LUND UNIVERSITY

EITG05 – Digital Communications

Lecture 3

Bandwidth of Transmitted Signals

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What did we do last week?

Concepts of M-ary digital signaling:

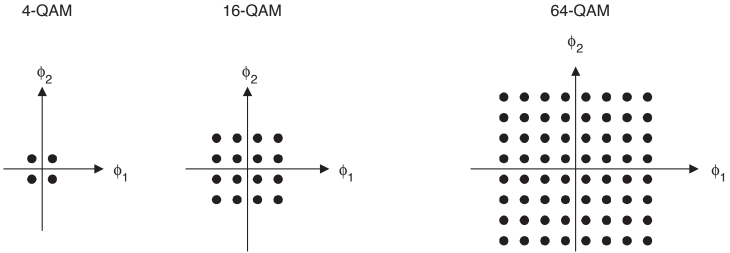
- Modulation of amplitude, phase or both: PAM, PSK, QAM
- Orthogonal signaling: FSK, OFDM
- Pulse position and width: PPM, PWM

We have paid special attention to:

- Average symbol energy $\overline{E}_s$
- Euclidean distance $D_{i,j}$
- Both values could be related to the energy E_g of the pulse $g(t)$

Signal space representation of QAM

- Now we can describe each signal alternative $s_\ell(t)$ as a point with coordinates $(s_{\ell,1}, s_{\ell,2})$ within a constellation diagram



$$s_{\ell,1} = A_\ell \sqrt{E_g/2}\,, \qquad s_{\ell,2} = B_\ell \sqrt{E_g/2}$$

- The signal energy E_ℓ and the Euclidean distance $D^2_{i,j}$ can be determined in the signal space

Geometric interpretation

- It is possible to describe QAM signals as two-dimensional vectors in a so-called signal space
- For this the signal

$$s_\ell(t) = A_\ell \; g(t) \; \cos(2\pi f_c \; t) - B_\ell \; g(t) \; \sin(2\pi f_c \; t)$$

is written as

$$s_\ell(t) = s_{\ell,1} \; \phi_1(t) + s_{\ell,2} \; \phi_2(t)$$

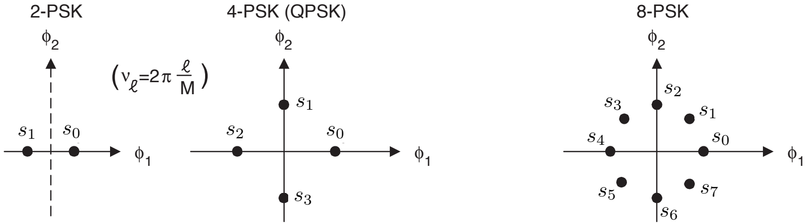
- Here $s_{\ell,1} = A_\ell \sqrt{E_g/2}$ and $s_{\ell,2} = B_\ell \sqrt{E_g/2}$ are the coordinates
- The functions $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal basis of a vector space that spans all possible transmit signals:

$$\phi_1(t) = \frac{g(t) \; \cos(2\pi f_c \; t)}{\sqrt{E_g/2}} \; , \quad \phi_2(t) = -\frac{g(t) \; \sin(2\pi f_c \; t)}{\sqrt{E_g/2}}$$

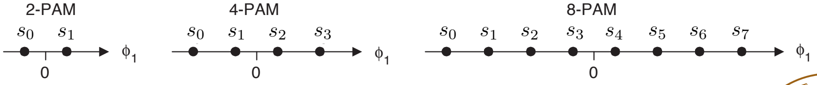
This looks abstract, but can be very useful!

