EITG05 – Digital Communications

Lecture 4

Bandwidth of Transmitted Signals

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Fourier transform

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\; e^{-j2\pi f t}\; dt$$
$$= X_{Re}(f) + j\, X_{Im}(f)$$
$$= |X(f)|\, e^{j\varphi(f)}$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\; e^{+j2\pi f t}\; df$$
$$= \int_{-\infty}^{\infty} |X(f)|\; e^{+j(2\pi f t + \varphi(f))}\; df$$

Some useful Fourier transform properties

$$g(at) \leftrightarrow \frac{1}{|a|}\, G(f/a) \qquad g^*(T-t) \leftrightarrow G^*(f)e^{-j2\pi fT}$$

$$g(-t) \leftrightarrow G(-f) \qquad \delta(t) \leftrightarrow 1$$

$$G(t) \leftrightarrow g(-f) \qquad 1(dc) \leftrightarrow \delta(f)$$

$$g(t-t_0) \leftrightarrow G(f)e^{-j2\pi f t_0} \qquad e^{j2\pi f_c t} \leftrightarrow \delta(f-f_c)$$

$$g(t)e^{j2\pi f_c t} \leftrightarrow G(f-f_c) \qquad \cos(2\pi f_c t) \leftrightarrow \frac{1}{2}\,(\delta(f+f_c)+\delta(f-f_c))$$

$$\frac{d}{dt}\, g(t) \leftrightarrow j2\pi f\; G(f) \qquad \sin(2\pi f_c t) \leftrightarrow \frac{j}{2}\,(\delta(f+f_c)-\delta(f-f_c))$$

$$g^*(t) \leftrightarrow G^*(-f) \qquad \alpha e^{-\pi\alpha^2 t^2} \leftrightarrow e^{-\pi f^2/\alpha^2}$$

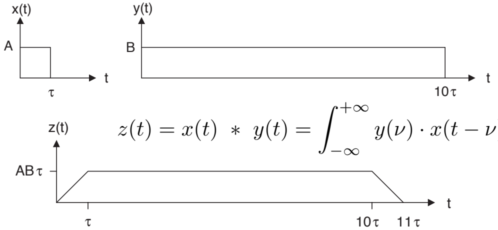
$\rightarrow$ full list in Appendix C of the compendium

Some useful Fourier transform properties

- Consider two signals $x(t)$ and $y(t)$ and their Fourier transforms

$$x(t) \longleftrightarrow X(f)\,, \quad y(t) \longleftrightarrow Y(f)$$

- Recall the convolution operation $z(t) = x(t) * y(t)$:



$$z(t) = x(t) * y(t) = \int_{-\infty}^{+\infty} y(\nu) \cdot x(t-\nu)\, d\nu$$

- **Filtering:**

$$x(t) * y(t) \longleftrightarrow X(f) \cdot Y(f)$$

- **Multiplication:**

$$x(t) \cdot y(t) \longleftrightarrow X(f) * Y(f)$$

Spectrum of time-limited signals

- Consider some time-limited signal $s_T(t)$ of duration T, with $s_T(t) = 0$ for $t < 0$ and $t > T$
- Assume that within the interval $0 \leq t \leq T$, the signal $s_T(t)$ is equal to some signal $s(t)$, i.e.,

$$s_T(t) = s(t) \cdot g_{rec}(t) \ ,$$

  where $g_{rec}(t)$ is the rectangular pulse of amplitude $A = 1$
- Taking the Fourier transform on both sides we get

$$S_T(f) = S(f) \ * \ G_{rec}(f) = S(f) \ * \ AT \ \frac{\sin(\pi f T)}{\pi f T} \ e^{-j\pi f T}$$

- Since $G_{rec}(f)$ is unlimited along the frequency axis, this is the case for $S_T(f)$ as well (convolution increases length)

