



LUND
UNIVERSITY

EITG05 – Digital Communications

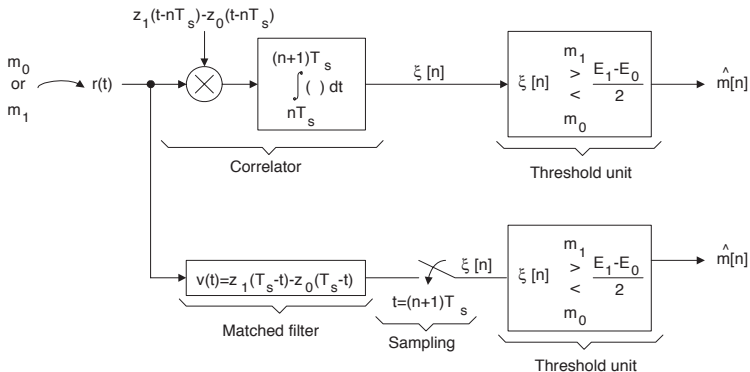
Lecture 6

Receivers continued:
System design criteria, Performance for M -ary signaling

Michael Lentmaier
Monday, September 23, 2019

Last week: Analysis Binary Signaling

- Only **one correlator** or **one matched filter** is now required:

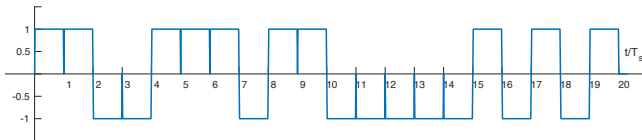


- Matched filter output needs to be sampled at correct time



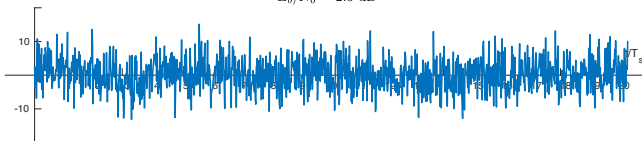
Example: (see Matlab demo)

$z(t)$ (random data, rectangular pulse)

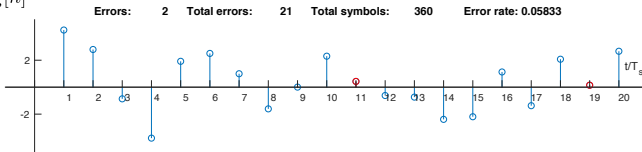


$r(t) = z(t) + N(t)$

$E_b/N_0 = 2.0$ dB



$\xi[n]$



An energy efficiency perspective

- ▶ Consider the case $P_0 = P_1 = 1/2$
- ▶ The average **received** energy per bit is then

$$\mathcal{E}_b = \frac{1}{2} \int_0^{T_b} z_0^2(t) dt + \frac{1}{2} \int_0^{T_b} z_1^2(t) dt = \frac{E_0 + E_1}{2}$$

- ▶ We can then introduce the **normalized** squared Euclidean distance

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\mathcal{E}_b} = \frac{1}{2\mathcal{E}_b} \int_0^{T_b} (z_1(t) - z_0(t))^2 dt$$

- ▶ With this the bit error probability becomes

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2 \frac{\mathcal{E}_b}{N_0}}\right)$$

- ▶ The parameter $d_{0,1}^2$ is a measure of **energy efficiency**



Special case 1: antipodal signals

- ▶ In case of antipodal signals we have $z_1(t) = -z_0(t)$ and

$$D_{0,1}^2 = \int_0^{T_b} (z_1(t) - z_0(t))^2 dt = 4 \int_0^{T_b} z_1^2(t) dt = 4E$$

- ▶ From $E_0 = E_1 = E$ follows

$$\mathcal{E}_b = \frac{E + E}{2} = E$$

and

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\mathcal{E}_b} = \frac{4E}{2E} = 2$$

- ▶ The bit error probability for **any pair of antipodal signals** becomes

$$P_b = Q\left(\sqrt{2 \frac{\mathcal{E}_b}{N_0}}\right)$$



Special case 2: orthogonal signals

- In case of orthogonal signals we have

$$\int_0^{T_b} z_0(t) z_1(t) dt = 0$$

and hence (compare page 28)

$$D_{0,1}^2 = \int_0^{T_b} (z_1(t) - z_0(t))^2 dt = E_0 + E_1$$

- This gives

$$\mathcal{E}_b = \frac{E_0 + E_1}{2}$$

and

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\mathcal{E}_b} = \frac{E_0 + E_1}{E_0 + E_1} = 1$$

