EITG05 – Digital Communications

Lecture 1

Introduction, Overview, Basic Concepts (p. 1–32)

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Monday, September 3, 2018

Digital Communications

We are in a global digital (r)evolution

- Mobile data and telephony (GSM, EDGE, 3G, 4G, 5G)
- Digital radio and television, Bluetooth, WLAN
- Data storage, CD, DVD, Flash, magnetic storage
- Optical fiber, DSL (long range, high rate)
- Cloud computing, big data, distributed storage
- Connected devices, Internet of things, machine-to-machine communication, distributed control, cyber physical systems

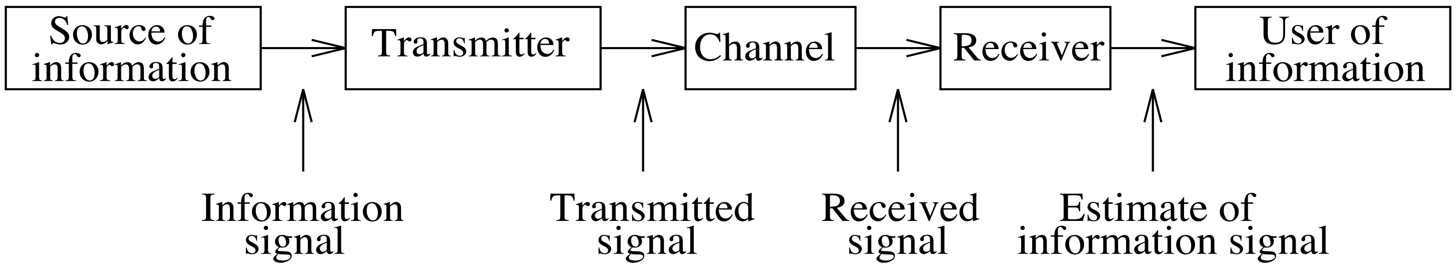
The large number of different application scenarios require flexible communication solutions (data rate / delay / reliability / complexity)

Remark storage of data falls also into the category of a communication system (why?)

What is communication?

- The purpose of a communication system is to transmit messages (information) from a source to a destination

  Examples: sound, picture, movie, text, etc.
- The messages are converted into signals that are suitable for transmission
- The physical medium for transmission is called the channel



- The received signal is used to estimate the messages

What are analog / digital signals?

Analog versus digital

- **Analog communication:**
  both source and processing are analog
- **Digital communication:**
  the source messages are digital, i.e., can be represented by discrete numbers (digits)

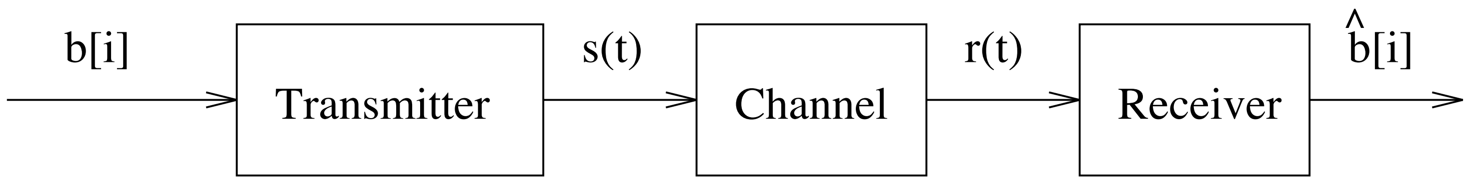
Example 1: I speak and you listen to the acoustic sound wave

Example 2: I record my speech to MP3 and send it to you, who plays it back on your computer or phone

Example 3: I use morse code and a flashlight to transmit a message to my neighbor

In all cases some analog medium has to be used during the transmission at some point

Scope of this course



- Transmitter principles: bits to analog signals (Chap. 2)
- Receiver principles: analog noisy signals to bits (Chap. 4,5,6)
- Characteristics of the communication link (Chap. 3,6)

Requirements:

- Data should arrive correctly at the receiver
- High bit rates are desirable
- Energy/power efficiency
- Bandwidth efficiency

What are the technical solutions and challenges?

Not in this course

- Analog to digital conversion, sampling theorem, quantization
  $\Rightarrow$ basic signals & systems or signal processing course
- Source coding (compression)
  $\Rightarrow$ covered in information theory course (elective)
- Channel coding (robust and reliable communication)
  $\Rightarrow$ covered in separate course (elective)
- Cryptography (secure communication)
  $\Rightarrow$ covered in separate course (elective))

There exist a large number of specialized courses that can be taken after this basic course.

There is also a project course in wireless communications.

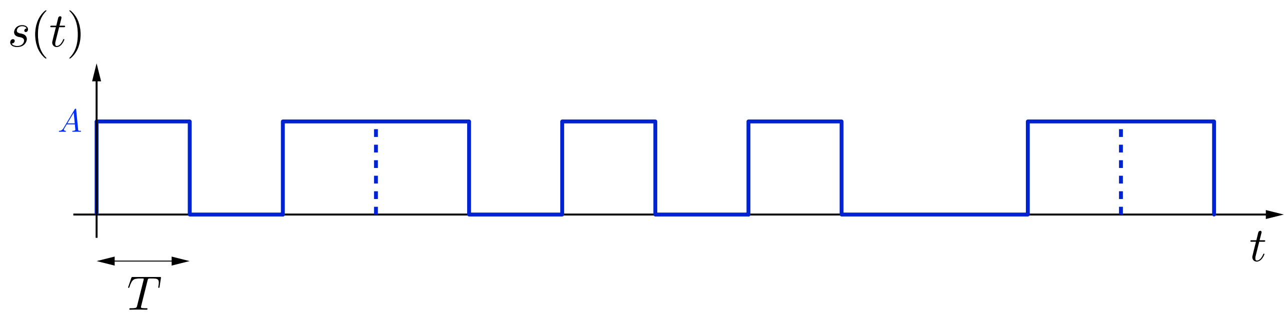
The Transmitter

How can we map digital data to analog signals?

$$b[i] = 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1$$

A simple approach:

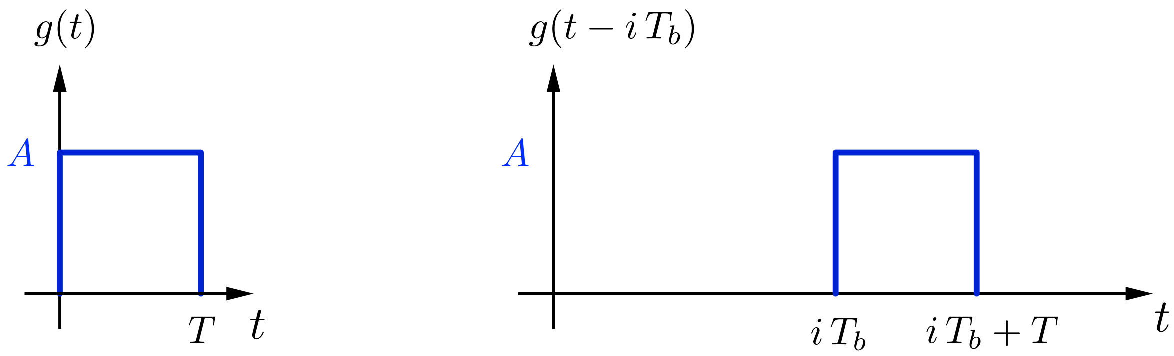
apply some voltage A during transmission of a 1



Basic operation: (more general)
represent the sequence of information bits $b[i]$ by a sequence of analog waveforms, resulting in the transmit signal $s(t)$

The Transmitter

- The analog waveform corresponding to the bit $b[i]$ can be written as a time-shifted version of an elementary pulse $g(t)$

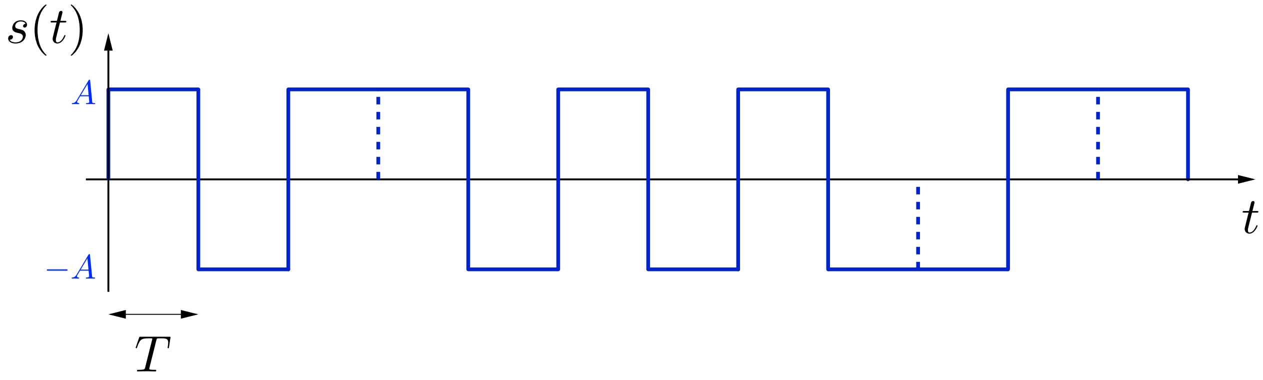


- T_b is the information bit interval, while T is the pulse duration
- For now we assume that $T \leq T_b$, i.e., the pulses do not overlap
- We can now represent the transmit sequence $s(t)$ as follows

$$s(t) = b[0]\,g(t) + b[1]\,g(t - T_b) + b[2]\,g(t - 2\,T_b) + \cdots$$

Variations of our signaling example

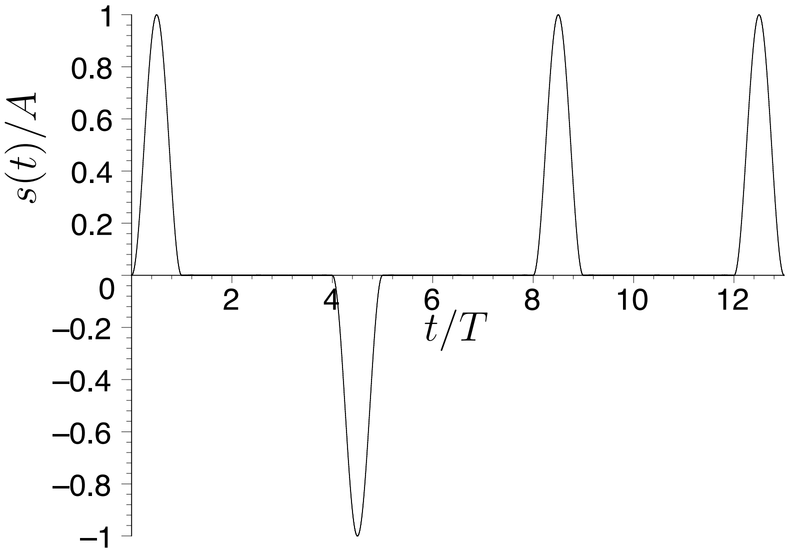
- In our example we only send a signal when $b[i] = 1$
  This modulation type is called on-off signaling
- Instead we could send a pulse with amplitude $-A$ for $b[i] = 0$:

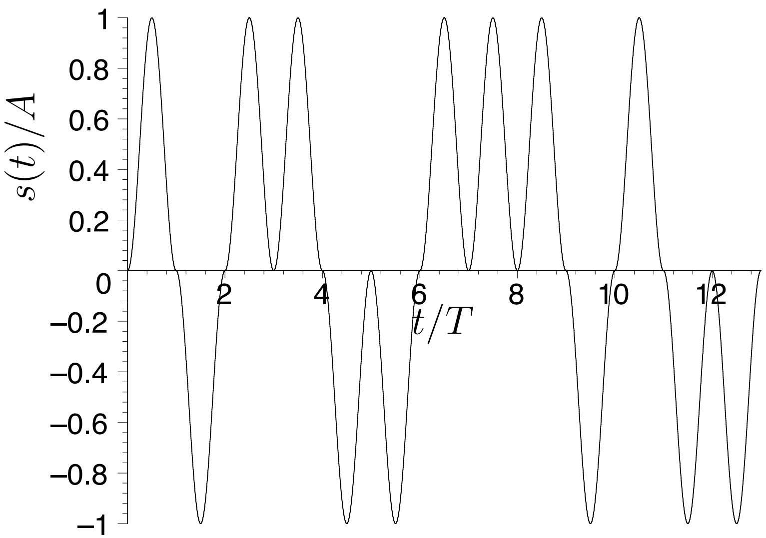


  This modulation type is called antipodal signaling
- We could also choose a different pulse shape $g(t)$

In this chapter: different modulation types and their properties

Another pulse example ($\rightarrow$ p. 10)

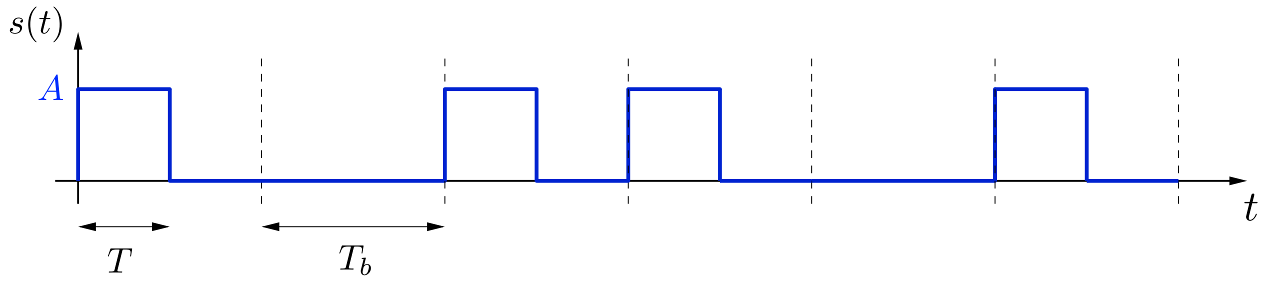




What are the input sequences $b[i]$ here?

What data rate can we achieve?

- We could also choose a shorter pulse, with $T < T_b$ (what for?)



- An important parameter is the information bit rate

$$R_b = \frac{B}{\tau} \text{ [bps] (bits per second)} ,$$

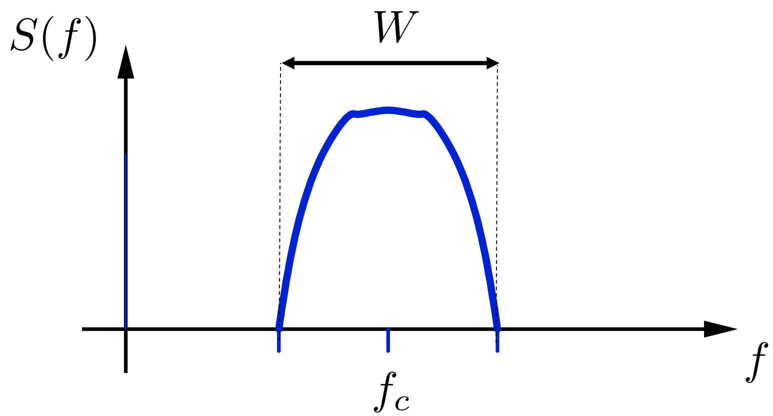
  if the source produces B information bits during τ seconds
- If we avoid overlapping pulses we need $T \leq T_b$ and

$$R_b = \frac{1}{T_b} \leq \frac{1}{T}$$

Observe: T determines the pulse length and T_b the rate

What bandwidth is required?

- The bandwidth W of the transmit signal is a valuable resource



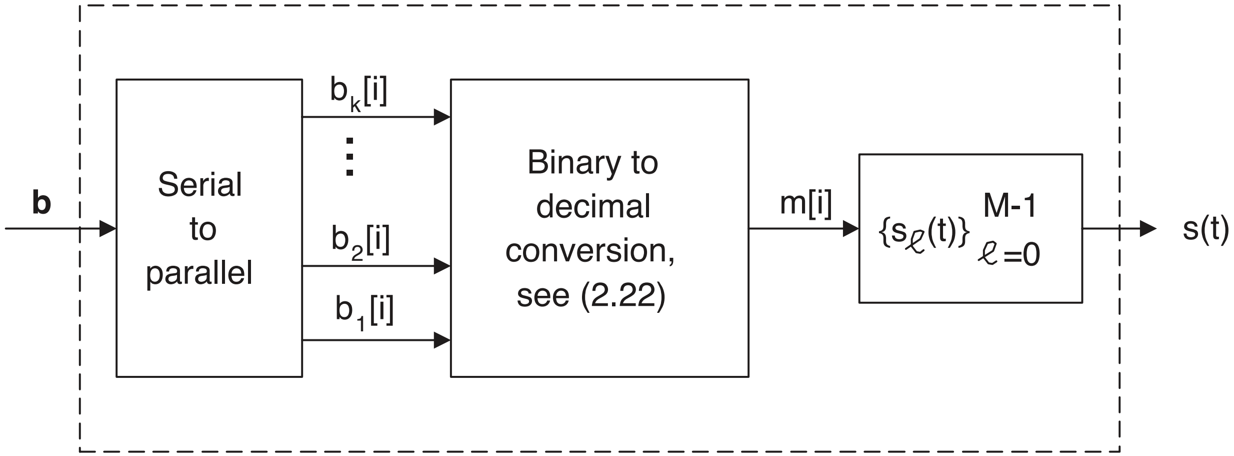
- For typical pulses $g(t)$ the bandwidth W is proportional to $\frac{1}{T}$
- More details about the bandwidth of $s(t)$ follow next week
- A challenging goal is to achieve a large bandwidth efficiency

$$\rho = \frac{R_b}{W} \quad \left[\frac{\mathrm{b/s}}{\mathrm{Hz}}\right]$$

Question: What happens when the pulse duration gets small?

Increasing the message alphabet

- Up to this point we have considered binary signaling only
- Each bit $b[i]$ was mapped to one of two signals $s_0(t)$ or $s_1(t)$
- More generally, we can combine k bits $b_1[i], b_2[i], \ldots b_k[i]$ to a single message $m[i]$, which then is mapped to a signal $s_\ell(t)$



- In case of M-ary signaling, one of $M = 2^k$ messages $m[i]$ is transmitted by its corresponding signal alternative

$$s_\ell(t) \in \{s_0(t), s_1(t), \ldots, s_{M-1}(t)\}$$

M-ary signaling

Example: $k = 2$, $M = 2^2 = 4$

The binary sequence

$$b_n[i] = 1 \quad 0 \quad 1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 1$$

is mapped by

$$m[i] = \sum_{n=1}^{k} b_n[i]\, 2^{n-1} = b_1[i] + b_2[i] \cdot 2$$

to $M = 4$ signal alternatives

$$b[i] = 00 \leftrightarrow m[i] = 0 \leftrightarrow s_0(t) \qquad b[i] = 10 \leftrightarrow m[i] = 1 \leftrightarrow s_1(t)$$
$$b[i] = 01 \leftrightarrow m[i] = 2 \leftrightarrow s_2(t) \qquad b[i] = 11 \leftrightarrow m[i] = 3 \leftrightarrow s_3(t)$$

The message sequence becomes

$$m[i] = 1 \quad 3 \quad 2 \quad 2 \quad 0 \quad 3$$

With $k = 14$ there are $M = 16\,384$ signal alternatives

Symbol rate versus bit rate

- Since k information bits are transmitted with each symbol, the symbol interval (symbol time) becomes

$$T_s = kT_b$$

- Accordingly, the symbol rate (signaling rate) is given by

$$R_s = \frac{1}{T_s}\left[\frac{\text{symbols}}{\text{s}}\right] = \frac{R_b}{k}$$

- When the message equals $m[i] = j$ then $s_j(t - iT_s)$ is sent

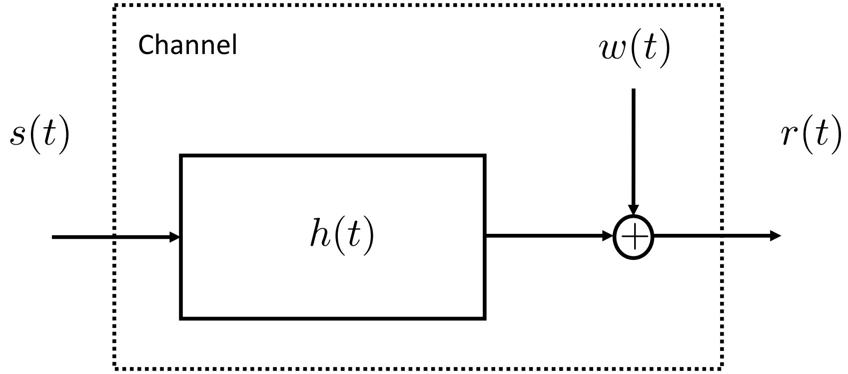
$$s(t) = s_{m[0]}(t) + s_{m[1]}(t - T_s) + s_{m[2]}(t - 2T_s) + \cdots$$

How does k affect the bandwidth efficiency ρ?