Time-limited signals can never be strictly band-limited

Some definitions of bandwidth

- **Main-lobe definition:**
  W_{lobe} is defined by the width of the main-lobe of $R(f)$
  This is how we have defined bandwidth in previous examples
- In baseband we use the one-sided width, while in bandpass applications the two-sided width is used (positive frequencies)
- **Percentage definition:**
  W_{99} is defined according to the location of 99% of the power
- For bandpass signals W_{99} is found as the value that satisfies

$$\int_{f_c-W_{99}/2}^{f_c+W_{99}/2} R(f)\,df = 0.99 \int_0^\infty R(f)\,df$$

- Other percentages can be used as well: W_{90}, $W_{99.9}$
- **Nyquist bandwidth**
  Assuming an ideal pulse with finite bandwidth (see Chapter 6)

$$W_{nyq} = \frac{R_s}{2}\ [\mathrm{Hz}]$$

Some definitions of bandwidth

Pulse shape	W_{lobe}	% power in W_{lobe}	W_{90}	W_{99}	$W_{99.9}$	Asymptotic decay
rec	2/T	90.3	1.70/T	20.6/T	204/T	f^{-2}
tri	4/T	99.7	1.70/T	2.60/T	6.24/T	f^{-4}
hcs	3/T	99.5	1.56/T	2.36/T	5.48/T	f^{-4}
rc	4/T	99.95	1.90/T	2.82/T	3.46/T	f^{-6}
Nyquist	R_s	100	$0.9R_s$	$0.99R_s$	$0.999R_s$	ideal

Table 2.1: Double-sided bandwidth results for power spectral densities according to (2.212). The $g_{rec}(t)$, $g_{tri}(t)$, $g_{hcs}(t)$ and $g_{rc}(t)$ pulse shapes are defined in Appendix D, and T denotes the duration of the pulse. The Nyquist pulse shape is not limited in time and it is defined in (D.49) with parameters $\beta = 0$ and $\mathcal{T} = T_s$.

- This table is useful for PAM, PSK, and QAM constellations
- Except bandwidth W, the asymptotic decay is also relevant

Pulse spectrum examples

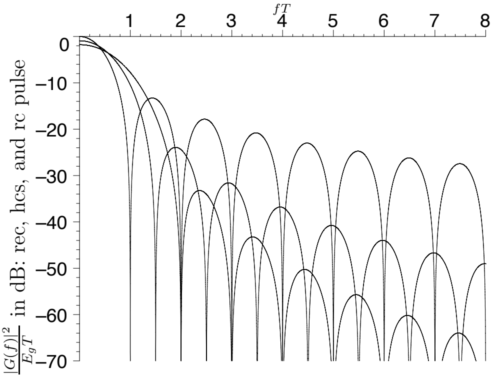


Figure 2.19: $10\log_{10}\left(\frac{|G(f)|^2}{E_g T}\right)$ for the $g_{rec}(t)$, $g_{hcs}(t)$, and $g_{rc}(t)$ pulse shapes. See also Example 2.26.

From last lecture: $R(f)$ for Binary Signaling

- In the general binary case, i.e., $M = 2$, we have

$$A(f) = P_0 \; S_0(f) + P_1 \; S_1(f)$$

- This simplifies the expression for the power spectral density to

$$\begin{aligned} R(f) &= R_c(f) && + R_d(f) \\ &= \frac{P_0 P_1}{T_b} |S_0(f) - S_1(f)|^2 && + \frac{|P_0 \; S_0(f) + P_1 \; S_1(f)|^2}{T_b^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_b) \end{aligned}$$

(derivation in Ex. 2.20)

- We will now consider some examples from the compendium

Example 2.21

Assume equally likely antipodal signal alternatives, such that

$$s_1(t) = -s_0(t) = g(t)$$

where $g(t) = g_{rec}(t)$*, and* $g_{rec}(t)$ *is given in (D.1). Assume also that* $T \leq T_b$*.*

i) *Calculate the power spectral density* $R(f)$*.*

ii) *Calculate* ***the bandwidth*** $\boldsymbol{W}$ ***defined as the one-sided width of the mainlobe of*** $\boldsymbol{R(f)}$*, if the information bit rate is 10* [kbps]*, and if* $T = T_b/2$*.*
Calculate also the bandwidth efficiency ρ*.*

iii) *Estimate the attenuation in* dB *of the first sidelobe of* $R(f)$ *compared to* $R(0)$*.*