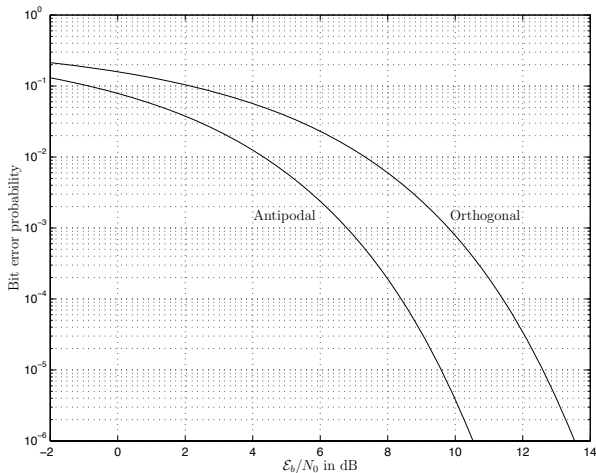
- The bit error probability for **any pair of orthogonal signals** is

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$



Comparison

Antipodal vs orthogonal signaling:



Larger values of $d_{0,1}^2$ give better energy efficiency



Antipodal vs orthogonal signaling

- ▶ There is a constant gap between the two curves
- ▶ We can measure the difference in energy efficiency by the ratio

$$\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}} = \frac{d_{0,1,ort}^2}{d_{0,1,atp}^2} = \frac{1}{2}$$

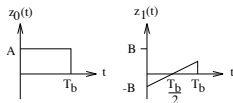
- ▶ In terms of dB this corresponds to

$$10\log_{10}\left(\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}}\right) = 10\log_{10}\left(\frac{d_{0,1,ort}^2}{d_{0,1,atp}^2}\right) = -3 \text{ [dB]}$$

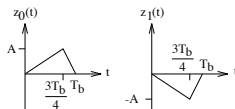
⇒ antipodal signaling requires 3 dB less energy for equal P_b



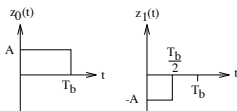
Example 4.11: rank pairs with respect to $d_{0,1}^2$



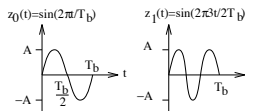
Pair 1



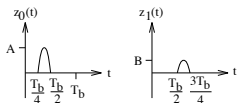
Pair 2



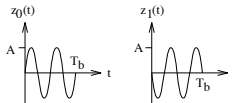
Pair 3



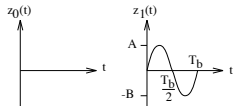
Pair 4



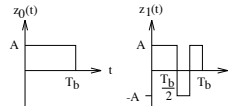
Pair 5



Pair 6



Pair 7



Pair 8



Can we do better?

- ▶ It is possible to show that for two equally likely signal alternatives we always have

$$d_{0,1}^2 \leq 2$$

- ▶ Antipodal signaling is hence optimal for binary signaling ($M = 2$)

Remark:

- ▶ Channel coding can be used to further increase $d_{0,1}^2$
- ▶ Sequences of binary pulses with large separation are designed
- ▶ This does not contradict the result from above:
coded binary signals correspond to uncoded signals with $M > 2$

Channel coding can be used for improving energy efficiency
Cost: complexity, latency, (bandwidth)



Relationship between parameters

- ▶ The bit error probability can be expressed in different ways

$$P_b = Q \left(\sqrt{\frac{D_{0,1}^2}{2N_0}} \right) = Q \left(\sqrt{d_{0,1}^2 \frac{\mathcal{E}_b}{N_0}} \right) = Q \left(\sqrt{d_{0,1}^2 \frac{P_z}{R_b N_0}} \right)$$

- ▶ Assuming $z_0(t) = \alpha s_0(t)$ and $z_1(t) = \alpha s_1(t)$ we also get

$$P_b = Q \left(\sqrt{d_{0,1}^2 \frac{\alpha^2 \bar{P}_{sent}}{R_b N_0}} \right) = Q \left(\sqrt{\frac{d_{0,1}^2}{\rho} \cdot \frac{\alpha^2 \bar{P}_{sent}}{N_0 W}} \right)$$

- ▶ Recall that $\rho = R_b/W$ is the bandwidth efficiency and $N_0 W$ is the noise power within the bandwidth W

