EITG05 – Digital Communications

Lecture 11

Intersymbol Interference
Nyquist condition, Spectral raised cosine, Equalizers

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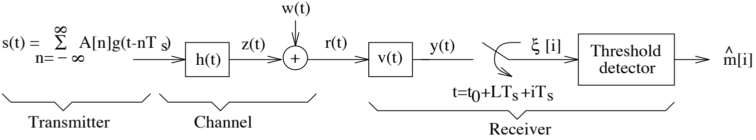
Thursday, October 10, 2019

Intersymbol Interference (ISI)

- For $R_s = 1/T_s < 1/T_u$ we can use the ML receiver from Chapter 4
- **Question:** can we use such a receiver for larger rates $R_s \geq 1/T_u$?
- Consider the following receiver structure (compare to last slide)



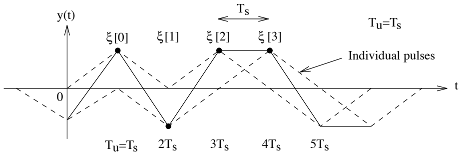
- Note that $z(t)$ now is a superposition of overlapping pulses $u(t)$
- The signal $y(t)$ after the receiver filter $v(t)$ is

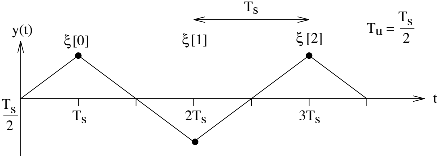
$$y(t) = \sum_{n=-\infty}^{\infty} A[n]\, x(t - n\,T_s) + w_c(t) \ ,$$

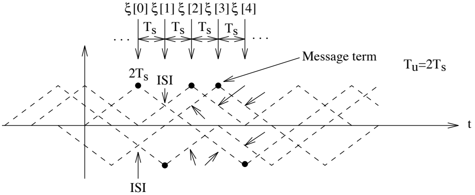
  where $w_c(t)$ is a filtered Gaussian process
- The decision variable is obtained after sampling

$$\xi[i] = y(\mathcal{T} + i\,T_s) \ , \quad \mathcal{T} = t_0 + LT_s \ , \text{ where } LT_s \geq T_u$$

Illustration of ISI in the receiver







Discrete time model for ISI

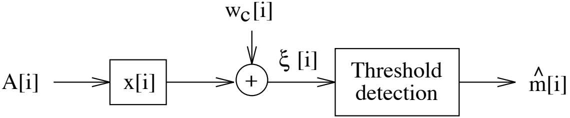
- According to our model the decision variable can be written as

$$\xi[i] = y(\mathcal{T} + iT_s) = \sum_{n=-\infty}^{\infty} A[n]\, x(\mathcal{T} + iT_s - nT_s) + w_c(\mathcal{T} + iT_s)$$

- Let us introduce the discrete sequences

$$x[i] = x(\mathcal{T} + iT_s)\ , \quad w_c[i] = w_c(\mathcal{T} + iT_s)$$

- This leads to the following discrete-time model of our system

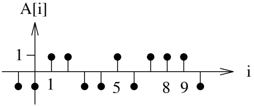


$$\xi[i] = \sum_{n=-\infty}^{\infty} A[n]\, x[i-n] + w_c[i] = A[i] * x[i] + w_c[i]$$

Remark: the discrete-time impulse response $x[i]$ represents pulse shape $g(t)$, channel filter $h(t)$, and receiver filter $v(t)$

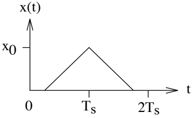
Example 6.1

The transmitted sequence of amplitudes $A[i]$ is given as,

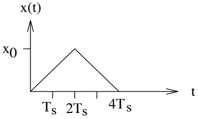


Calculate, and plot, the sequence of decision variables $\xi[i]$ in Figure 6.2, for $0 \leq i \leq 8$, in the noiseless case (i.e. $w(t) = 0$) if $t_0 = 0$ and if the output pulse $x(t)$ is:

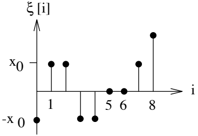
i) L=1 and x(t) as below.



ii) L=2 and x(t) as below.