Signal space representation of PSK and PAM

- PSK and PAM can be seen as a special cases of QAM:



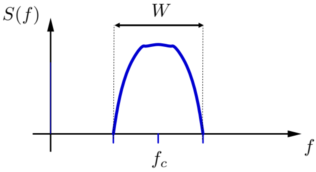
$$s_{\ell,1} = \cos(\nu_\ell)\sqrt{E_g/2}\,, \qquad s_{\ell,2} = \sin(\nu_\ell)\sqrt{E_g/2}$$



$$s_{\ell,1} = (-M + 1 + 2\,\ell)\ \sqrt{E_g}$$

Bandwidth of Transmitted Signal

- The bandwidth W of a signal is the width of the frequency range where most of the signal energy or power is located



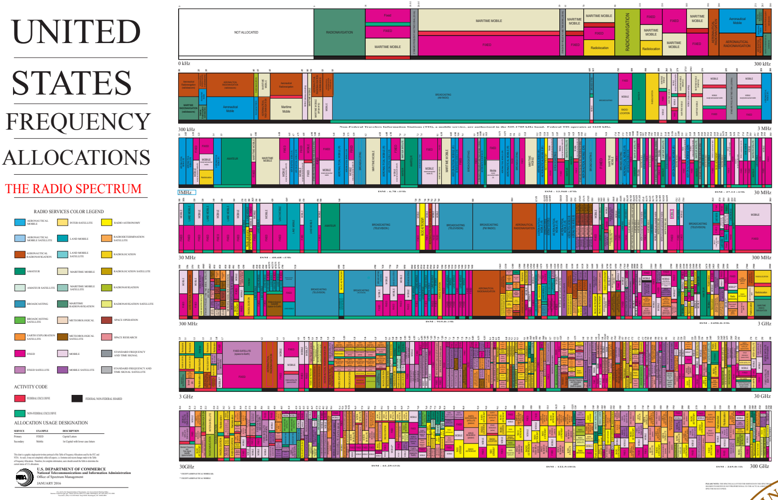
- W is measured on the positive frequency axis
- The bandwidth is a limited and precious resource
- We must have control of the bandwidth and use it efficiently

Questions:

What is the relationship between information bit rate and required bandwidth?

How does the bandwidth depend on the signaling method?

United States Frequency Allocations (2016)



Source: https://www.ntia.doc.gov/category/spectrum-management

Energy Spectrum

- We have seen last week that the energy of a signal $x(t)$ can be determined as

$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt$$

- The function $x^2(t)$ shows how the energy E_x is distributed along the time axis
- According to Parseval's relation we can alternatively express the energy as

$$E_x = \int_{-\infty}^{\infty} |X(f)|^2\, df \ ,$$

  where $X(f)$ denotes the Fourier transform of the signal $x(t)$
- The function $|X(f)|^2$ shows how the energy E_x is distributed in the frequency domain

$\Rightarrow$ We need the Fourier transform as a tool for finding the bandwidth of our signals

Fourier Transform

- The Fourier transform of a signal $x(t)$ is given by

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\; e^{-j2\pi f\,t}\; dt = X_{Re}(f) + j\, X_{Im}(f)\ ,$$

  where $j = \sqrt{-1}$, i.e., the solution to $j^2 = -1$

- We can also express $X(f)$ in terms of magnitude $|X(f)|$ and phase $\varphi(f) = \arg X(f)$ (argument)

$$X(f) = |X(f)|\; e^{j\,\varphi(f)}$$

- Then

$$|X(f)| = \sqrt{X_{Re}^2(f) + X_{Im}^2(f)}$$
$$X_{Re}(f) = |X(f)|\cos(\varphi(f))$$
$$X_{Im}(f) = |X(f)|\sin(\varphi(f))$$

Fourier Transform

- The original signal $x(t)$ can then be expressed in terms of the inverse Fourier transform as

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df = \int_{-\infty}^{\infty} |X(f)|\, e^{+j(2\pi f t + \varphi(f))}\, df$$

- **Interpretation:** any signal $x(t)$ can be decomposed into sinusoidal components at different frequencies and phase offsets
- The magnitude $|X(f)|$ measures the strength of the signal component at frequency f
- Assuming $x(t)$ is a real-valued signal this can be written as

$$x(t) = 2\int_{0}^{\infty} |X(f)|\, \cos(2\pi f t + \varphi(f))\, df$$

and it can be shown that

$$|X(f)| = |X(-f)|\ , \text{ (even)} \quad \varphi(f) = -\varphi(-f)\ , \text{ (odd)}$$

Example: rectangular pulse

- Let us compute the Fourier transform of the following signal:

$$x_{rec}(t) = \begin{cases} A & -\frac{T}{2} \le t \le \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