The expression that is most appropriate to use depends on the specific problem to be solved



A "typical" type of problem

- ▶ The bit error probability must not exceed a certain level,

$$P_b \leq P_{b,req} = Q(\sqrt{\mathcal{X}})$$

- ▶ **Example:** if $P_{b,req} = 10^{-9}$ then $\mathcal{X} \approx 36$

- ▶ **Consequences:**

$$d_{0,1}^2 \frac{\mathcal{E}_b}{N_0} \geq \mathcal{X}$$

$$R_b \leq \frac{d_{0,1}^2}{\mathcal{X}} \cdot \frac{P_z}{N_0}$$

$$R_b \leq \frac{d_{0,1}^2}{\mathcal{X}} \cdot \frac{\alpha^2 \bar{P}_{sent}}{N_0}$$

- ▶ **Note:** the received signal power P_z decreases with communication distance



Example 4.12: transmission hidden in noise

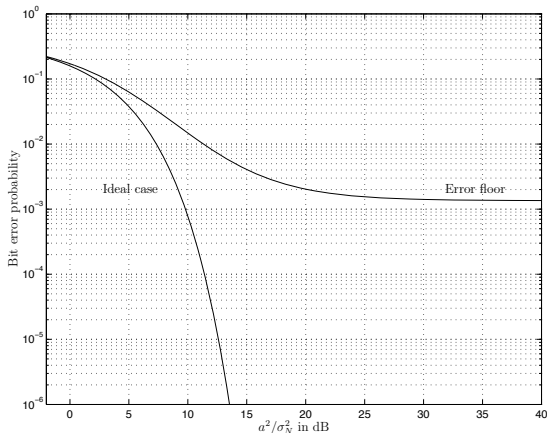
In a specific application equally likely binary antipodal signals are used, and the pulse shape is $g_{rc}(t)$ with amplitude A and duration $T \leq T_b$. AWGN with power spectral density $N_0/2$, and the ML receiver is assumed. It is required that the bit error probability must not exceed 10^{-9} . It is also required that the power spectral density satisfies $R(f) \leq N_0/2$ for all frequencies f (the information signal is intentionally “hidden” in the noise). Determine system and signal parameters above such that these two requirements are satisfied.

- ▶ $P_b = Q\left(\sqrt{2\mathcal{E}_b/N_0}\right) \leq 10^{-9} \Rightarrow \mathcal{E}_b/N_0 \geq 18$
- ▶ $R(f) = R_b |G_{rc}(f)|^2$ has maximum at $f = 0$
- ▶ $R(0) = R_b A^2 T^2 / 4 \leq N_0/2$ (check pulse shape)
- ▶ $\mathcal{E}_b/N_0 = 3/8 A^2 T / N_0 \geq 18$
- ▶ Hidden in noise: $A^2 T / N_0 \leq 2/(R_b T)$
- ▶ P_b requirement: $A^2 T / N_0 \geq 48$
- ▶ **Solution:**
choose $T \leq T_b/24$ and $A^2 = 48N_0/T$



Non-ideal receiver conditions

Example 4.15: unexpected additional noise w_x , i.e., $w = w_N + w_x$

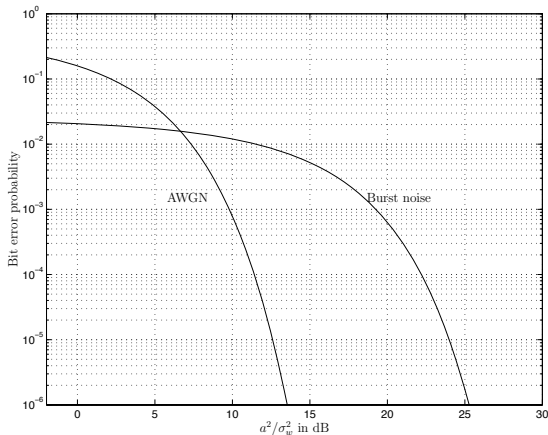


Can be analyzed with our methods



Non-ideal receiver conditions

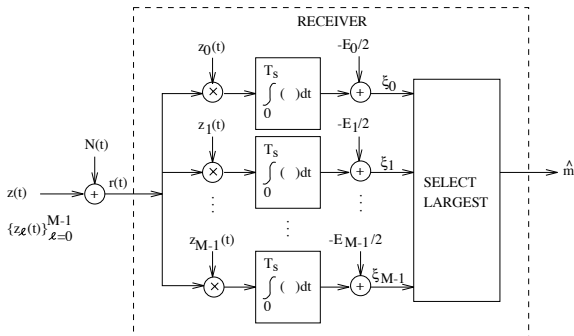
Example 4.16: hostile bursty interference, active with $p_{on} = 0.05$