Remark: Be careful with the different definitions of time:
t: time variable T: pulse duration T_b: bit time T_s: symbol time

The Channel

- The channel is often modeled as time-invariant filter with noise



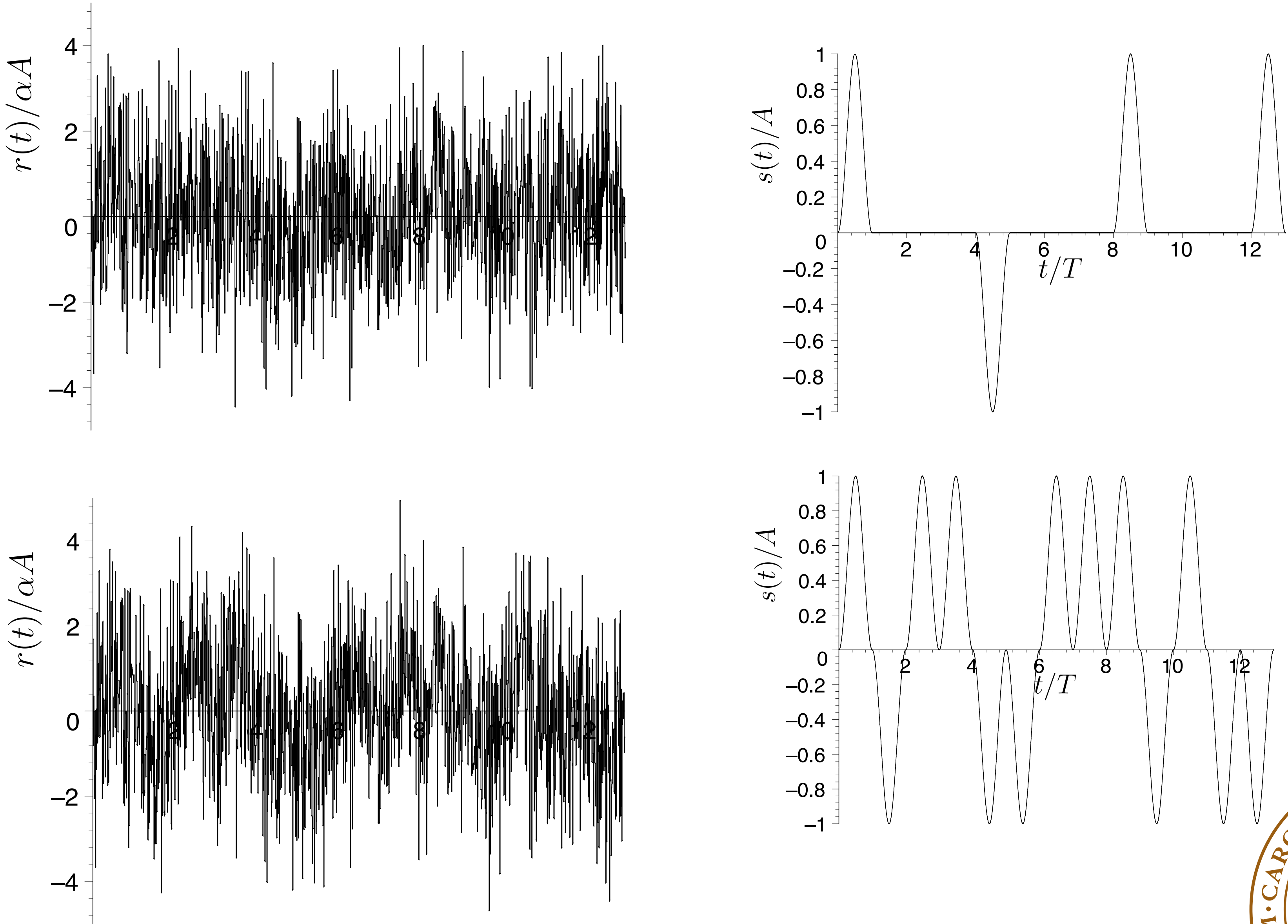
- $h(t)$ is the channel impulse response and $w(t)$ the additive noise
- The received signal becomes

$$r(t) = s(t) * h(t) + w(t) = \int_{-\infty}^{\infty} h(\tau)\, s(t-\tau)\, d\tau + w(t)$$

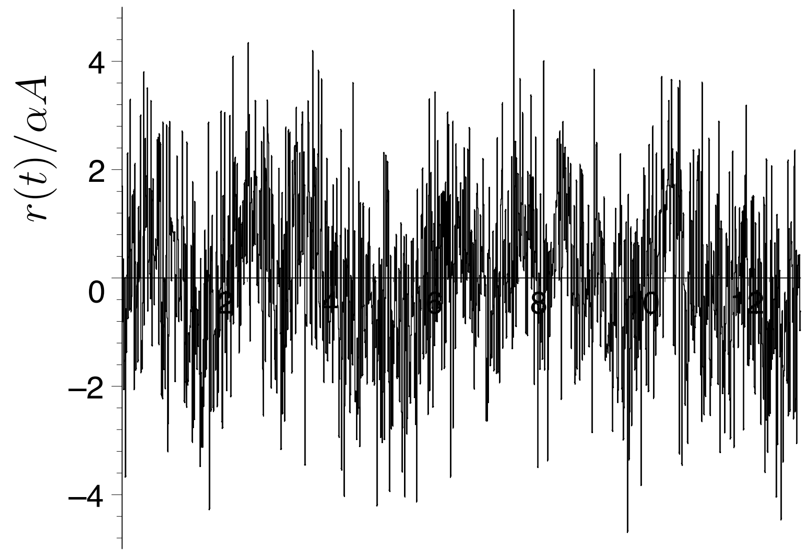
- For now we assume the simple case (α: attenuation)

$$h(t) = \alpha\, \delta(t) \qquad \Rightarrow r(t) = \alpha\, s(t) + w(t)$$

Example: noisy signal at the receiver (p. 13)



The Receiver



- Due to the attenuation α during transmission, the noise $w(t)$ has a strong impact on the received signal $r(t)$
- A well designed receiver can still detect the symbols correctly!
  In this example, only 1 of 10^5 bits will be wrong in average
- We will learn about the receiver and its performance later, in Chapters 4 and 5

Bit Errors

- The bit error probability is an important measure of communication performance
- It is defined as the average number of information bit errors per detected information bit

$$P_b = \frac{E\{B_{err}\}}{B}$$

Example:

- Assume a bit rate of 1 Mbps and that 10 bit errors occur per hour *on the average*. What is the bit error probability?
- The total number of bits in an hour is

$$B = 1\,000\,000 \cdot 60 \cdot 60 = 3.6 \cdot 10^9$$

This gives

$$P_b = \frac{10}{B} = 2.78 \cdot 10^{-9}$$

$\Rightarrow$ Computer simulations become very time consuming!

Signal energy and power

- The symbol energy E_ℓ of a signal alternative $s_\ell(t)$ is given by

$$E_\ell = \int_0^{T_s} s_\ell^2(t)\, dt < \infty \,, \qquad \ell = 0, 1, \ldots, M-1$$

- An important system parameter is the average symbol energy

$$\overline{E}_s = \sum_{\ell=0}^{M-1} P_\ell \, E_\ell \,, \qquad P_\ell = \Pr\{m[i] = \ell\}$$

and the average signal energy per information bit

$$\overline{E}_b = \frac{\overline{E}_s}{k}$$

- The average signal power is then given by

$$\overline{P} = R_s \overline{E}_s = \frac{R_b}{k} \cdot k \overline{E}_b = R_b \overline{E}_b$$

Signal energy and power

- The attenuation α and the noise $w(t)$ determine the quality of a communication link

$$r(t) = \alpha\, s(t) + w(t)$$

Example:
If a transmitted signal $s(t)$ has energy $\overline{E}_b$, how much energy $\mathcal{E}_b$ is then in the received signal $z(t) = \alpha \cdot s(t)$ if $\alpha = 0.001$?

- Using $z^2(t) = \alpha^2 s^2(t)$ we obtain

$$\overline{P}_z = \alpha^2\, \overline{P} = \alpha^2 R_b \overline{E}_b$$

$$\text{and} \quad \mathcal{E}_b = \frac{\overline{P}_z}{R_b} = \alpha^2 \frac{\overline{P}}{R_b} = \alpha^2 \overline{E}_b$$

- If $\alpha = 0.001$ then the power is reduced by a factor 10^{-6}

This will increase the bit error probability!

How well can we distinguish two signals?

- The squared Euclidean distance between two signals $s_i(t)$ and $s_j(t)$ is defined as

$$
\begin{aligned}
D_{i,j}^2 &= \int_0^{T_s} \left(s_i(t) - s_j(t)\right)^2 dt \\
&= \int_0^{T_s} s_i^2(t) + s_j^2(t) - 2\, s_i(t) s_j(t)\, dt \\
&= E_i + E_j - 2 \int_0^{T_s} s_i(t) s_j(t)\, dt
\end{aligned}
$$

- Two signals are antipodal if

$$
s_i(t) = -s_j(t) \,, \quad 0 \le t \le T_s
$$

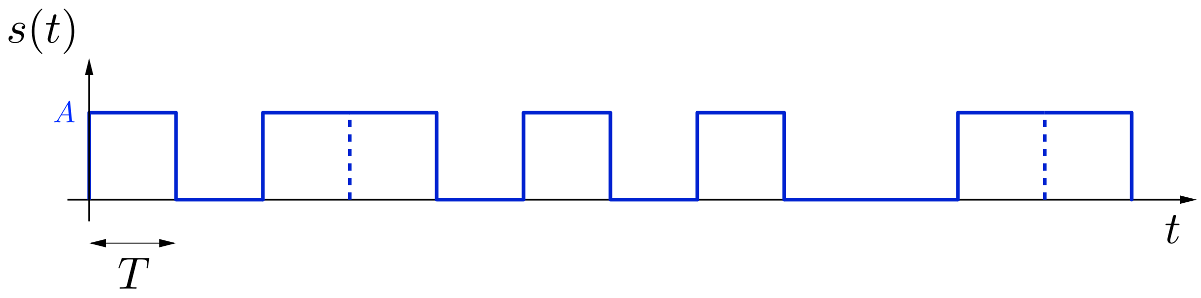
- Two signals are orthogonal if

$$
\int_0^{T_s} s_i(t) s_j(t)\, dt = 0
$$

Antipodal signals have larger Euclidean distance

Euclidean distance example $M = 2$

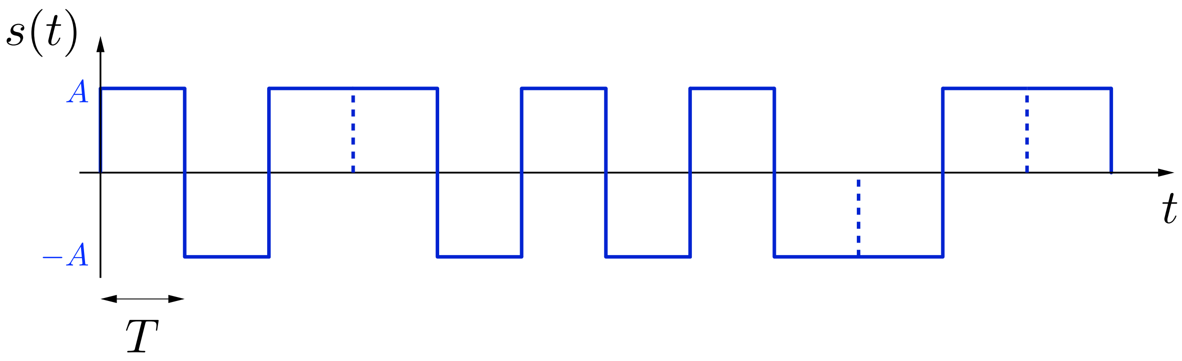
Case 1: on-off signaling



$s_0(t) = A$ and $s_1(t) = 0$ for $0 < t < T_s = T$, which gives $D^2_{0,1} = 2\overline{E}_b$

Observe: on-off signaling is orthogonal

Case 2: antipodal signaling



$s_0(t) = A$ and $s_1(t) = -A$ for $0 < t < T_s = T$, and $D^2_{0,1} = 4\overline{E}_b$

Examples of pulse shapes: Appendix D

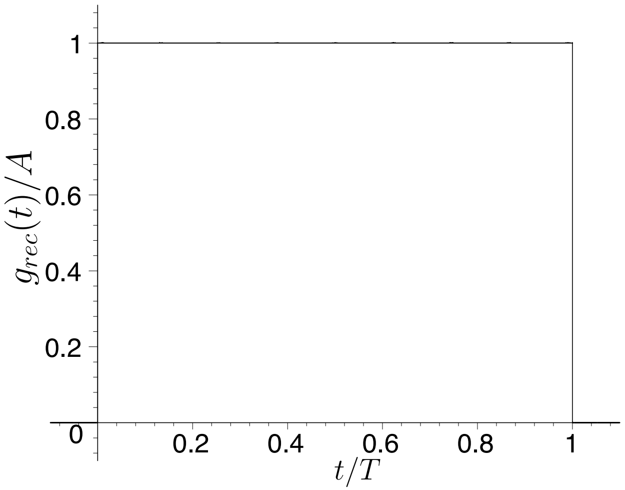


Figure D.1: $g_{rec}(t)/A$.

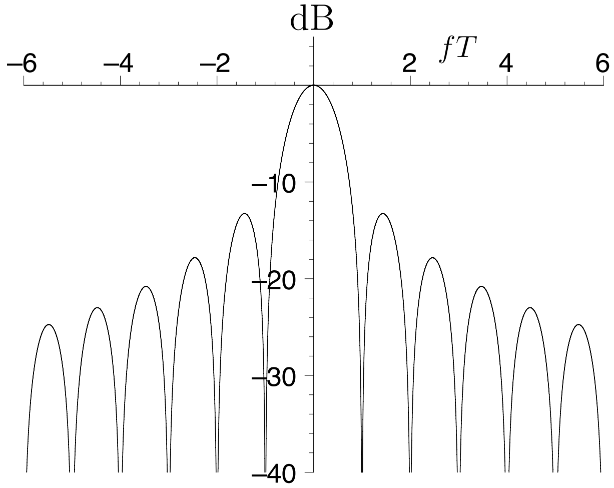


Figure D.2: $\frac{|G_{rec}(f)|^2}{E_g T}$ in dB.

1. The rectangular pulse:

$$g_{rec}(t) = \begin{cases} A & , \quad 0 \leq t \leq T \\ 0 & , \quad \text{otherwise} \end{cases} \tag{D.1}$$

$$E_g = \int_0^T g_{rec}^2(t)dt = \int_{-\infty}^{\infty} |G_{rec}(f)|^2 df = A^2 T \tag{D.2}$$

Examples of pulse shapes: Appendix D

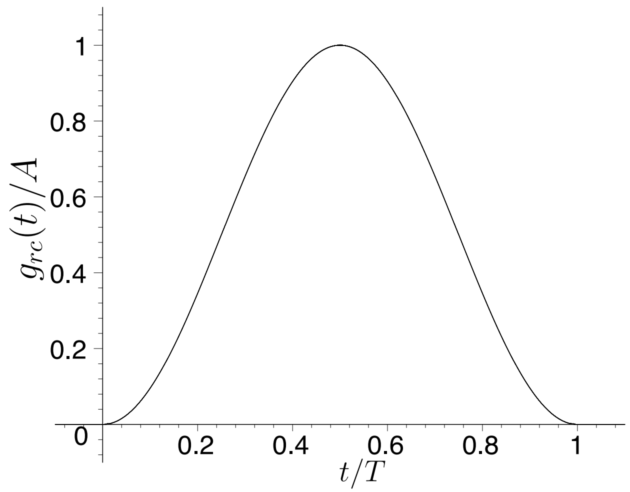


Figure D.9: $g_{rc}(t)/A$.

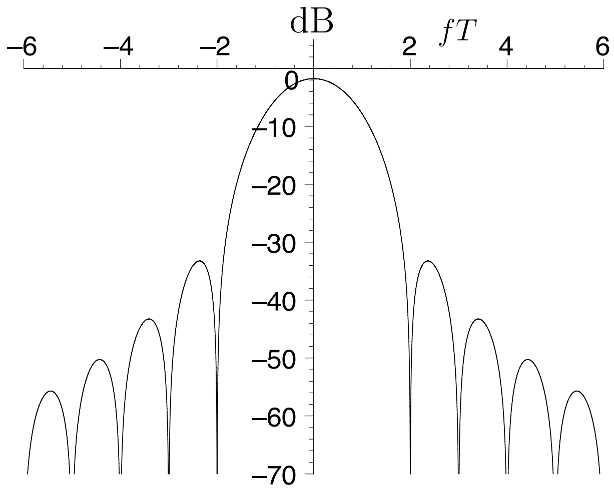


Figure D.10: $\frac{|G_{rc}(f)|^2}{E_g T}$ in dB.

5. The time raised cosine pulse:

$$g_{rc}(t) = \begin{cases} \frac{A}{2}\left(1 - \cos(2\pi t/T)\right) & , \quad 0 \le t \le T \\ 0 & , \quad \text{otherwise} \end{cases} \tag{D.18}$$

$$E_g = 3A^2T/8 \tag{D.19}$$

EITG05 – Digital Communications

Lecture 2

Signal Constellations (p. 31–55)

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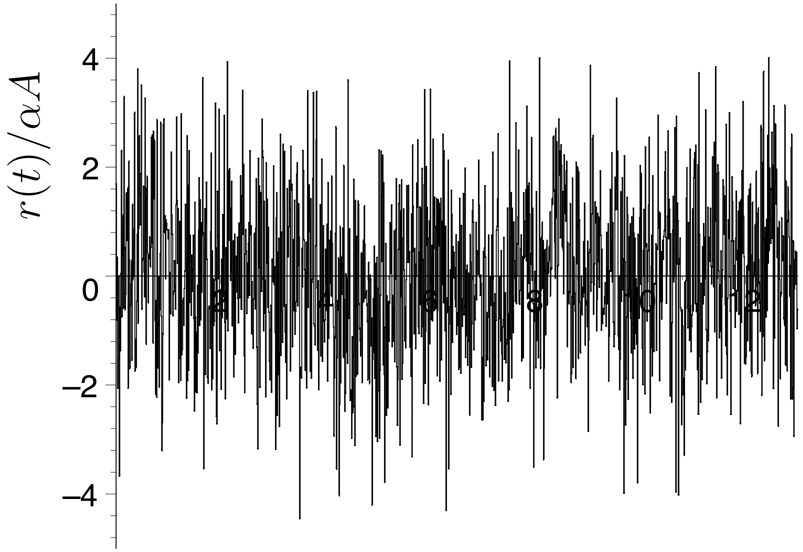
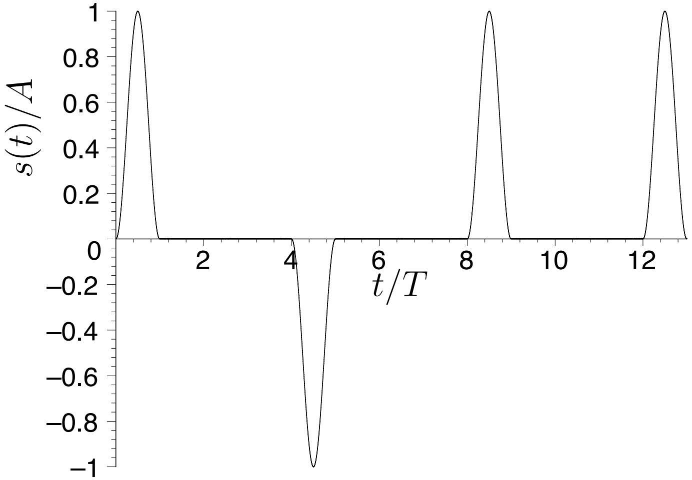
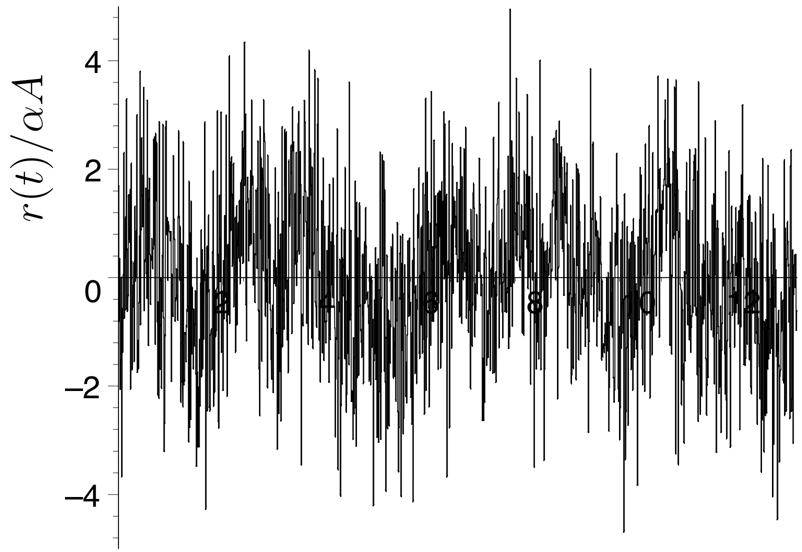
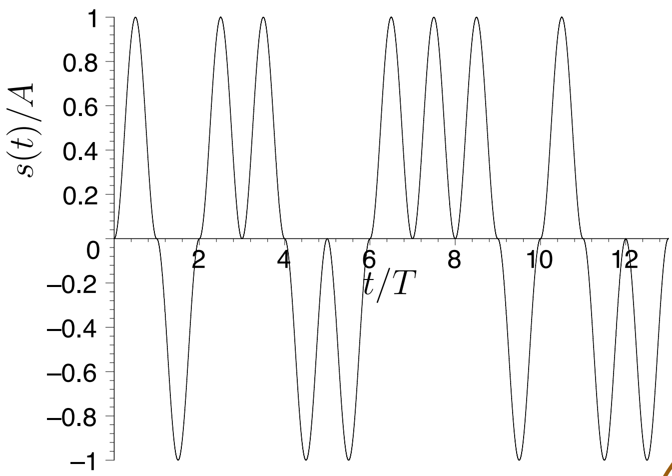
Thursday, September 6, 2018

Example: noisy signal at the receiver (p. 13)



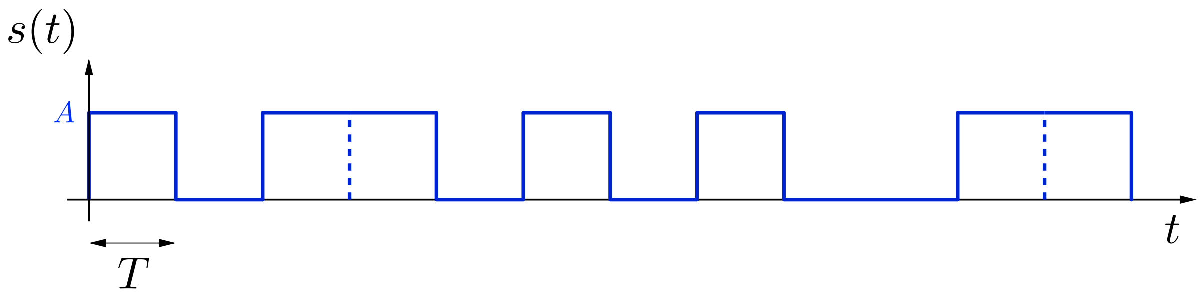






Euclidean distance example $M = 2$

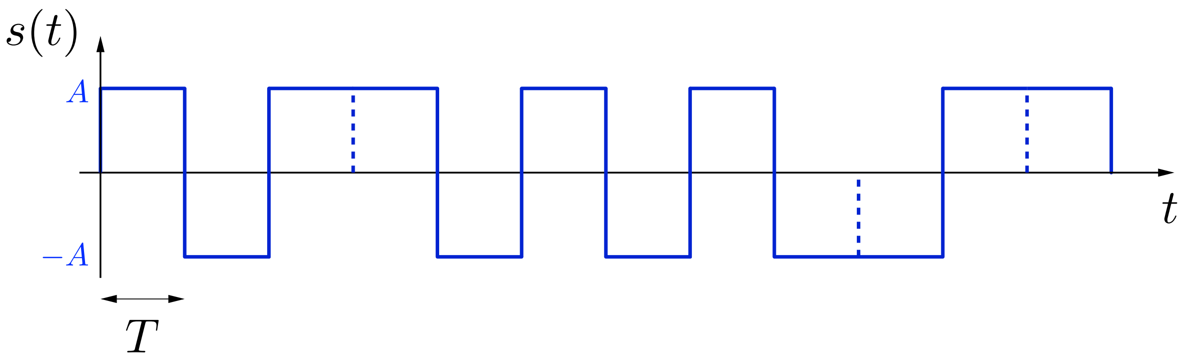
Case 1: on-off signaling



$s_0(t) = A$ and $s_1(t) = 0$ for $0 < t < T_s = T$, which gives $D^2_{0,1} = 2\overline{E}_b$

Observe: on-off signaling is orthogonal

Case 2: antipodal signaling



$s_0(t) = A$ and $s_1(t) = -A$ for $0 < t < T_s = T$, and $D^2_{0,1} = 4\overline{E}_b$

How well can we distinguish two signals?

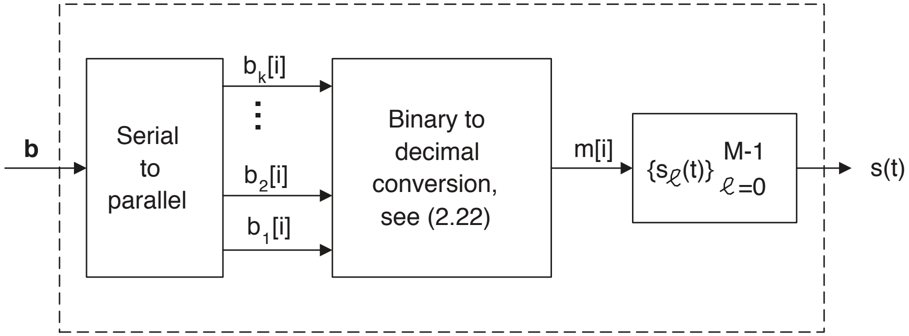
- The squared Euclidean distance between two signals $s_i(t)$ and $s_j(t)$ is defined as

$$D_{i,j}^2 = \int_0^{T_s} \left(s_i(t) - s_j(t)\right)^2 dt$$
$$= \int_0^{T_s} s_i^2(t) + s_j^2(t) - 2\, s_i(t) s_j(t)\, dt$$
$$= E_i + E_j - 2 \int_0^{T_s} s_i(t) s_j(t)\, dt$$

- The symbol energy E_ℓ of a signal alternative $s_\ell(t)$ is given by

$$E_\ell = \int_0^{T_s} s_\ell^2(t)\, dt < \infty\,, \quad \ell = 0, 1, \ldots, M-1$$

Signal constellations



- In case of M-ary signaling, one of $M = 2^k$ messages $m[i]$ is transmitted by its corresponding signal alternative

$$s_\ell(t) \in \{s_0(t), s_1(t), \ldots, s_{M-1}(t)\}$$

- The signal constellation is the set of possible signal alternatives
- The mapping defines which message is assigned to which signal
- When the message equals $m[i] = j$ then $s_j(t - iT_s)$ is sent

$$s(t) = s_{m[0]}(t) + s_{m[1]}(t - T_s) + s_{m[2]}(t - 2T_s) + \cdots$$

Question: how should we choose M distinguishable signals?