- We get

$$X_{rec}(f) = \mathcal{F}\{x_{rec}(t)\} = \int_{-\infty}^{\infty} x_{rec}(t)\ e^{-j2\pi f t}\ dt$$

$$= \int_{-T/2}^{+T/2} A\ e^{-j2\pi f t}\ dt\ =\ \left[-\frac{A e^{-j2\pi f t}}{j2\pi f} \right]_{-T/2}^{+T/2}$$

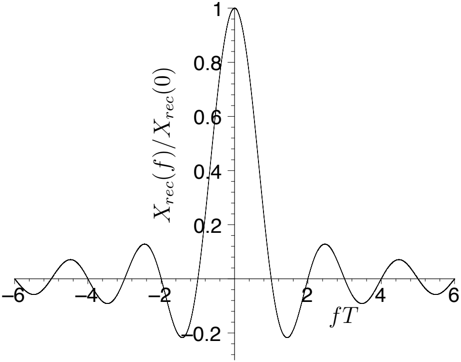
$$= \frac{A}{\pi f}\ \frac{e^{j\pi f T} - e^{-j\pi f T}}{2j} = AT\ \frac{\sin(\pi f T)}{\pi f T}$$

- We have found that

$$x_{rec}(t) \longleftrightarrow AT\ \frac{\sin(\pi f T)}{\pi f T} = AT \operatorname{sinc}(fT)$$

Notation: $x(t) \longleftrightarrow \mathcal{F}\{x(t)\}$

Example 2.17: sketch of $X_{rec}(f)$



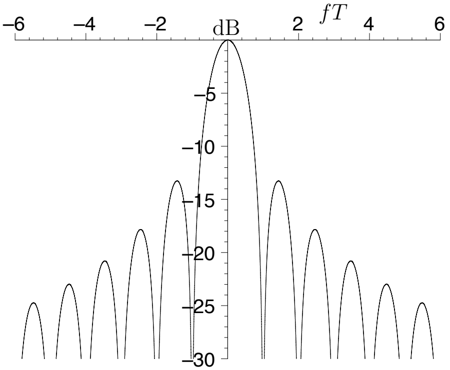
- the Fourier transform $X(f)$ is centered around $f = 0$: baseband
- we observe a main-lobe and several side-lobes
- Note: $fT = 2$ means that $f = 2 \cdot 1/T$

Sketch the function for $T = 10^{-6}\,s$ and $T = 2 \cdot 10^{-6}\,s$

Example 2.17: sketch of $|X_{rec}(f)|^2$

- Consider now the normalized energy spectrum in dB

$$10\log_{10}\left(\frac{|X_{rec}(f)|^2}{E_x T}\right) = 10\log_{10}\left(\frac{\sin(\pi f T)}{\pi f T}\right)^2$$



$\Rightarrow$ most energy is contained in the main-lobe (90.3 %)

Fourier transform of time-shifted signals

- Did you notice the difference between $x_{rec}(t)$ in this example and the elementrary pulse $g_{rec}(t)$ which we used last week?

$$x_{rec}(t) = \begin{cases} A & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}, \qquad g_{rec}(t) = \begin{cases} A & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

- The pulse $g_{rec}(t) = x_{rec}(t - T/2)$ is a time-shifted version of $x_{rec}(t)$
- In general, the Fourier transform of a signal $y(t) = x(t - t_d)$ with a constant delay t_d becomes

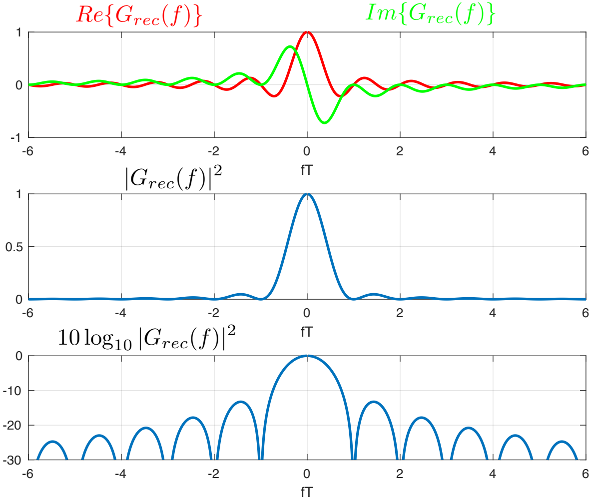
$$Y(f) = \int_{-\infty}^{\infty} x(t - t_d)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f (\tau + t_d)}\, d\tau = X(f)\, e^{-j2\pi f t_d}$$

