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1 Lecture 9 and 10

Special relativity

Inertial systems

An inertial system is a system that is either at rest or moves with a constant velocity.

Postulates

1. The principle of relativity. The laws of physics apply to all inertial systems.
2. The universal speed of light. The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

The γ factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \tag{1.1}$$

We also introduce

$$\beta = \frac{v}{c} \tag{1.2}$$

The Lorentz transformation

Let S and $\bar{S}$ be two inertial systems where $\bar{S}$ moves with velocity $v\hat{\boldsymbol{x}}$ seen from S. At the same time S is seen to move with speed $-v\hat{\boldsymbol{x}}$ relative $\bar{S}$. We choose the coordinate systems that follow S and $\bar{S}$ such that their origin coincide at time $t = \bar{t} = 0$. An event that occurs at (x, y, z, t) in S occurs at $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ in $\bar{S}$ where

$$\bar{x} = \gamma(x - vt) \tag{1.3}$$
$$\bar{y} = y \tag{1.4}$$
$$\bar{z} = z \tag{1.5}$$
$$\bar{t} = \gamma\left(t - \frac{v}{c^2}x\right) \tag{1.6}$$

An event $(\bar{x}, \bar{y}, \bar{z}, \bar{t})$ in $\bar{S}$ occurs at (x, y, z, t) in S where

$$x = \gamma(\bar{x} + v\bar{t}) \tag{1.7}$$
$$y = \bar{y} \tag{1.8}$$
$$z = \bar{z} \tag{1.9}$$
$$t = \gamma\left(\bar{t} + \frac{v}{c^2}\bar{x}\right) \tag{1.10}$$

Time dilatation

Consider that we have a clock at $x = 0$ in S and a clock at $\bar{x} = 0$ in $\bar{S}$ and that the origins of S and $\bar{S}$ coincide when $t = \bar{t} = 0$. An observer in S watch the clock in $\bar{S}$. At time Δt in S the clock in $\bar{S}$ will, according to (1.10), be $\Delta\bar{t} = \gamma^{-1}\Delta t$. Notice that it easier to use (1.10) than (1.6) since in (1.6) we need to first determine the position x of the clock located in $\bar{S}$.

Lorentz contraction

Let's again have a clock at $x = 0$ in S and a clock at $\bar{x} = 0$ in $\bar{S}$ and that the origins coincide when $t = \bar{t} = 0$. An observer in S now observes a stick in $\bar{S}$ that has one end at $\bar{\boldsymbol{r}} = (0, 0, 0)$ and the other at $\bar{\boldsymbol{r}} = (\Delta\bar{x}, 0, 0)$. At $t = 0$ the observer, according to (1.3), then sees one end at $\boldsymbol{r} = (0, 0, 0)$ and the other at $\boldsymbol{r} = \gamma^{-1}(\Delta\bar{x}, 0, 0)$. Thus the stick is γ^{-1} times shorter seen from an observer in S than what it is for an observer in $\bar{S}$. A stick in $\bar{S}$ that has one end at $\bar{\boldsymbol{r}} = (0, 0, 0)$ and the other at $\bar{\boldsymbol{r}} = (0, \Delta\bar{y}, 0)$ has, according to (1.6), the same length in both S and $\bar{S}$.

Four vectors

Introduce the vector

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \tag{1.11}$$

The Lorentz transformation reads

$$\bar{x}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu \tag{1.12}$$

where Λ is the matrix

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.13}$$

The upper index μ in Λ^μ_ν is the the colon and the lower index is the row.