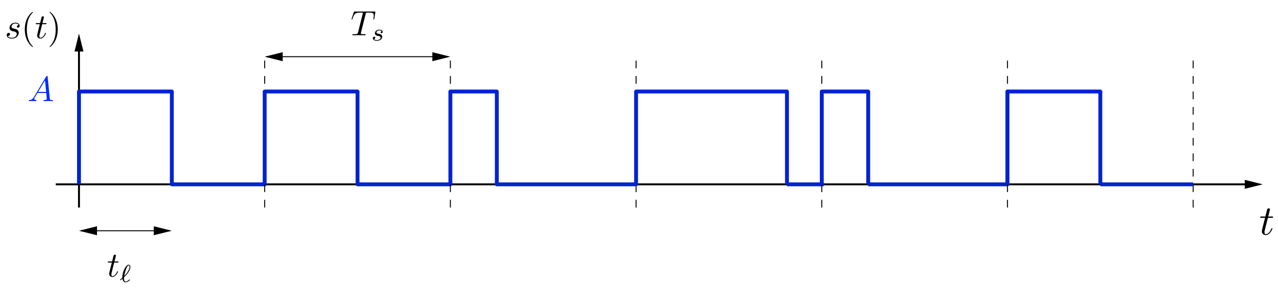
Pulse Width Modulation (PWM)

- In pulse width modulation the message modulates the duration T of a pulse $c(t)$ within the symbol interval T_s

$$s_\ell(t) = c\left(\frac{t}{t_\ell}\right) \;, \quad \ell = 0, 1, \ldots, M-1$$

- The duration of the pulse $c(t)$ is equal to $T = 1$
- It follows that $s_\ell(t)$ is zero outside the interval $0 \leq t \leq t_\ell$
- It is assumed that $t_\ell < T_s$
- Average symbol energy: $\overline{E}_s = E_c \, \bar{t}_\ell$

Example:



Used in control applications, not much for data transmission (e.g., speed of CPU fan, LED intensity)

Pulse Position Modulation (PPM)

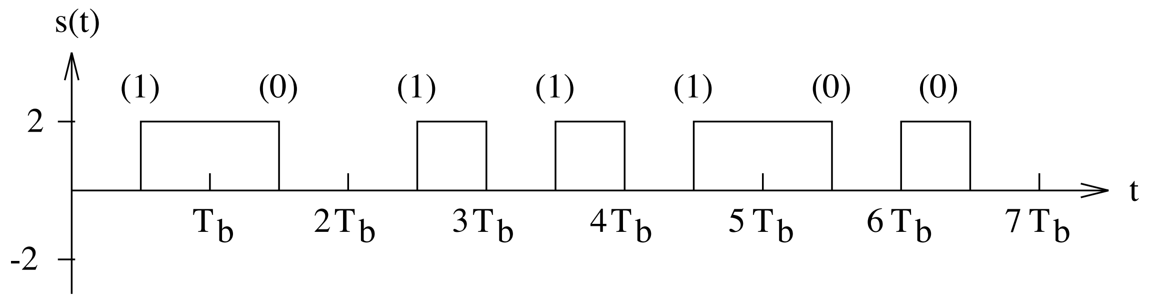
- In pulse position modulation the message modulates the position of a short pulse $c(t)$ within the symbol interval T_s

$$s_\ell(t) = c\left(t - \ell \, \frac{T_s}{M}\right) \; , \quad \ell = 0, 1, \ldots, M-1$$

- The duration T of the pulse $c(t)$ has to satisfy $T \leq T_s/M$
- The pulses are orthogonal and we get

$$\overline{E}_s = E_c \; , \qquad D_{i,j}^2 = E_i + E_j = 2\, E_c$$

Example:



Used for low-power optical links (e.g. IR remote controls)

Pulse Amplitude Modulation (PAM)

- In pulse amplitude modulation the message is mapped into the amplitude only:

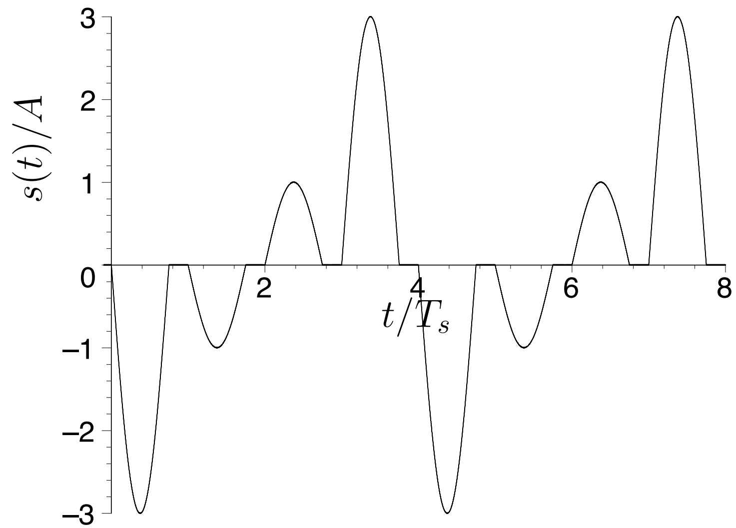
$$s_\ell(t) = A_\ell \, g(t) \;, \quad \ell = 0, 1, \ldots, M-1$$

- PAM is a natural generalization of binary on-off signaling and antipodal signaling, which are special cases for $M = 2$
- A common choice are equidistant amplitudes located symmetrically around zero:

$$A_\ell = -M + 1 + 2\ell \;, \quad \ell = 0, 1, \ldots, M-1$$

Example of 4-ary PAM

- **Example:** $M = 4$, $A_0 = -3$, $A_1 = -1$, $A_2 = +1$, $A_3 = +3$



- The same constellation, defined by the amplitudes

$$\{A_\ell\}_{\ell=0}^{M-1} = \{\pm 1, \pm 3, \pm 5, \ldots, \pm(M-1)\}$$

could also be used with other mappings

What is the message sequence $m[i]$?

Symbol Energy of PAM

- The symbol energy of a PAM signal is

$$E_\ell = \int_0^{T_s} s_\ell^2(t)\, dt = \int_0^{T_s} A_\ell^2\, g^2(t)\, dt$$

- Using

$$E_g = \int_0^{T_s} g^2(t)\, dt$$

we can write the average symbol energy as

$$\overline{E}_s = E_g \sum_{\ell=0}^{M-1} P_\ell\, A_\ell^2$$

- Often the messages are equally likely, i.e., $P_\ell = \frac{1}{M} = 2^{-k}$, and for the symmetric constellation from above we get

$$\overline{E}_s = E_g \frac{M^2 - 1}{3} \,.$$

Euclidean distances of PAM signals

- The squared Euclidean distance between two PAM signal alternatives is

$$D_{i,j}^2 = \int_0^{T_s} \left(s_i(t) - s_j(t)\right)^2 \, dt = E_g \, (A_i - A_j)^2$$

- With $A_\ell = -M + 1 + 2\ell$ this becomes

$$D_{i,j}^2 = 4E_g \, (i - j)^2$$

  Compare this with Example 2.7 on page 28

- We will later see that the minimum Euclidean distance $\min_{i,j} D_{i,j}$ strongly influences the error probability of the receiver
- For this reason, equidistant constellations are often used

Bandpass Signals

- In many applications we want to transmit signals at high frequencies, centered around a carrier frequency f_c
- A typical bandpass signal has the form

$$s(t) = A(t) \cdot \cos\big(2\pi f(t)\, t \,+\, \varphi(t)\big)$$

- The general idea of carrier modulation techniques is to map the messages $m[i]$ to the different signal parameters:
  - **PAM**: amplitude $A(t)$
  - **PSK**: phase $\varphi(t)$
  - **FSK**: frequency $f(t)$
  - **QAM**: amplitude $A(t)$ and phase $\varphi(t)$
  - **OFDM**: amplitude $A(t)$, phase $\varphi(t)$, and frequency $f(t)$

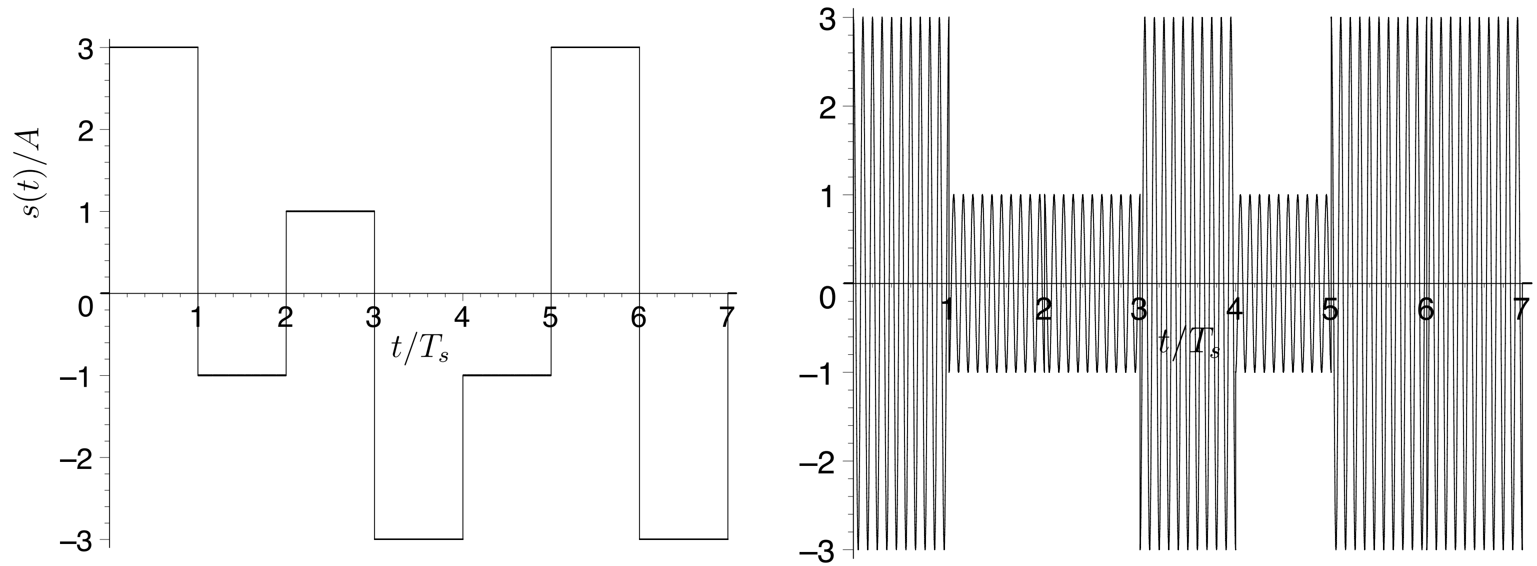
Remark:
analog modulation (AM or FM) changes the parameters by means of a continuous input signal

Bandpass M-ary PAM

- To modulate the pulse amplitude, we can multiply the original PAM signal $s(t)$ with a sinusoidal signal

$$s_{bp}(t) = s(t) \cdot \cos(2\pi f_c\, t) = \sum_{i=0}^{\infty} A_{m[i]}\, g(t - i\, T_s) \cdot \cos(2\pi f_c\, t)$$

Example:



Phase Shift Keying (PSK)

- We have seen that with PAM signaling the message modulates the amplitude A_ℓ of the signal $s_\ell(t)$
- The idea of phase shift keying signaling is to modulate instead the phase v_ℓ of $s_\ell(t)$

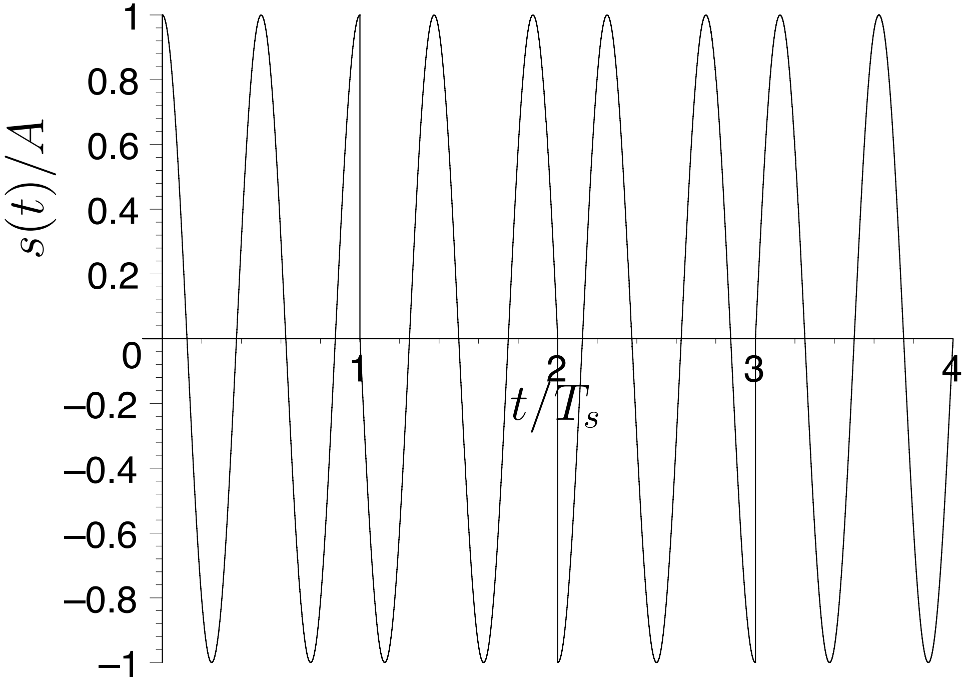
$$s_\ell(t) = g(t)\ \cos(2\pi f_c\, t\ +\ v_\ell)\ , \quad \ell = 0, 1, \dots, M-1\ ,$$

- $M = 2$**:** binary PSK (BPSK) with $v_0 = 0$ and $v_1 = \pi$ is equivalent to binary PAM with $A_0 = +1$ and $A_1 = -1$
- $M = 4$**:** 4-ary PSK is also called quadrature PSK (QPSK)
- If we choose

$$f_c = n\, R_s$$

for some positive integer n, then n full cycles of the carrier wave are contained within a symbol interval T_s

Example of QPSK



$$f_c = 2\,R_s\ , \quad \nu_0 = 0,\ \nu_1 = \pi/2,\ \nu_2 = \pi,\ \text{and}\ \nu_3 = 3\pi/2$$

What is the message sequence $m[i]$?

Symmetric M-ary PSK

- Normally, the phase alternatives are located symmetrically on a circle
$$v_\ell = \frac{2\pi\,\ell}{M} + v_{const} \ , \quad \ell = 0, 1, \ldots, M-1 \ ,$$
where v_{const} is a contant phase offset value

- If $P_\ell = \frac{1}{M}$, and $f_c \gg R_s$, then the average symbol energy is
$$\overline{E}_s = \frac{E_g}{2}$$
and
$$D_{i,j}^2 = E_g \left(1 - \cos(v_i - v_j)\right)$$

- PSK has a constant symbol energy

Frequency Shift Keying (FSK)

- Instead of amplitude and phase, the message can modulate the frequency f_ℓ

$$s_\ell(t) = A \cos(2\pi f_\ell\, t + \nu) , \quad \ell = 0, 1, \ldots, M-1$$

- Amplitude A and phase ν are constants
- In many applications the frequency alternatives f_ℓ are chosen such that the signals are orthogonal, i.e.,

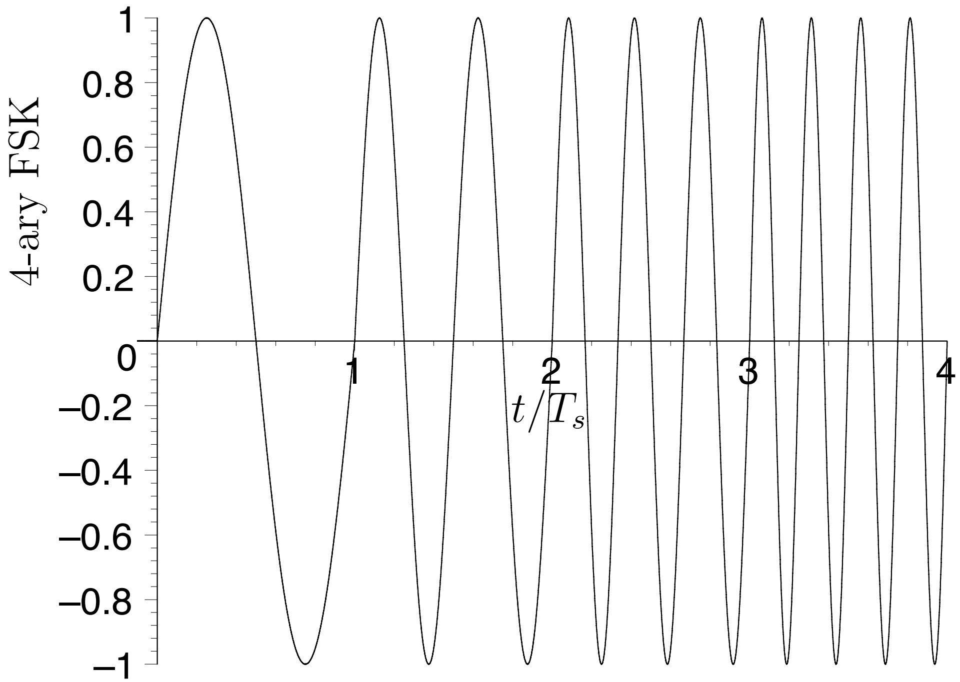
$$\int_0^{T_s} s_i(t)\, s_j(t)\, dt = 0 , \quad i \neq j$$

- If $\nu = 0$ or $\nu = -\pi/2$ (often used), then we can choose

$$f_\ell = n_0 \frac{R_s}{2} + \ell\, I\, \frac{R_s}{2} \stackrel{\text{def}}{=} f_0 + \ell\, f_\Delta , \quad \ell = 0, 1, \ldots, M-1 ,$$

where n_0 and I are positive integers

Example of 4-ary FSK



$$\nu = -\frac{\pi}{2}\ , \quad f_0 = R_s,\ f_1 = 2R_s,\ f_2 = 3R_s,\ \text{and}\ f_3 = 4R_s$$

What is the message sequence $m[i]$?

Quadrature Amplitude Modulation (QAM)

- With QAM signaling the message modulates the amplitudes of two orthogonal signals (inphase and quadrature component)

$$s_\ell(t) = A_\ell \; g(t) \; \cos(2\pi f_c \; t) - B_\ell \; g(t) \; \sin(2\pi f_c \; t) \; , \quad \ell = 0, 1, \ldots, M-1$$

- We can interpret $s_\ell(t)$ as the sum of two bandpass PAM signals
- **Motivation:** We can transmit two signals independently using the same carrier frequency and bandwidth

With QAM we can change both amplitude and phase

Quadrature Amplitude Modulation (QAM)

$$s_\ell(t) = A_\ell \, g(t) \, \cos(2\pi f_c \, t) - B_\ell \, g(t) \, \sin(2\pi f_c \, t)$$

- The signal $s_\ell(t)$ can also be expressed as

$$s_\ell(t) = g(t)\sqrt{A_\ell^2 + B_\ell^2} \, \cos(2\pi f_c \, t \, + \, \nu_\ell)$$

- It follows that QAM is a generalization of PSK:

  selecting $A_\ell^2 + B_\ell^2 = 1$ we can put the information into ν_ℓ and get

$$A_\ell = \cos(\nu_\ell) \, , \quad B_\ell = \sin(\nu_\ell)$$

Energy and Distance of M-ary QAM

- Choosing $f_c \gg R_s$ it can be shown that

$$E_\ell = \left(A_\ell^2 + B_\ell^2\right) \frac{E_g}{2}$$

$$D_{i,j}^2 = \left((A_i - A_j)^2 + (B_i - B_j)^2\right) \frac{E_g}{2}$$

- A common choice are equidistant amplitudes located symmetrically around zero: (two $\sqrt{M}$-ary PAM with $k/2$ bits each)

$$\{A_\ell\}_{\ell=0}^{\sqrt{M}-1} = \{B_\ell\}_{\ell=0}^{\sqrt{M}-1} = \left\{\pm 1, \pm 3, \pm 5, \ldots, \pm\left(\sqrt{M}-1\right)\right\}$$

- For equally likely messages $P_\ell = \frac{1}{M}$, this results in the average energy

$$\overline{E}_s = \sum_{\ell=0}^{M-1} \frac{1}{M} E_\ell = \frac{2(M-1)}{3} \frac{E_g}{2}$$

Geometric interpretation

- It is possible to describe QAM signals as two-dimensional vectors in a so-called signal space
- For this the signal

$$s_\ell(t) = A_\ell \, g(t) \, \cos(2\pi f_c \, t) - B_\ell \, g(t) \, \sin(2\pi f_c \, t)$$

  is written as

$$s_\ell(t) = s_{\ell,1} \, \phi_1(t) + s_{\ell,2} \, \phi_2(t)$$

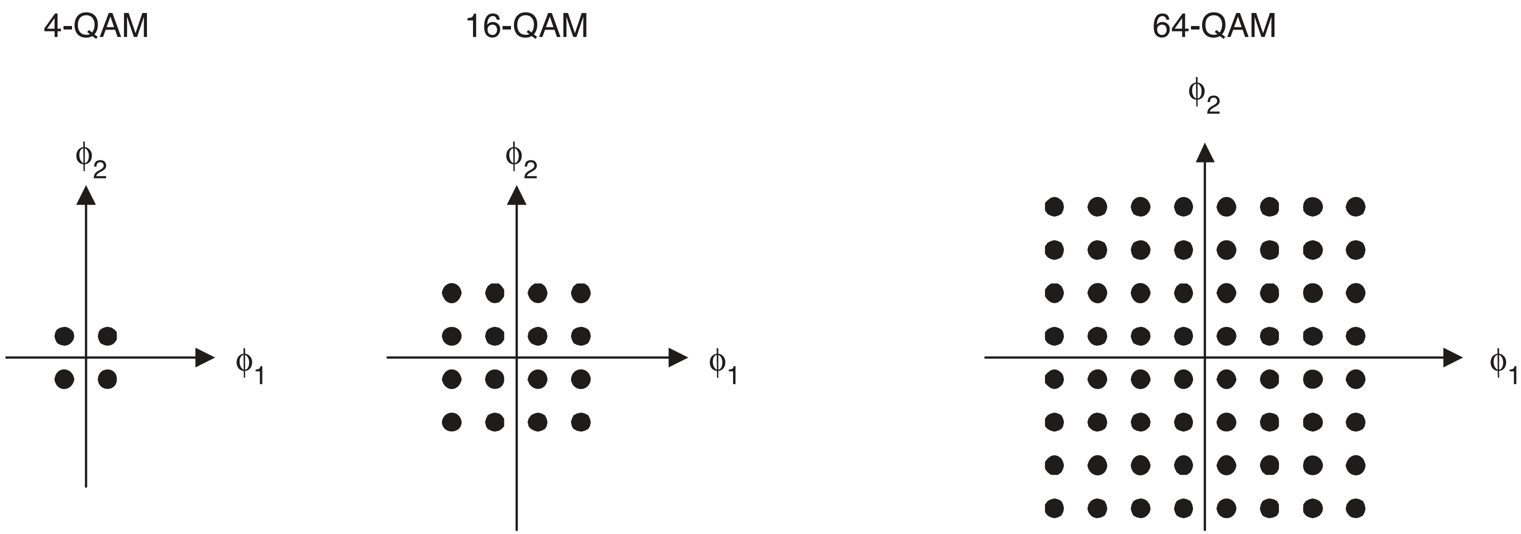
- Here $s_{\ell,1} = A_\ell \sqrt{E_g/2}$ and $s_{\ell,2} = B_\ell \sqrt{E_g/2}$ are the coordinates
- The functions $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal basis of a vector space that spans all possible transmit signals:

$$\phi_1(t) = \frac{g(t) \, \cos(2\pi f_c \, t)}{\sqrt{E_g/2}} \, , \quad \phi_2(t) = -\frac{g(t) \, \sin(2\pi f_c \, t)}{\sqrt{E_g/2}}$$

This looks abstract, but can be very useful!