- **Observe:** the delay t_d changes only the phase of $Y(f)$
- The energy spectrum is not affected by time-shifts

$$|X_{rec}(f)|^2 = |G_{rec}(f)|^2 \quad \text{(compare App. D.1)}$$

A simple Matlab exercise

Let us plot the spectrum of the pulse $g_{rec}(t)$



A simple Matlab exercise

And this is how it was done:

```
% Example: rect pulse spectrum


x=-6:0.01:6;
G=sin(pi.*x)./(pi.*x).*exp(-j*pi*x); % T=1

figure(2)
subplot(3,1,1);
plot(x,real(G),'r',x,imag(G),'g'); xlabel('fT');
grid on;

subplot(3,1,2);
plot(x,abs(G).^2); xlabel('fT');
grid on;

subplot(3,1,3);
plot(x,10.*log10(abs(G).^2)); xlabel('fT');
set(gca,'YLim',[-30 0]);
grid on;
```

script Ln 13 Col 34

Fourier transform of other pulses

- The Fourier transforms $G(f)$ and sketches of the energy spectra $|G(f)|^2$ are given for a number of different elementary pulses $g(t)$ in Appendix D
- **Example:** half cycle sinusoidal pulse

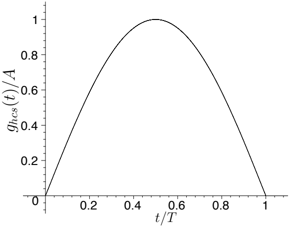


Figure D.7: $g_{hcs}(t)/A$.

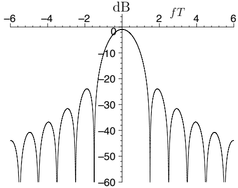


Figure D.8: $\frac{|G_{hcs}(f)|^2}{E_g T}$ in dB.

$$g_{hcs}(t) = \begin{cases} A\sin(\pi t/T) & , \quad 0 \le t \le T \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$E_g = A^2T/2$$

$$G_{hcs}(f) = \mathcal{F}\{g_{hcs}(t)\} = \frac{2AT}{\pi} \frac{\cos(\pi fT)}{1-(2fT)^2} e^{-j\pi fT}$$

$$G_{hcs}(f = \pm 1/2T) = \mp jAT/2$$

$$G_{hcs}(n/T) = 0 \text{ if } n = \pm 3/2, \pm 5/2, \pm 7/2, \ldots$$

Frequency shift operations

- We have seen the effect of a time shift on the Fourier transform

$$g(t-t_d) \longleftrightarrow G(f)\, e^{-j2\pi f t_d}$$

- In a similar way we can characterize a frequency shift f_c by

$$g(t)\, e^{j2\pi f_c t} \longleftrightarrow G(f-f_c)$$

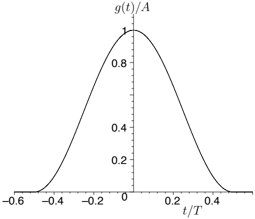
- Let us make use of the relation $e^{j2\pi f_c t} = \cos(2\pi f_c t) + j\, \sin(2\pi f_c t)$
- We can now express this in terms of cosine and sine functions,

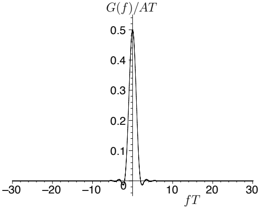
$$g(t)\, \cos(2\pi f_c t) \longleftrightarrow \frac{G(f+f_c)+G(f-f_c)}{2}$$

$$g(t)\, \sin(2\pi f_c t) \longleftrightarrow j\, \frac{G(f+f_c)-G(f-f_c)}{2}$$

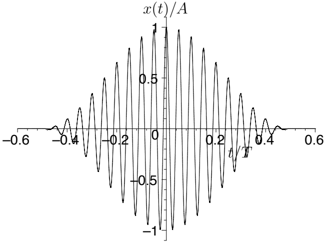
$\Rightarrow$ by simply changing the carrier frequency f_c we can move our signals to a suitable location along the frequency axis