- $M = 2$ with equally likely antipodal signaling $s_1(t) = -s_0(t) = g(t)$
- With $P_0 = P_1 = 1/2$ and $S_1(f) = -S_0(f) = G(f)$ we get

$$R(f) = R_b\,|S_1(f)|^2 = R_b\,|S_0(f)|^2 = R_b\,|G(f)|^2$$

- Details for the pulse in Appendix D

Example 2.23

Assume equally likely antipodal signal alternatives below. Assume that $s_1(t) = -s_0(t) = g_{rc}(t)$, where the time raised cosine pulse $g_{rc}(t)$ is defined in (D.18). Assume also that $T = T_b$.

Find an expression for the power spectral density $R(f)$. Calculate the bandwidth W, defined as the one-sided width of the mainlobe of $R(f)$, if R_b is 10 [kbps]. Calculate also the bandwidth efficiency ρ.

- Same as Example 2.21, but with $g_{rc}(t)$ pulse
- Analogously we get

$$R(f) = R_b \left| G_{rc}(f) \right|^2$$

- From the one-sided main-lobe we get

$$W = 2/T \text{ [Hz]}$$

- Bandwidth efficiency $\rho = 1/2$ [bps/Hz] is the same (why?)

Example 2.24

Assume $P_0 = P_1$ *and that,*

$$s_1(t) = -s_0(t) = g_{rc}(t)\cos(2\pi f_c t)$$

with $T = T_b$, *and* $f_c \gg 1/T$. *Hence, a version of binary PSK signaling is considered here (alternatively binary antipodal bandpass PAM). Calculate* ***the bandwidth W, defined as the double-sided width of the mainlobe around the carrier frequency*** $\boldsymbol{f_c}$. *Assume that the information bit rate is 10* [kbps]. *Calculate also the bandwidth*

- This corresponds to the bandpass case
- Let $g_{hf}(t)$ denote the high-frequency pulse

$$g_{hf}(t) = g_{rc}(t)\cos(2\pi f_c t) \quad \text{and} \quad R(f) = R_b\,|G_{hf}(f)|^2$$

- Using shift operations we get

$$R(f) = R_b \left| \frac{G_{rc}(f+f_c)}{2} + \frac{G_{rc}(f-f_c)}{2} \right|^2$$

- From the two-sided main-lobe we get

$$W = 4/T\ [\mathrm{Hz}]$$

Example: discrete frequencies in $R(f)$

- Assume $M = 2$
- Let $s_0(t) = 0$ and $s_1(t) = 5$ with a pulse duration $T = T_b/2$
- With this the average signal becomes

$$a(t) = \frac{s_0(t) + s_1(t)}{2} = 2.5 \ , \quad 0 \leq t \leq T$$

- We can then write (within the pulse duration T)

$$s_0(t) = -2.5 + a(t) \ , \quad s_1(t) = +2.5 + a(t)$$

Observe:

1. this method is a waste of signal energy since $a(t)$ does not carry any information
2. repetition of $a(t)$ in every symbol interval creates some periodic signal component in the time domain, which leads to discrete frequencies in the frequency domain

From last lecture: general $R(f)$

- The power spectral density $R(f)$ can be divided into a continuous part $R_c(f)$ and a discrete part $R_d(f)$

$$R(f) = R_c(f) + R_d(f)$$

- The general expression for the continuous part is

$$\begin{aligned} R_c(f) &= \frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f) - A(f)|^2 \\ &= \left(\frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 \right) - \frac{|A(f)|^2}{T_s} \end{aligned}$$

- For the discrete part we have

$$R_d(f) = \frac{|A(f)|^2}{T_s^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_s)$$

$R(f)$: M-ary PAM signals

- With M-ary PAM signaling we have

$$s_\ell = A_\ell \, g(t) \ , \quad \ell = 0, 1, \ldots, M-1$$

- Then

$$S_\ell(f) = A_\ell G(f) \ , \quad \text{and} \quad A(f) = \sum_{\ell=0}^{M-1} P_\ell A_\ell G(f)$$

