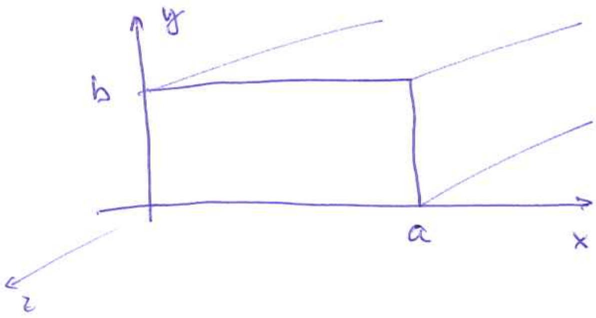
Rectangular waveguides



$a \geq b$

TE-waves

$E_z = 0, \quad H_z = w(x,y) e^{ik_z z}$

Eigenvalue problem

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + k_t^2 w = 0 \qquad (1)$$

$$\hat{n} \cdot \nabla_T w = 0 \Leftrightarrow \quad \frac{\partial w}{\partial x}(0,y) = \frac{\partial w}{\partial x}(a,y) = \frac{\partial w}{\partial y}(\cancel{x},0) = \frac{\partial w}{\partial y}(x,a) = 0$$

Separation of variables

$$w(x,y) = X(x) Y(y)$$

$$(1) \Rightarrow \quad Y \cdot X'' + X Y'' + k_t^2 X Y = 0$$

Divide by $X \cdot Y \Rightarrow$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} - k_t^2 \qquad (2)$$

$$\left.\begin{array}{ll} \text{LHS} & \text{independent of } y \\ \text{RHS} & \text{———— '' ————} \quad x \end{array}\right\} - \quad \text{RHS} = \text{LHS} = \text{constant} = -k_x^2$$

$$\Rightarrow \begin{cases} X''(x) + k_x^2 X(x) = 0 & (3) \\ X'(0) = X'(a) = 0 \end{cases}$$

$(3) \Rightarrow \quad X(x) = A \cos k_x x + B \sin k_x x$

$X'(0) = 0 \Rightarrow \quad B k_x \cos(k_x \cdot 0) = 0 \Rightarrow \quad B = 0$

$X'(a) = 0 \Rightarrow \quad A\, k_x \sin(k_x a) = 0 \Rightarrow \quad k_x a = m \cdot \pi \qquad m = 0, 1, 2 \ldots$

$\therefore \; X(x) = A \cos \frac{m\pi x}{a}, \qquad k_{xm} = \frac{m\pi}{a}$

$(2) \Rightarrow \quad Y''(y) + (k_t^2 - k_{xm}^2)\, Y = 0$

$Y'(0) = Y'(b) = 0$

$\Rightarrow \quad Y(y) = D \cos \frac{n\pi x}{b}, \qquad k_t^2 - k_{xm}^2 = k_y^2 = \left(\frac{n\pi}{b}\right)^2$

$\therefore$

$$\boxed{\begin{array}{l} \text{Eigenfunctions} \\ w_{mn}(x,y) = A_{mn} \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right) \\ \\ \text{Eigenvalues} \\ k_{tmn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \end{array}}$$

$m = 0, 1, 2 \ldots$

$n = 0, 1, 2 \ldots$

$\underline{(m, n) \neq (0, 0)}$ note!

Normalization

$$\int_0^a \int_0^b \left(U_{mn}(x,y)\right)^2 \, dy \, dx = 1$$

$$\Rightarrow \quad A_{mn} = \begin{cases} \sqrt{\dfrac{2}{ab}} & \text{if } m=0 \text{ or } n=0 \\[2ex] \dfrac{2}{\sqrt{ab}} & \text{if } m>0 \text{ and } n>0 \end{cases}$$

TE-modes

$$H_{z_{mn}}(\vec{r}) = \sqrt{\frac{\epsilon_m \epsilon_n}{ab}} \cos\frac{m\pi x}{a} \cdot \cos\frac{n\pi y}{b} \cdot e^{i k_{zmn} z}$$

$$\epsilon_m = \begin{cases} 1 & \text{if } m=0 \\ 2 & \text{if } m>0 \end{cases}$$

$$k_{tmn}^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2; \quad k_{zmn} = \sqrt{k^2 - k_{tmn}^2}\,; \quad k = \frac{\omega}{c}$$

Cut-off frequency

Three cases

1. $\frac{\omega}{c} < k_{tmn} \Rightarrow f < \frac{c}{2\pi} k_{tmn}$

   $\Rightarrow$ k_{zmn} imaginary $\Rightarrow$ $e^{i k_{zmn} z} = e^{-|k_{zmn}| z}$

   $\Rightarrow$ non-propagating mode

2. $\frac{\omega}{c} > k_{tmn} \Rightarrow f > \frac{c}{2\pi} k_{tmn}$

   $\Rightarrow$ k_{zmn} real $\Rightarrow$ propagating mode

3. $\frac{\omega}{c} = k_{tmn} \Rightarrow k_{zmn} = 0$

   $\Rightarrow$ Cut-off frequency $\boxed{f = f_c = \frac{c}{2\pi} k_{tmn}}$

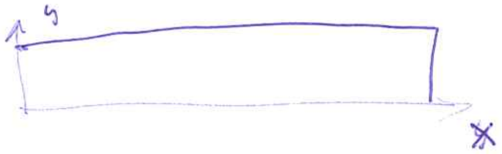
   $\Rightarrow$ Standing wave $e^{ik_z z} = 1$

Comsol example

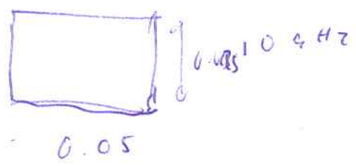
The fundamental mode TE_{10}

width = 0.5

height = 0.05



+ 2D



Plot E_z

H_x

H_y

2 GHz . . . 12 GHz

$\Rightarrow f_c = \cancel{3} \cdot 10\,c =$

Phase speed

$$H_z(\bar{r}, t) = w(\bar{\rho}) \cos(k_z z - \omega t) = w(\bar{\rho}) \cos\left(k_z\left(z - \frac{\omega}{k_z} t\right)\right)$$

$\Rightarrow v_p = \frac{\omega}{k_z}$ = phase speed

The wave pattern is moving with v_p.

Notice $k_{z_{mn}} < k = \frac{\omega}{c} \Rightarrow \boxed{v_p > c}$

At cut-off $v_p \to \infty$ as $f \to f_c$

$v_p \to c$ as $f \to \infty$

Wavelength

$\lambda_z = \frac{2\pi}{k_z}$ = wavelength along z

$\lambda_z \to \infty$ as $f \to f_c$

$\lambda_z \to \lambda = \frac{2\pi}{k}$ as $f \to \infty$