<u>Definition:</u> A vector a^μ with four components is a four vector if it transforms according to (1.14), i.e.

$$\bar{a}^\mu = \sum_{\nu=0}^{3} \Lambda^\mu_\nu a^\nu \tag{1.14}$$

Invariants under Lorentz transformations

If a^μ, $\mu = 0, 1, 2, 3$ is a four vector then it follows that

$$-(\bar{a}^0)^2 + (\bar{a}^1)^2 + (\bar{a}^2)^2 + (\bar{a}^3)^2 = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2 \tag{1.15}$$

We can write the left hand side with a short hand notation by introducing contravariant and covariant four vectors. The contravarint vector is a^μ whereas the covariant is a_μ where

$$a_\mu = (a_0, a_1, a_2, a_3) \equiv (-a^0, a^1, a^2, a^3) \tag{1.16}$$

Einstein introduced a summation rule to get rid of the summation sign

$$a^\mu b_\mu = a_\mu b^\mu = \sum_{n=0}^{3} a^n b_n \tag{1.17}$$

We call $a^\mu b_\mu$ the four-dimensional scalar product. From (1.15) we have that $a^\mu a_\mu$ is invariant under the Lorentz transformation. It means that it has the same value in all inertial systems.

If two events occur at x_A^μ and x_B^μ then the difference $\Delta x^\mu = x_A^\mu - x_B^\mu$ is a four vector. The invariant interval between the two events is

$$\Delta x^\mu \Delta x_\mu = -c^2 \Delta t^2 + d^2 \tag{1.18}$$

where d is the spatial distance. Then

$$-c^2 \Delta \bar{t}^2 + \bar{d}^2 = -c^2 \Delta t^2 + d^2 \tag{1.19}$$

We have three cases (here we let $\Delta t > 0$ and $\Delta \bar{t} > 0$):

1. The events are space like if $\Delta x^\mu \Delta x_\mu > 0$. It means that the distance d is larger than $c\Delta t$. There is no way that an observer at x_A^μ has got the information about the event at x_B^μ since the information cannot run faster than the speed of light.

2. The events are light like if $\Delta x^\mu \Delta x_\mu = 0$. It means that the distance d equals $c\Delta t$. An observer at x_A^μ may just receive the information about the event at x_B^μ.

3. The events are time like if $\Delta x^\mu \Delta x_\mu < 0$. It means that the distance d is less than $c\Delta t$. It is possible for observer at x_A^μ to have received the information about the event at x_B^μ.

2 Lecture 10

2.1 Energy and momentum

Assume that we are in inertial system S and we see an object with mass m that travels with velocity $\boldsymbol{u}$, as measured in S. The relativistic momentum of the object is the vector

$$\boldsymbol{p} = (p^1, p^2, p^3) = m\gamma\boldsymbol{u} \tag{2.1}$$

where $\gamma = 1/\sqrt{1-\beta^2}$, and $\beta = |\boldsymbol{u}|/c$. Now introduce $p^0 = m\gamma c$. It turns out that p^μ is a four vector. The invariant is

$$p^\mu p_\mu = -m^2c^2 \tag{2.2}$$

Einstein identified p^0c as the relativistic energy

$$E \equiv m\gamma c^2 \tag{2.3}$$

The energy for an object in rest is the famous $E = mc^2$. It also follows from (2.3) that

$$E^2 - p^2c^2 = m^2c^4 \tag{2.4}$$

This is a useful relation between the relativistic momentum of a particle and its energy.

2.2 Equation of motion

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \tag{2.5}$$

where $\boldsymbol{p}$ is the relativistic momentum.

Example

A charged particle, with mass m and charge q, moves in a uniform magnetc flux density $\boldsymbol{B} = B\hat{\boldsymbol{z}}$. The particle moves in the xy plane with constant speed u. The Lorentz force is then $F = quB$, directed perpendicular to the velocity. With the same argument as in classical mechanics the particle moves in a circular orbit with radius R. It is straightforward to see that all formulas are the same except that in the relativistic case we have $m\gamma$ everywhere where we in classical mechanics have m. The classical radius is given by

$$F = \frac{mu^2}{R} \tag{2.6}$$

$$R = \frac{mu^2}{F} = \frac{mu}{qB} \tag{2.7}$$

The relativistic expression of the orbit's radius is

$$R = \frac{m\gamma u}{qB} = \frac{p}{qB} \tag{2.8}$$

The rest of this lecture is on the solution of the equation of motion using Matlab. It can be found in Chapter 7 of the exercise book.