- With this we obtain the simplified expression

$$R(f) = \frac{\sigma_A^2}{T_s} |G(f)|^2 + \frac{m_A^2}{T_s^2} |G(f)|^2 \sum_{n=-\infty}^{\infty} \delta(f - n/T_s) \ ,$$

  where m_A denotes the mean and $\sigma_A^2 = \overline{E}_s / E_g - m_A^2$ the variance of the amplitudes A_ℓ

- Assuming zero average amplitude $m_A = 0$ and using $\overline{P} = \sigma_A^2 E_g R_s$ this reduces to

$$R(f) = R_c(f) = \frac{\sigma_A^2}{T_s} |G(f)|^2 = \frac{\overline{P}}{E_g} |G(f)|^2$$

Example 2.28

Assume the bit rate $R_b = 9600$ [bps], M-ary PAM transmission and that $m_A = 0$. Determine the (baseband) bandwidth W, defined as the one-sided width of the mainlobe of the power spectral density $R(f)$, if $M = 2$, $M = 4$ and $M = 8$, respectively. Furthermore, assume a rectangular pulse shape with amplitude A_g, and duration $T = T_s$. Calculate also the bandwidth efficiency ρ.

- What is W for a given pulse shape and different M?
- Using $T = T_s$, $m_A = 0$ and $g(t) = g_{rec}(t)$, we have

$$R(f) = \frac{\sigma_A^2}{T_s} |G_{rec}(f)|^2$$

- For the given pulse we get $W = 1/T_s$, where $T_s = k\ T_b$

$$\begin{aligned} k = 1 \quad &\Rightarrow \quad M = 2 \quad \Rightarrow \quad W = 9600[\text{Hz}] \\ k = 2 \quad &\Rightarrow \quad M = 4 \quad \Rightarrow \quad W = 4800[\text{Hz}] \\ k = 3 \quad &\Rightarrow \quad M = 8 \quad \Rightarrow \quad W = 3200[\text{Hz}] \end{aligned}$$

- Bandwidth efficiency: $\rho = R_b/W = k\ T_b/T_b = k$

What does bandwidth efficiency tell us?

In the previous example we had a bandwidth efficiency of

$$\rho = \frac{R_b}{W} = k$$

Saving bandwidth

- The previous example showed that the bandwidth W can be reduced by increasing M
- $T = T_s = k\,T_b$ increases with M
- $W = 1/T = R_b/k$ decreases accordingly

Improving bit rate

- Assume instead that the bandwidth W is fixed in the same example, i.e., the symbol duration $T_s = T$ is fixed
- Then $R_b = k\,W$ increases with M
- Assume for example $W = 1\,\mathrm{MHz}$:
  $R_b = 1\,\mathrm{Mbps}$ if $M = 2$ $(k = 1)$
  $R_b = 10\,\mathrm{Mbps}$ if $M = 1024$ $(k = 10)$

$R(f)$: M-ary QAM signals

- With M-ary QAM signaling the signal alternatives are

$$s_\ell(t) = A_\ell \, g(t) \, \cos(2\pi f_c \, t) - B_\ell \, g(t) \, \sin(2\pi f_c \, t) \, , \quad \ell = 0, 1, \ldots, M-1$$

- Then the Fourier transform becomes

$$\begin{aligned} S_\ell(f) &= A_\ell \, \frac{G(f+f_c) + G(f-f_c)}{2} - j \, B_\ell \, \frac{G(f+f_c) - G(f-f_c)}{2} \\ &= (A_\ell - jB_\ell) \, \frac{G(f+f_c)}{2} + (A_\ell + jB_\ell) \, \frac{G(f-f_c)}{2} \end{aligned}$$

- Assuming a zero average signal $a(t) = 0$ and $f_c T \geq 1$ this simplifies to

$$R(f) = R_c(f) = \overline{P} \, \frac{|G(f+f_c)|^2 + |G(f-f_c)|^2}{2 E_g}$$

$R(f)$: M-ary QAM signals

- Remember that M-ary QAM signals contain M-ary PSK and M-ary bandpass PAM signals as special cases:

$$\text{BP-PAM:} \quad B_\ell = 0$$
$$\text{PSK:} \quad A_\ell = \cos(\nu_\ell)\;, \quad B_\ell = \sin(\nu_\ell)$$