Observe: at low power an interference in bursts is more severe than continuous interference



M-ary Signaling



- ▶ The receiver computes M **decision variables** $\xi_0, \xi_1, \dots, \xi_{M-1}$
- ▶ The selected message \hat{m} is based on the **largest value**

$$\hat{m} = m_\ell, \quad \ell = \arg \max_i \xi_i$$

- ▶ **Question:** when do we make a wrong decision?



Probability of a wrong decision

- ▶ For $M = 2$ we have considered **two** error probabilities P_F and P_M
- ▶ For a **given message** $m = m_j$, in general there are **$M - 1$ ways** (events) to make a wrong decision,

$$\{\xi_i > \xi_j \mid m = m_j\}, \quad i \neq j$$

- ▶ The probability of a **wrong decision** can be upper bounded by

$$\begin{aligned} Pr\{\hat{m} \neq m_j \mid m = m_j\} &= Pr\left\{ \bigcup_{\substack{i=0 \\ i \neq j}}^{M-1} \xi_i > \xi_j \mid m = m_j \right\} \\ &\leq \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\} \quad (\text{union bound}) \end{aligned}$$

- ▶ **Note:** given some events A and B , the union bound states that

$$Pr\{A \cup B\} \leq Pr\{A\} + Pr\{B\},$$

where **equality** holds if A and B are **independent**



Symbol error probability

- ▶ The symbol error probability can be upper bounded by

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\}$$

- ▶ From the binary case $M = 2$ we know that (pick $i = 0$ and $j = 1$)

$$Pr\{\xi_i > \xi_j \mid m = m_j\} = Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

where $D_{i,j}$ is the Euclidean distance between $z_i(t)$ and $z_j(t)$

- ▶ We obtain the following **main result** for M -ary signaling:

$$\max_{\substack{i \\ i \neq j}} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \leq P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$



Example: orthogonal signaling

- ▶ Consider M orthogonal signals with equal energy E
- ▶ **Examples:** FSK, PPM
- ▶ For each pair $z_i(t)$ and $z_j(t)$ we get

$$D_{i,j}^2 = E + E = 2E$$

- ▶ From the union bound we obtain

$$\begin{aligned} P_s &\leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Q \left(\sqrt{\frac{D_{i,j}^2}{2N_0}} \right) \\ &= (M-1) Q \left(\sqrt{\frac{2E}{2N_0}} \right) = (M-1) Q \left(\sqrt{\frac{E}{N_0}} \right) \end{aligned}$$

- ▶ This generalizes the binary case considered previously



Distances $D_{i,j}$ are important

- ▶ P_s is determined by the distances $D_{i,j}$ between the signal pairs
- ▶ Let us sort these distances

$$D_{min} < D_1 < D_2 < \dots < D_{max}$$

- ▶ Then the upper bound on P_s can be written as

$$P_s \leq c \mathcal{Q} \left(\sqrt{\frac{D_{min}^2}{2N_0}} \right) + c_1 \mathcal{Q} \left(\sqrt{\frac{D_1^2}{2N_0}} \right) + \dots + c_x \mathcal{Q} \left(\sqrt{\frac{D_{max}^2}{2N_0}} \right)$$

- ▶ The coefficients are

$$c_\ell = \sum_{j=1}^{M-1} P_j \cdot n_{j,\ell} , \quad \ell = 0, 1, 2, \dots, x$$

- ▶ $n_{j,\ell}$: number of signals at distance D_ℓ from signal $z_j(t)$

How many distinct terms do exist for 4-PAM?



A useful approximation of P_s

- ▶ The union bound is easy to compute if we know all distances D_ℓ
- ▶ At large signal-to-noise ratio (small N_0), i.e., when P_s is small, the **first term** provides a good **approximation**

$$P_s \approx c Q \left(\sqrt{\frac{D_{\min}^2}{2N_0}} \right)$$

- ▶ We see that the minimum distance D_{\min}^2 and the average number of closest signals c dominate the performance in this case
- ▶ **Explanation:**
the function $Q(x)$ decreases very fast as x increases (faster than exponentially). The other terms become negligible at some point.