Signal space representation of QAM

- Now we can describe each signal alternative $s_\ell(t)$ as a point with coordinates $(s_{\ell,1}, s_{\ell,2})$ within a constellation diagram

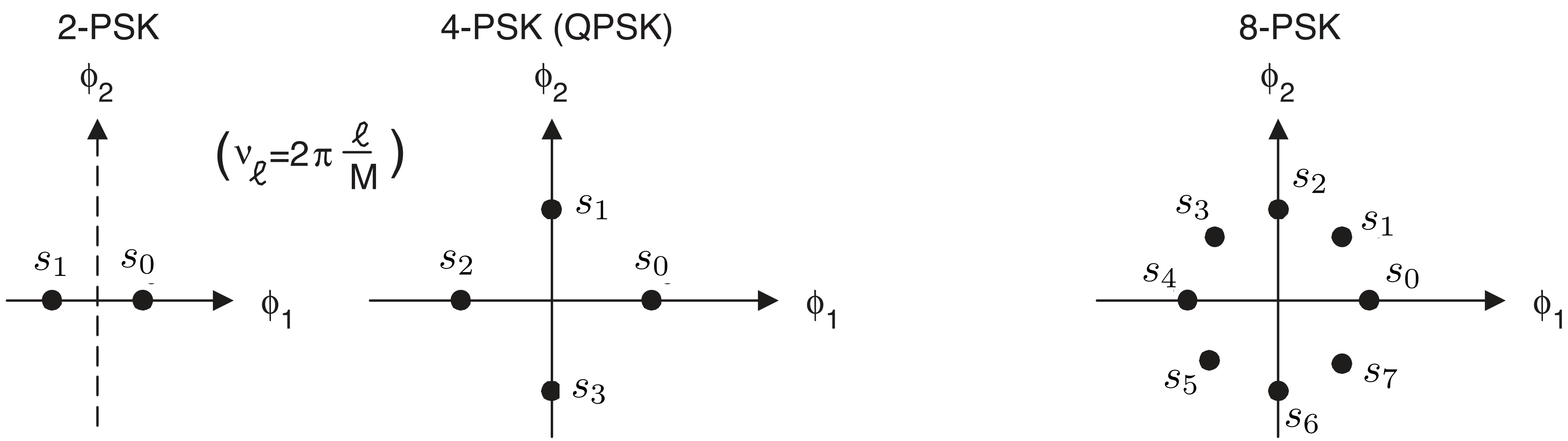


$$s_{\ell,1} = A_\ell \sqrt{E_g/2}\ , \qquad s_{\ell,2} = B_\ell \sqrt{E_g/2}$$

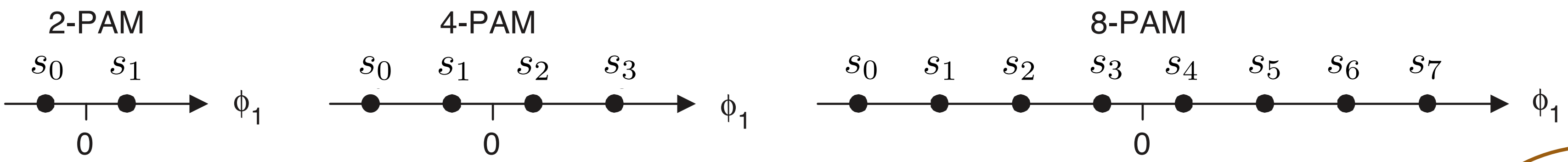
- The signal energy E_ℓ and the Euclidean distance $D^2_{i,j}$ can be determined in the signal space

Signal space representation of PSK and PAM

- PSK and PAM can be seen as a special cases of QAM:



$$s_{\ell,1} = \cos(\nu_\ell)\sqrt{E_g/2}\ , \qquad s_{\ell,2} = \sin(\nu_\ell)\sqrt{E_g/2}$$



$$s_{\ell,1} = (-M + 1 + 2\,\ell)\ \sqrt{E_g}$$

Multitone Signaling: OFDM

- With FSK signaling, orthogonal signal alternatives are transmitted at different frequencies
- Disadvantage: only one frequency can be used at the same time
- **Orthogonal Frequency Division Multiplexing (OFDM)**: use QAM at N orthogonal frequencies and transmit the sum
- OFDM is widely used in modern communication systems: WLAN, LTE, DAB (radio), DVB (TV), DSL

Example:

$N = 4096$
64-ary QAM at each frequency (carrier)

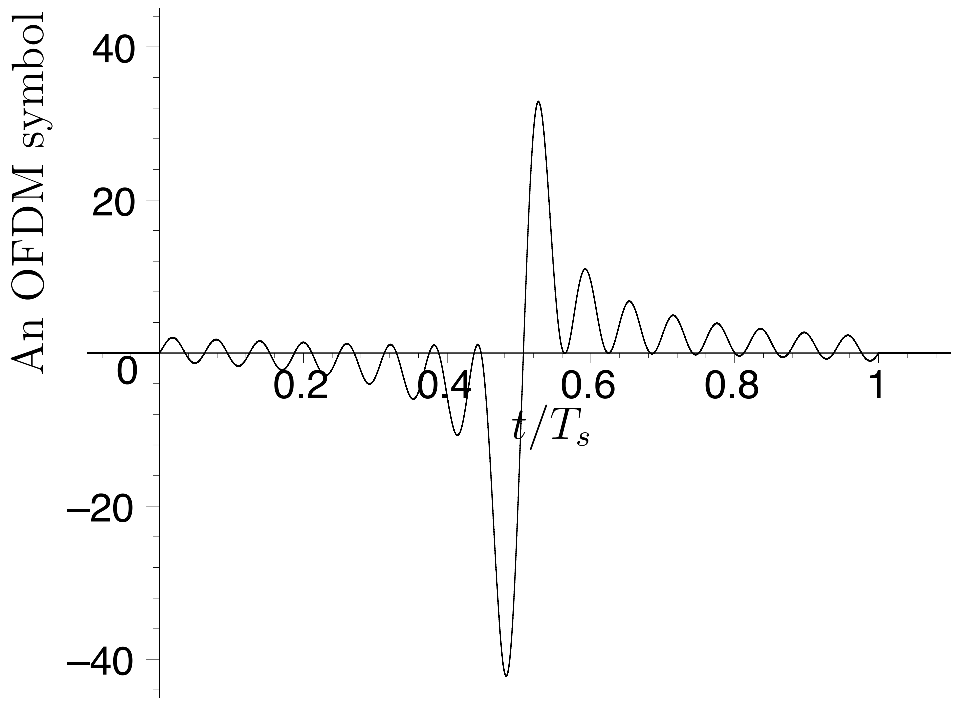
Then an OFDM signal carries $4096 \cdot 6 = 24\,576$ bits

How does a typical OFDM signal look like?

How can such a system be realized in practice?
$\Rightarrow$ OFDM will be explained in detail in the advanced course

Example of an OFDM symbol

$N = 16$, 16-ary QAM in each subcarrier (p. 52)



$$x(t) = \sum_{n=0}^{N-1} \left(a_I[n]\; g(t)\; \cos(2\pi f_n\, t) - a_Q[n]\; g(t)\; \sin(2\pi f_n\, t) \right) , \quad 0 \le t \le T_s$$

In this example the symbol $x(t)$ carries $16 \cdot 4 = 64$ bits

EITG05 – Digital Communications

Lecture 3

Bandwidth of Transmitted Signals

Michael Lentmaier
Monday, September 10, 2018

What did we do last week?

Concepts of M-ary digital signaling:

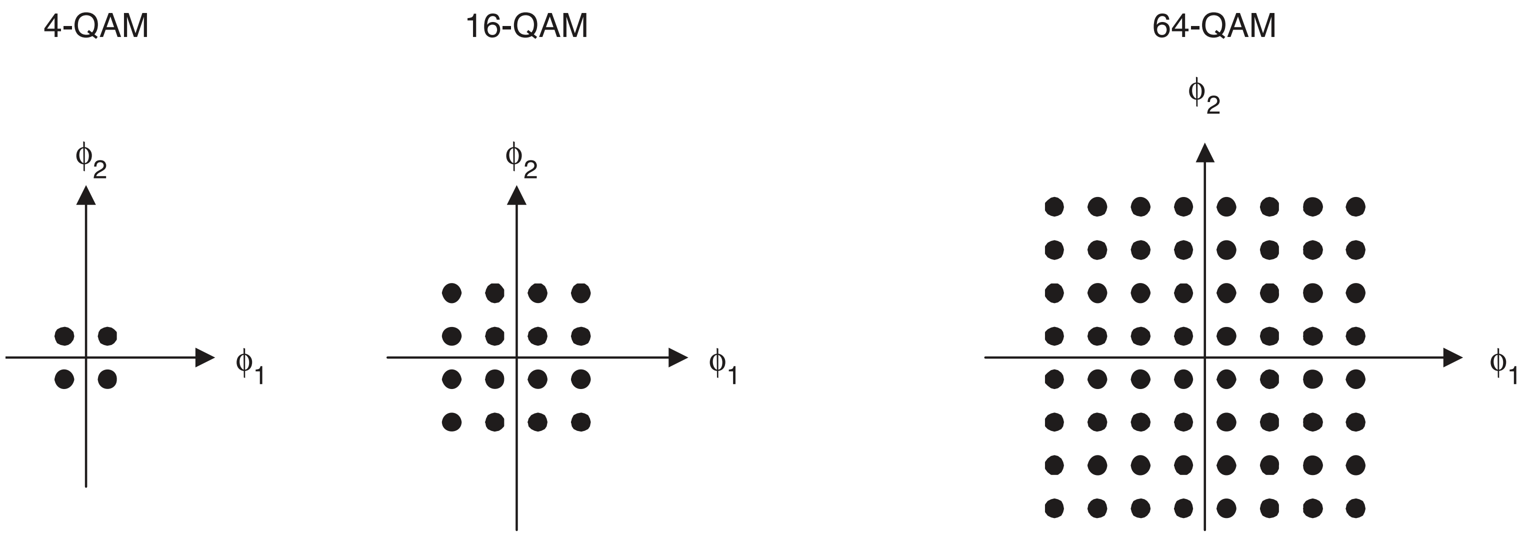
- Modulation of amplitude, phase or both: PAM, PSK, QAM
- Orthogonal signaling: FSK, OFDM
- Pulse position and width: PPM, PWM

We have paid special attention to:

- Average symbol energy $\overline{E}_s$
- Euclidean distance $D_{i,j}$
- Both values could be related to the energy E_g of the pulse $g(t)$

Signal space representation of QAM

- Now we can describe each signal alternative $s_\ell(t)$ as a point with coordinates $(s_{\ell,1}, s_{\ell,2})$ within a constellation diagram



$$s_{\ell,1} = A_\ell \sqrt{E_g/2}\,, \qquad s_{\ell,2} = B_\ell \sqrt{E_g/2}$$

- The signal energy E_ℓ and the Euclidean distance $D^2_{i,j}$ can be determined in the signal space

Geometric interpretation

- It is possible to describe QAM signals as two-dimensional vectors in a so-called signal space
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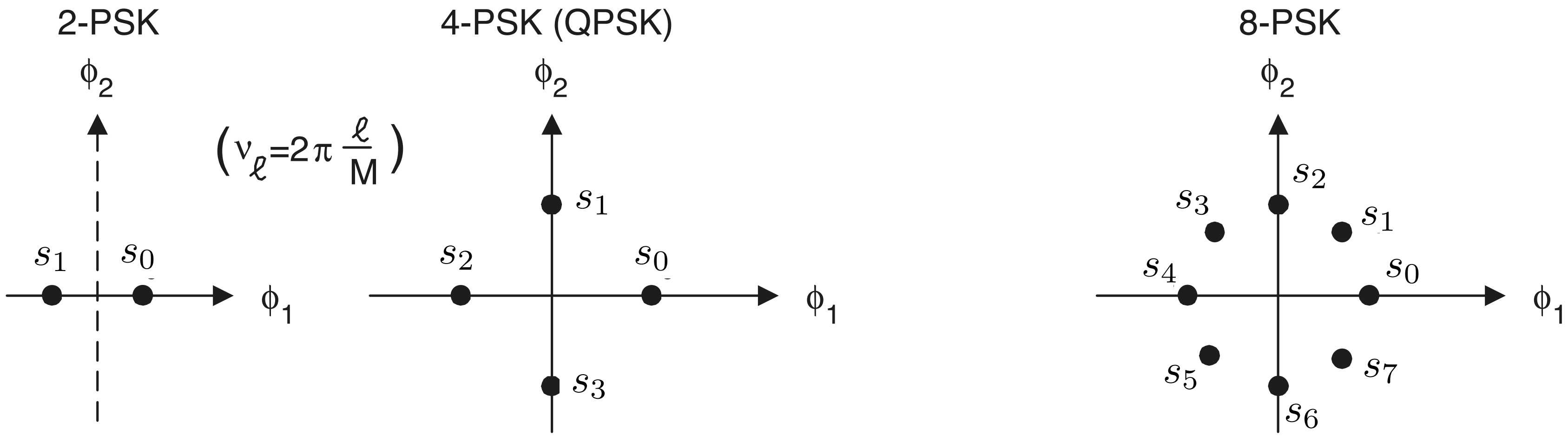
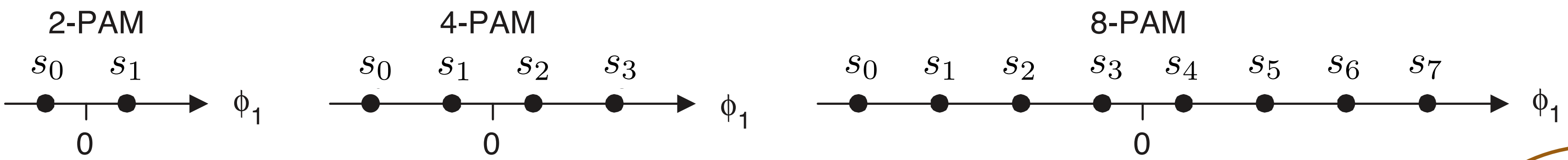


$$s_{\ell,1} = \cos(\nu_\ell)\sqrt{E_g/2}\ , \qquad s_{\ell,2} = \sin(\nu_\ell)\sqrt{E_g/2}$$



$$s_{\ell,1} = (-M + 1 + 2\,\ell)\ \sqrt{E_g}$$

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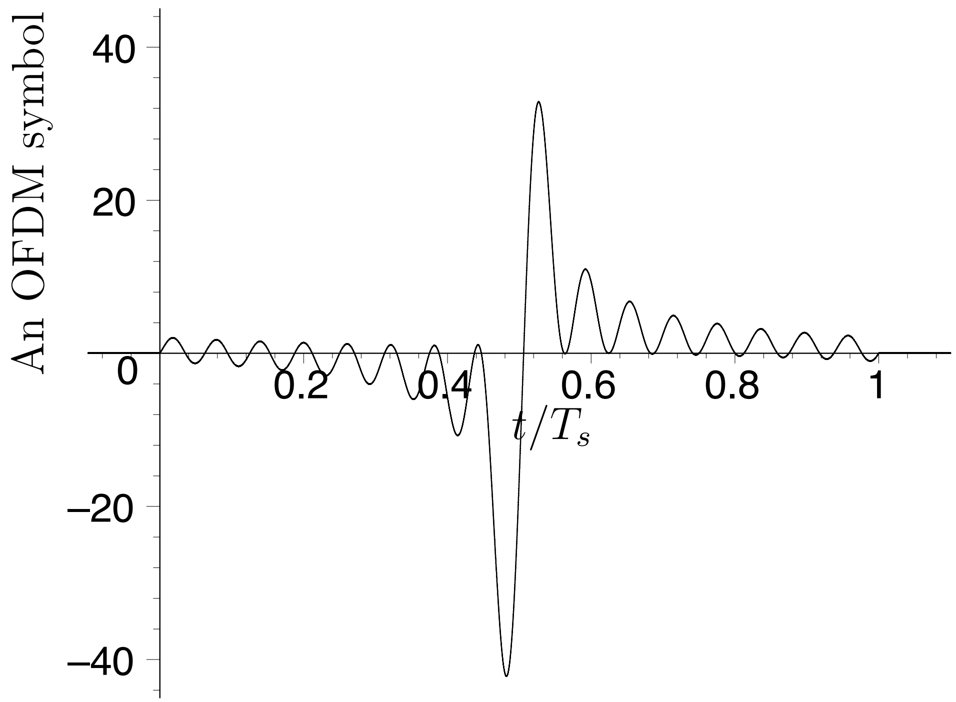
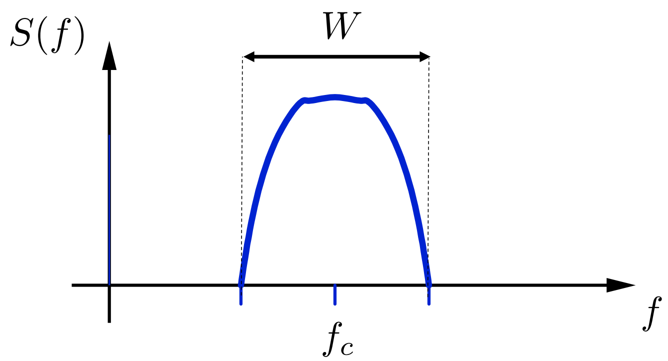


$$x(t) = \sum_{n=0}^{N-1} \left(a_I[n]\; g(t)\; \cos(2\pi f_n\, t) - a_Q[n]\; g(t)\; \sin(2\pi f_n\, t)\right)\;, \quad 0 \le t \le T_s$$

In this example the symbol $x(t)$ carries $16 \cdot 4 = 64$ bits

Bandwidth of Transmitted Signal

- The bandwidth W of a signal is the width of the frequency range where most of the signal energy or power is located



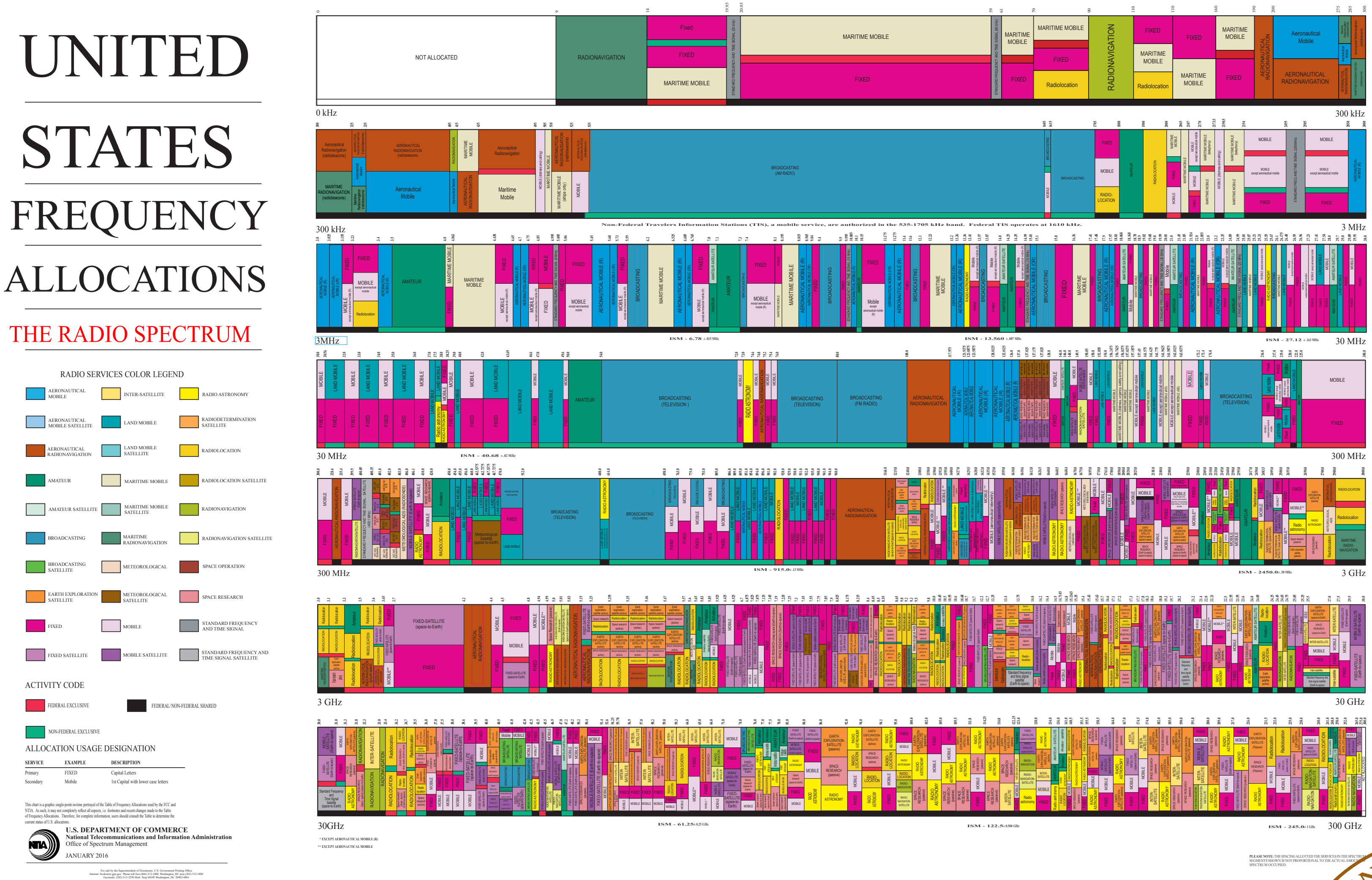
- W is measured on the positive frequency axis
- The bandwidth is a limited and precious resource
- We must have control of the bandwidth and use it efficiently

Questions:

What is the relationship between information bit rate and required bandwidth?

How does the bandwidth depend on the signaling method?

