Project in Wireless Communication
Lecture 4: OFDM

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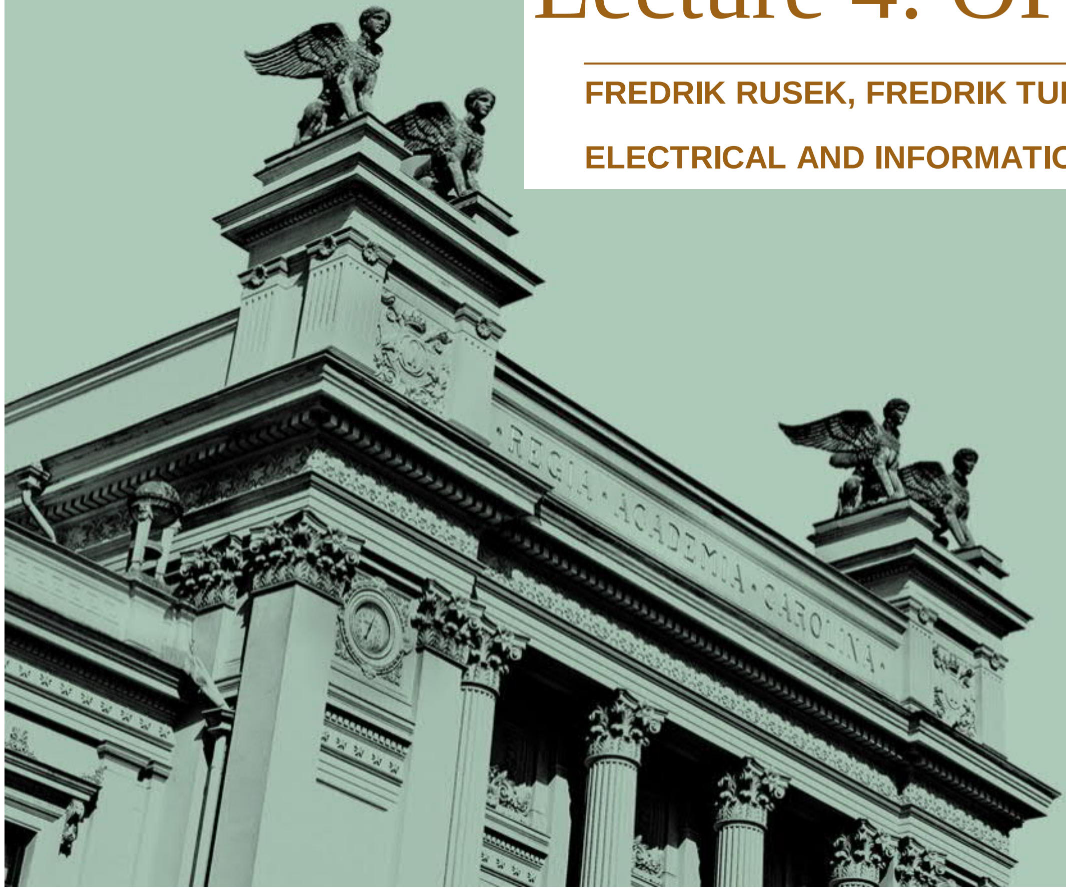
ELECTRICAL AND INFORMATION TECHNOLOGY



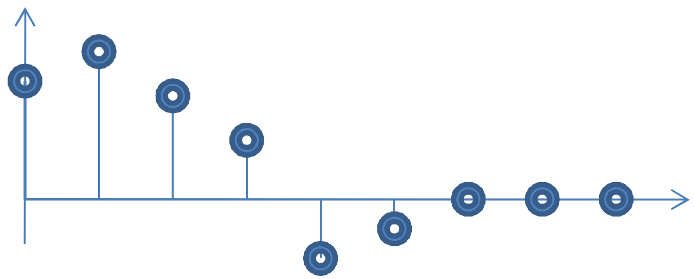
ISI channels

As seen in previous lecture, after pulse shaping and carrier modulation at the TX, and carrier demodulation, matched filtering and sampling at the Rx, the channel model can be described through

$$y = h * a + n$$

The data symbols are denoted a, h denotes the channel impulse response, and n denotes WGN. All variables are here complex valued.

