LUND UNIVERSITY

EITG05 – Digital Communications

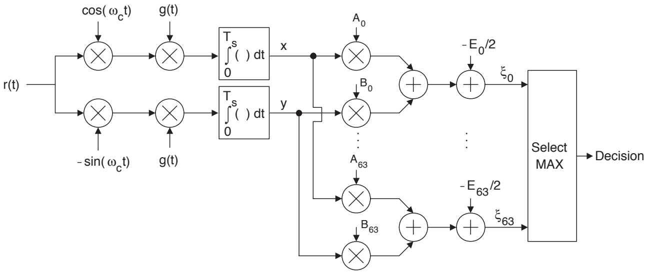
Lecture 7

Receivers continued:
Geometric representation, Capacity,
Multiuser receiver, Non-coherent receiver

Michael Lentmaier
Thursday, September 26, 2019

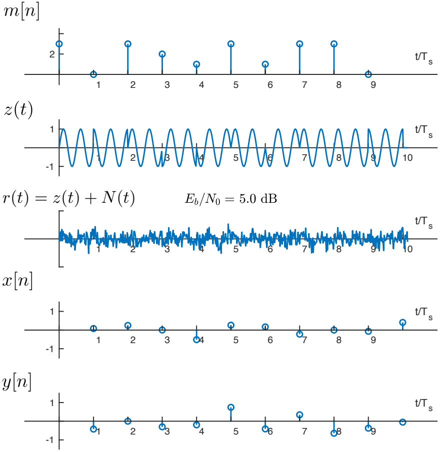
Recall: QAM receiver (Example 4.4)

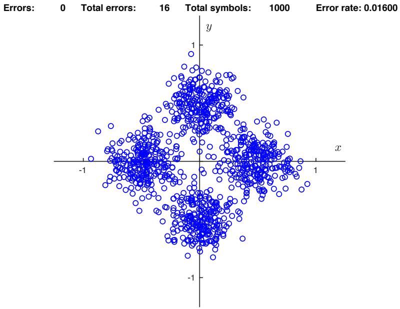
The implementation of this receiver is shown below:



The complexity of this receiver is significantly reduced compared to the receiver in Figure 4.8 on page 241! Only two integrators are here used, instead of 64 (= M) in Figure 4.8.

Example: QPSK (see Matlab demo)





Distances $D_{i,j}$ are important

- P_s is determined by the distances $D_{i,j}$ between the signal pairs
- Let us sort these distances

$$D_{min} < D_1 < D_2 < \cdots < D_{max}$$

- Then the upper bound on P_s can be written as

$$P_s \leq c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) + c_1\, Q\left(\sqrt{\frac{D_1^2}{2N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{\frac{D_{max}^2}{2N_0}}\right)$$

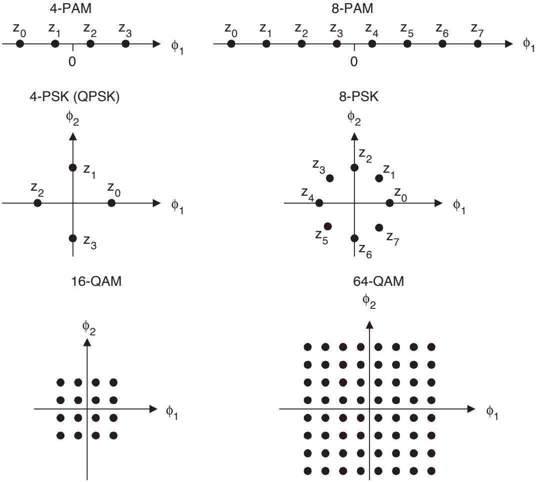
- The coefficients are

$$c_\ell = \sum_{j=1}^{M-1} P_j \cdot n_{j,\ell}\ , \quad \ell = 0, 1, 2, \ldots, x$$

- $n_{j,\ell}$: number of signals at distance D_ℓ from signal $z_j(t)$

How many distinct terms do exist for QPSK?

