t_\mathrm{r})\delta^3(\boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r}))}{|\boldsymbol{r} - \boldsymbol{r}'|}\,\mathrm{d}v' \tag{3.4}$$

where $\boldsymbol{v}(t_\mathrm{r}) = \dfrac{\partial \boldsymbol{w}(t_\mathrm{r})}{\partial t_\mathrm{r}}$ is the velocity of the particle at the retarded time.

We need to to handle substitution of variables the delta functions in a correct way. First consider the integral

$$I = \int_{-\infty}^{\infty} \delta(g(x'))\,\mathrm{d}x' \tag{3.5}$$

where we assume $g(x)$ to be a monotonic function of x with one zero $x = x_1$ in the interval $-\infty < x < \infty$. To solve it we make a substitution of variables so that $\xi = g(x')$. Then the integral becomes

$$I = \int_{g(-\infty)}^{g(\infty)} \left(\frac{g(x'(\xi))}{dx'}\right)^{-1} \delta(\xi)\,\mathrm{d}\xi \tag{3.6}$$

Then

$$I = \frac{1}{\left|\dfrac{g(x)}{dx}\right|_{x=x_1}} \tag{3.7}$$

It means that the delta function in (3.5) can be written as

$$\delta(g(x)) = \frac{\delta(x - x_1)}{\left|\dfrac{g(x)}{dx}\right|_{x=x_1}} \tag{3.8}$$

where $g(x_1) = 0$.[2] Now we do the substitution of variables in a 3-dimensional integral of the form

$$\int_V \delta(\boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r}))\,\mathrm{d}v' = \int_V \delta(\boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r}))\,\mathrm{d}x'\mathrm{d}y'\mathrm{d}z' \tag{3.9}$$

We let $\boldsymbol{r}'' = \boldsymbol{r}' - \boldsymbol{w}(t_\mathrm{r})$ and make a change of variables so that

$$\int_V (\mathbf{J})^{-1}\,\delta(\boldsymbol{r}'')\,\mathrm{d}x''\mathrm{d}y''\mathrm{d}z'' \tag{3.10}$$

where $\mathbf{J}$ is the Jacobian, i.e., the determinant

$$\mathbf{J} = \begin{vmatrix} \frac{\partial x''}{\partial x'} & \frac{\partial x''}{\partial y'} & \frac{\partial x''}{\partial z'} \\ \frac{\partial y''}{\partial x'} & \frac{\partial y''}{\partial y'} & \frac{\partial y''}{\partial z'} \\ \frac{\partial z''}{\partial x'} & \frac{\partial z''}{\partial y'} & \frac{\partial z''}{\partial z'} \end{vmatrix} \tag{3.11}$$

We use that

$$\frac{\partial x''}{\partial x'} = 1 - \frac{\partial w_x(t_\mathrm{r})}{\partial t_\mathrm{r}}\frac{\partial t_\mathrm{r}}{\partial x'} \tag{3.12}$$

and

$$\frac{\partial t_\mathrm{r}}{\partial x'} = -\frac{1}{c}\frac{x - x'}{|\boldsymbol{r} - \boldsymbol{r}'|} \tag{3.13}$$

Also

$$\frac{\partial w_x(t_\mathrm{r})}{\partial t_\mathrm{r}} = v_x(t_\mathrm{r}) \tag{3.14}$$

is the $x-$component of the velocity of the charge at the retarded time. Then

$$\frac{\partial x''}{\partial x'} = 1 - \frac{v_x(t_\mathrm{r})(x - x')}{c|\boldsymbol{r} - \boldsymbol{r}'|} \tag{3.15}$$

$$\frac{\partial y''}{\partial x'} = -\frac{v_y(t_\mathrm{r})(x - x')}{c|\boldsymbol{r} - \boldsymbol{r}'|} \tag{3.16}$$

$$\tag{3.17}$$

and so on. We can then evaluate the Jacobian. After some algebra it can be written as

$$\mathbf{J} = 1 - \frac{\boldsymbol{v}\cdot(\boldsymbol{r} - \boldsymbol{r}')}{c|\boldsymbol{r} - \boldsymbol{r}'|} \tag{3.18}$$

Notice that $\mathbf{J} > 0$ since $|\boldsymbol{v}| < c$. After the integration the delta function $\delta(\boldsymbol{r}'')$ makes $\boldsymbol{r}' = \boldsymbol{w}(t_r)$.