⇒ at small P_s (small N_0) we can compare different signal constellations by means of D_{\min}^2 , similarly to the binary case



Energy efficiency and normalized distances

- ▶ Consider the case $P_\ell = 1/M$, $\ell = 0, 1, \dots, M-1$
- ▶ The average **received** energy per bit is given by

$$\mathcal{E}_b = \frac{1}{k} \sum_{i=0}^{M-1} \frac{1}{M} \int_0^{T_s} z_i^2(t) dt = \frac{1}{k} \frac{E_0 + E_1 + \dots + E_{M-1}}{M}$$

- ▶ Using the **normalized** squared Euclidean distances

$$d_\ell^2 = \frac{D_\ell^2}{2\mathcal{E}_b},$$

the union bound can be written as

$$P_s \leq c Q\left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}}\right) + c_1 Q\left(\sqrt{d_1^2 \frac{\mathcal{E}_b}{N_0}}\right) + \dots + c_x Q\left(\sqrt{d_{\max}^2 \frac{\mathcal{E}_b}{N_0}}\right)$$

- ▶ The parameters d_ℓ^2 determine the **energy efficiency**



Approximate P_s for some constellations

- ▶ Considering the dominating term in the union bound we obtain

$$P_s \approx c Q \left(\sqrt{d_{\min}^2 \frac{\mathcal{E}_b}{N_0}} \right)$$

- ▶ This approximation is valid if $\frac{\mathcal{E}_b}{N_0}$ is sufficiently large

	c	d_{\min}^2
M-ary PAM	$2(1 - 1/M)$	$\frac{6 \log_2(M)}{M^2 - 1}$
M-ary PSK ($M > 2$)	2	$2 \log_2(M) \sin^2(\pi/M)$
M-ary FSK	$M - 1$	$\log_2(M)$
M-ary QAM	$4(1 - 1/\sqrt{M})$	$\frac{3 \log_2(M)}{M - 1}$

Table 4.1: The coefficient c , and d_{\min}^2 , for some common signal constellations. Equally likely signal alternatives are assumed. See Subsection 2.4.1.1 for the M-ary PAM case, and Subsection 2.4.5.1 for the M-ary QAM case. M equal energy orthogonal FSK signals are also assumed.



Example 4.19

Assume two signal constellations, denoted A and B respectively, with corresponding parameters $d_{\min,A}^2$ and $d_{\min,B}^2$. From the equality (see e.g. the dominating term in the union bound),

$$d_{\min,A}^2 \mathcal{E}_{b,A} / N_0 = d_{\min,B}^2 \mathcal{E}_{b,B} / N_0$$

we find that the difference (in dB) in received energy per information bit is (compare with (2.13) on page 16),

$$10 \log_{10}(\mathcal{E}_{b,B}) - 10 \log_{10}(\mathcal{E}_{b,A}) = 10 \log_{10} \left(\frac{d_{\min,A}^2}{d_{\min,B}^2} \right)$$

Calculate the value $10 \log_{10} \left(\frac{d_{\min,A}^2}{d_{\min,B}^2} \right)$ if “A” is binary antipodal PAM, and if “B” is 4-ary PAM. Assume, that the conditions leading to (2.50) are satisfied.

- For M -ary PAM we have (Table 4.1 or Table 5.1)

$$d_{\min}^2 = 6 \log_2(M) / (M^2 - 1) \quad \Rightarrow \quad d_{\min,A}^2 = 2, \quad d_{\min,B}^2 = 4/5$$

- $10 \log_{10} d_{\min,A}^2 / d_{\min,B}^2 = 10 \log_{10} 5/2 = 3.98 \text{ dB}$

Binary PAM is 3.98 dB more energy efficient than 4-ary PAM!



Short summary

- ▶ The binary receiver in Fig. 4.10 (one correlator / matched filter) is equivalent to the general receiver in Fig. 4.9 with $M = 2$
- ▶ For binary signaling ($M = 2$), the bit error probability P_b of the ML receiver can be computed using equation (4.53) on page 252
- ▶ For the M -ary case: union bound on P_s is given on p. 274
- ▶ Energy efficiency: measureable by normalized distances $d_{i,j}^2$
 $M = 2$: antipodal signaling with $d_{0,1}^2 = 2$ is optimal
- ▶ Study the relationships between P_b / P_s and rate, bandwidth, energy, power, given on p. 261–262 and p. 281–282