- $\Rightarrow$ our results for $R(f)$ of M-ary QAM signals include these cases

- For symmetric constellations, such that $a(t) = 0$, the simplified version applies
- The bandwidth W is determined by $|G(f - f_c)|^2$ and hence the two-sided main-lobe of $|G(f)|^2$

$\Rightarrow$ if the same pulse $g(t)$ is used then M-ary QAM, M-ary bandpass PAM and M-ary PSK have the same bandwidth W

Example

Bandwidth consumption for BPSK, QPSK and 16-QAM assuming equal R_b and $f_c = 100 R_b$

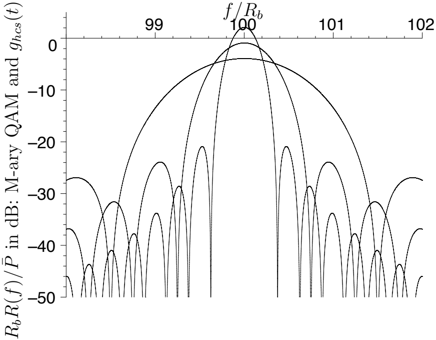


Figure 2.20: The power spectral density for binary QAM (BPSK, widest mainlobe), 4-ary QAM (QPSK), and 16-ary QAM (smallest mainlobe). The figure shows $10\log_{10}(R_b R(f)/\bar{P})$ [dB] in the frequency interval $98R_b \le f \le 102R_b$. The carrier frequency is $f_c = 100R_b$ [Hz], and a $T_s = kT_b$ long $g_{hcs}(t)$ pulse is assumed. See also (2.227) and (2.230).

$R(f)$: M-ary FSK signals

- With M-ary frequency shift keying (FSK) signaling the signal alternatives are

$$s_\ell(t) = A \cos(2\pi f_\ell\, t + \nu) \ , \quad 0 \leq t \leq T_s$$

- Choosing $\nu = -\pi/2$ this can be written as

$$s_\ell(t) = g_{rec}(t) \sin(2\pi f_\ell\, t) \ , \quad \text{with } T = T_s \ ,$$

since $s_\ell(t) = 0$ outside the symbol interval

- The Fourier transform is then

$$S_\ell(f) = j\, \frac{G_{rec}(f + f_\ell) - G_{rec}(f - f_\ell)}{2}$$

- The exact power spectral density $R(f)$ can now be computed by the general formula (2.202)–(2.204)

$R(f)$: M-ary FSK signals

- Let us find an approximate expression for the FSK bandwidth W
- Assume that

$$f_\ell = f_0 + \ell\, f_\Delta\,, \quad \ell = 0, \dots, M-1$$

- Then the bandwidth W can be approximated by

$$W \approx R_s + f_{M-1} - f_0 + R_s = (M-1) f_\Delta + 2R_s$$

- Consider now orthogonal FSK with $f_\Delta = I \cdot R_s/2$ for some $I > 0$
- The bandwidth efficiency is then

$$\rho = \frac{R_b}{W} \approx \frac{R_b}{(M-1)f_\Delta + 2R_s} = \frac{R_b}{\big((M-1)I/2+2\big)R_s} = \frac{\log_2 M}{(M-1)I/2+2}$$

Observe: the bandwidth efficiency of orthogonal M-ary FSK gets small if M is large

Last week we saw: M-ary FSK has good energy and Euclidean distance properties $\Rightarrow$ trade-off

Example 2.36

Assume that orthogonal M-ary FSK is used to communicate digital information in the frequency band $1.1 \le f \le 1.2$ [MHz].