Signal Space Representation



$$\phi_1(t) = \frac{g(t)}{\sqrt{E_g}}$$

$$\phi_1(t) = \frac{g(t)\cos(2\pi f_c t)}{\sqrt{E_g/2}}$$

$$\phi_2(t) = \frac{g(t)\sin(2\pi f_c t)}{\sqrt{E_g/2}}$$

A geometric description

- As we have seen in Chapter 2 we can represent our signal alternatives $z_j(t)$ as vectors (points) in signal space

$$\mathbf{z}_j = \left(z_{j,1}\right) = \left(A_j\sqrt{E_g}\right) \qquad \text{PAM}$$

$$\mathbf{z}_j = \left(z_{j,1} \quad z_{j,2}\right) = \left(A_j\sqrt{\frac{E_g}{2}} \quad B_j\sqrt{\frac{E_g}{2}}\right) \qquad \text{QAM, PSK}$$

- The signal energy can be written as

$$E_j = \int_0^{T_s} z_j^2(t)\, dt = z_{j,1}^2 + z_{j,2}^2$$

- Likewise, the squared Euclidean distance becomes

$$D_{i,j}^2 = \int_0^{T_s} \left(z_i(t) - z_j(t)\right)^2 dt = (z_{i,1} - z_{j,1})^2 + (z_{i,2} - z_{j,2})^2$$

Signal energies and distances have a geometric interpretation

Approximate P_s for some constellations

- Considering the dominating term in the union bound we obtain

$$P_s \approx c\, Q\left(\sqrt{d_{min}^2 \frac{\mathcal{E}_b}{N_0}}\right)$$

- This approximation is valid if $\frac{\mathcal{E}_b}{N_0}$ is sufficiently large

	c	$d^2_{\min}$
M-ary PAM	$2(1-1/M)$	$\frac{6\log_2(M)}{M^2-1}$
M-ary PSK $(M>2)$	2	$2\log_2(M)\sin^2(\pi/M)$
M-ary FSK	$M-1$	$\log_2(M)$
M-ary QAM	$4(1-1/\sqrt{M})$	$\frac{3\log_2(M)}{M-1}$

Table 4.1: The coefficient c, and $d^2_{\min}$, for some common signal constellations. Equally likely signal alternatives are assumed. See Subsection 2.4.1.1 for the M-ary PAM case, and Subsection 2.4.5.1 for the M-ary QAM case. M equal energy orthogonal FSK signals are also assumed.

Example 4.19

Assume two signal constellations, denoted A and B respectively, with corresponding parameters $d^2_{\min,A}$ and $d^2_{\min,B}$. From the equality (see e.g. the dominating term in the union bound),

$$d^2_{\min,A}\mathcal{E}_{b,A}/N_0 = d^2_{\min,B}\mathcal{E}_{b,B}/N_0$$

we find that the difference (in dB) in received energy per information bit is (compare with (2.13) on page 16),

$$10\log_{10}(\mathcal{E}_{b,B}) - 10\log_{10}(\mathcal{E}_{b,A}) = 10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$$

Calculate the value $10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$ if "A" is binary antipodal PAM, and if "B" is 4-ary PAM. Assume, that the conditions leading to (2.50) are satiesfied.

- For M-ary PAM we have (Table 4.1 or Table 5.1)

$$d^2_{min} = 6\log_2(M)/(M^2-1) \quad \Rightarrow \; d^2_{min,A} = 2, \; d^2_{min,B} = 4/5$$

- $10\log_{10} d^2_{min,A}/d^2_{min,B} = 10\log_{10} 5/2 = 3.98 \text{ dB}$

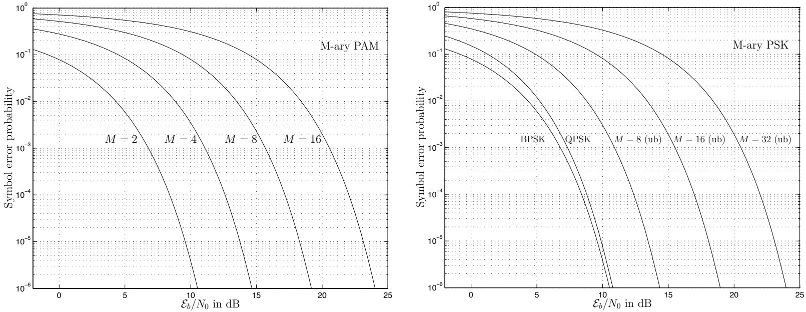
Binary PAM is 3.98 dB more energy efficient than 4-ary PAM!