- i) $\xi[i] = x_0 A[i]$ ii) $\xi[i] = \frac{x_0}{2} A[i+1] + x_0 A[i] + \frac{x_0}{2} A[i-1]$

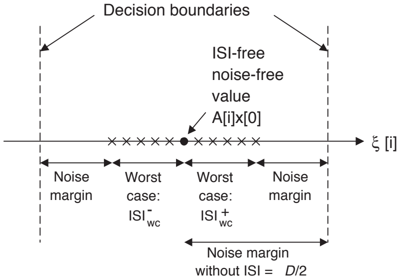


How much ISI can we tolerate?

- We can divide the decision variable $\xi[i]$ into a desired term (message) and an undesired term (interference plus noise)

$$\xi[i] = \underbrace{A[i]\,x[0]}_{\text{message}} + \underbrace{\sum_{\substack{n=-\infty \\ n\neq i}}^{\infty} A[n]\,x[i-n]}_{\text{ISI}} + \underbrace{w_c[i]}_{\text{noise}}$$

- The influence of ISI depends on its relative strength



Worst case ISI

- The ISI term can be written as

$$ISI = \sum_{\substack{n=-\infty \\ n\neq i}}^{\infty} A[n]\, x[i-n] = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A[i-n]\, x[n]$$

- **Question:** when does this term become largest?
- For symmetric M-ary PAM we have $\max|A[i]| = M-1$ and get

$$ISI_{wc}^{+} = \max(ISI) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \max\left(A[i-n]\, x[n]\right) = (M-1) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |x[n]|$$

- Similarly, the worst case minimal ISI becomes

$$ISI_{wc}^{-} = \min(ISI) = -(M-1) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |x[n]|$$

Observe: the worst case ISI occurs for a information sequence $A[i]$ consisting of a particular pattern of $\pm(M-1)$ values

Condition for ISI free reception

- Let us assume that $x[i]$ satisfies the following condition:

$$x[i] = x(\mathcal{T} + i\,T_s) = x_0\,\delta[i] = \begin{cases} x_0 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

- Then

$$\xi[i] = \sum_{n=-\infty}^{\infty} A[n]\,x[i-n] + w_c[i] = A[i]\,x[0] \; + w_c[i]$$

- Otherwise there always will exist some non-zero ISI term
- For this reason we are interested in signals

$$x(t) = g(t) \; * \; h(t) \; * \; v(t)$$

for which the above condition is satisfied

Which parts of $x(t)$ can we influence?

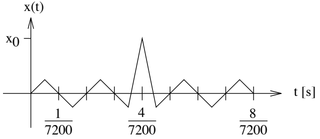
Symbol rates for ISI free reception

- Suppose that the ISI free condition is satisfied for symbol rate R_s^*
- Then it will be satisfied for rates

$$R_s = \frac{R_s^*}{\ell}\ , \quad \ell = 1, 2, 3, \ldots$$

Example 6.6:

Consider the overall pulse shape $x(t)$ below, and $\mathcal{T} = 4/7200$.



Assume the bitrate 14400 [b/s] and 16-ary PAM signaling. Does ISI occur in the receiver?

Representation in frequency domain

- The discrete sequence $x[i]$ can be obtained by sampling a non-causal pulse $x_{nc}(t)$ at times iT_s,

$$x[i] = x_{nc}(iT_s)\ , \quad \text{where } x_{nc}(t) = x(\mathcal{T} + t)\ ,$$

- The Fourier transform $\mathcal{X}(\nu)$ of $x[i]$ can then be expressed in terms of the Fourier transform $X_{nc}(f)$ of the signal $x_{nc}(t)$:

$$\mathcal{X}(\nu) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi\nu n} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X_{nc}\left(\frac{\nu - n}{T_s}\right)\ ,$$

where

$$X_{nc}(f) = \int_{-\infty}^{\infty} x_{nc}(t) e^{-j2\pi f t}\, dt = G(f)\, H(f)\, V(f)\, e^{+j2\pi f \mathcal{T}}$$

Observe: the spectrum of the sampled sequence $x[i]$ consists of the periodically repeated spectrum of the continuous signal