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[2] If $g(x)$ has N zeros the expression is $\delta(g(x)) = \sum_{i=1}^{N} \frac{\delta(x - x_i)}{\left|\frac{g(x)}{dx}\right|_{x=x_i}}$ where $g(x_i) = 0$, $i = 1 \ldots N$.

Now insert the charge density, see (3.1), and current density, see (3.2), in the expressions for the vector and scalar potentials, see (2.16) and (2.20). Then use the Jacobian (3.18) to get

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}}{|\boldsymbol{r}-\boldsymbol{w}|-\boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{w})} \tag{3.19}$$

$$V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{q}{|\boldsymbol{r}-\boldsymbol{w}|-\boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{w})} \tag{3.20}$$

Here we have introduced $\boldsymbol{\beta}=\boldsymbol{v}/c$. We also see that

$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\boldsymbol{v}}{c^2}V(\boldsymbol{r},t) \tag{3.21}$$

We have to remember that $\boldsymbol{w}$ and $\boldsymbol{v}$ are the position and velocity of the charge at the retarded time t_{r} and that t_{r} is implicitly given by

$$t_{\mathrm{r}}=t-\frac{|\boldsymbol{r}-\boldsymbol{w}(t_{\mathrm{r}})|}{c} \tag{3.22}$$

The retarded time makes it somewhat cumbersome to obtain the expressions for the electric and magnetic fields for the charged particle, but as we will see it can be done.

3.1 The electric and magnetic fields

To evaluate the expressions (2.9) for $\boldsymbol{H}$ and (2.11) for $\boldsymbol{E}$ for a point charge we need to find the expressions for ∇V and $\dfrac{\partial \boldsymbol{A}}{\partial t}$.

$$\nabla V=-\frac{q}{4\pi\varepsilon_0(|\boldsymbol{r}-\boldsymbol{w}|-\beta\cdot(\boldsymbol{r}-\boldsymbol{w}))^2}\nabla(|\boldsymbol{r}-\boldsymbol{w}|-\boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{w})) \tag{3.23}$$

Since $|\boldsymbol{r}-\boldsymbol{w}|=c(t-t_{\mathrm{r}})$ we have $\nabla|\boldsymbol{r}-\boldsymbol{w}|=-c\nabla t_{\mathrm{r}}$. Use

$$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{r})=\boldsymbol{\beta}+(\nabla t_{\mathrm{r}})c^{-1}\boldsymbol{a}\cdot\boldsymbol{r} \tag{3.24}$$

and

$$\nabla(\boldsymbol{\beta}\cdot\boldsymbol{w})=(\nabla t_{\mathrm{r}})c^{-1}(\boldsymbol{a}\cdot\boldsymbol{w}-v^2) \tag{3.25}$$

Here $\boldsymbol{a}$ is the acceleration at time t_{r}. We also need an expression for ∇t_{r}

$$\nabla t_{\mathrm{r}}=-c^{-1}\nabla|\boldsymbol{r}-\boldsymbol{w}| \tag{3.26}$$

where

$$\nabla|\boldsymbol{r}-\boldsymbol{w}|=\frac{1}{2|\boldsymbol{r}-\boldsymbol{w}|}\nabla(r^2+w^2-2\boldsymbol{r}\cdot\boldsymbol{w}) \tag{3.27}$$

and

$$\nabla(r^2+w^2-2\boldsymbol{r}\cdot\boldsymbol{w})=2\boldsymbol{r}+2(\nabla t_{\mathrm{r}})\boldsymbol{w}\cdot\boldsymbol{v}-2\boldsymbol{w}-2(\nabla t_{\mathrm{r}})\boldsymbol{r}\cdot\boldsymbol{v} \tag{3.28}$$

From (3.26)–(3.28) we get

$$\nabla t_{\mathrm{r}} = -\frac{(\boldsymbol{r}-\boldsymbol{w})}{c|\boldsymbol{r}-\boldsymbol{w}| - (\boldsymbol{r}-\boldsymbol{w})\cdot\boldsymbol{v}} \tag{3.29}$$