For each M below, find the largest bit rate that can be used (use bandwidth approximations):

i) $M = 2$ *ii)* $M = 4$ *iii)* $M = 8$ *iv)* $M = 16$ *v)* $M = 32$

Which of the M-values above give a higher bit rate than the $M = 2$ *case?*

Solution:
It is given that $W_{M-FSK} = 100$ [kHz]. *From (2.245), the largest bit rate is obtained with* $I = 1$:

$$R_b \approx 10^5 \cdot \frac{\log_2(M)}{(M-1)/2+2}$$

M	$\frac{\log_2(M)}{(M-1)/2+2}$	R_b
2	$\frac{1}{5/2} = 0.4$	40 kbps
4	$\frac{2}{7/2} = \frac{4}{7} \approx 0.5714$	$\approx$ 57 kbps
8	$\frac{3}{11/2} = \frac{6}{11} \approx 0.5455$	$\approx$ 55 kbps
16	$\frac{4}{19/2} = \frac{8}{19} \approx 0.4211$	$\approx$ 42 kbps
32	$\frac{5}{35/2} = \frac{10}{35} \approx 0.2857$	$\approx$ 29 kbps

From this table it is seen that $M = 4, 8, 16$ *give a higher bit rate than* $M = 2$. $\square$

$R(f)$: OFDM-type signals

- An OFDM symbol (signal alternative) $x(t)$ can be modeled as a superposition of N orthogonal QAM signals, each carrying k_n bits, that are transmitted at different frequencies (sub-carriers)

$$x(t) = \sum_{n=0}^{N-1} s_{n,QAM}(t)$$

- Assuming each QAM signal has zero mean and that the different carriers have independent bit streams we get

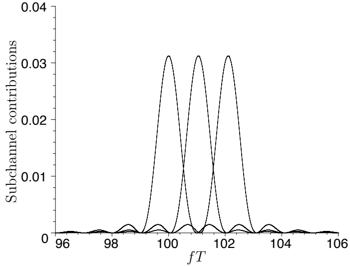
$$R(f) = R_c(f) = R_s \, E\{|X(f)|^2\} = \sum_{n=0}^{N-1} R_n(f)$$

- Using our previous results for QAM in each sub-carrier we get

$$R(f) = R_c(f) = \sum_{n=0}^{N-1} \overline{P} \; \frac{|G(f+f_c)|^2 + |G(f-f_c)|^2}{2\,E_g}$$

$R(f)$: OFDM-type signals

Illustration of $R_n(f)$ contributed by three neighboring sub-carriers:



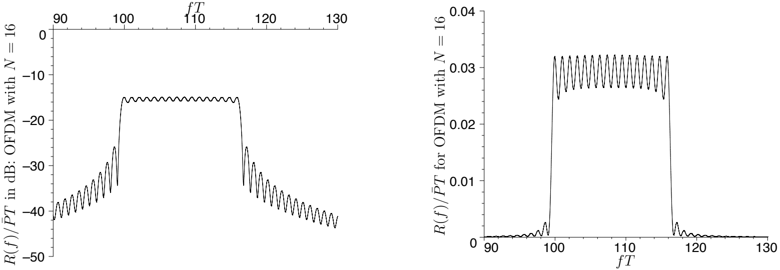
- Assuming $f_n = f_0 + n/(T_s - \Delta_h)$ we can estimate the bandwidth as

$$W \approx (N+1) f_\Delta = \frac{N+1}{1-\Delta_h/T_s} R_s \approx N \cdot R_s \ , \quad N \gg 1 \ , \ \Delta_h \ll T_s$$

- The bandwidth efficiency is then approximated by

$$\rho = \frac{R_b}{W} = \frac{R_s}{W} \sum_{k=0}^{N-1} k_n \approx \frac{1}{N} \sum_{k=0}^{N-1} k_n \ [\text{bps/Hz}]$$

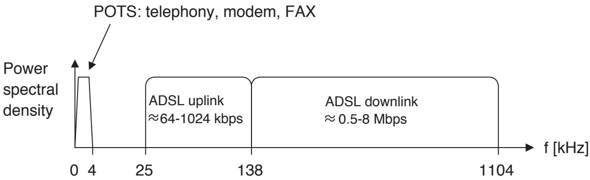
Example: $R(f)$ for OFDM



- $N = 16$ sub-carriers
- $T = T_s = 0.1$ [ms]
- $f_\Delta = R_s/0.95 = 10.53$ [kHz]
- $W \approx \frac{17}{0.95}\, R_s = 179$ [kHz]

Example 2.35

ADSL: uses plain telephone cable (twisted pair, copper)



In ADSL, a coded OFDM technique is used. The level of the power spectral density in the downstream is roughly -73 dB. As a basic example, let us here assume that the OFDM symbol rate in the downlink is 4000 [symbol/s], and that the subchannel carrier spacing is 5 kHz. Furthermore, it is here also assumed that uncoded 16-ary QAM is used in each subchannel (assumes a very "good" communication link).