For example, h could take the form



ISI channels

How should we estimate a from y?

Since there is memory is the signal y, it can be represented with a trellis and the Viterbi algorithm can be applied.

How many states are there in the Viterbi algorithm?

This depends on the length of h but also on the modulation format used.

Definition 2

The memory of the ISI channel is said to be the length of the channel, L_{ISI}, minus 1.

Property 4

Assume that the data symbols are belongs to a M-QAM constellation.

Then the number of states in the Viterbi is $M^{L_{ISI}-1}$

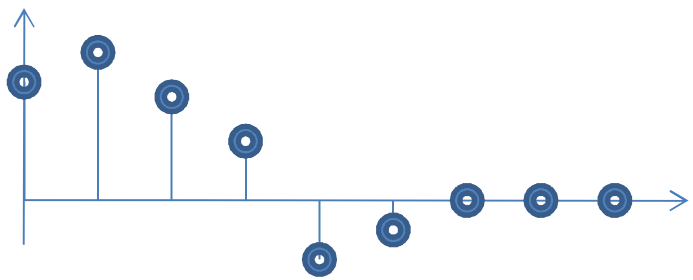
ISI channels

This number quickly grows large and the Viterbi cannot be used. For the channel given previously and with QPSK we get

$$L_{ISI} = 9 \qquad M = 4 \qquad \#\,states = 4^{8} = 2^{16} = 65536$$

In high speed real-time applications this workload is extremely demanding.

What can be done?



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ISI Channels - Zero-forcing equalization

Basic idea: A convolution is a linear operation and linear operations have inverses. Seek to find an impulse response c so that

$$\mathrm{c} * \mathrm{h} = \delta \qquad (1)$$

Then we would get

$$\mathrm{r} = \mathrm{c} * \mathrm{y} = (\mathrm{c} * \mathrm{h}) * \mathrm{a} + \mathrm{c} * \mathrm{n} = \mathrm{a} + \mathrm{w}$$

Note that there is no ISI in r, which allow trivial estimation of a. How should we find a filter c so that (1) is fulfilled? We have

$$(\mathrm{c} * \mathrm{h})[n] = \sum_{l} c_l h_{n-l} = \delta_n$$

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ISI Channels - Zero-forcing equalization

For n=0 this yields

$$n = 0: \quad h_0 c_0 = 1 \quad \Rightarrow \quad c_0 = 1/h_0$$

For n=1 we get

$$n = 1: \quad h_0 c_1 + h_1 c_0 = 0 \quad \Rightarrow \quad c_1 = -h_1 / h_0^2$$

This can be repeated for n=2,3,4,.........

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ZFE – noise enhancement

Consider a sample of the filtered noise w. It equals

$$w_k = \sum_l c_l n_{k-l}$$

Each sample has variance

$$V(w_k) = \sum_l c_l^2 V(n_{k-l}) = \sum_l c_l^2 N_0$$

where it has been assuemed that the variance of each noise sample n_k has variance N_0 (each real dimension of the noise has half the variance).

Hence, the noise gets amplified due to the filtering. In general, the sum $\sum_l c_l^2$ is much larger than 1. This situation is named noise enhancement.

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ISI Channels, summary

ISI channels appear frequently in practice.

The optimal decoder is the Viterbi algorithm, but it often fails due to complexity Issues

The simple ZF equlizer fails due to noise enhancement

There are intermediate schemes as well (MMSE, DFE, etc)

OFDM,
Orthogonal Frequency Division Multiplexing

Main idea: Convert the ISI channel into a set of independent parallel channels.

This enables optimal detection with trivial complexity!

Standard confusion:

OFDM is not better than ”non-OFDM + Viterbi-algorithm”.

There is no need for OFDM unless there is ”large” ISI.