Comparisons

$M = 2$	P_b	$Q\left(\sqrt{d^2_{\min}\ \frac{\mathcal{E}_b}{N_0}}\right)$, (4.55)
	$d^2_{\min}$	$0 \le d^2_{\min} \le 2$, (4.57)
	ρ	ρ_{bin} , (2.21)
M-ary PAM	P_s	$2\left(1-\frac{1}{M}\right)Q\left(\sqrt{d^2_{\min}\ \frac{\mathcal{E}_b}{N_0}}\right)$, (5.35)
	$d^2_{\min}$	$\frac{6\log_2(M)}{M^2-1}$, Table 4.1 on page 281, (2.50)
	ρ	$\rho_{2-PAM} \cdot \log_2(M)$, (2.220)
M-ary PSK	P_s	$< 2Q\left(\sqrt{d^2_{\min}\ \frac{\mathcal{E}_b}{N_0}}\right)$, (5.43)
	$d^2_{\min}$	$2\sin^2(\pi/M)\log_2(M)$, Table 4.1, Fig. 5.11
	ρ	$\rho_{BPSK} \cdot \log_2(M)$, (2.229)
M-ary QAM (rect., k even) (QPSK with $M = 4$)	P_s	$4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{d^2_{\min}\ \frac{\mathcal{E}_b}{N_0}}\right) - -4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{d^2_{\min}\ \frac{\mathcal{E}_b}{N_0}}\right)$, (5.50)
	$d^2_{\min}$	$\frac{3\log_2(M)}{M-1}$, Table 4.1, Subsection 2.4.5.1
	ρ	$\rho_{BPSK} \cdot \log_2(M)$, (2.229)
M-ary FSK (orthogonal FSK)	P_s	$\le (M-1)Q\left(\sqrt{d^2_{\min}\ \frac{\mathcal{E}_b}{N_0}}\right)$, Example 4.18c, Table 4.1
	$d^2_{\min}$	$\log_2(M)$, Table 4.1 on page 281
	ρ	See (2.245)

Table 5.1, p. 361

Symbol error probability comparison



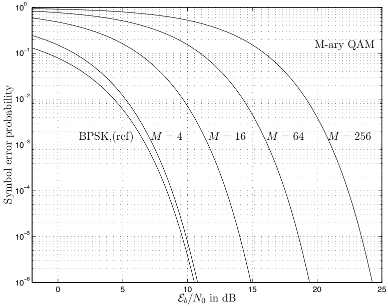
M-ary PAM, $M = 2, 4, 8, 16$

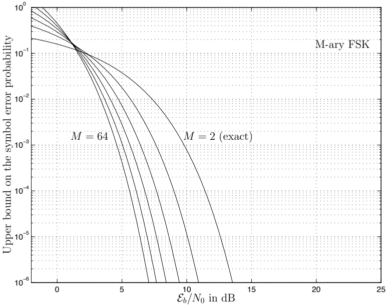
M-ary PSK, $M = 2, 4, 8, 16, 32$

$$d_{min}^2 = 6 \cdot \frac{\log_2 M}{M^2 - 1}$$

$$d_{min}^2 = 2 \sin^2(\pi/M) \, \log_2 M$$

Symbol error probability comparison





M-ary QAM, $M = 4, 16, 64, 256$

$$d_{min}^2 = 3 \cdot \frac{\log_2 M}{M-1}$$

M-ary FSK, $M = 2, 4, 8, 16, 32, 64$

$$d_{min}^2 = \log_2 M$$

Gain in d_{min}^2 compared with binary antipodal

Antipodal	$M = 2$	0[dB]
Orthogonal	$M = 2$	-3.01
M-ary PAM	$M = 2$	0
	$M = 4$	-3.98
	$M = 8$	-8.45
	$M = 16$	-13.27
	$M = 32$	-18.34
	$M = 64$	-23.57
M-ary PSK	$M = 2$	0
	$M = 4$	0
	$M = 8$	-3.57
	$M = 16$	-8.17
	$M = 32$	-13.18
	$M = 64$	-18.40
M-ary QAM	$M = 4$	0
	$M = 16$	-3.98
	$M = 64$	-8.45
	$M = 256$	-13.27
	$M = 1024$	-18.34
	$M = 4096$	-23.57