We combine (3.23)–(3.25) with (3.29) to get

$$\nabla V = \frac{qc}{4\pi\varepsilon_0}\left[\frac{\boldsymbol{v}}{(|\boldsymbol{r}-\boldsymbol{w}|c - (\boldsymbol{r}-\boldsymbol{w})\cdot\boldsymbol{v})^2} - \frac{(c^2 - v^2 + (\boldsymbol{r}-\boldsymbol{w})\cdot\boldsymbol{a})(\boldsymbol{r}-\boldsymbol{w})}{(|\boldsymbol{r}-\boldsymbol{w}|c - (\boldsymbol{r}-\boldsymbol{w})\cdot\boldsymbol{v})^3}\right] \tag{3.30}$$

The time derivative of $\boldsymbol{A}$ is

$$\frac{\partial \boldsymbol{A}}{\partial t} = \frac{\partial t_{\mathrm{r}}}{\partial t}\frac{\partial \boldsymbol{A}}{\partial t_{\mathrm{r}}} \tag{3.31}$$

Here

$$\frac{\partial t_{\mathrm{r}}}{\partial t} = \left(\frac{\partial t}{\partial t_{\mathrm{r}}}\right)^{-1} \tag{3.32}$$

(remember that $t = t_{\mathrm{r}} + c^{-1}|\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})|$). By that we get

$$\frac{\partial t_{\mathrm{r}}}{\partial t} = \frac{c|\boldsymbol{r}-\boldsymbol{w}|}{c|\boldsymbol{r}-\boldsymbol{w}| - (\boldsymbol{r}-\boldsymbol{w})\cdot\boldsymbol{v}} \tag{3.33}$$

Also

$$\frac{\partial \boldsymbol{A}}{\partial t_{\mathrm{r}}} = c^{-2}\boldsymbol{a}V + \frac{\boldsymbol{v}}{c^2}\frac{\partial V}{\partial t_{\mathrm{r}}} \tag{3.34}$$

and

$$\frac{\partial V}{\partial t_{\mathrm{r}}} = -\frac{q}{4\pi\varepsilon_0(|\boldsymbol{r}-\boldsymbol{w}| - \beta\cdot(\boldsymbol{r}-\boldsymbol{w}))^2}\frac{\partial}{\partial t_{\mathrm{r}}}(|\boldsymbol{r}-\boldsymbol{w}| - \boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{w})) \tag{3.35}$$

where (again remember that $|\boldsymbol{r}-\boldsymbol{w}| = c(t - t_{\mathrm{r}})$)

$$\frac{\partial}{\partial t_{\mathrm{r}}}(|\boldsymbol{r}-\boldsymbol{w}| - \boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{w})) = c^{-1}\left(v^2 - c^2 - c^2\frac{\partial t}{\partial t_{\mathrm{r}}} - \boldsymbol{a}\cdot(\boldsymbol{r}-\boldsymbol{w})\right) \tag{3.36}$$

Then

$$\begin{aligned}\frac{\partial \boldsymbol{A}}{\partial t} =& \frac{q}{4\pi\varepsilon_0 c^3(|\boldsymbol{r}-\boldsymbol{w}| - \boldsymbol{\beta}\cdot(\boldsymbol{r}-\boldsymbol{w}))^3}\big((\boldsymbol{a}|\boldsymbol{r}-\boldsymbol{w}| - c\boldsymbol{v})(c|\boldsymbol{r}-\boldsymbol{w}| - \boldsymbol{v}\cdot(\boldsymbol{r}-\boldsymbol{w})) \\ &+ \boldsymbol{v}|\boldsymbol{r}-\boldsymbol{w}|(c^2 - v^2 + \boldsymbol{a}\cdot(\boldsymbol{r}-\boldsymbol{w})\big)\end{aligned} \tag{3.37}$$

Let $\boldsymbol{\mathscr{r}} = \boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})$, $|\boldsymbol{\mathscr{r}}| = |\boldsymbol{r}-\boldsymbol{w}| = c(t - t_{\mathrm{r}})$, and $\boldsymbol{u} = c\hat{\boldsymbol{\mathscr{r}}} - \boldsymbol{v}$, then from $\boldsymbol{E} = -\nabla V - \partial_t\boldsymbol{A}$, (3.30) and (3.37)

$$\boldsymbol{E}(\boldsymbol{r}, t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2 - v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})] \tag{3.38}$$