Preliminaries from Math

Definition 1, Unitary matrix

A matrix U is said to be unitary if $UU^* = U^*U = I$

Property 1

A unitary matrix does not change the magnitude of a vector:

$$\|\mathrm{x}\|^2 = \sum_k |x_k|^2 = \|U\,\mathrm{x}\|^2$$

Property 2, variance of a sum of scaled variables

Let X and Y be two independent random variables with variance V(X) and V(Y). Then, the random variable Z=aX+bY has variance

$$V(Z) = a^2V(X) + b^2V(Y)$$

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Preliminaries from Math

Property 3, Distribution of a sum of Gaussian varibles

Let $\mathrm{x} = \left[x_1 \; x_2 \; \cdots \; x_N\right]^T$ be a vector of Gaussian random variables, i.e.

$$x_k \in N(0, \sigma^2)$$

$$\mathrm{x} \in N(0, \sigma^2 I)$$

Then, if we multiply x with a matrix H, $\mathrm{y} = H\,\mathrm{x}$

we get that y has a Gaussian distribution with correlation matrix H*H, i.e.

$$\mathrm{y} \in N(0, \sigma^2 H^* H)$$

If H is unitary, then H*H=I, and it follows that y and x have the same distributions

Preliminaries from Math

Singular value decompostion

Any matrix H (dimension KxL) can be decomposed as

$$H = U\Sigma V^*$$

where U is KxK unitary, V is LxL unitary, and $\sum$ is a KxL matrix with the singular values along the main diagonal and zeros elsewhere, i.e.

$$\Sigma_{kk} = \lambda_k \quad and \quad \Sigma_{kl} = 0, k \neq l$$

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Matrix representation of convolution

The convolution $\mathrm{y} = \mathrm{h} * \mathrm{a} + \mathrm{n}$ can in matrix

form be represented as $Y = HA + N$

where $Y = \left[y_0, y_1, \ldots, y_{N+L_{ISI}-2}\right]^T$

$$N = \left[n_0, n_1, \ldots, n_{N+L_{ISI}-2}\right]^T$$

$$A = \left[a_0, a_1, \ldots, a_{N-1}\right]^T$$

There are N channel inputs and $N + L_{ISI} - 1$ channel outputs.

The size of H is $\left(N + L_{ISI} - 1\right) \times N$

$$H = \begin{bmatrix}
h_0 & 0 & 0 & & & & 0 \\
h_1 & h_0 & 0 & & & & \\
\ldots & h_1 & h_0 & & & & \\
\ldots & & h_1 & \ldots & & & \\
\ldots & & & \ldots & & & \\
h_{L_{ISI}-1} & & & & & & \\
0 & h_{L_{ISI}-1} & & & & & \\
0 & 0 & \ldots & & \ldots & 0 & \\
 & 0 & \ldots & & \ldots & h_0 & 0 \\
 & & & & & h_1 & h_0 \\
 & & & & & & h_1 \\
 & & & & & & \ldots \\
 & & & & \ldots & & \ldots \\
 & & & & 0 & h_{L_{ISI}-1} & \ldots \\
0 & & & & 0 & 0 & h_{L_{ISI}-1}
\end{bmatrix}$$

Using the EVD

Express the covolution as $Y = U\Sigma V^* A + N$

The receiver can remove U as

$$R = U^* U\Sigma V^* A + U^* N = \Sigma V^* A + W$$

where W is some new noise vector. From Property 3 it follows that W is statistically distributed exactly as N.

If the transmitter constructs A as $A = VX$
where X are QAM symbols carrying the actual data, then R becomes

$$R = \Sigma V^* VX + W = \Sigma X + W$$

From Property 2 it follows that the energy in X equals that in A.

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EVD generates parallel channels

Now recall that Σ is not square, but diagonal.

The kth element of R equals $r_k = \Sigma_{kk} a_k + w_k \quad 1 \le k \le N$

THE ISI CHANNEL HAS VANISHED AND OPTIMAL DETECTION CAN BE MADE
WITHOUT MEANS OF THE VITERBI ALGORITHM

Major Problem

The approach just described can not work in practice since the Tx does not know the matrix V.

The solution will be to add a cyclic prefix so that U and V both are constant for all channel matrices H.