M-ary FSK	$M = 2$	-3.01
	$M = 4$	0
	$M = 8$	1.76
	$M = 16$	3.01
	$M = 32$	3.98
	$M = 64$	4.77
M -ary bi-orthogonal	$M = 2$	0
	$M = 4$	0
	$M = 8$	1.76
	$M = 16$	3.01
	$M = 32$	3.98
	$M = 64$	4.77

Large values M reduce energy efficiency

Example scenario: M-ary QAM

- We want to ensure that $P_s \leq P_{s,req}$, where for M-ary QAM

$$P_s \leq 4\, Q\left(\sqrt{d_{min}^2 \frac{\mathcal{E}_b}{N_0}}\right) = 4\, Q\left(\sqrt{\mathcal{X}}\right) \;, \quad d_{min}^2 = 3\, \frac{\log_2 M}{M-1}$$

- The pulse shape $g(t)$ is chosen such that

$$\rho = \log_2(M)\, \rho_{BPSK} \;, \quad \text{where } \rho = \frac{R_b}{W} \leq \frac{d_{min}^2}{\mathcal{X}} \cdot \frac{\mathcal{P}_z}{N_0\, W}$$

- Combining these requirements we obtain

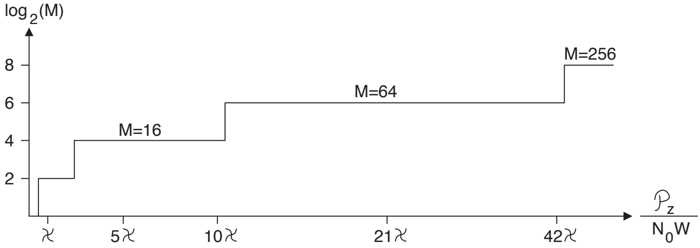
$$M \leq 1 + \frac{3}{\mathcal{X}\, \rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\, W} = 1 + \frac{3}{\mathcal{X}} \cdot \frac{\mathcal{P}_z\, T_s}{N_0}$$

- Hence we want to choose $M = 2^k$ such that (QAM: k even)

$$2^k \;\leq\; 1 + \frac{3}{\mathcal{X}\, \rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\, W} \;<\; 2^{k+2}$$

Example 4.22: adapting M to channel quality

Assume that an M-ary QAM system adapts between 4-ary QAM, 16-ary QAM, 64-ary QAM and 256-ary QAM. Show when a new M is chosen by plotting M (or $\log_2(M)$*) versus* $\mathcal{P}_z/N_0W$*. How large is the bit rate in each case? Assume that* $\rho_{BPSK} = 1/2$ [bps/Hz].



Depending on the channel quality we can achieve different bit rates $R_b = W$, $2W$, $3W$, or $4W$[bps]

Bit errors vs symbol errors

- Assume that S symbols are transmitted and S_{err} are in error
- If a symbol $\hat{m} \neq m$ is decided, this causes at least 1 bit error and at most $k = \log_2 M$ bit errors

$$S_{err} \leq B_{err} \leq k\, S_{err}$$

- This leads to the following relationship between P_b and P_s:

$$\frac{P_s}{k} = \frac{E\{S_{err}\}}{S \cdot k} \leq P_b \leq \frac{E\{S_{err} \cdot k\}}{S \cdot k} = P_s$$

- P_s depends on the signal constellation only
- The exact P_b depends on the mapping from bits to messages m_ℓ and hence signal alternatives $s_{m_\ell}(t)$

