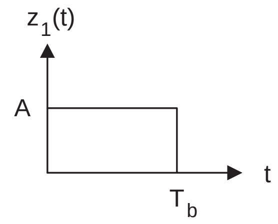
Answers/short solutions to
Examination in Digital Communications - October 16, 2006

Problem 1:

$z_0(t) = x \cdot z_1(t)$



$x = 0 \Rightarrow P_b = 10^{-3}$

a)

$$P_b = Q\left(\sqrt{\frac{D^2}{2N_0}}\right)$$

$$D^2 = \int_0^{T_b} (z_1(t) - z_0(t))^2 dt = \int_0^{T_b} z_1^2(t)(1-x)^2 dt = (1-x)^2 E_{z_1}$$

$$P_b = Q\left(\sqrt{\frac{(1-x)^2 E_{z_1}}{2N_0}}\right)$$

$$x = 0: \ P_b = Q\left(\sqrt{\frac{E_{z_1}}{2N_0}}\right) = 10^{-3} \Rightarrow \frac{E_{z_1}}{2N_0} = 3.0902^2$$

$$P_b = Q\left(\sqrt{(1-x)^2 3.0902^2}\right) = \begin{cases} Q(6.18) \approx 3 \cdot 10^{-10} & \text{i)} \\ Q(9.27) \approx 10^{-20} & \text{ii)} \end{cases}$$

b)

$$\begin{aligned} x = \ \ 0: &\quad \mathcal{E}_b = E_{z_1}/2 \\ x = -1: &\quad \mathcal{E}_b = E_{z_1} \qquad \text{(3 dB larger than the } x = 0 \text{ case)} \\ x = \ \ 4: &\quad \mathcal{E}_b = 17E_{z_1}/2 \text{ (12.3 dB larger than the } x = 0 \text{ case)} \end{aligned}$$

Problem 2:

1. True since: M-ary PPM:

$$D_{ij}^2 = 2E_c, \ E_s = k\mathcal{E}_b = E_c$$

$$d_{\min}^2 = \frac{2E_c}{2E_s/k} = k$$

So, $d_{\min}^2 = 4$ for 16-PPM, and $d_{\min}^2 = 1$ for binary FSK.

2. True since:

   1024-QAM has $d_{\min}^2 = \frac{3 \cdot 10}{1023}$, and $d_{\min}^2 = 1$ for binary FSK.

$$10 \log_{10} \frac{3 \cdot 10}{1023} = -15.33 \text{ dB}$$

3. True since;

$$W = \frac{3}{2T} = \frac{3 \cdot 6}{2 \cdot 5 T_s} = \frac{3 \cdot 6}{2 \cdot 5 \cdot 6 \cdot T_b} = \frac{3 \cdot 400 \cdot 10^3}{10} = 120 \text{ kHz}$$

4. True since:

$$\mathcal{E}_b = \frac{3A^2T}{8} = \frac{3A^2 5T_s/6}{8} = \frac{5A^2T_b}{16} = \bar{P}T_b$$

5. True since:

$$T_z = T + 0.25 \cdot 10^{-6}$$

$$T_s \geq T_z = 0.75 \cdot 10^{-6} + 0.25 \cdot 10^{-6} = 10^{-6}$$

Problem 3:

a)

$$P_s \leq 4 \cdot Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right) \leq 4 \cdot 10^{-6}$$

$$d_{\min}^2 \frac{\mathcal{E}_b}{N_0} \geq 4.7534^2$$

$$\frac{3 \log_2(M)}{M-1} \cdot \frac{R_b \mathcal{E}_b}{R_b N_0} \geq 4.7534^2$$

$$\frac{3T_s P_z/N_0}{M-1} \geq 4.7534^2$$

$$M - 1 \leq \frac{3T_s}{4.7534^2} \cdot \frac{P_z}{N_0}$$

$$T_s = k \cdot T_b = \frac{k}{R_b}: \quad \frac{2}{192} = \frac{1}{96}, \ \frac{4}{384} = \frac{1}{96}, \ \frac{6}{576} = \frac{1}{96}, \ \frac{8}{768} = \frac{1}{96}, \ \frac{10}{960} = \frac{1}{96}$$

$$\because \ T_s = \frac{1}{96 \cdot 10^3}$$

$$M - 1 \leq \frac{3}{4.7534^2} \cdot \frac{1}{96 \cdot 10^3} \cdot 6 \cdot 10^8 = 830$$