Circulant Matrices

Definition 3

A circulant matrix has the form

i.e., every row is a cyclic shift of the first.

$$T = \begin{bmatrix} t_0 & t_1 & \dots & \dots & t_{N-2} & t_{N-1} \\ t_{N-1} & t_0 & t_1 & & & t_{N-2} \\ \dots & t_{N-1} & t_0 & t_1 & & \dots \\ \dots & & \dots & \dots & \dots & \dots \\ \dots & & & \dots & \dots & t_1 \\ t_{N-1} & t_{N-2} & \dots & \dots & t_1 & t_0 \end{bmatrix}$$

Property 5

The Eigenvalue decomposition of a circulant matrix is

$$T = F^* \Sigma F$$

where F is the DFT matrix, i.e.

$$f_{kl} = \frac{1}{\sqrt{N}} \exp\left(-\frac{i2\pi kl}{N}\right), \qquad 0 \le k, l \le N-1$$

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Converting the channel convolution into a cyclic convolution

Property 5 is precisly what is needed in order to make OFDM work.

How can we make use of it?

Consider the toy N=4 case with a 3-tap channel. The channel matrix is

$$H = \begin{bmatrix} h_0 & 0 & 0 & 0 \\ h_1 & h_0 & 0 & 0 \\ h_2 & h_1 & h_0 & 0 \\ 0 & h_2 & h_1 & h_0 \\ 0 & 0 & h_2 & h_1 \\ 0 & 0 & 0 & h_2 \end{bmatrix}$$

which resembles the circulant matrix

$$H_{Circ} = \begin{bmatrix} h_0 & 0 & h_2 & h_1 \\ h_1 & h_0 & 0 & h_2 \\ h_2 & h_1 & h_0 & 0 \\ 0 & h_2 & h_1 & h_0 \end{bmatrix}$$

It can be seen that the tail of the channel convolution has been cut off and been put in the begining of the output.

Cyclic Prefix

For the previous toy example, if we transmit

This is the cyclic prefix

$$A = \begin{bmatrix} a_2 & a_3 & a_0 & a_1 & a_2 & a_3 \end{bmatrix}^T$$

We will receive 8 output symbols

$$Y = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 \end{bmatrix}^T$$

Consider the middle 4 outputs, they can be described as

$$\begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & h_2 & h_1 \\ h_1 & h_0 & 0 & h_2 \\ h_2 & h_1 & h_0 & 0 \\ 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} n_2 \\ n_3 \\ n_4 \\ n_5 \end{bmatrix}$$

$$\Leftrightarrow$$

$$Y = H_{Circ} A + N$$

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OFDM

If A=F*X and R=FY, we get

$$\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \end{bmatrix} = \begin{bmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

In general, for an ISI channel of length L_{ISI} the cyclic prefix length needs to be $L_{ISI} - 1$. Since this number cannot be known beforehand, some fairly large number is chosen so that it covers most channels.

OFDM

Property 6

The vector of eigenvalues λ equals the FFT of the first row of the circulant matrix.

Hence, for the circulant representation of convolution, it follows that the eigenvalues are given by the FFT of the channel impulse response.

Consequently, the data symbols X are transmitted on the Frequency response of the channel. This explains the name Orthogonal Frequency Division Multiplexing

Orthogonal because there is no ISI and Frequency division since the different symbols are transmitted on the frequency response.

OFDM

The structure of the signaling is

Data symbols: $X = [x_0 \quad \ldots \quad x_{N-1} \quad x_N \quad \ldots \quad x_{2N-1} \quad \ldots \quad \ldots]^T$

symbol 1 symbol 2 etc etc

After IFFT: $A = [a_0 \quad \ldots \quad a_{N-1} \quad a_N \quad \ldots \quad a_{2N-1} \quad \ldots \quad \ldots]^T$

symbol 1 symbol 2 etc etc

With CP:

$$\tilde{A} = [a_{N-Ncp} \quad \ldots \quad a_{N-1} \quad a_0 \quad \ldots \quad a_{N-1} \quad a_{2N-Ncp} \quad \ldots \quad a_{2N-1} \quad a_N \quad \ldots \quad a_{2N-1} \quad \ldots \quad]^T$$