Example: Which mapping is better for 4-PAM? (and why?)

$$(1) \quad m_0 = 00,\ m_1 = 11,\ m_2 = 01,\ m_3 = 10$$

$$(2) \quad m_0 = 00,\ m_1 = 01,\ m_2 = 11,\ m_3 = 10$$

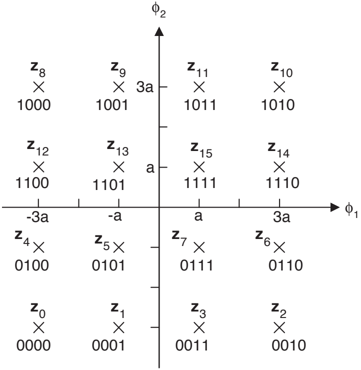
Gray code mappings

- We have seen that for small N_0 we can approximate

$$P_s \approx c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

- This motivates the use of Gray code mappings:

Example:
16-QAM



How can we achieve large data rates?

- The bit rate R_b can be increased in different ways
- We can select a signal constellation with large M
  $\Rightarrow$ this typically increases the error probability P_s
  exception: orthogonal signals (FSK): require more bandwidth W
- Achieving equal P_s with larger M is possible by increasing $\mathcal{E}_b/N_0$
  $\Rightarrow$ this reduces the energy efficiency
- We can also increase R_b by increasing the bandwidth W
  $\Rightarrow$ this does not improve the bandwidth efficiency $\rho = R_b/W$

Question:
what is the largest achievable rate R_b for a given error probability P_s, channel quality $\mathcal{E}_b/N_0$ and bandwidth W?

This question was answered by Claude Shannon in 1948:
"A mathematical theory of communication"
Course EITN45: Information Theory (VT2)

A fundamental limit: channel capacity

- Consider a single-path channel ($|H(f)|^2 = \alpha^2$) with finite bandwidth W and additive white Gaussian noise (AWGN) $N(t)$
- The capacity for this channel is given by

$$\mathcal{C} = W \log_2 \left(1 + \frac{\mathcal{P}_z}{N_0 W} \right) \text{ [bps]}$$

- Shannon showed that reliable communication requires that

$$R_b \leq \mathcal{C}$$

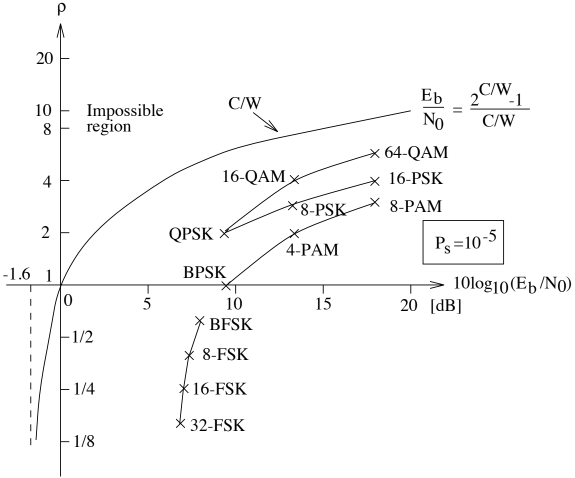
- **Observe:** the capacity formula does not include P_s (why?)
- Shannon also showed that if $R_b < \mathcal{C}$, then the probability of error P_s can be made arbitrarily small

$$P_s \to 0$$

if messages are coded in blocks of length $N \to \infty$

Bandwidth efficiency and gap to capacity

(p. 369)



- $\rho \leq C/W$: reliable communication is impossible above
- this limit can be approached with channel coding

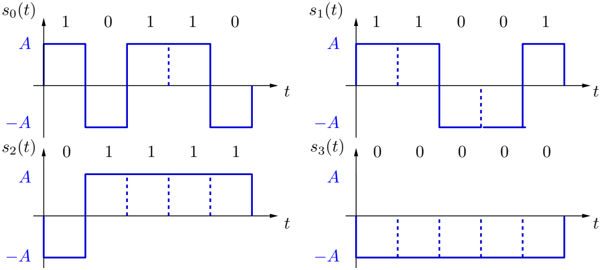
How does channel coding work?