$\Rightarrow$ Choose $M = 256 \Rightarrow R_b = 768$ kbps

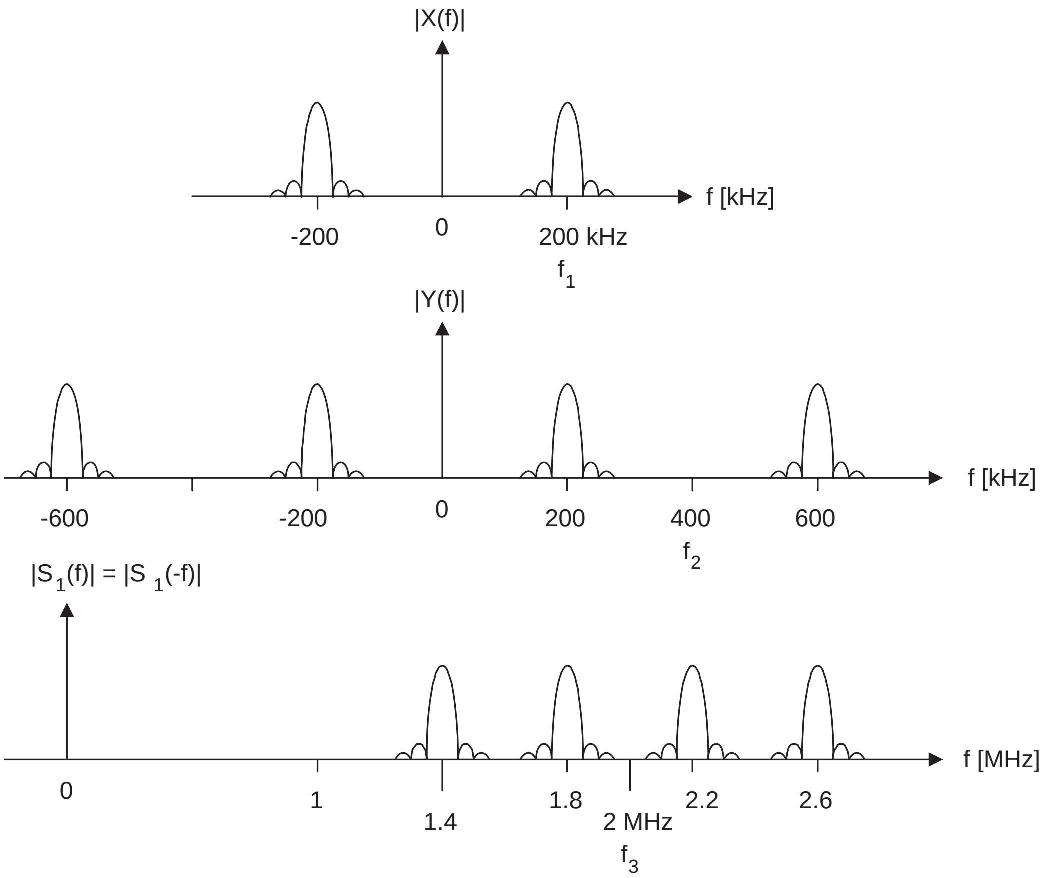
b)

$$W_{99} = \frac{2.82}{T} = \frac{2.82}{3T_s/4} = \frac{2.82 \cdot 4}{3} \cdot 96 \cdot 10^3 = 361 \text{ kHz}$$

Problem 4:

a)

$$W_{lobe} = \frac{4}{T} = \frac{4}{80 \cdot 10^{-6}} = 50 \text{ kHz}$$



b)

$$T'_s = \frac{3}{R'_b} = \frac{3}{384 \cdot 10^3}$$

i)

$$T''_s = \frac{8}{R''_b} = 4T'_s = \frac{12}{R'_b}$$

$$R''_b = \frac{8R'_b}{12} = \frac{2}{3} \cdot 384 \cdot 10^3 = 256 \text{ kbps}$$

$$T''_s = \frac{8}{R''_b} = (7+n)T'_s = (7+n)\ \frac{3}{R'_b},\ n = 0, 1, 2, \ldots$$

$$R''_b = \frac{8}{3(7+n)} \cdot 384 \cdot 10^3 = 146.3 \text{ kbps},\ 128 \text{ kbps}$$

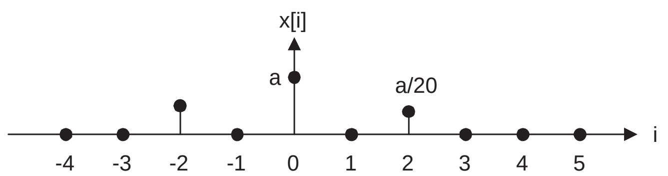
$$113.8 \text{ kbps},\ 102.4 \text{ kbps}$$

$$93.1 \text{ kbps}, \ldots$$

ii)

$$T_s''' = \frac{6}{R_b'''} = \frac{6}{256 \cdot 10^3} = \frac{2 \cdot 3}{\frac{2}{3} \cdot 384 \cdot 10^3} =$$

$$= 3 \cdot \frac{3}{384 \cdot 10^3} = 3 \cdot T_s'$$



ISI worst case $= \pm 63 \cdot \frac{2a}{20} = \pm 6.3a$

Problem 5: $D_{i,j}^2 = E_i + E_j$ since all pairs are orthogonal.