Nyquist condition in frequency domain

- Let us now formulate the ISI free condition in frequency domain:

$$x[i] = x_0\,\delta[i] \quad \Rightarrow \mathcal{X}(\nu) = \mathcal{F}\{x[i]\} = x_0 \quad \forall\,\nu$$

- Choosing $\nu = f\,T_s$ this leads to the equivalent Nyquist condition

$$\frac{\mathcal{X}(f\,T_s)}{R_s} = \sum_{n=-\infty}^{\infty} X_{nc}(f - n\,R_s) = \frac{x_0}{R_s}\;, \quad R_s = \frac{1}{T_s}$$

- Let W_{lp} denote the baseband bandwidth of $x_{nc}(t)$,

$$X_{nc}(f) = 0, \quad |f| > W_{lp}$$

- Then ISI always will be present if the symbol rate satisfies

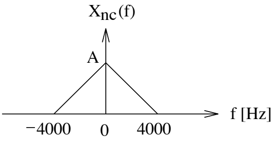
$$R_s > 2\,W_{lp}$$

(non-overlapping spectrum cannot add up to a constant)

- If we have $R_s \leq 2\,W_{lp}$:
ISI-free reception is possible if $X_{nc}(f)$ has a proper shape

Example 6.7

Assume that $X_{nc}(f)$ is given below.



a) *Sketch the left hand side of (6.33),* $\sum_{n=-\infty}^{\infty} X_{nc}(f - nR_s)$, *if* $R_s = 12000$ *symbols per second.*

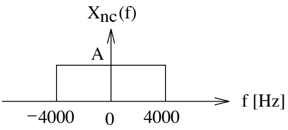
b) *Does ISI occur in the receiver?*

What happens if $R_s = 8000$?

And $R_s = 4000$?

Example 6.8

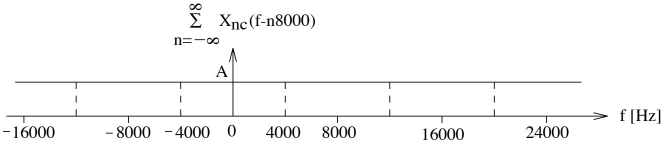
Assume that $X_{nc}(f)$ *is,*



$A = x_0 T_s$.

Show that there is no ISI if the symbol rate is $R_s = 8000$ *[symbol/s].*

Solution:



Since $\sum_{n=-\infty}^{\infty} X_{nc}(f - n8000) = x_0/R_s$, *for all* f, *there is no ISI in the receiver.*

Ideal Nyquist pulse

- The maximum possible signaling rate for ISI-free reception is

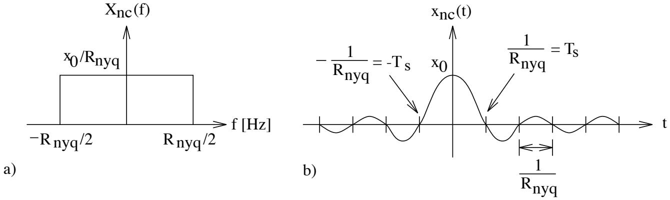
$$R_{nyq} = R_s = \frac{1}{T_s} = 2\,W_{lp} \quad \text{(Nyquist rate)}$$

- With ideal Nyquist signaling, the bandwidth efficiency is

$$\rho_{nyq} = \frac{R_b}{W_{lp}} = \frac{R_{nyq}\,\log_2(M)}{R_{nyq}/2} = 2\log_2 M = 2\,k \;\text{[bps/Hz]}$$

- The ideal Nyquist pulse must have rectangular spectrum

$$X_{nc}(f) = \begin{cases} x_0/R_{nyq}\,, & \text{if } |f| \le R_{nyq}/2 \\ 0\,, & \text{else} \end{cases} \qquad \Rightarrow x_{nc}(t) = x_0\,\frac{\sin(\pi R_{nyq}\,t)}{\pi R_{nyq}\,t}$$