CP1 Symbol 1 CP2 Symbol 2

OFDM receiver

Received signal

$$R = \begin{bmatrix} r_0 & r_1 & \dots & r_{Ncp-1} & r_{Ncp} & \dots & r_{Ncp+N-1} & \dots & \dots & \dots & \dots \end{bmatrix}^T$$

Cyclic prefix
Discard

Data
Send to FFT

Cyclic prefix
Discard

Etc etc

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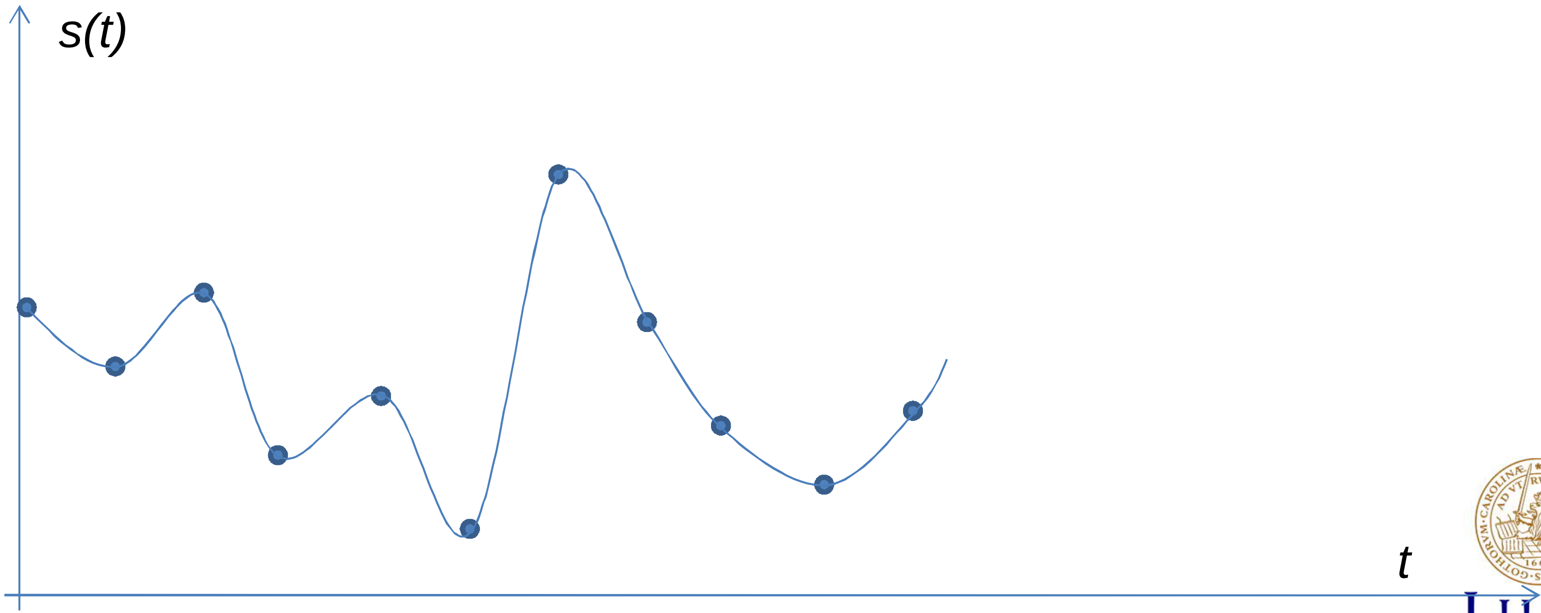
D/A conversion

So far only the discrete-time models have been discussed.

After the IFFT, adding CP, carrier modulation we have a vector s that should be transmitted. We need to construct a time continuous signal s(t).