- We have seen that a large minimum distance d^2_{min} between signals is required to improve the energy efficiency
- For binary signaling ($M = 2$) we have seen that $d^2_{min} \leq 2$

Idea of coding:

- generate M binary sequences of length N
- use binary antipodal signaling to create M signals $s_\ell(t)$

Example: $N = 5$, $M = 4$, $g_{rec}(t)$ pulse with $T = T_s/N$ (what is D^2_{min}?)



Increasing d_{min}^2 with coding

- In our example we have

$$D_{min}^2 = 4A^2\,T \cdot 3 = 4\,E_g\,3 = 12\,E_g$$

- Normalizing by the average energy $\mathcal{E}_b = N\,E_g/k$ this gives

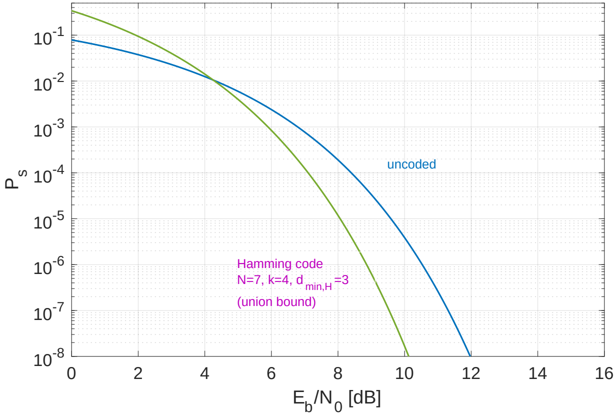
$$d_{min}^2 = \frac{D_{min}^2}{2\mathcal{E}_b} = \frac{12\,E_g}{2N/k\,E_g} = 6 \cdot \frac{k}{N} = \frac{12}{5} = 2.4$$

- Let $d_{min,H}$ denote the minimum Hamming distance between the binary code sequences $\Rightarrow$ in our example: $d_{min,H} = 3$
- Then we can write

$$d_{min}^2 = 2\,\frac{k}{N}\,d_{min,H}$$

  where $R = k/N$ is called the code rate
- Larger $d_{min,H}$ values can be achieved with larger N

Example: symbol error probability

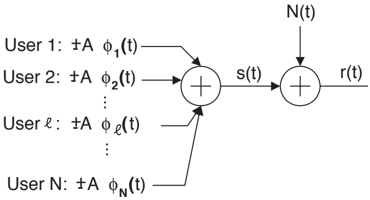


- Hamming code, $N = 7$, $k = 4$, $d_{min,H} = 3 \Rightarrow d^2_{min} = 3.43$
- How can we construct good codes?

EITN70: Channel Coding for Reliable Communication (HT2)

Multiuser Communication

(p. 395/396)



A simple model:

- N users transmit at same time with orthonormal waveforms $\phi_\ell(t)$
- Binary antipodal signaling is used in this example, such that

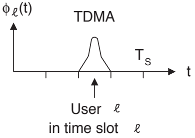
$$s(t) = \sum_{n=1}^{N} A_n \phi_n(t) \ , \quad A_n \in \pm A$$

- The orthonormal waveforms satisfy

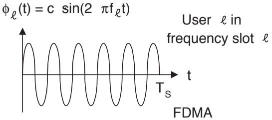
$$\int_0^{T_s} \phi_i(t)\, \phi_j(t)\ dt = \begin{cases} 0 & \text{if } i \neq j \ , \\ 1 & \text{if } i = j \end{cases}$$

Multiuser Communication

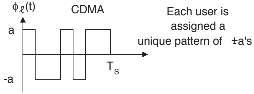
- The separation of users can be achieved in different ways
- **TDMA:** (time-division multiple access)



- **FDMA / OFDMA:** (frequency-division multiple access)