Also from $\boldsymbol{B} = \nabla \times \boldsymbol{A}$, $\boldsymbol{A} = c^{-2}\boldsymbol{v}V$,

$$\nabla \times \boldsymbol{A} = c^{-2}(\nabla V \times \boldsymbol{v} - V\nabla \times \boldsymbol{v}) \tag{3.39}$$

and

$$\nabla \times \boldsymbol{v} = (\nabla t_{\mathrm{r}}) \times \boldsymbol{a} \tag{3.40}$$

one obtains

$$\boldsymbol{B}(\boldsymbol{r}, t) = \frac{1}{c}\hat{\boldsymbol{\imath}} \times \boldsymbol{E}(\boldsymbol{r}, t) \tag{3.41}$$

The expression for $\boldsymbol{B}$ is the familiar right hand rule that is valid for planar and spherical waves. It holds also here, but notice that the direction $\hat{\boldsymbol{\imath}}$ is the unit vector from the position of the charge at the retarded time, $\boldsymbol{w}(t_{\mathrm{r}})$ to the field point $\boldsymbol{r}$.

The relation (3.41) is not obvious since the first term in the right hand side of (3.38) is a near field that drops off as r^{-2}. The electric and magnetic near fields of antennas do not satisfy (3.41). The far field of an antenna drops off as r^{-1} and satisfies (3.41). The second term in the right hand side of (3.38) also drops off as r^{-1} and constitute the radiated field. It not a surprise that (3.41) holds for that field.

3.2 Final expressions

Here are the final expressions that we will use further on.

A charge q travels along the curve $\boldsymbol{w}(t)$, where t is time. At a fixed position $\boldsymbol{r}$ the vector and scalar potentials $\boldsymbol{A}$, V, the electric field $\boldsymbol{E}$, and the magnetic flux density field $\boldsymbol{B}$, generated by q, and measured at $\boldsymbol{r}$ at time t are given by:

Retarded time

$$t_{\mathrm{r}} = t - \frac{|\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})|}{c}$$

Vector potential $\boldsymbol{A}$ and scalar potential V

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{r}, t) &= \frac{\mu_0}{4\pi}\frac{q\boldsymbol{v}(t_{\mathrm{r}})}{|\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})| - \boldsymbol{\beta}\cdot(\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}}))} \\
V(\boldsymbol{r}, t) &= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}})| - \boldsymbol{\beta}\cdot(\boldsymbol{r} - \boldsymbol{w}(t_{\mathrm{r}}))} \\
\boldsymbol{A}(\boldsymbol{r}, t) &= \frac{\boldsymbol{v}(t_{\mathrm{r}}))}{c^2}V(\boldsymbol{r}, t)
\end{aligned}$$

Electric field $\boldsymbol{E}$ and magnetic flux density $\boldsymbol{B}$

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{q|\boldsymbol{\mathscr{r}}|}{4\pi\varepsilon_0(\boldsymbol{\mathscr{r}}\cdot\boldsymbol{u})^3}[(c^2-v^2)\boldsymbol{u} + \boldsymbol{\mathscr{r}}\times(\boldsymbol{u}\times\boldsymbol{a})]$$
$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\hat{\boldsymbol{\mathscr{r}}}\times\boldsymbol{E}(\boldsymbol{r},t)$$

$$\begin{aligned}
\boldsymbol{\mathscr{r}} &= \boldsymbol{r} - \boldsymbol{w}(t_\mathrm{r}) \\
|\boldsymbol{\mathscr{r}}| &= |\boldsymbol{r} - \boldsymbol{w}(t_\mathrm{r})| = c(t - t_\mathrm{r}) \\
\boldsymbol{u} &= c\hat{\boldsymbol{\mathscr{r}}}(t_\mathrm{r}) - \boldsymbol{v}(t_\mathrm{r}) \\
\boldsymbol{v} &= \frac{d\boldsymbol{w}(t_\mathrm{r})}{dt_\mathrm{r}} \\
\boldsymbol{a} &= \frac{d\boldsymbol{v}(t_\mathrm{r})}{dt_\mathrm{r}}
\end{aligned}$$