Some comments on bandwidth

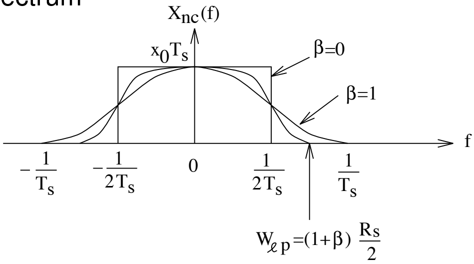
- **Remember:** in Chapter 2 we have seen that strictly band-limited signals always have to be unlimited in time
- In practice we have to find compromises, which was leading to different definitions of bandwidth for time-limited signals

Pulse shape	W_{lobe}	% power in W_{lobe}	W_{90}	W_{99}	$W_{99.9}$	Asymptotic decay
rec	2/T	90.3	1.70/T	20.6/T	204/T	f^{-2}
tri	4/T	99.7	1.70/T	2.60/T	6.24/T	f^{-4}
hcs	3/T	99.5	1.56/T	2.36/T	5.48/T	f^{-4}
rc	4/T	99.95	1.90/T	2.82/T	3.46/T	f^{-6}
Nyquist	R_s	100	$0.9R_s$	$0.99R_s$	$0.999R_s$	ideal

- We can see that time-limited signals need at least about twice the Nyquist bandwidth
- For OFDM with many sub-carriers N this is negligible (why?)
- For single-carrier systems, some close-to-Nyquist pulses are typically used in practice

Spectral Raised Cosine Pulses

- The spectral raised cosine pulse shape is defined by the following spectrum



- The name refers to the way the shape is composed

$$X_{nc}(f) = \begin{cases} x_0\, T_s\ , & 0 \leq |f| \leq \frac{1-\beta}{2T_s} \\ \frac{x_0\, T_s}{2}\left[1+\cos\left(\frac{\pi |f| T_s}{\beta} - \frac{\pi}{2}\cdot\frac{1-\beta}{\beta}\right)\right]\ , & \frac{1-\beta}{2T_s} \leq |f| \leq W_{lp} \\ 0 & |f| > W_{lp} \end{cases}$$

where $W_{lp} = \dfrac{1+\beta}{2T_s} = (1+\beta)\dfrac{R_s}{2}\ , \quad 0 \leq \beta \leq 1$

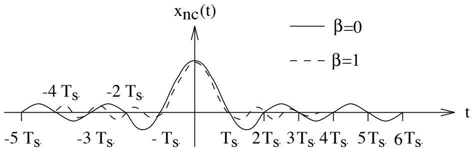
Spectral Raised Cosine Pulses

- The parameter β, $0 \leq \beta \leq 1$, is called the rolloff factor and can be used to smoothly control the bandwidth efficiency

$$\rho_{src} = \frac{R_b}{W_{lp}} = \frac{R_s \log_2 M}{(1+\beta)R_s/2} = \frac{2\log_2 M}{1+\beta} = \frac{2k}{1+\beta}$$

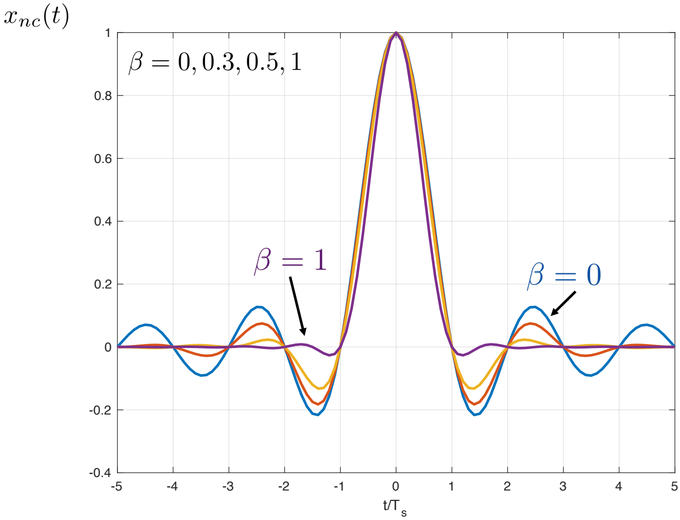
- In time domain the signal can be expressed as

$$x_{nc}(t) = x_0 \frac{\sin(\pi t/T_s)}{\pi t/T_s} \cdot \frac{\cos(\pi\beta t/T_s)}{1-(2\beta t/T_s)^2}\ , \quad -\infty \leq t \leq \infty$$