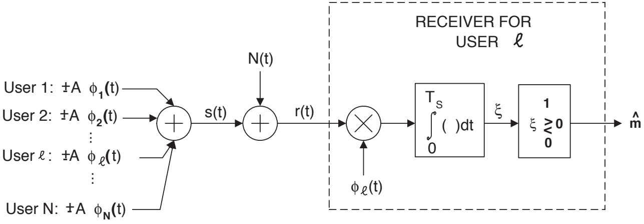


- **CDMA:** (code-division multiple access)



- **MC-CDMA:** (multi-carrier CDMA) combined OFDM/CDMA

Receiver for Multiuser Communication



- This permits a simple receiver structure for each user ℓ
- The decision variable becomes

$$\xi = \int_0^{T_s} \phi_\ell(t)\, r(t)\; dt = \int_0^{T_s} \phi_\ell(t) \left(\sum_{n=1}^{N} A_n\, \phi_n(t) + N(t) \right)\; dt$$

$$= A_\ell + \int_0^{T_s} \phi_\ell(t)\, N(t)\; dt = A_\ell + \mathcal{N}$$

$\Rightarrow$ receiver is only disturbed by noise and not by other users!

Non-coherent receivers

- With phase-shift keying (PSK) the message $m[n]$ at time nT_s is put into the phase θ_n of the transmit signal

$$s(t) = g(t)\,\sqrt{2E}\,\cos(2\pi f_c t + \theta_n)\,, \quad nT_s \le t \le (n+1)T_s$$

- The channel introduces some attenuation α, some additive noise $N(t)$ and also some phase offset ν into the received signal

$$r(t) = \alpha\; g(t)\,\sqrt{2E}\,\cos(2\pi f_c t + \theta_n + \nu) + N(t)$$

- **Challenge:** the optimal receiver needs to know α and ν
- In some applications an accurate estimation of ν is infeasible (cost, complexity, size)
- **Non-coherent receivers:**
  receiver structures that can work well without knowledge of the exact phase offset

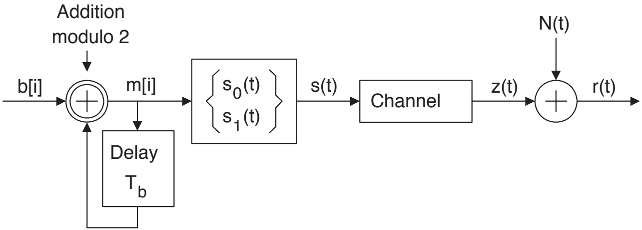
How can we modify our PSK transmission accordingly?

Differential Phase Shift Keying

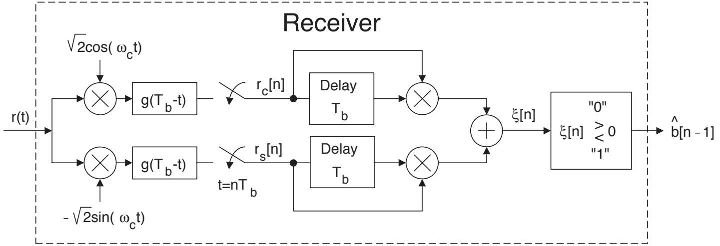
- With differential PSK, the message $m[n] = m_\ell$ is mapped to the phase according to

$$\theta_n = \theta_{n-1} + \frac{2\pi\ell}{M} \qquad \ell = 0, \dots, M-1$$

- The transmitted phase θ_n depends on both θ_{n-1} and $m[n]$
- This differential encoding introduces memory and the transmitted signal alternatives become dependent
- **Example 5.25:** binary DPSK



Differential Phase Shift Keying ($M = 2$)



- The receiver uses no phase offset v in the carrier waveforms
- Without noise, the decision variable is

$$\begin{aligned}\xi[n] &= r_c[n]\, r_c[n-1] + r_s[n]\, r_s[n-1] \\ &= A\cos(\theta_{n-1}+v)\, A\cos(\theta_{n-2}+v) + A\sin(\theta_{n-1}+v)\, A\sin(\theta_{n-2}+v) \\ &= A^2\cos(\theta_{n-1}-\theta_{n-2}) \quad \Rightarrow \text{ independent of } v\end{aligned}$$

- Note: non-coherent reception increases variance of noise