For the ADSL downlink above, determine the bit rate in each subchannel, the total bit rate, and the bandwidth efficiency.

What about filtering away the side-lobes?

- Let us use a spectral rectangular pulse $X_{srec}(f)$ of amplitude $A = 1$ and width f_Δ to strictly limit the bandwidth
- Similar to the time-limited case we can write

$$S_{f_\Delta}(f) = S(f) \cdot X_{srec}(f)$$

- Taking the inverse Fourier transform on both sides we get

$$s_{f_\Delta}(t) = s(t) \, * \, x_{srec}(t) = s(t) \, * \, A f_0 \, \frac{\sin(\pi f_0 \, t)}{\pi f_0 \, t}$$

- Since $x_{srec}(t)$ is unlimited along the time axis, this is the case for the filtered signal $s_{f_\Delta}(t)$ as well
- The signal $x_{srec}(t)$ defines the ideal Nyquist pulse

As a consequence of filtering, the transmitted symbols will overlap in time domain $\Rightarrow$ inter-symbol-interference (ISI)

Nyquist Pulse

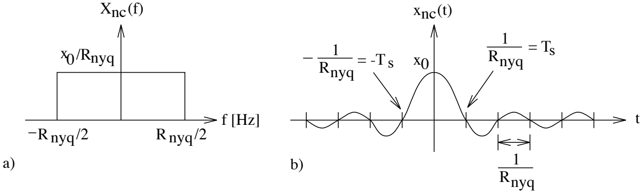


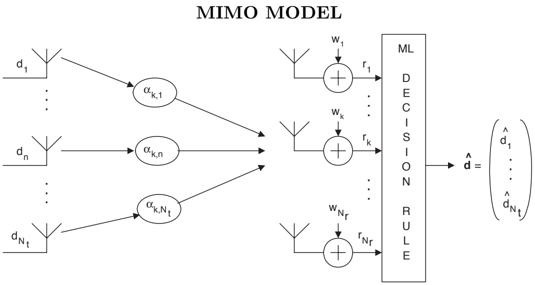
Figure 6.6: a) Ideal Nyquist spectrum; b) Ideal Nyquist pulse.

$$x_{nc}(t) = x_0 \frac{\sin(\pi R_{nyq} t)}{\pi R_{nyq} t} \;,\; -\infty \leq t \leq \infty \tag{6.39}$$

$$X_{nc}(f) = \begin{cases} x_0/R_{nyq} & , \; |f| \leq R_{nyq}/2 \\ 0 & , \; |f| > R_{nyq}/2 \end{cases} \tag{6.40}$$

The Nyquist pulse and the effect of ISI will be studied in Chapter 6

How can we further improve ρ?



$$r_k = \sum_{n=1}^{N_t} \alpha_{k,n} d_n + w_k$$

- MIMO: multiple-input multiple output
- transmission over multiple antennas in the same frequency band
- challenge: the individual wireless channels interfere
- **5G world record 2016:** (team from Lund involved) spectral efficiency of 145.6 bps/Hz with 128 antennas

Short summary

- Check different definitions bandwidth (one-sided and two-sided), time-limited signals cannot be strictly band-limited and vice versa
- Filtering and multiplication:
  linearity of convolution can be useful for combined signals
- A summary of bandwidth W and asymptotic decay of $R(f)$ for different pulse shapes is given in Table 2.1
- The power spectral density $R(f)$ and bandwidth efficiency ρ for M-PAM, M-QAM, M-FSK and OFDM are studied on p. 88–102