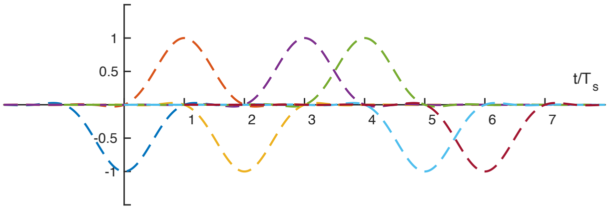
- Larger rolloff factors $\beta \Rightarrow$ faster amplitude decay of $x_{nc}(t)$

Spectral Raised Cosine Pulses

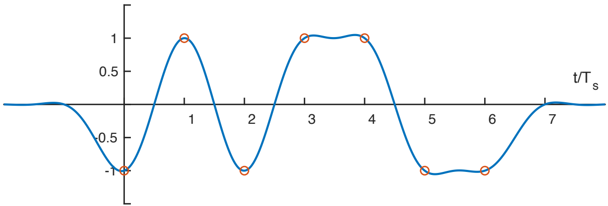


Signaling with overlapping pulses: $\beta = 1$

$A[n]\,x(t - nTs)$

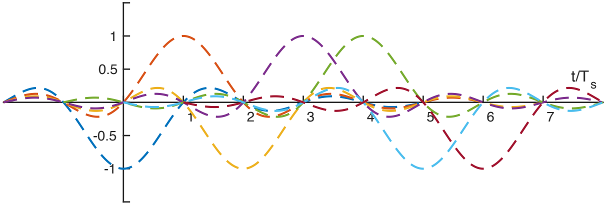


$y(t)$

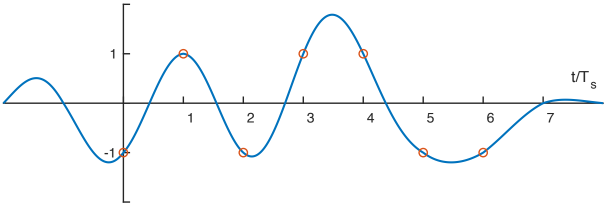


Signaling with overlapping pulses: $\beta = 0$

$A[n]\, x(t - nTs)$



$y(t)$



Spectral Root Raised Cosine Pulse

- When analyzing the Nyquist condition we have considered the output signal of the receiver filter $v(t)$, i.e.,

$$x_{nc}(t) = g(t) \ * \ h(t) \ * \ v(t) = u(t) \ * \ v(t)$$

- The matched filter for our receiver structure with delay $\mathcal{T} = LT_s$ should be equal to

$$v(t) = u(LT_s - t)$$

- As a consequence, we need to choose pulse shape $g(t)$ and receiver filter $v(t)$ in such a way that

$$|V(f)| = \sqrt{X_{nc}^{rc}(f)} \quad \text{and} \quad |G(f)\,H(f)| = \sqrt{X_{nc}^{rc}(f)}$$

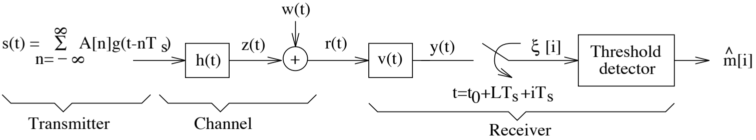
  in order to ensure a raised cosine spectrum for

$$X_{nc}(f) = |G(f)\,H(f)|^2 = |V(f)|^2 = X_{nc}^{rc}(f)$$

- Hence $v(t)$ is a pulse with root-raised cosine spectrum

Introduction to equalizers

- We have considered the receiver structure



- When ISI occurs this receiver is suboptimal and is no longer equivalent to the ML rule (sequence estimation, Viterbi algorithm)
- **Equalization:**
  instead of tolerating the ISI in the above structure, an equalizer can be used for removing (or reducing) the effect of ISI
- Linear equalizer: zero-forcing, MMSE
  can be implemented by linear filters, low complexity
- Decision feedback equalizer:
  non-linear device with feedback, aims at subtracting the estimated ISI from the signal

Introduction to equalizers