a)

$$D_{\min}^2 = D_{1,3}^2 = 2 \cdot \frac{A^2}{4} \frac{T_s}{4} = \frac{A^2 T_s}{8}$$

$$\bar{E}_s = 2\mathcal{E}_b = \frac{2 \cdot A^2 T_s/4 + 2 \cdot A^2 T_s/16}{4} = \frac{A^2 T_s}{8} + \frac{A^2 T_s}{32} = \frac{5A^2 T_s}{32}$$

$$d_{\min}^2 = \frac{D_{\min}^2}{2\mathcal{E}_b} = \frac{A^2 T_s/8}{\frac{A^2 T_s}{8} + \frac{A^2 T_s}{32}} = \frac{4}{5}$$

4-PAM:

$$d_{\min}^2 = \frac{6 \cdot 2}{15} = \frac{4}{5}$$

$d_{\min}^2$ is the same $\Rightarrow$ 0 dB difference.

b)

$$D_{0,1}^2 = \frac{A^2 T_s}{4} + \frac{A^2 T_s}{16} = D_1^2, \quad D_{1,2}^2 = D_{0,1}^2, \quad D_{2,3}^2 = D_{0,1}^2$$

$$D_{0,2}^2 = 2 \cdot \frac{A^2 T_s}{4} = D_2^2, \quad D_{1,3}^2 = \frac{A^2 T_s}{8} = D_{\min}^2$$

$$D_{0,3}^2 = D_{0,1}^2$$

So,

$$D_{\min}^2 = \frac{A^2 T_s}{8}, \quad D_1^2 = D_{0,1}^2 = \frac{5A^2 T_s}{16}, \; D_2^2 = \frac{A^2 T_s}{2}$$

$$d_{\min}^2 = \frac{4}{5}, \quad d_1^2 = \frac{5}{2} \cdot \frac{4}{5} = 2, \quad d_2^2 = 4 \cdot \frac{4}{5} = \frac{16}{5}$$

Now let

$$Q_{i,j} = Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \text{ and } Q_i = Q\left(\sqrt{\frac{D_i^2}{2N_0}}\right)$$

Union bound =

$$= \frac{1}{4}\sum_{i=0}^{3}\sum_{\substack{j=0\\ j\neq i}}^{3} Q_{i,j} = \frac{1}{4}(Q_{0,1} + Q_{0,2} + Q_{0,3} + Q_{1,0} + Q_{1,2} + Q_{1,3} +$$

$$+ Q_{2,0} + Q_{2,1} + Q_{2,3} + Q_{3,0} + Q_{3,1} + Q_{3,2}) =$$

$$= \frac{1}{4}\ (2Q_1 + Q_2 + Q_{d_{\min}} + 2Q_1)2 =$$

$$= \frac{1}{2}\ Q_{d_{\min}} + 2Q_1 + \frac{1}{2}\ Q_2, \quad \frac{1}{2} + 2 + \frac{1}{2} = M - 1 = 3, \text{ see (4.110)}$$

Alternatively from (4.109):

$$c = c_0 = \frac{1}{4}\ (n_{0,0} + n_{1,0} + n_{2,0} + n_{3,0}) = \frac{1}{4}\ (0 + 1 + 0 + 1) = \frac{1}{2}$$

$$c_1 = \frac{1}{4}\ (n_{0,1} + n_{1,1} + n_{2,1} + n_{3,1}) = \frac{1}{4}\ (2 + 2 + 2 + 2) = 2$$

$$c_2 = \frac{1}{4}\ (n_{0,2} + n_{1,2} + n_{2,2} + n_{3,2}) = \frac{1}{4}\ (1 + 0 + 1 + 0) = \frac{1}{2}$$

$\mathcal{E}_b/N_0 = 11.246$

Union bound =

$$= \frac{1}{2}\ Q\left(\sqrt{\frac{4}{5}\cdot 11.246}\right) + 2Q(\sqrt{2\cdot 11.246}) + \frac{1}{2}\ Q\left(\sqrt{\frac{16}{5}\cdot 11.246}\right) =$$

$$= \frac{1}{2}\ \underbrace{Q(3)}_{1.35\cdot 10^{-3}} + 2\underbrace{Q(4.74)}_{<1.3\cdot 10^{-6}} + \frac{1}{2}\ \underbrace{Q(6)}_{9.9\cdot 10^{-10}} \approx \frac{1}{2}\cdot 1.35\cdot 10^{-3} = 0.68\cdot 10^{-3}$$