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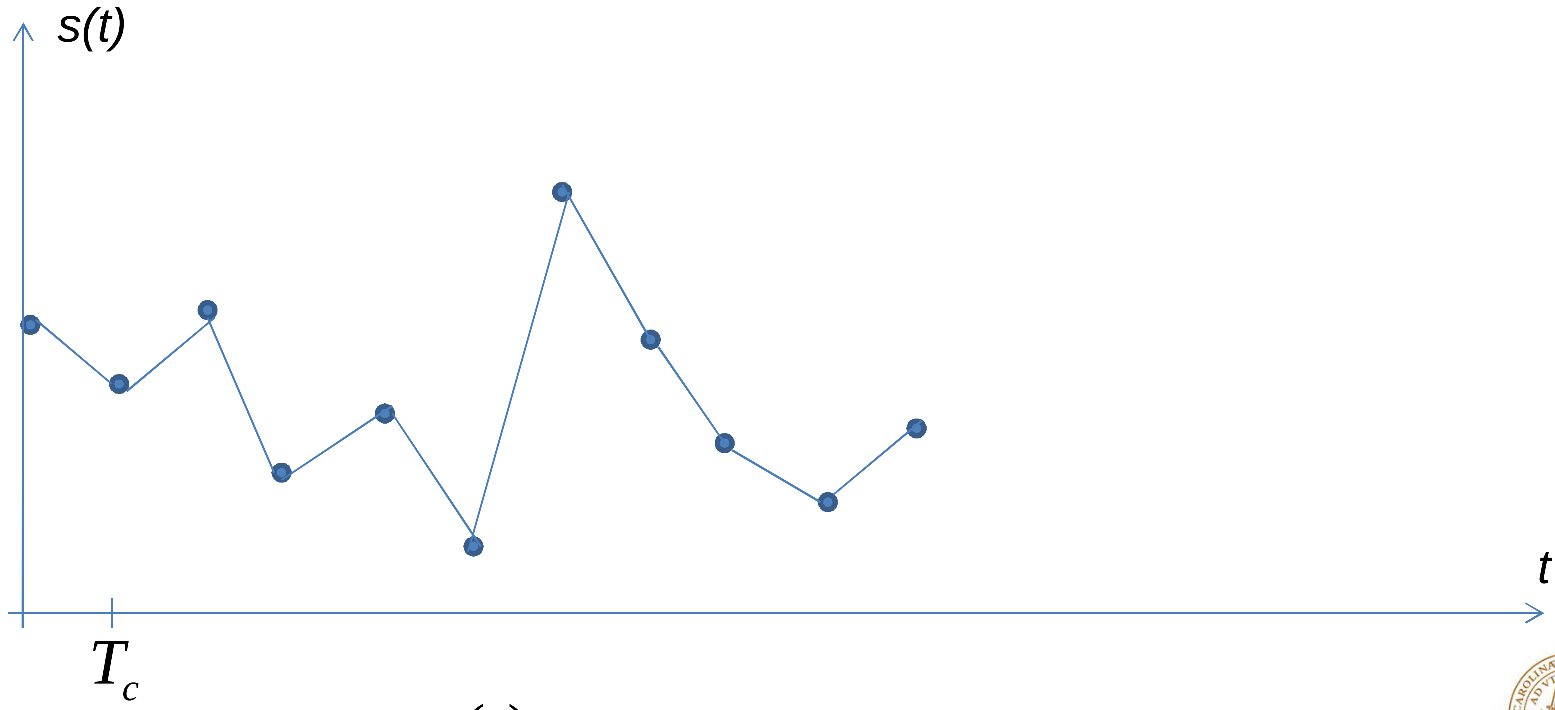
D/A

One way to accomplish this is to use linear interpolation followed by a low pass filter



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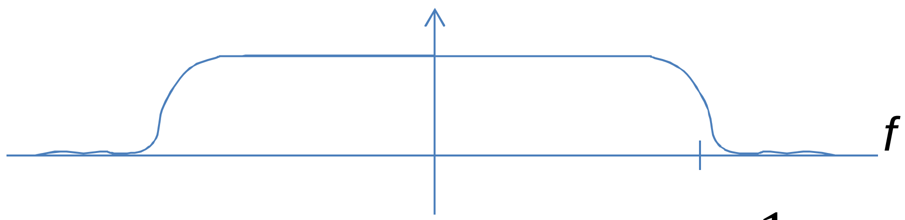
D/A – the sampling theorem