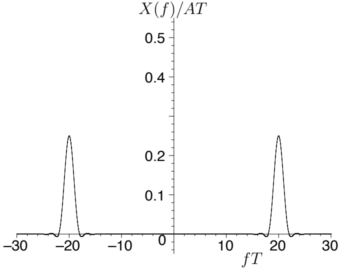
Example: time raised cosine pulse





$$x(t) = g(t) \cdot \cos(2\pi f_c t) = g_{rc}(t + T/2) \cdot \cos(2\pi f_c t) \ , \quad f_c = 20/T$$





Back to the transmitted signal

- We have seen how the Fourier transform can be used to calculate the energy spectrum $|X(f)|^2$ of a given signal $x(t)$
- Let us now look at the transmitted signal for M-ary modulation

$$s(t) = s_{m[0]}(t) + s_{m[1]}(t - T_s) + s_{m[2]}(t - 2T_s) + \cdots = \sum_{i=0}^{\infty} s_{m[i]}(t - i\,T_s)$$

- Message $m[i]$ selects the signal alternative to be sent at time iT_s
- Since the information bit stream is random, the transmitted signal $s(t)$ consists of a sequence of random signal alternatives

How can we determine the bandwidth W of the transmitted signal?

Does the information sequence influence the spectrum? How?

Power Spectral Density

- Since the signal has no predefined length the energy is not a good measure (could be infinite according to our model)
- On the other hand, we know that the signal has finite power
- The power spectral density $R(f)$ shows how the average signal power $\overline{P}$ is distributed along the frequency axis on average

$$\overline{P} = \overline{E}_b R_b = \int_{-\infty}^{\infty} R(f) \, df$$

- Most of the average signal power $\overline{P}$ $[\mathrm{V}^2]$ will be contained within the main-lobe of $R(f)$ $[\mathrm{V}^2/\mathrm{Hz}]$

  $\Rightarrow$ we can determine the signal bandwidth from $R(f)$

Our aim is to find $R(f)$ for a given modulation order M and set of M signal alternatives (constellation)

Power Spectral Density

Assumptions:

- The random M-ary sequence of messages $m[i]$ consists of independent, identically distributed (i.i.d) M-ary symbols
- The probability for each of the $M = 2^k$ symbols (messages) is denoted by $P_\ell, \ell = 0, 1, \ldots, M-1$
- All signal alternatives $s_\ell(t)$ in the constellation have finite energy
- The average signal over all signal alternatives is denoted $a(t)$, i.e.,

$$a(t) = \sum_{\ell=0}^{M-1} P_\ell \, s_\ell(t)$$

$$A(f) = \sum_{n=0}^{M-1} P_n \, S_n(f)$$

Remark: Source coding (compression) can be used to remove or reduce correlations in the information stream

$R(f)$: Main Result

- The power spectral density $R(f)$ can be divided into a continuous part $R_c(f)$ and a discrete part $R_d(f)$

$$R(f) = R_c(f) + R_d(f)$$

- The general expression for the continuous part is

$$R_c(f) = \frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f) - A(f)|^2$$
$$= \left(\frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 \right) - \frac{|A(f)|^2}{T_s}$$

- For the discrete part we have

$$R_d(f) = \frac{|A(f)|^2}{T_s^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_s)$$

$R(f)$: Main Result

- Assume now that the average signal $a(t) = 0$ for all t
- It follows that $A(f) = 0$ for all f
- This simplifies the result to

$$R(f) = R_c(f) = R_s \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 = R_s \, E\{|S_{m[n]}(f)|^2\}$$

- These **general results** can also be used to study the consequences that technical errors or impairments in the transmitter can have on the frequency spectrum
- We will now consider various **special cases** used in practice