United States Frequency Allocations (2016)



Source: https://www.ntia.doc.gov/category/spectrum-management

Energy Spectrum

- We have seen last week that the energy of a signal $x(t)$ can be determined as
$$E_x = \int_{-\infty}^{\infty} x^2(t)\, dt$$
- The function $x^2(t)$ shows how the energy E_x is distributed along the time axis
- According to Parseval's relation we can alternatively express the energy as
$$E_x = \int_{-\infty}^{\infty} |X(f)|^2\, df \ ,$$
where $X(f)$ denotes the Fourier transform of the signal $x(t)$
- The function $|X(f)|^2$ shows how the energy E_x is distributed in the frequency domain

$\Rightarrow$ We need the Fourier transform as a tool for finding the bandwidth of our signals

Fourier Transform

- The Fourier transform of a signal $x(t)$ is given by

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt = X_{Re}(f) + j\, X_{Im}(f) \ ,$$

  where $j = \sqrt{-1}$, i.e., the solution to $j^2 = -1$

- We can also express $X(f)$ in terms of magnitude $|X(f)|$ and phase $\varphi(f) = \arg X(f)$ (argument)

$$X(f) = |X(f)|\, e^{j\,\varphi(f)}$$

- Then

$$|X(f)| = \sqrt{X_{Re}^2(f) + X_{Im}^2(f)}$$
$$X_{Re}(f) = |X(f)| \cos(\varphi(f))$$
$$X_{Im}(f) = |X(f)| \sin(\varphi(f))$$

Fourier Transform

- The original signal $x(t)$ can then be expressed in terms of the inverse Fourier transform as

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df = \int_{-\infty}^{\infty} |X(f)|\, e^{+j(2\pi f t + \varphi(f))}\, df$$

- **Interpretation:** any signal $x(t)$ can be decomposed into sinusoidal components at different frequencies and phase offsets
- The magnitude $|X(f)|$ measures the strength of the signal component at frequency f
- Assuming $x(t)$ is a real-valued signal this can be written as

$$x(t) = 2\int_{0}^{\infty} |X(f)|\, \cos(2\pi f t + \varphi(f))\, df$$

and it can be shown that

$$|X(f)| = |X(-f)|\ , \text{ (even)} \quad \varphi(f) = -\varphi(-f)\ , \text{ (odd)}$$

Example: rectangular pulse

- Let us compute the Fourier transform of the following signal:

$$x_{rec}(t) = \begin{cases} A & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases}$$

- We get

$$X_{rec}(f) = \mathcal{F}\{x_{rec}(t)\} = \int_{-\infty}^{\infty} x_{rec}(t)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-T/2}^{+T/2} A\, e^{-j2\pi f t}\, dt \;=\; \left[-\frac{Ae^{-j2\pi f t}}{j2\pi f} \right]_{-T/2}^{+T/2}$$

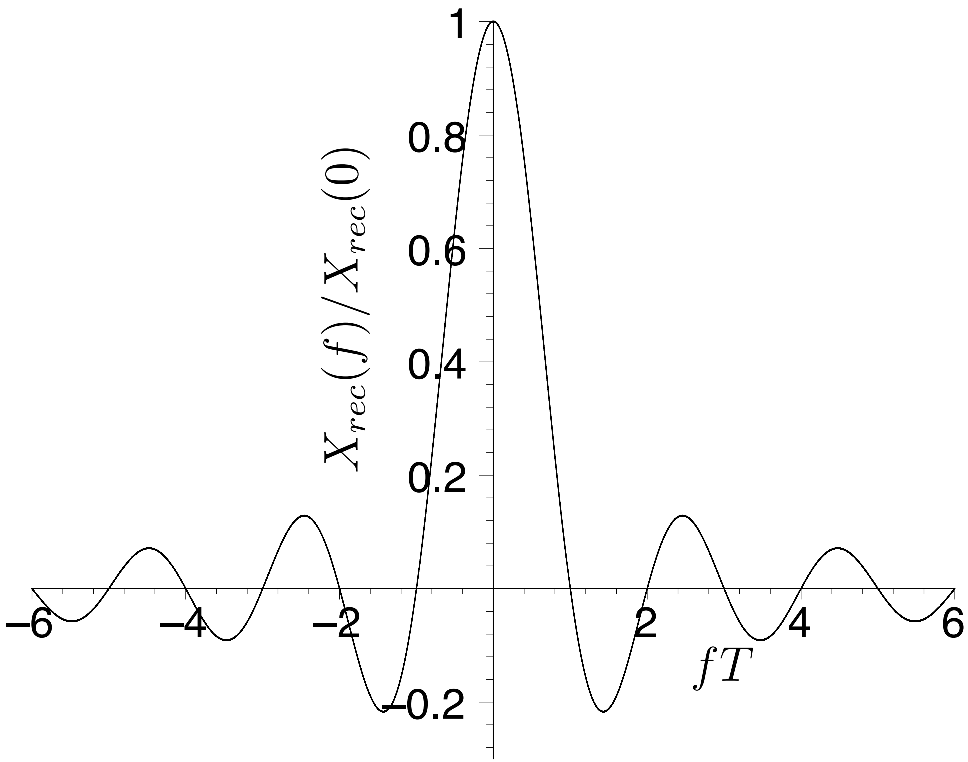
$$= \frac{A}{\pi f}\, \frac{e^{j\pi f T} - e^{-j\pi f T}}{2j} = AT\, \frac{\sin(\pi f T)}{\pi f T}$$

- We have found that

$$x_{rec}(t) \longleftrightarrow AT\, \frac{\sin(\pi f T)}{\pi f T} = AT \operatorname{sinc}(fT)$$

Notation: $x(t) \longleftrightarrow \mathcal{F}\{x(t)\}$

Example 2.17: sketch of $X_{rec}(f)$



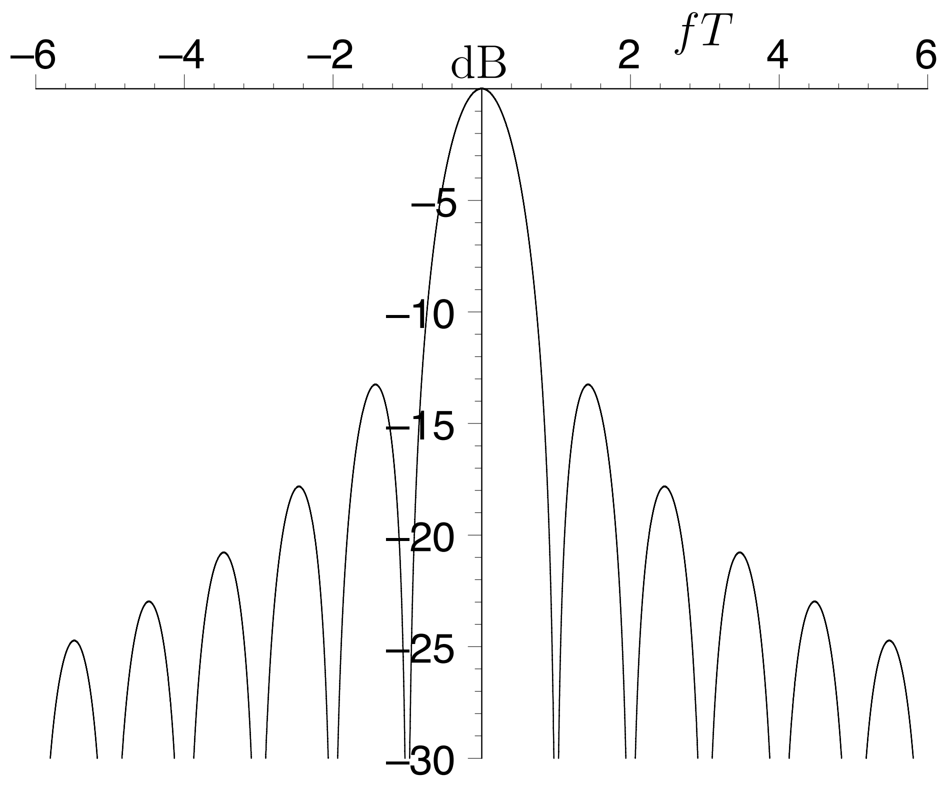
- the Fourier transform $X(f)$ is centered around $f = 0$: baseband
- we observe a main-lobe and several side-lobes
- Note: $fT = 2$ means that $f = 2 \cdot 1/T$

Sketch the function for $T = 10^{-6}\,s$ and $T = 2 \cdot 10^{-6}\,s$

Example 2.17: sketch of $|X_{rec}(f)|^2$

- Consider now the normalized energy spectrum in dB

$$10\log_{10}\left(\frac{|X_{rec}(f)|^2}{E_x T}\right) = 10\log_{10}\left(\frac{\sin(\pi f T)}{\pi f T}\right)^2$$



$\Rightarrow$ most energy is contained in the main-lobe (90.3 %)

Fourier transform of time-shifted signals

- Did you notice the difference between $x_{rec}(t)$ in this example and the elementrary pulse $g_{rec}(t)$ which we used last week?

$$x_{rec}(t) = \begin{cases} A & -\frac{T}{2} \leq t \leq \frac{T}{2} \\ 0 & \text{otherwise} \end{cases} , \qquad g_{rec}(t) = \begin{cases} A & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

- The pulse $g_{rec}(t) = x_{rec}(t - T/2)$ is a time-shifted version of $x_{rec}(t)$
- In general, the Fourier transform of a signal $y(t) = x(t - t_d)$ with a constant delay t_d becomes

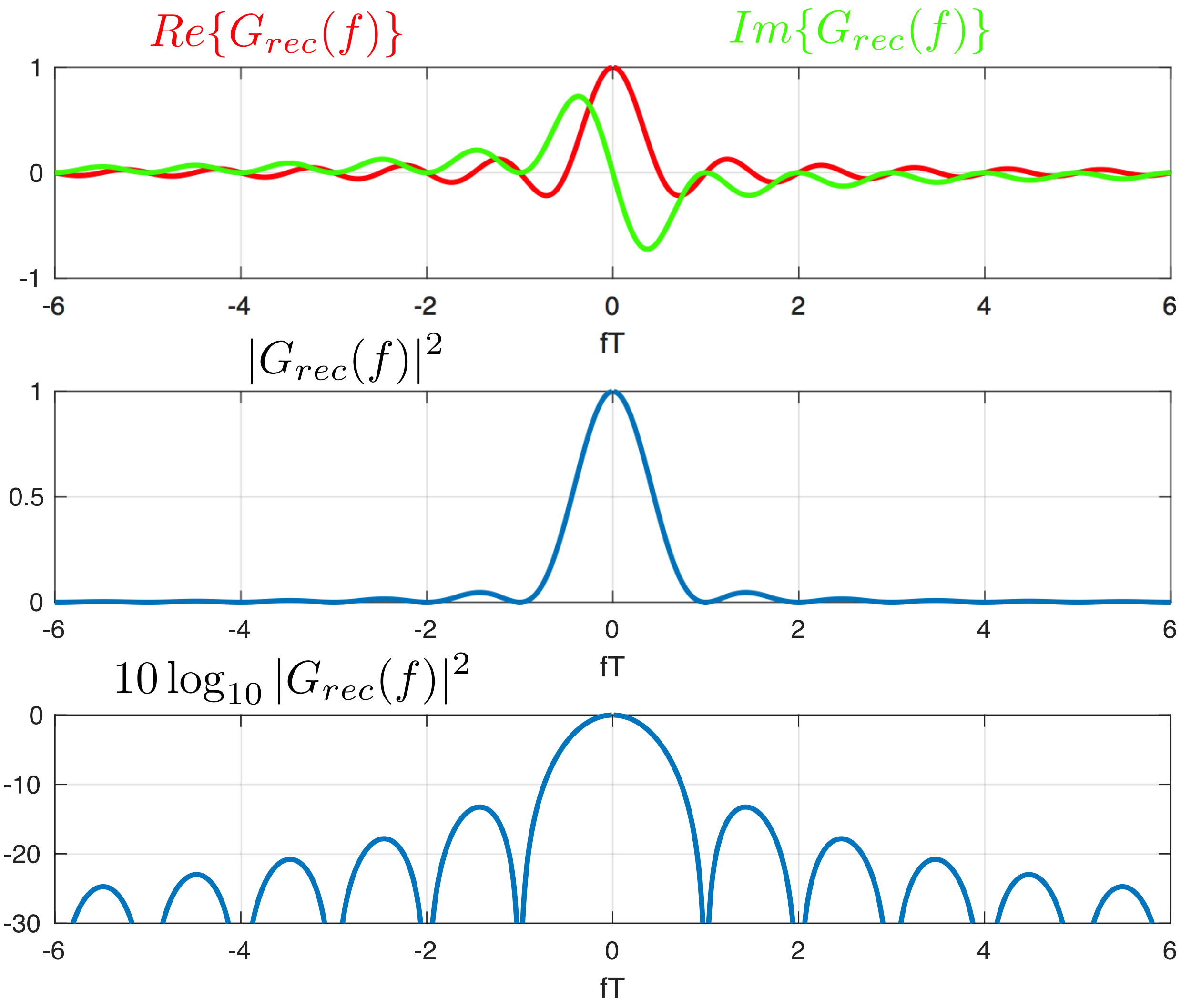
$$Y(f) = \int_{-\infty}^{\infty} x(t - t_d)\, e^{-j2\pi f t}\, dt = \int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f (\tau + t_d)}\, d\tau = X(f)\, e^{-j2\pi f t_d}$$

- **Observe:** the delay t_d changes only the phase of $Y(f)$
- The energy spectrum is not affected by time-shifts

$$|X_{rec}(f)|^2 = |G_{rec}(f)|^2 \quad \text{(compare App. D.1)}$$

A simple Matlab exercise

Let us plot the spectrum of the pulse $g_{rec}(t)$



A simple Matlab exercise

And this is how it was done:

```
% Example: rect pulse spectrum


x=-6:0.01:6;
G=sin(pi.*x)./(pi.*x).*exp(-j*pi*x); % T=1

figure(2)
subplot(3,1,1);
plot(x,real(G),'r',x,imag(G),'g'); xlabel('fT');
grid on;

subplot(3,1,2);
plot(x,abs(G).^2); xlabel('fT');
grid on;

subplot(3,1,3);
plot(x,10.*log10(abs(G).^2)); xlabel('fT');
set(gca,'YLim',[-30 0]);
grid on;
```

script | Ln 13 | Col 34

Fourier transform of other pulses

- The Fourier transforms $G(f)$ and sketches of the energy spectra $|G(f)|^2$ are given for a number of different elementary pulses $g(t)$ in Appendix D
- **Example:** half cycle sinusoidal pulse

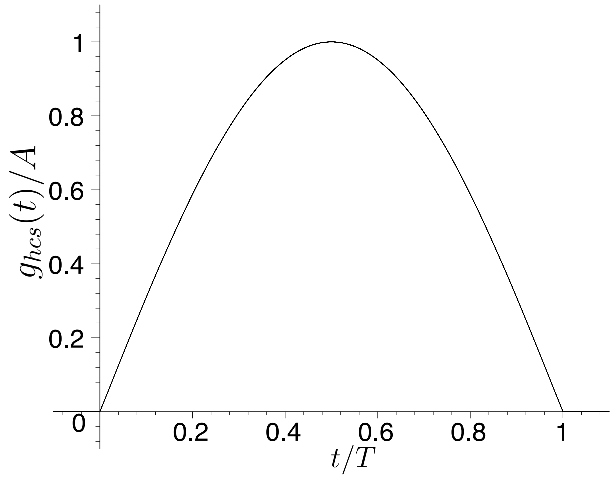


Figure D.7: $g_{hcs}(t)/A$.

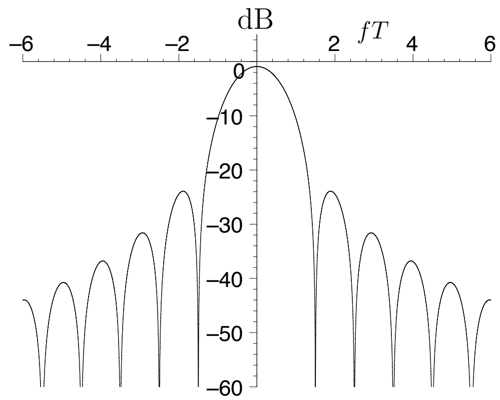


Figure D.8: $\frac{|G_{hcs}(f)|^2}{E_g T}$ in dB.

$$g_{hcs}(t) = \begin{cases} A\sin(\pi t/T) & , \quad 0 \leq t \leq T \\ 0 & , \quad \text{otherwise} \end{cases}$$

$$E_g = A^2T/2$$

$$G_{hcs}(f) = \mathcal{F}\{g_{hcs}(t)\} = \frac{2AT}{\pi} \, \frac{\cos(\pi fT)}{1-(2fT)^2} \, e^{-j\pi fT}$$

$$G_{hcs}(f = \pm 1/2T) = \mp jAT/2$$

$$G_{hcs}(n/T) = 0 \text{ if } n = \pm 3/2, \pm 5/2, \pm 7/2, \ldots$$

Frequency shift operations

- We have seen the effect of a time shift on the Fourier transform

$$g(t-t_d) \;\longleftrightarrow\; G(f)\, e^{-j2\pi f t_d}$$

- In a similar way we can characterize a frequency shift f_c by

$$g(t)\, e^{j2\pi f_c t} \;\longleftrightarrow\; G(f-f_c)$$

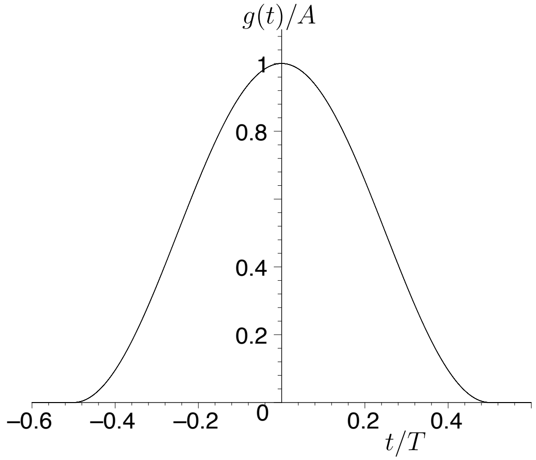
- Let us make use of the relation $e^{j2\pi f_c t} = \cos(2\pi f_c t) + j\, \sin(2\pi f_c t)$
- We can now express this in terms of cosine and sine functions,

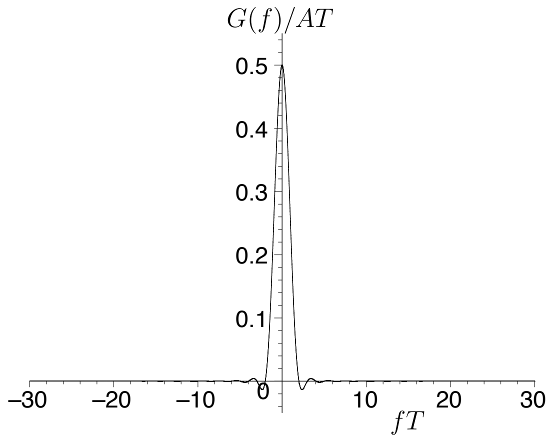
$$g(t)\, \cos(2\pi f_c t) \;\longleftrightarrow\; \frac{G(f+f_c)+G(f-f_c)}{2}$$

$$g(t)\, \sin(2\pi f_c t) \;\longleftrightarrow\; j\, \frac{G(f+f_c)-G(f-f_c)}{2}$$

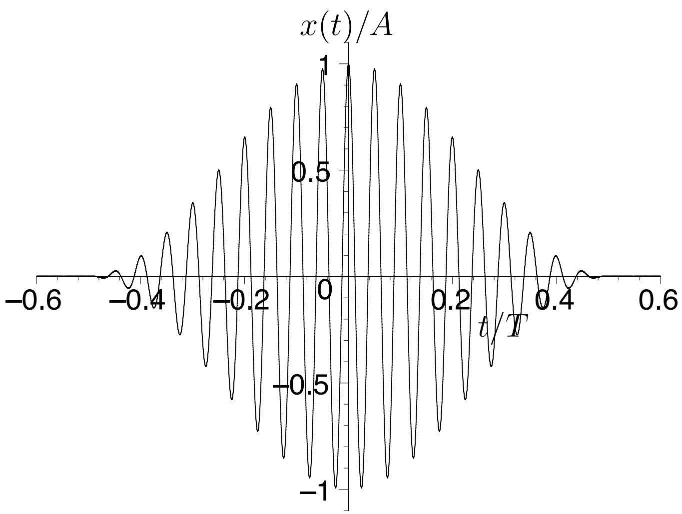
$\Rightarrow$ by simply changing the carrier frequency f_c we can move our signals to a suitable location along the frequency axis

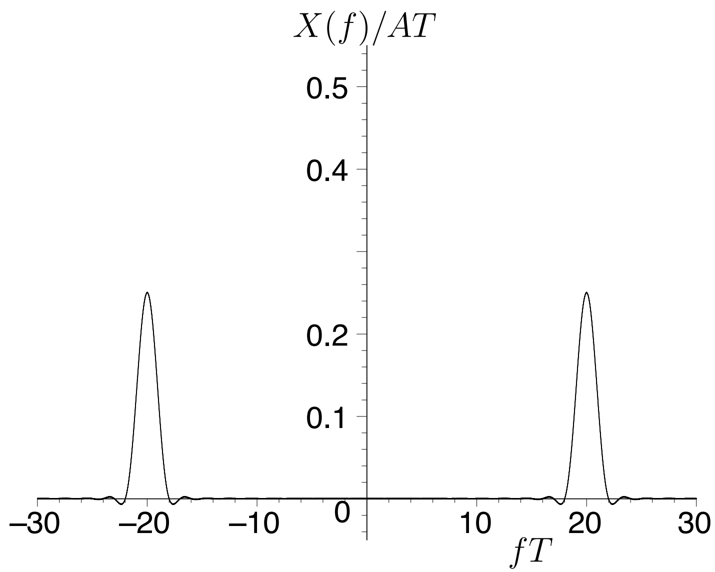
Example: time raised cosine pulse





$$x(t) = g(t) \cdot \cos(2\pi f_c\, t) = g_{rc}(t + T/2) \cdot \cos(2\pi f_c\, t)\ , \quad f_c = 20/T$$





Back to the transmitted signal

- We have seen how the Fourier transform can be used to calculate the energy spectrum $|X(f)|^2$ of a given signal $x(t)$
- Let us now look at the transmitted signal for M-ary modulation

$$s(t) = s_{m[0]}(t) + s_{m[1]}(t - T_s) + s_{m[2]}(t - 2T_s) + \cdots = \sum_{i=0}^{\infty} s_{m[i]}(t - iT_s)$$

- Message $m[i]$ selects the signal alternative to be sent at time iT_s
- Since the information bit stream is random, the transmitted signal $s(t)$ consists of a sequence of random signal alternatives

How can we determine the bandwidth W of the transmitted signal?

Does the information sequence influence the spectrum? How?

Power Spectral Density

- Since the signal has no predefined length the energy is not a good measure (could be infinite according to our model)
- On the other hand, we know that the signal has finite power
- The power spectral density $R(f)$ shows how the average signal power $\overline{P}$ is distributed along the frequency axis on average

$$\overline{P} = \overline{E}_b R_b = \int_{-\infty}^{\infty} R(f)\, df$$

- Most of the average signal power $\overline{P}$ $[\mathrm{V}^2]$ will be contained within the main-lobe of $R(f)$ $[\mathrm{V}^2/\mathrm{Hz}]$

  $\Rightarrow$ we can determine the signal bandwidth from $R(f)$

Our aim is to find $R(f)$ for a given modulation order M and set of M signal alternatives (constellation)

Power Spectral Density

Assumptions:

- The random M-ary sequence of messages $m[i]$ consists of independent, identically distributed (i.i.d) M-ary symbols
- The probability for each of the $M = 2^k$ symbols (messages) is denoted by $P_\ell, \ell = 0, 1, \ldots, M-1$
- All signal alternatives $s_\ell(t)$ in the constellation have finite energy
- The average signal over all signal alternatives is denoted $a(t)$, i.e.,

$$a(t) = \sum_{\ell=0}^{M-1} P_\ell \, s_\ell(t)$$

$$A(f) = \sum_{n=0}^{M-1} P_n \, S_n(f)$$

Remark: Source coding (compression) can be used to remove or reduce correlations in the information stream

$R(f)$: Main Result

- The power spectral density $R(f)$ can be divided into a continuous part $R_c(f)$ and a discrete part $R_d(f)$

$$R(f) = R_c(f) + R_d(f)$$

- The general expression for the continuous part is

$$\begin{aligned} R_c(f) &= \frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f) - A(f)|^2 \\ &= \left(\frac{1}{T_s} \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 \right) - \frac{|A(f)|^2}{T_s} \end{aligned}$$

- For the discrete part we have

$$R_d(f) = \frac{|A(f)|^2}{T_s^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_s)$$

$R(f)$: Main Result

- Assume now that the average signal $a(t) = 0$ for all t
- It follows that $A(f) = 0$ for all f
- This simplifies the result to

$$R(f) = R_c(f) = R_s \sum_{n=0}^{M-1} P_n \, |S_n(f)|^2 = R_s \; E\{|S_{m[n]}(f)|^2\}$$

- These **general results** can also be used to study the consequences that technical errors or impairments in the transmitter can have on the frequency spectrum
- We will now consider various **special cases** used in practice

$R(f)$: Binary Signaling

- In the general binary case, i.e., $M = 2$, we have

$$A(f) = P_0 \, S_0(f) + P_1 \, S_1(f)$$

- This simplifies the expression for the power spectral density to

$$\begin{aligned} R(f) &= R_c(f) && + R_d(f) \\ &= \frac{P_0 P_1}{T_b} |S_0(f) - S_1(f)|^2 && + \frac{|P_0 \, S_0(f) + P_1 \, S_1(f)|^2}{T_b^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_b) \end{aligned}$$

(derivation in Ex. 2.20)

- We will now consider some examples from the compendium

Example 2.21

Assume equally likely antipodal signal alternatives, such that

$$s_1(t) = -s_0(t) = g(t)$$

where $g(t) = g_{rec}(t)$, *and* $g_{rec}(t)$ *is given in (D.1). Assume also that* $T \leq T_b$.

i) *Calculate the power spectral density* $R(f)$.

ii) *Calculate* ***the bandwidth W defined as the one-sided width of the mainlobe of R(f)***, *if the information bit rate is 10* [kbps], *and if* $T = T_b/2$. *Calculate also the bandwidth efficiency* ρ.

iii) *Estimate the attenuation in* dB *of the first sidelobe of* $R(f)$ *compared to* $R(0)$.