It is known that the necessary (one-sided) bandwidth of s(t) is $1/2T_c$ Hz

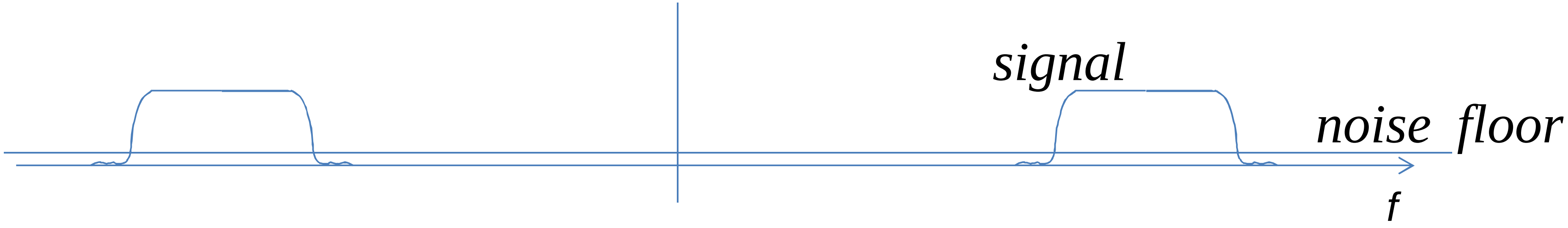


$$cutoff \approx \frac{1}{2T_c}\ Hz$$

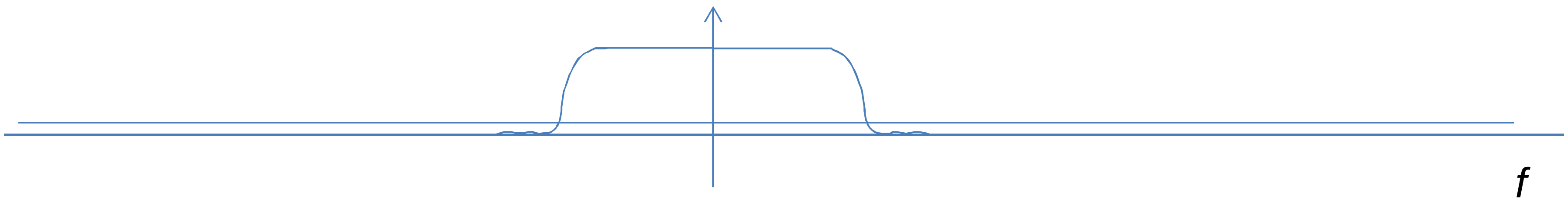
This is essentially the statement of the sampling theorem.

The RF receiver

Spectrally, we receive the following signal



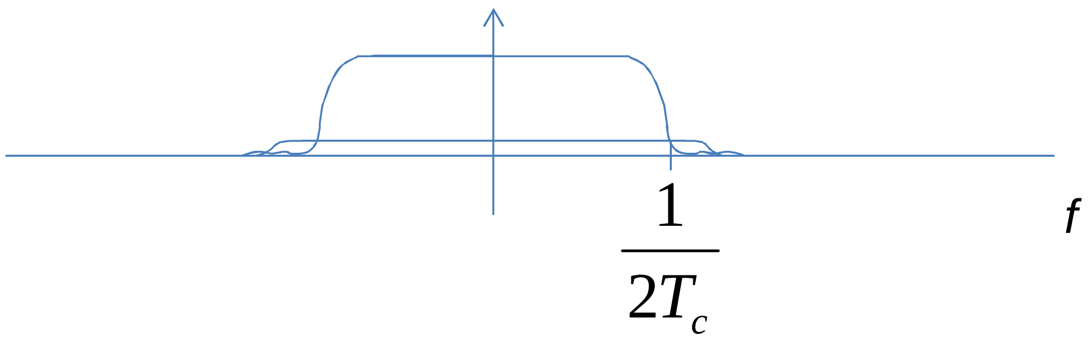
After carrier de-modulation we get



The RF receiver

If this signal is sampled, the noise will be folded back on the signal (aliasing).

To avoid this, low pass filter the signal to get



This signal can now be sampled with spacing T_c

In practice the outer sub-channels, close to the edges, are set to zero to have some margin and make implementation easier

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