$R(f)$: Binary Signaling

- In the general binary case, i.e., $M = 2$, we have

$$A(f) = P_0\ S_0(f) + P_1\ S_1(f)$$

- This simplifies the expression for the power spectral density to

$$\begin{aligned} R(f) &= R_c(f) && + R_d(f) \\ &= \frac{P_0 P_1}{T_b} |S_0(f) - S_1(f)|^2 && + \frac{|P_0\ S_0(f) + P_1\ S_1(f)|^2}{T_b^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_b) \end{aligned}$$

(derivation in Ex. 2.20)

- We will now consider some examples from the compendium

Example 2.21

Assume equally likely antipodal signal alternatives, such that

$$s_1(t) = -s_0(t) = g(t)$$

where $g(t) = g_{rec}(t)$, *and* $g_{rec}(t)$ *is given in (D.1). Assume also that* $T \leq T_b$.

i) *Calculate the power spectral density* $R(f)$.

ii) *Calculate* ***the bandwidth*** $\boldsymbol{W}$ ***defined as the one-sided width of the mainlobe of*** $\boldsymbol{R(f)}$, *if the information bit rate is 10* [kbps], *and if* $T = T_b/2$.
Calculate also the bandwidth efficiency ρ.

iii) *Estimate the attenuation in* dB *of the first sidelobe of* $R(f)$ *compared to* $R(0)$.

- $M = 2$ with equally likely antipodal signaling $s_1(t) = -s_0(t) = g(t)$
- With $P_0 = P_1 = 1/2$ and $S_1(f) = -S_0(f) = G(f)$ we get

$$R(f) = R_b \left|S_1(f)\right|^2 = R_b \left|S_0(f)\right|^2 = R_b \left|G(f)\right|^2$$

- Details for the pulse in Appendix D

Example 2.23

Assume equally likely antipodal signal alternatives below. Assume that $s_1(t) = -s_0(t) = g_{rc}(t)$, *where the time raised cosine pulse* $g_{rc}(t)$ *is defined in (D.18). Assume also that* $T = T_b$.

Find an expression for the power spectral density $R(f)$. *Calculate the bandwidth* W, *defined as the one-sided width of the mainlobe of* $R(f)$, *if* R_b *is 10* [kbps]. *Calculate also the bandwidth efficiency* ρ.

- Same as Example 2.21, but with $g_{rc}(t)$ pulse
- Analogously we get

$$R(f) = R_b \left|G_{rc}(f)\right|^2$$

- From the one-sided main-lobe we get

$$W = 2/T \text{ [Hz]}$$

- Bandwidth efficiency $\rho = 1/2$ [bps/Hz] is the same (why?)

Example 2.24

Assume $P_0 = P_1$ *and that,*

$$s_1(t) = -s_0(t) = g_{rc}(t)\cos(2\pi f_c t)$$

with $T = T_b$*, and* $f_c \gg 1/T$*. Hence, a version of binary PSK signaling is considered here (alternatively binary antipodal bandpass PAM). Calculate* ***the bandwidth*** $\boldsymbol{W}$***, defined as the double-sided width of the mainlobe around the carrier frequency*** $\boldsymbol{f_c}$***.*** *Assume that the information bit rate is 10* [kbps]*. Calculate also the bandwidth*

- This corresponds to the bandpass case
- Let $g_{hf}(t)$ denote the high-frequency pulse

$$g_{hf}(t) = g_{rc}(t)\cos(2\pi f_c t) \quad \text{and} \quad R(f) = R_b\,|G_{hf}(f)|^2$$

- Using shift operations we get

$$R(f) = R_b \left| \frac{G_{rc}(f+f_c)}{2} + \frac{G_{rc}(f-f_c)}{2} \right|^2$$

- From the two-sided main-lobe we get

$$W = 4/T \text{ [Hz]}$$

Short summary

- Energy spectrum: $|X(f)|^2$ shows how the energy E_x of a given signal $x(t)$ is distributed across different frequencies f
- $X(f)$ is computed by the Fourier transform of $x(t)$. The Fourier transform $G(f)$ of different elementary pulses $g(t)$ can be found in Appendix D
- Get familiar with the Fourier transform of time-shifted signals and the frequency shift operations
- The power spectral density $R(f)$ shows how signal power is distributed across f if random messages $m[i]$ are sent
- A general formula for $R(f)$ is given on pages 77-78, followed by a series of examples