- $M = 2$ with equally likely antipodal signaling $s_1(t) = -s_0(t) = g(t)$
- With $P_0 = P_1 = 1/2$ and $S_1(f) = -S_0(f) = G(f)$ we get

$$R(f) = R_b\,|S_1(f)|^2 = R_b\,|S_0(f)|^2 = R_b\,|G(f)|^2$$

- Details for the pulse in Appendix D

Example 2.23

Assume equally likely antipodal signal alternatives below. Assume that $s_1(t) = -s_0(t) = g_{rc}(t)$*, where the time raised cosine pulse* $g_{rc}(t)$ *is defined in (D.18). Assume also that* $T = T_b$.

Find an expression for the power spectral density $R(f)$*. Calculate the bandwidth* W*, defined as the one-sided width of the mainlobe of* $R(f)$*, if* R_b *is 10* [kbps]*. Calculate also the bandwidth efficiency* ρ.

- Same as Example 2.21, but with $g_{rc}(t)$ pulse
- Analogously we get

$$R(f) = R_b \, |G_{rc}(f)|^2$$

- From the one-sided main-lobe we get

$$W = 2/T \ [\mathrm{Hz}]$$

- Bandwidth efficiency $\rho = 1/2 \ [\mathrm{bps/Hz}]$ is the same (why?)

Example 2.24

Assume $P_0 = P_1$ *and that,*

$$s_1(t) = -s_0(t) = g_{rc}(t)\cos(2\pi f_c t)$$

with $T = T_b$, *and* $f_c \gg 1/T$. *Hence, a version of binary PSK signaling is considered here (alternatively binary antipodal bandpass PAM). Calculate* ***the bandwidth W, defined as the double-sided width of the mainlobe around the carrier frequency*** $\boldsymbol{f_c}$. *Assume that the information bit rate is 10* [kbps]. *Calculate also the bandwidth*

- This corresponds to the bandpass case
- Let $g_{hf}(t)$ denote the high-frequency pulse

$$g_{hf}(t) = g_{rc}(t)\cos(2\pi f_c t) \quad \text{and} \quad R(f) = R_b\,|G_{hf}(f)|^2$$

- Using shift operations we get

$$R(f) = R_b \left| \frac{G_{rc}(f+f_c)}{2} + \frac{G_{rc}(f-f_c)}{2} \right|^2$$

- From the two-sided main-lobe we get

$$W = 4/T \text{ [Hz]}$$

EITG05 – Digital Communications

Lecture 4

Bandwidth of Transmitted Signals

Michael Lentmaier
Thursday, September 13, 2018

Fourier transform

$$X(f) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t}\, dt$$
$$= X_{Re}(f) + j\, X_{Im}(f)$$
$$= |X(f)|\, e^{j\varphi(f)}$$

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi f t}\, df$$
$$= \int_{-\infty}^{\infty} |X(f)|\, e^{+j(2\pi f t + \varphi(f))}\, df$$

Some useful Fourier transform properties

$$g(at) \leftrightarrow \frac{1}{|a|}\, G(f/a) \qquad g^*(T-t) \leftrightarrow G^*(f)e^{-j2\pi fT}$$

$$g(-t) \leftrightarrow G(-f) \qquad \delta(t) \leftrightarrow 1$$

$$G(t) \leftrightarrow g(-f) \qquad 1(dc) \leftrightarrow \delta(f)$$

$$g(t-t_0) \leftrightarrow G(f)e^{-j2\pi f t_0} \qquad e^{j2\pi f_c t} \leftrightarrow \delta(f-f_c)$$

$$g(t)e^{j2\pi f_c t} \leftrightarrow G(f-f_c) \qquad \cos(2\pi f_c t) \leftrightarrow \frac{1}{2}\,(\delta(f+f_c)+\delta(f-f_c))$$

$$\frac{d}{dt}\, g(t) \leftrightarrow j2\pi f\; G(f) \qquad \sin(2\pi f_c t) \leftrightarrow \frac{j}{2}\,(\delta(f+f_c)-\delta(f-f_c))$$

$$g^*(t) \leftrightarrow G^*(-f) \qquad \alpha e^{-\pi\alpha^2 t^2} \leftrightarrow e^{-\pi f^2/\alpha^2}$$

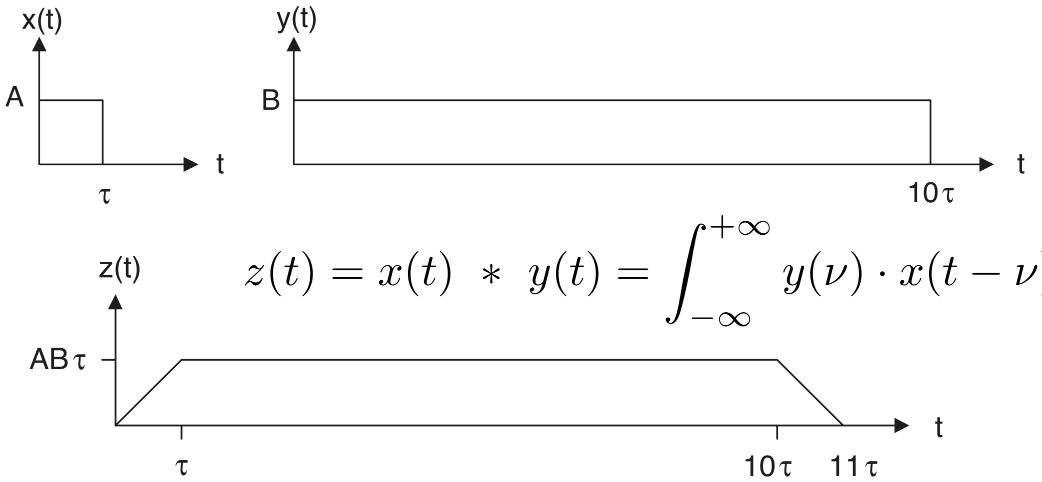
$\rightarrow$ full list in Appendix C of the compendium

Some useful Fourier transform properties

- Consider two signals $x(t)$ and $y(t)$ and their Fourier transforms

$$x(t) \longleftrightarrow X(f) \;, \quad y(t) \longleftrightarrow Y(f)$$

- Recall the convolution operation $z(t) = x(t) * y(t)$:



$$z(t) = x(t) * y(t) = \int_{-\infty}^{+\infty} y(\nu) \cdot x(t-\nu)\, d\nu$$

- **Filtering:**

$$x(t) * y(t) \longleftrightarrow X(f) \cdot Y(f)$$

- **Multiplication:**

$$x(t) \cdot y(t) \longleftrightarrow X(f) * Y(f)$$

Spectrum of time-limited signals

- Consider some time-limited signal $s_T(t)$ of duration T, with $s_T(t) = 0$ for $t < 0$ and $t > T$
- Assume that within the interval $0 \leq t \leq T$, the signal $s_T(t)$ is equal to some signal $s(t)$, i.e.,

$$s_T(t) = s(t) \cdot g_{rec}(t) \ ,$$

  where $g_{rec}(t)$ is the rectangular pulse of amplitude $A = 1$
- Taking the Fourier transform on both sides we get

$$S_T(f) = S(f) \ * \ G_{rec}(f) = S(f) \ * \ AT \ \frac{\sin(\pi f T)}{\pi f T} \ e^{-j\pi f T}$$

- Since $G_{rec}(f)$ is unlimited along the frequency axis, this is the case for $S_T(f)$ as well (convolution increases length)

Time-limited signals can never be strictly band-limited

Some definitions of bandwidth

- **Main-lobe definition:**
  W_{lobe} is defined by the width of the main-lobe of $R(f)$
  This is how we have defined bandwidth in previous examples
- In baseband we use the one-sided width, while in bandpass applications the two-sided width is used (positive frequencies)
- **Percentage definition:**
  W_{99} is defined according to the location of 99% of the power
- For bandpass signals W_{99} is found as the value that satisfies

$$\int_{f_c-W_{99}/2}^{f_c+W_{99}/2} R(f)\,df = 0.99 \int_0^\infty R(f)\,df$$

- Other percentages can be used as well: W_{90}, $W_{99.9}$
- **Nyquist bandwidth**
  Assuming an ideal pulse with finite bandwidth (see Chapter 6)

$$W_{nyq} = \frac{R_s}{2}\ [\mathrm{Hz}]$$

Some definitions of bandwidth

Pulse shape	W_{lobe}	% power in W_{lobe}	W_{90}	W_{99}	$W_{99.9}$	Asymptotic decay
rec	2/T	90.3	1.70/T	20.6/T	204/T	f^{-2}
tri	4/T	99.7	1.70/T	2.60/T	6.24/T	f^{-4}
hcs	3/T	99.5	1.56/T	2.36/T	5.48/T	f^{-4}
rc	4/T	99.95	1.90/T	2.82/T	3.46/T	f^{-6}
Nyquist	R_s	100	$0.9R_s$	$0.99R_s$	$0.999R_s$	ideal

Table 2.1: Double-sided bandwidth results for power spectral densities according to (2.212). The $g_{rec}(t)$, $g_{tri}(t)$, $g_{hcs}(t)$ and $g_{rc}(t)$ pulse shapes are defined in Appendix D, and T denotes the duration of the pulse. The Nyquist pulse shape is not limited in time and it is defined in (D.49) with parameters $\beta = 0$ and $\mathcal{T} = T_s$.

- This table is useful for PAM, PSK, and QAM constellations
- Except bandwidth W, the asymptotic decay is also relevant

Pulse spectrum examples

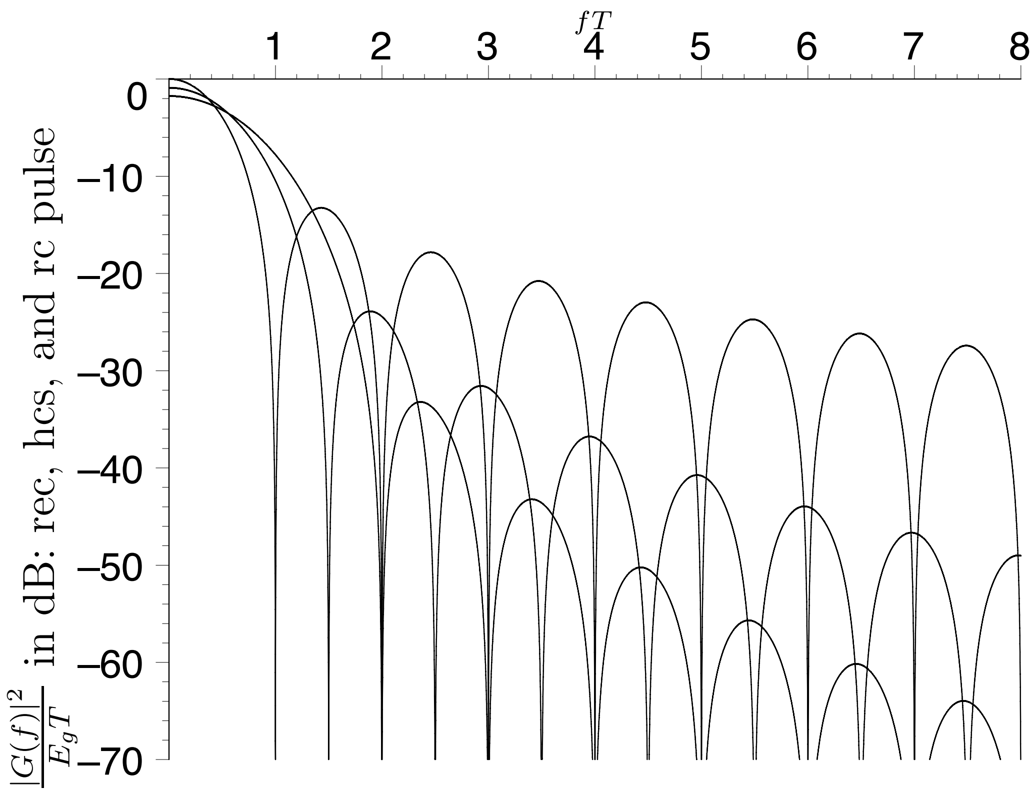


Figure 2.19: $10\log_{10}\left(\frac{|G(f)|^2}{E_g T}\right)$ for the $g_{rec}(t)$, $g_{hcs}(t)$, and $g_{rc}(t)$ pulse shapes. See also Example 2.26.

From last lecture: $R(f)$ for Binary Signaling

- In the general binary case, i.e., $M=2$, we have

$$A(f) = P_0 \; S_0(f) + P_1 \; S_1(f)$$

- This simplifies the expression for the power spectral density to

$$\begin{aligned} R(f) &= R_c(f) && + R_d(f) \\ &= \frac{P_0 P_1}{T_b} |S_0(f) - S_1(f)|^2 && + \frac{|P_0 \; S_0(f) + P_1 \; S_1(f)|^2}{T_b^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_b) \end{aligned}$$

(derivation in Ex. 2.20)

- We will now consider some examples from the compendium

Example 2.21

Assume equally likely antipodal signal alternatives, such that

$$s_1(t) = -s_0(t) = g(t)$$

where $g(t) = g_{rec}(t)$*, and* $g_{rec}(t)$ *is given in (D.1). Assume also that* $T \leq T_b$*.*

i) *Calculate the power spectral density* $R(f)$*.*

ii) *Calculate* ***the bandwidth W defined as the one-sided width of the mainlobe of*** $\boldsymbol{R(f)}$*, if the information bit rate is 10* [kbps]*, and if* $T = T_b/2$*.*
Calculate also the bandwidth efficiency ρ*.*

iii) *Estimate the attenuation in* dB *of the first sidelobe of* $R(f)$ *compared to* $R(0)$*.*

- $M = 2$ with equally likely antipodal signaling $s_1(t) = -s_0(t) = g(t)$
- With $P_0 = P_1 = 1/2$ and $S_1(f) = -S_0(f) = G(f)$ we get

$$R(f) = R_b\,|S_1(f)|^2 = R_b\,|S_0(f)|^2 = R_b\,|G(f)|^2$$

- Details for the pulse in Appendix D

Example 2.23

Assume equally likely antipodal signal alternatives below. Assume that $s_1(t) = -s_0(t) = g_{rc}(t)$*, where the time raised cosine pulse* $g_{rc}(t)$ *is defined in (D.18). Assume also that* $T = T_b$.

Find an expression for the power spectral density $R(f)$*. Calculate the bandwidth* W*, defined as the one-sided width of the mainlobe of* $R(f)$*, if* R_b *is 10* [kbps]*. Calculate also the bandwidth efficiency* ρ.

- Same as Example 2.21, but with $g_{rc}(t)$ pulse
- Analogously we get

$$R(f) = R_b \, |G_{rc}(f)|^2$$

- From the one-sided main-lobe we get

$$W = 2/T \ [\mathrm{Hz}]$$

- Bandwidth efficiency $\rho = 1/2 \ [\mathrm{bps/Hz}]$ is the same (why?)

Example 2.24

Assume $P_0 = P_1$ *and that,*

$$s_1(t) = -s_0(t) = g_{rc}(t)\cos(2\pi f_c t)$$

with $T = T_b$, *and* $f_c \gg 1/T$. *Hence, a version of binary PSK signaling is considered here (alternatively binary antipodal bandpass PAM). Calculate* ***the bandwidth*** $\boldsymbol{W}$***, defined as the double-sided width of the mainlobe around the carrier frequency*** $\boldsymbol{f_c}$. *Assume that the information bit rate is 10* [kbps]. *Calculate also the bandwidth*

- This corresponds to the bandpass case
- Let $g_{hf}(t)$ denote the high-frequency pulse

$$g_{hf}(t) = g_{rc}(t)\cos(2\pi f_c t) \quad \text{and} \quad R(f) = R_b\,|G_{hf}(f)|^2$$

- Using shift operations we get

$$R(f) = R_b \left| \frac{G_{rc}(f+f_c)}{2} + \frac{G_{rc}(f-f_c)}{2} \right|^2$$

- From the two-sided main-lobe we get

$$W = 4/T \;[\text{Hz}]$$

Example: discrete frequencies in $R(f)$

- Assume $M = 2$
- Let $s_0(t) = 0$ and $s_1(t) = 5$ with a pulse duration $T = T_b/2$
- With this the average signal becomes

$$a(t) = \frac{s_0(t) + s_1(t)}{2} = 2.5 \ , \quad 0 \leq t \leq T$$

- We can then write (within the pulse duration T)

$$s_0(t) = -2.5 + a(t) \ , \quad s_1(t) = +2.5 + a(t)$$

Observe:

1. this method is a waste of signal energy since $a(t)$ does not carry any information
2. repetition of $a(t)$ in every symbol interval creates some periodic signal component in the time domain, which leads to discrete frequencies in the frequency domain

From last lecture: general $R(f)$

- The power spectral density $R(f)$ can be divided into a continuous part $R_c(f)$ and a discrete part $R_d(f)$

$$R(f) = R_c(f) + R_d(f)$$

- The general expression for the continuous part is

$$R_c(f) = \frac{1}{T_s} \sum_{n=0}^{M-1} P_n \left| S_n(f) - A(f) \right|^2$$
$$= \left(\frac{1}{T_s} \sum_{n=0}^{M-1} P_n \left| S_n(f) \right|^2 \right) - \frac{|A(f)|^2}{T_s}$$

- For the discrete part we have

$$R_d(f) = \frac{|A(f)|^2}{T_s^2} \sum_{n=-\infty}^{\infty} \delta(f - n/T_s)$$

$R(f)$: M-ary PAM signals

- With M-ary PAM signaling we have

$$s_\ell = A_\ell \, g(t) \; , \quad \ell = 0, 1, \ldots, M-1$$

- Then

$$S_\ell(f) = A_\ell \, G(f) \; , \quad \text{and} \quad A(f) = \sum_{\ell=0}^{M-1} P_\ell \, A_\ell \, G(f)$$

- With this we obtain the simplified expression

$$R(f) = \frac{\sigma_A^2}{T_s} \, |G(f)|^2 \; + \; \frac{m_A^2}{T_s^2} \, |G(f)|^2 \sum_{n=-\infty}^{\infty} \delta(f - n/T_s) \; ,$$

  where m_A denotes the mean and $\sigma_A^2 = \overline{E}_s/E_g - m_A^2$ the variance of the amplitudes A_ℓ
- Assuming zero average amplitude $m_A = 0$ and using $\overline{P} = \sigma_A^2 \, E_g \, R_s$ this reduces to

$$R(f) = R_c(f) = \frac{\sigma_A^2}{T_s} \, |G(f)|^2 = \frac{\overline{P}}{E_g} \, |G(f)|^2$$

Example 2.28

Assume the bit rate $R_b = 9600$ [bps], M-ary PAM transmission and that $m_A = 0$. Determine the (baseband) bandwidth W, defined as the one-sided width of the mainlobe of the power spectral density $R(f)$, if $M = 2$, $M = 4$ and $M = 8$, respectively. Furthermore, assume a rectangular pulse shape with amplitude A_g, and duration $T = T_s$. Calculate also the bandwidth efficiency ρ.

- What is W for a given pulse shape and different M?
- Using $T = T_s$, $m_A = 0$ and $g(t) = g_{rec}(t)$, we have

$$R(f) = \frac{\sigma_A^2}{T_s} |G_{rec}(f)|^2$$

- For the given pulse we get $W = 1/T_s$, where $T_s = k\, T_b$

$$\begin{aligned} k = 1 \quad &\Rightarrow \quad M = 2 \quad \Rightarrow \quad W = 9600[\text{Hz}] \\ k = 2 \quad &\Rightarrow \quad M = 4 \quad \Rightarrow \quad W = 4800[\text{Hz}] \\ k = 3 \quad &\Rightarrow \quad M = 8 \quad \Rightarrow \quad W = 3200[\text{Hz}] \end{aligned}$$

- Bandwidth efficiency: $\rho = R_b/W = k\, T_b/T_b = k$

What does bandwidth efficiency tell us?

In the previous example we had a bandwidth efficiency of

$$\rho = \frac{R_b}{W} = k$$

Saving bandwidth

- The previous example showed that the bandwidth W can be reduced by increasing M
- $T = T_s = kT_b$ increases with M
- $W = 1/T = R_b/k$ decreases accordingly

Improving bit rate

- Assume instead that the bandwidth W is fixed in the same example, i.e., the symbol duration $T_s = T$ is fixed
- Then $R_b = kW$ increases with M
- Assume for example $W = 1\,\mathrm{MHz}$:

  $R_b = 1\,\mathrm{Mbps}$ if $M = 2$ $(k = 1)$

  $R_b = 10\,\mathrm{Mbps}$ if $M = 1024$ $(k = 10)$

$R(f)$: M-ary QAM signals

- With M-ary QAM signaling the signal alternatives are

$$s_\ell(t) = A_\ell \; g(t) \; \cos(2\pi f_c \; t) - B_\ell \; g(t) \; \sin(2\pi f_c \; t) \; , \quad \ell = 0, 1, \ldots, M-1$$

- Then the Fourier transform becomes

$$\begin{aligned} S_\ell(f) &= A_\ell \, \frac{G(f+f_c) + G(f-f_c)}{2} - j\, B_\ell \, \frac{G(f+f_c) - G(f-f_c)}{2} \\ &= (A_\ell - jB_\ell) \, \frac{G(f+f_c)}{2} + (A_\ell + jB_\ell) \, \frac{G(f-f_c)}{2} \end{aligned}$$

- Assuming a zero average signal $a(t) = 0$ and $f_c T \geq 1$ this simplifies to

$$R(f) = R_c(f) = \overline{P} \, \frac{|G(f+f_c)|^2 + |G(f-f_c)|^2}{2\, E_g}$$

$R(f)$: M-ary QAM signals

- Remember that M-ary QAM signals contain M-ary PSK and M-ary bandpass PAM signals as special cases:

$$\text{BP-PAM:} \quad B_\ell = 0$$

$$\text{PSK:} \quad A_\ell = \cos(\nu_\ell)\ , \quad B_\ell = \sin(\nu_\ell)$$

- $\Rightarrow$ our results for $R(f)$ of M-ary QAM signals include these cases

- For symmetric constellations, such that $a(t) = 0$, the simplified version applies
- The bandwidth W is determined by $|G(f - f_c)|^2$ and hence the two-sided main-lobe of $|G(f)|^2$

$\Rightarrow$ if the same pulse $g(t)$ is used then M-ary QAM, M-ary bandpass PAM and M-ary PSK have the same bandwidth W

Example

Bandwidth consumption for BPSK, QPSK and 16-QAM assuming equal R_b and $f_c = 100\,R_b$

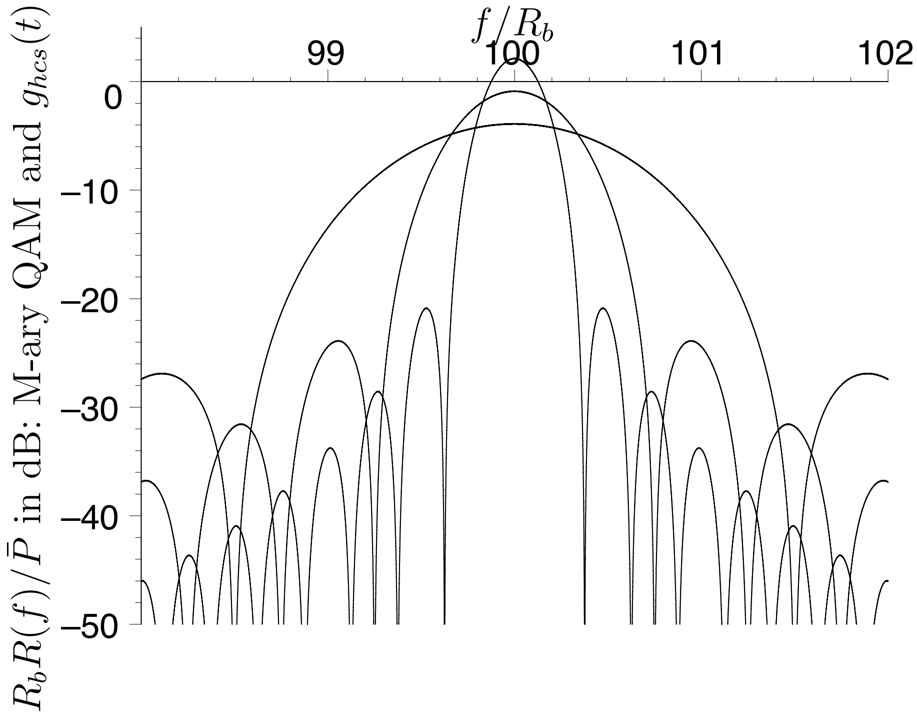


Figure 2.20: The power spectral density for binary QAM (BPSK, widest mainlobe), 4-ary QAM (QPSK), and 16-ary QAM (smallest mainlobe). The figure shows $10\log_{10}(R_b R(f)/\bar{P})$ [dB] in the frequency interval $98R_b \leq f \leq 102R_b$. The carrier frequency is $f_c = 100R_b$ [Hz], and a $T_s = kT_b$ long $g_{hcs}(t)$ pulse is assumed. See also (2.227) and (2.230).

$R(f)$: M-ary FSK signals

- With M-ary frequency shift keying (FSK) signaling the signal alternatives are

$$s_\ell(t) = A \cos(2\pi f_\ell \, t + \nu) \; , \quad 0 \le t \le T_s$$

- Choosing $\nu = -\pi/2$ this can be written as

$$s_\ell(t) = g_{rec}(t) \, \sin(2\pi f_\ell \, t) \; , \quad \text{with } T = T_s \; ,$$

  since $s_\ell(t) = 0$ outside the symbol interval

- The Fourier transform is then

$$S_\ell(f) = j \, \frac{G_{rec}(f + f_\ell) - G_{rec}(f - f_\ell)}{2}$$

- The exact power spectral density $R(f)$ can now be computed by the general formula (2.202)–(2.204)

$R(f)$: M-ary FSK signals

- Let us find an approximate expression for the FSK bandwidth W
- Assume that

$$f_\ell = f_0 + \ell\, f_\Delta \ , \quad \ell = 0, \ldots, M-1$$

- Then the bandwidth W can be approximated by

$$W \approx R_s + f_{M-1} - f_0 + R_s = (M-1) f_\Delta + 2R_s$$

- Consider now orthogonal FSK with $f_\Delta = I \cdot R_s/2$ for some $I > 0$
- The bandwidth efficiency is then

$$\rho = \frac{R_b}{W} \approx \frac{R_b}{(M-1) f_\Delta + 2R_s} = \frac{R_b}{\big((M-1)I/2+2\big)R_s} = \frac{\log_2 M}{(M-1)I/2+2}$$

Observe: the bandwidth efficiency of orthogonal M-ary FSK gets small if M is large
Last week we saw: M-ary FSK has good energy and Euclidean distance properties $\Rightarrow$ trade-off

Example 2.36

Assume that orthogonal M-ary FSK is used to communicate digital information in the frequency band $1.1 \leq f \leq 1.2$ [MHz].

For each M *below, find the largest bit rate that can be used (use bandwidth approximations):*

i) $M = 2$ *ii)* $M = 4$ *iii)* $M = 8$ *iv)* $M = 16$ *v)* $M = 32$

Which of the M*-values above give a higher bit rate than the* $M = 2$ *case?*

Solution:
It is given that $W_{M-FSK} = 100$ [kHz]. *From (2.245), the largest bit rate is obtained with* $I = 1$*:*

$$R_b \approx 10^5 \cdot \frac{\log_2(M)}{(M-1)/2+2}$$

M	$\frac{\log_2(M)}{(M-1)/2+2}$	R_b
2	$\frac{1}{5/2} = 0.4$	40 kbps
4	$\frac{2}{7/2} = \frac{4}{7} \approx 0.5714$	$\approx$ 57 kbps
8	$\frac{3}{11/2} = \frac{6}{11} \approx 0.5455$	$\approx$ 55 kbps
16	$\frac{4}{19/2} = \frac{8}{19} \approx 0.4211$	$\approx$ 42 kbps
32	$\frac{5}{35/2} = \frac{10}{35} \approx 0.2857$	$\approx$ 29 kbps

From this table it is seen that $M = 4, 8, 16$ *give a higher bit rate than* $M = 2$. $\square$

$R(f)$: OFDM-type signals

- An OFDM symbol (signal alternative) $x(t)$ can be modeled as a superposition of N orthogonal QAM signals, each carrying k_n bits, that are transmitted at different frequencies (sub-carriers)

$$x(t) = \sum_{n=0}^{N-1} s_{n,QAM}(t)$$

- Assuming each QAM signal has zero mean and that the different carriers have independent bit streams we get

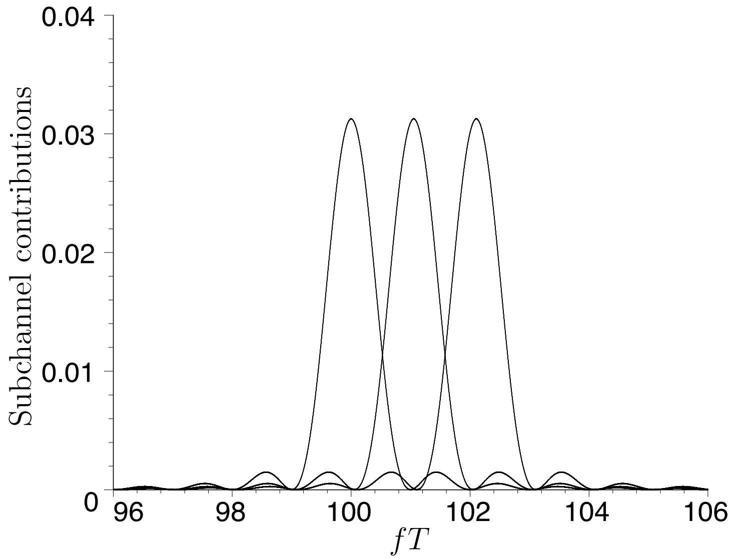
$$R(f) = R_c(f) = R_s\, E\{|X(f)|^2\} = \sum_{n=0}^{N-1} R_n(f)$$

- Using our previous results for QAM in each sub-carrier we get

$$R(f) = R_c(f) = \sum_{n=0}^{N-1} \overline{P}\, \frac{|G(f+f_c)|^2 + |G(f-f_c)|^2}{2E_g}$$

$R(f)$: OFDM-type signals

Illustration of $R_n(f)$ contributed by three neighboring sub-carriers:



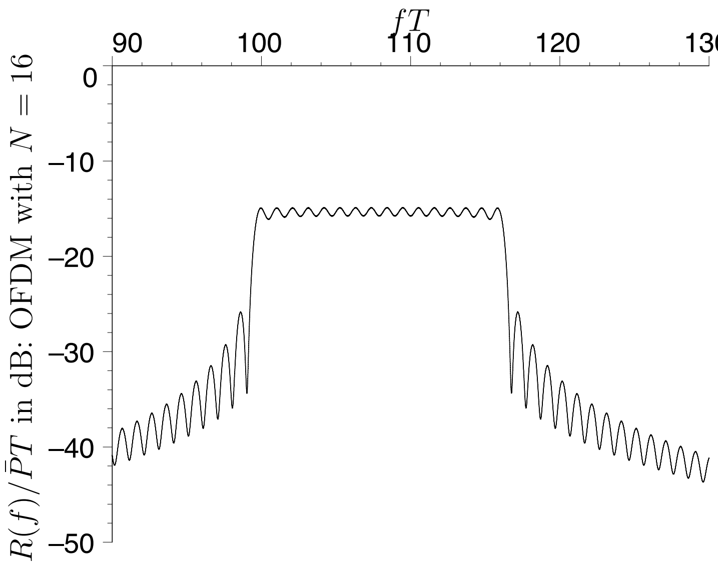
- Assuming $f_n = f_0 + n/(T_s - \Delta_h)$ we can estimate the bandwidth as

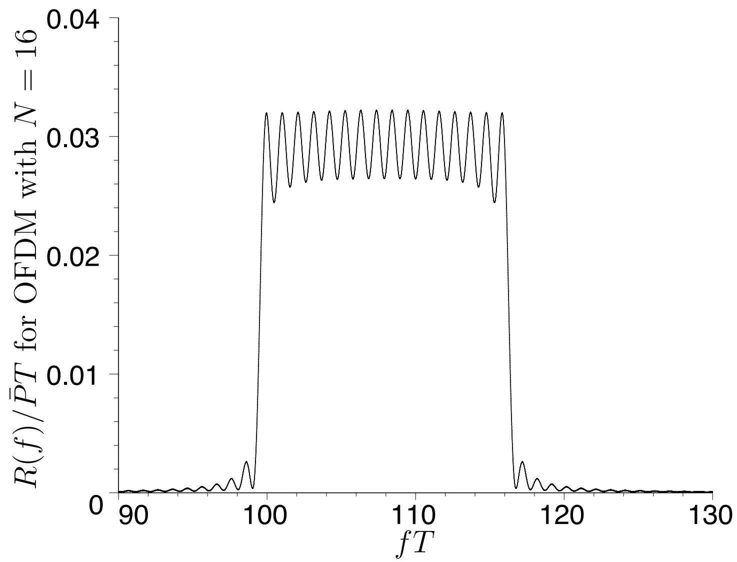
$$W \approx (N+1) f_\Delta = \frac{N+1}{1-\Delta_h/T_s} R_s \approx N \cdot R_s \ , \quad N \gg 1 \ , \ \Delta_h \ll T_s$$

- The bandwidth efficiency is then approximated by

$$\rho = \frac{R_b}{W} = \frac{R_s}{W} \sum_{k=0}^{N-1} k_n \approx \frac{1}{N} \sum_{k=0}^{N-1} k_n \ [\text{bps/Hz}]$$

Example: $R(f)$ for OFDM

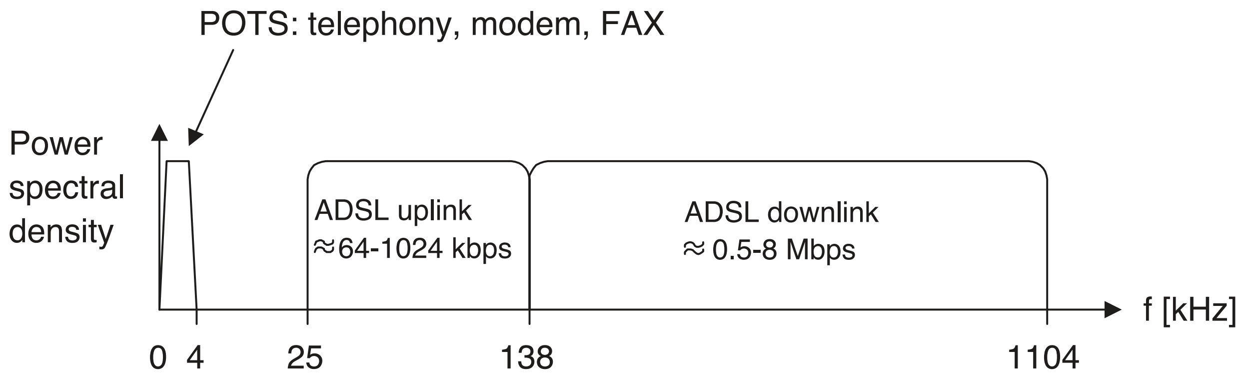




- $N = 16$ sub-carriers
- $T = T_s = 0.1$ [ms]
- $f_\Delta = R_s/0.95 = 10.53$ [kHz]
- $W \approx \frac{17}{0.95} R_s = 179$ [kHz]

Example 2.35

ADSL: uses plain telephone cable (twisted pair, copper)



In ADSL, a coded OFDM technique is used. The level of the power spectral density in the downstream is roughly -73 dB. As a basic example, let us here assume that the OFDM symbol rate in the downlink is 4000 [symbol/s], and that the subchannel carrier spacing is 5 kHz. Furthermore, it is here also assumed that uncoded 16-ary QAM is used in each subchannel (assumes a very "good" communication link).

For the ADSL downlink above, determine the bit rate in each subchannel, the total bit rate, and the bandwidth efficiency.

What about filtering away the side-lobes?

- Let us use a spectral rectangular pulse $X_{srec}(f)$ of amplitude $A = 1$ and width f_Δ to strictly limit the bandwidth
- Similar to the time-limited case we can write

$$S_{f_\Delta}(f) = S(f) \cdot X_{srec}(f)$$

- Taking the inverse Fourier transform on both sides we get

$$s_{f_\Delta}(t) = s(t) \, * \, x_{srec}(t) = s(t) \, * \, A f_0 \, \frac{\sin(\pi f_0 \, t)}{\pi f_0 \, t}$$

- Since $x_{srec}(t)$ is unlimited along the time axis, this is the case for the filtered signal $s_{f_\Delta}(t)$ as well
- The signal $x_{srec}(t)$ defines the ideal Nyquist pulse

As a consequence of filtering, the transmitted symbols will overlap in time domain $\Rightarrow$ inter-symbol-interference (ISI)

Nyquist Pulse

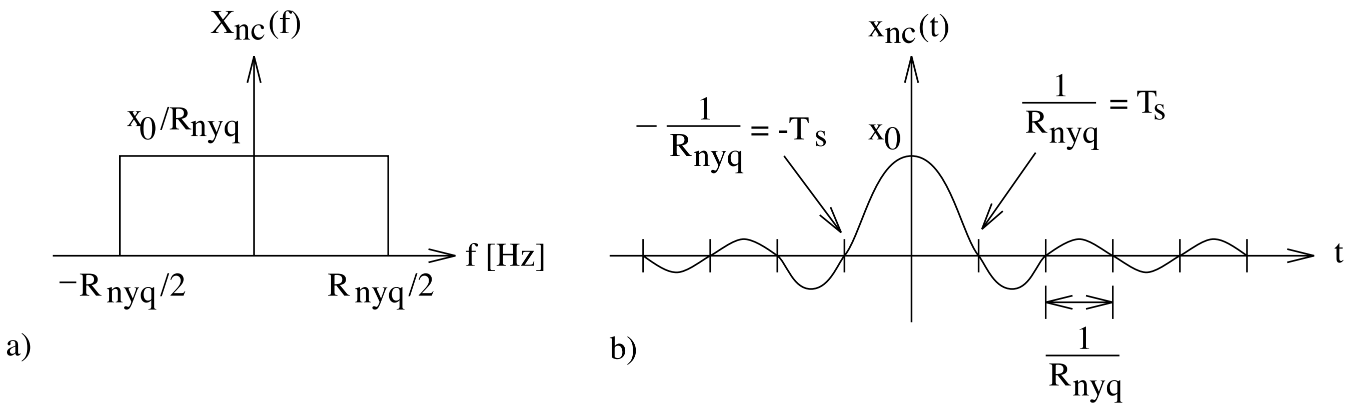


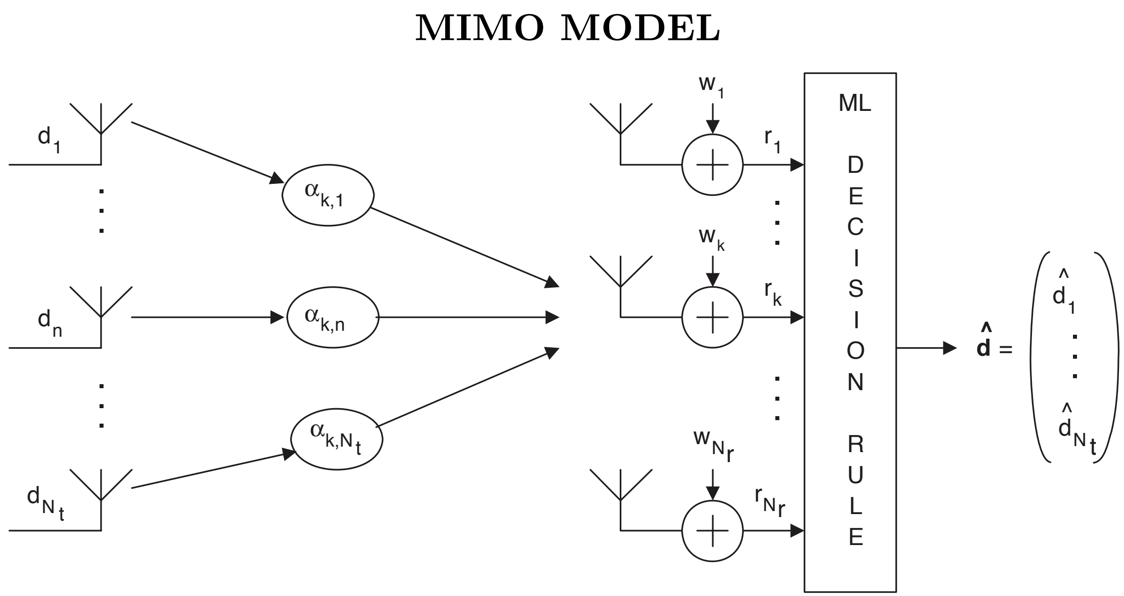
Figure 6.6: a) Ideal Nyquist spectrum; b) Ideal Nyquist pulse.

$$x_{nc}(t) = x_0 \frac{\sin(\pi R_{nyq} t)}{\pi R_{nyq} t}, \quad -\infty \leq t \leq \infty \tag{6.39}$$

$$X_{nc}(f) = \begin{cases} x_0/R_{nyq} & , \quad |f| \leq R_{nyq}/2 \\ 0 & , \quad |f| > R_{nyq}/2 \end{cases} \tag{6.40}$$

The Nyquist pulse and the effect of ISI will be studied in Chapter 6

How can we further improve ρ?



$$r_k = \sum_{n=1}^{N_t} \alpha_{k,n} d_n + w_k$$

- MIMO: multiple-input multiple output
- transmission over multiple antennas in the same frequency band
- challenge: the individual wireless channels interfere
- **5G world record 2016:** (team from Lund involved) spectral efficiency of 145.6 bps/Hz with 128 antennas

EITG05 – Digital Communications

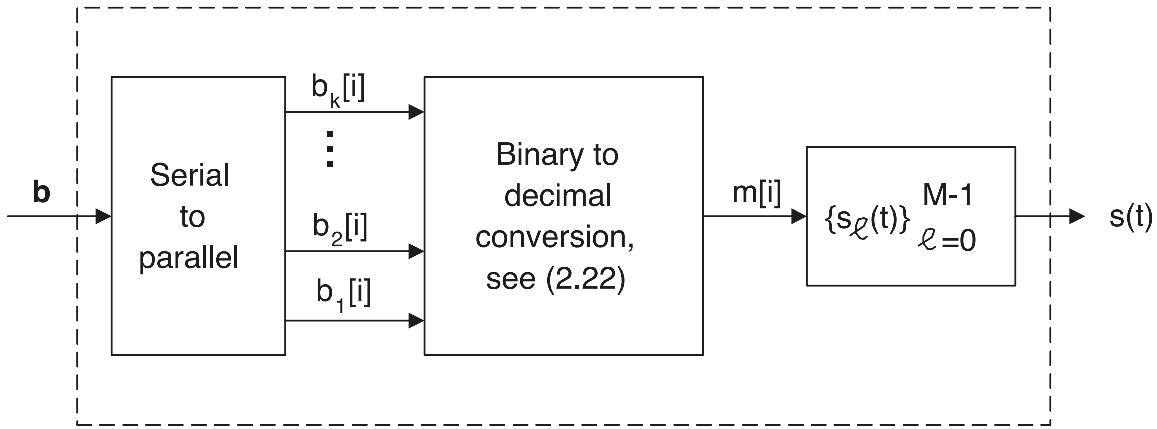
Lecture 5

Receivers in Digital Communication Systems

Michael Lentmaier
Monday, September 17, 2018

Where are we now?

What we have done so far: (Chapter 2)



- Concepts of digital signaling: bits to analog signals
- Average symbol energy $\overline{E}_s$, Euclidean distance $D_{i,j}$
- Bandwidth of the transmit signal

Chapter 4: Receivers

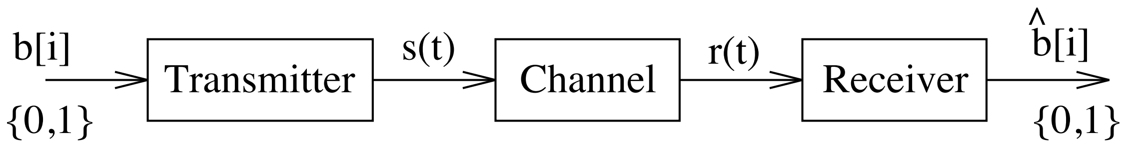
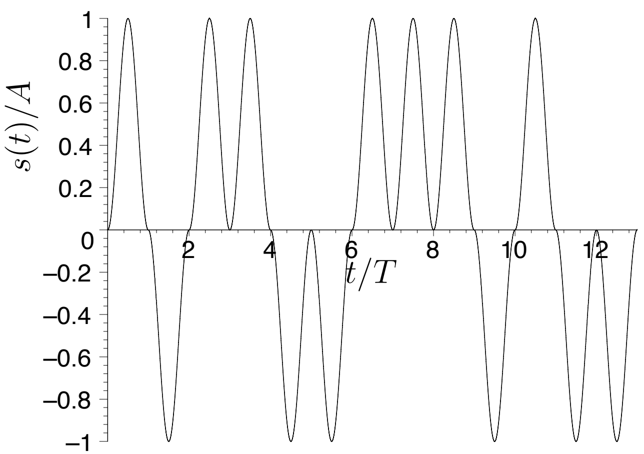
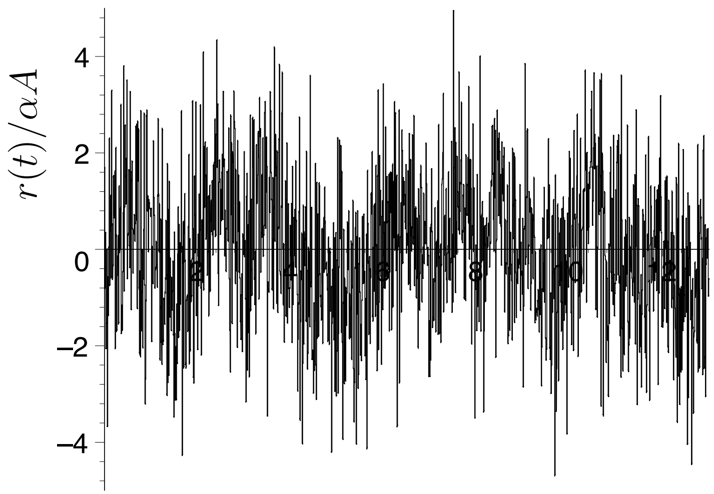


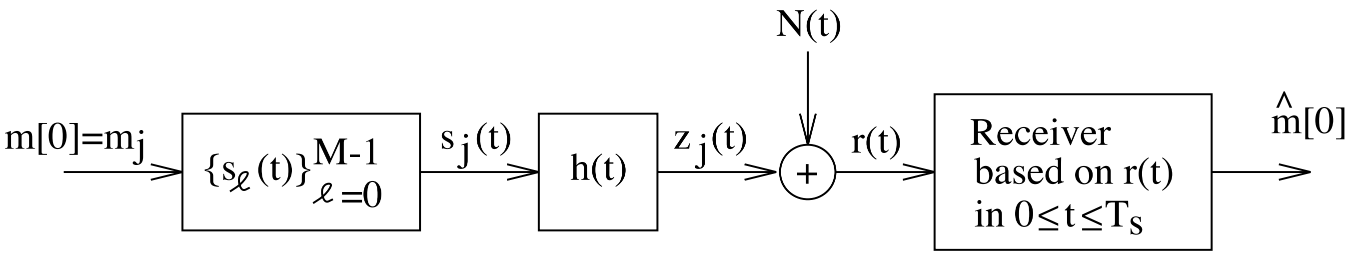
Figure 4.1: A digital communication system.





- How can we estimate the transmitted sequence?
- Is there an optimal way to do this?

The Detection Problem



Assumptions:

- A random (i.i.d.) sequence of messages $m[i]$ is transmitted
- There are $M = 2^k$ possible messages, i.e., k bits per message
- All signal alternatives $z_\ell(t)$, $\ell = 1, \ldots, M$ are known by the receiver
- T_s is chosen such that the signal alternatives $z_\ell(t)$ do not overlap
- $N(t)$ is additive white Gaussian noise (AWGN) with $R_N(f) = N_0/2$

Questions:

- How should decisions be made at the receiver?
- What is the resulting bit error probability P_b?

An optimal decision strategy

- Suppose we want to minimize the symbol error probability P_s
- That means we maximize the probability of a correct decision

$$Pr\{m = \hat{m}(r(t)) \mid r(t)\}$$

  where m denotes the transmitted message

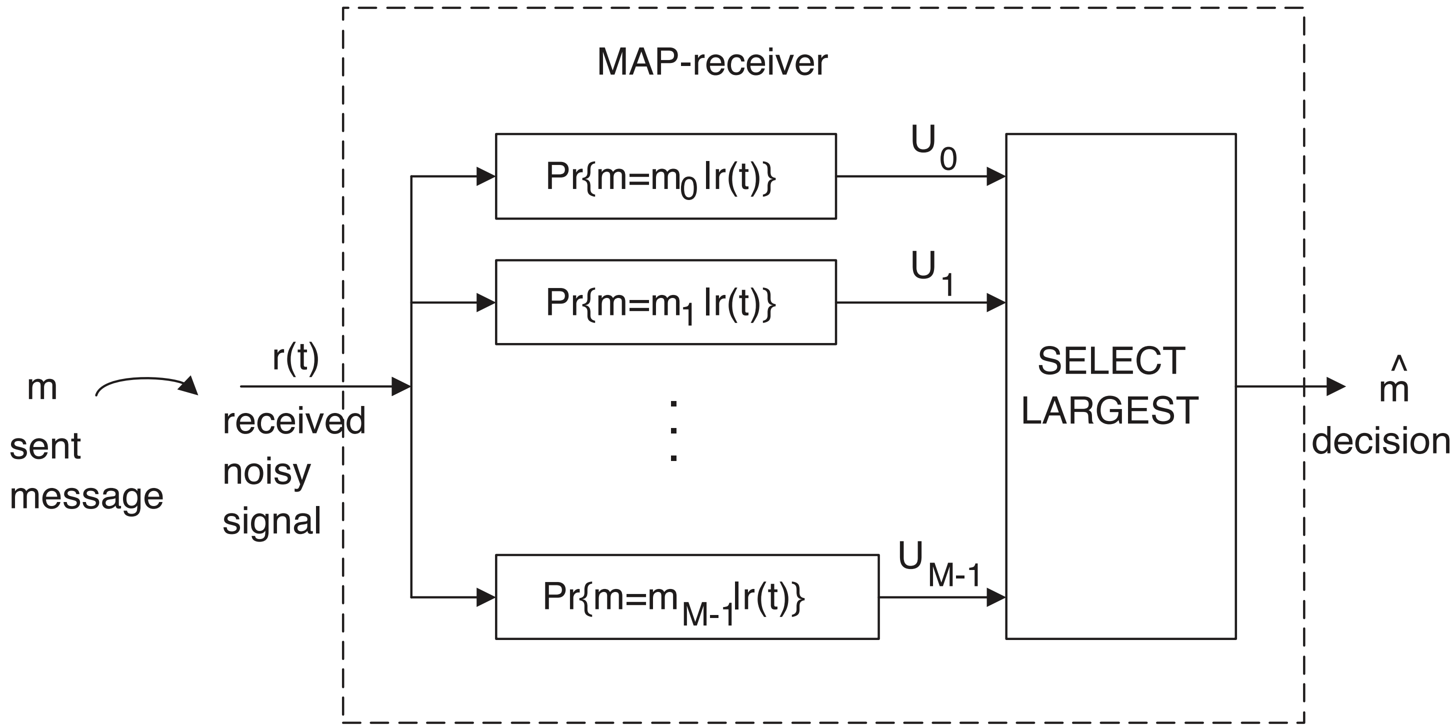
- This leads to the following decision rule:

$$\hat{m}(r(t)) = m_\ell \ ,$$
$$\text{where } \ \ell = \arg\max_i Pr\{m = m_i | r(t)\}$$

- We decide for the message that maximizes the probability above
- A receiver that is based on this decision rule is called maximum-a-posteriori probability (MAP) receiver

Structure of the general MAP receiver

- We know that one of the M messages must be the best
- Hence we can simply test each m_ℓ, $\ell = 0, 1, \ldots, M-1$



This receiver minimizes the symbol error probability P_s

A slightly different decision strategy

- The maximum likelihood (ML) receiver is based on a slightly different decision rule:

$$\hat{m}(r(t)) = m_\ell : \quad \ell = \arg\max_i Pr\{r(t) \,|\, m_i \text{ sent}\}$$

- Using the Bayes rule we can write

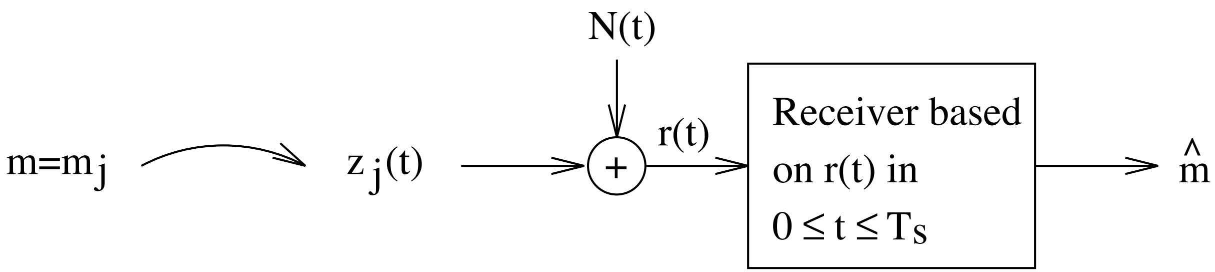
$$Pr\{m = m_i \mid r(t)\} = \frac{Pr\{r(t) \,|\, m_i \text{ sent}\} \cdot P_i}{Pr\{r(t)\}}$$

- The decision rule of the MAP receiver can be formulated as

$$\hat{m}(r(t)) = m_\ell : \quad \ell = \arg\max_i Pr\{r(t) \,|\, m_i \text{ sent}\} \cdot P_i$$

- It follows that the ML receiver is equivalent to the MAP receiver for equally likely messages, $P_i = 1/M, i = 0, 1, \ldots, M-1$.

The Minimum Euclidean Distance Receiver



- For our considered scenario with Gaussian noise: maximizing $Pr\{r(t)\,|\,m_i$ sent$\}$ is equivalent to minimizing the squared Euclidean distance $D^2_{r,i}$.
- The received signal is compared with all noise-free signals $z_i(t)$ in terms of the squared Euclidean distance

$$D^2_{r,i} = \int_0^{T_s} \big(r(t) - z_i(t)\big)^2 \, dt$$

- The message is selected according to the decision rule:

$$\hat{m}(r(t)) = m_\ell : \quad \ell = \arg\min_i \; D^2_{r,i}$$

The Minimum Euclidean Distance Receiver

- The squared Euclidean distance is a measure of similarity
- An implementation is often based on correlators with output

$$\int_0^{T_s} r(t)\, z_i(t)\; dt\;, \quad i = 0, 1, \ldots, M-1$$

- Using

$$D^2_{r,i} = \int_0^{T_s} \big(r(t) - z_i(t)\big)^2\; dt = E_r - 2\int_0^{T_s} r(t)\, z_i(t)\; dt + E_i$$

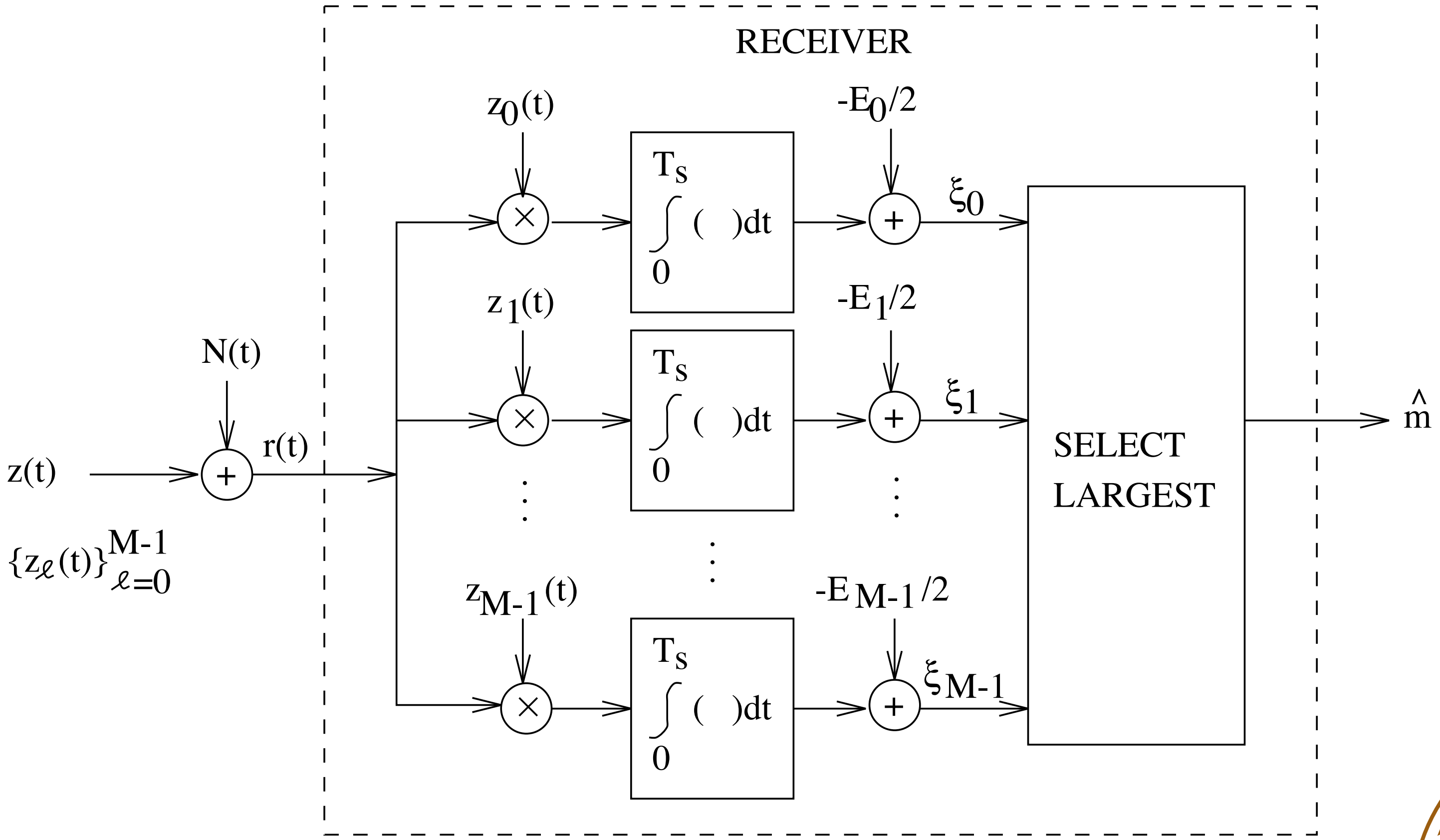
  we can write

$$\ell = \arg\min_i\; D^2_{r,i} = \arg\max_i\; \int_0^{T_s} r(t)\, z_i(t)\; dt - E_i/2$$

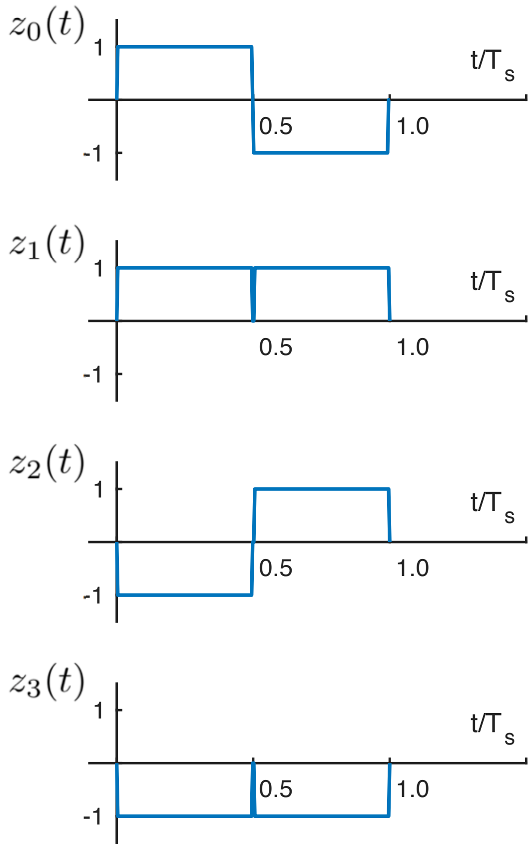
- The received signal is compared with all possible noise-free signal alternatives $z_i(t)$
  The receiver needs to know the channel!

Correlation based implementation

$$\ell = \arg\min_i \; D^2_{r,i} = \arg\max_i \; \int_0^{T_s} r(t)\, z_i(t)\; dt - E_i/2$$

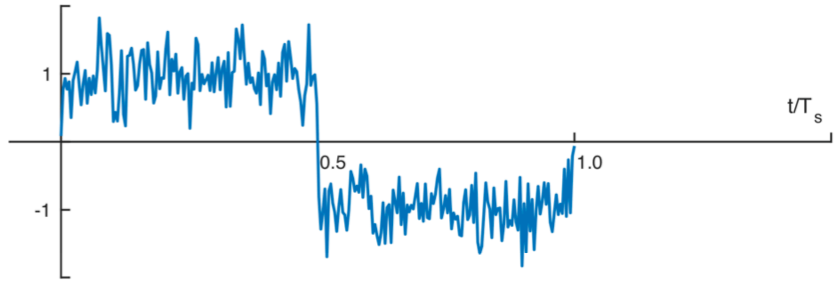


Example: $M = 4$



$E_0 = E_1 = E_2 = E_3 = E$

$r(t) = z_0(t) + N(t)$



$\xi_0 = 0.4778\,E$

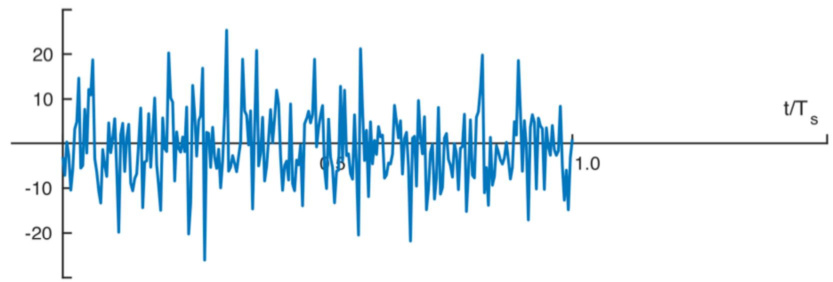
$\xi_1 = -0.5011\,E$

$\xi_2 = -1.4754\,E$

$\xi_3 = -0.4989\,E$

Stronger noise:

$r(t) = z_0(t) + N(t)$



$\xi_0 = 0.2187\,E$

$\xi_1 = -1.4575\,E$

$\xi_2 = -1.2447\,E$

$\xi_3 = 0.4575\,E$

$\Rightarrow$ wrong decision: $\hat{m} = 3$

Example 4.4: 64-QAM receiver

Assume that $\{z_\ell(t)\}_{\ell=0}^{M-1}$ *is a 64-ary QAM signal constellation. Draw a block-diagram of a minimum Euclidean distance receiver that uses only* ***two*** *integrators.*

Solution:
A QAM signal alternative can be written as $z_i(t) = A_i g(t)\cos(\omega_c t) - B_i g(t)\sin(\omega_c t)$, *where* $g(t)$ *is a baseband pulse. The output value from the i:th correlator in Figure 4.8 is,*

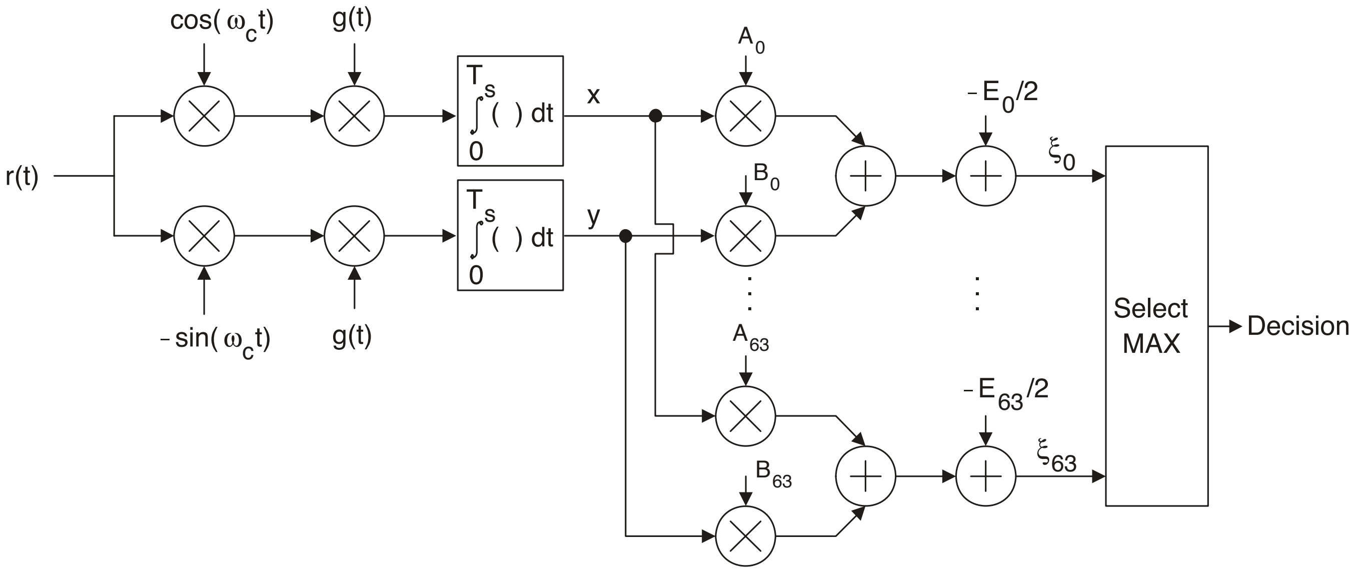
$$\begin{aligned}\int_0^{T_s} r(t)z_i(t)dt &= A_i \underbrace{\int_0^{T_s} r(t)g(t)\cos(\omega_c t)dt}_{x} - B_i \underbrace{\int_0^{T_s} r(t)g(t)\sin(\omega_c t)dt}_{-y} = \\ &= A_i x + B_i y\end{aligned}$$

Observe that x *and* y *do not depend on the index* i.
Hence, a possible implementation of the receiver is to ***first*** *generate* x *and* y, *and then calculate the* M *correlations* $A_i x + B_i y$, $i = 0,\ i, \ldots, M-1$. *By subtracting the value* $E_i/2$ *from the i:th correlation, the decision variables* $\xi_0, \ldots, \xi_{M-1}$ *are finally obtained.*

For M-ary constellations with fixed pulse shape $g(t)$ the implementation can be further simplified

Example 4.4: 64-QAM receiver

The implementation of this receiver is shown below:



The complexity of this receiver is significantly reduced compared to the receiver in Figure 4.8 on page 241! Only two integrators are here used, instead of 64 ($= M$) in Figure 4.8.

- pulse shape and carrier waveform are recreated at the receiver
  $\Rightarrow$ these parts are very similar to the transmitter
- integration and comparison can be performed separately

A geometric interpretation

- Our receiver computes: (maximum correlation)

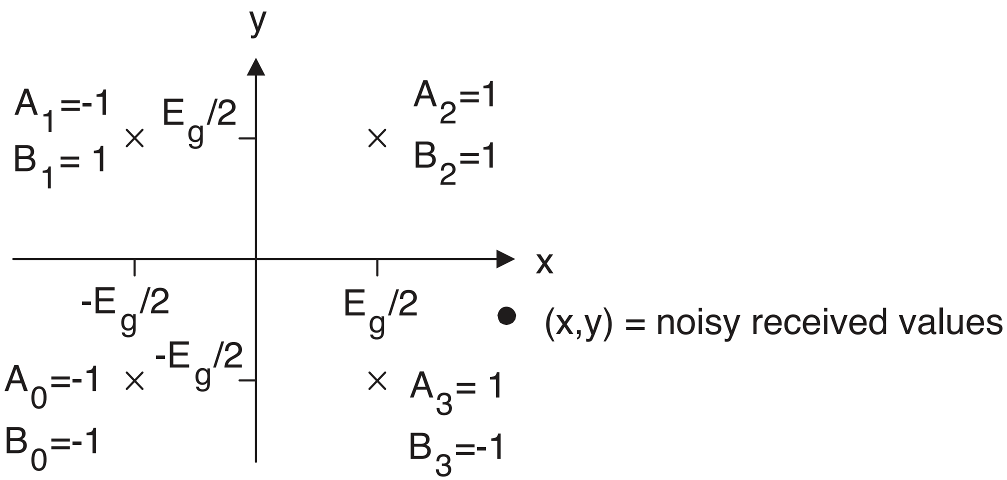
$$\max_i \{xA_i + yB_i - E_g/2\}$$

- Equivalently we can compute: (minimum Euclidean distance)

$$\min_i \left\{ \left(x - \frac{A_i E_g}{2}\right)^2 + \left(y - \frac{B_i E_g}{2}\right)^2 \right\}$$

Ex. QPSK: received point (x,y) is closest to the point of message m_3

x = *message points*, $\bullet$ = *noisy received values* (x,y)



Matched filter implementation

- A filter with impulse response $q(t)$ is matched to a signal $z_i(t)$ if

$$q(t) = z_i(-t + T_s) = z_i(-(t - T_s))$$

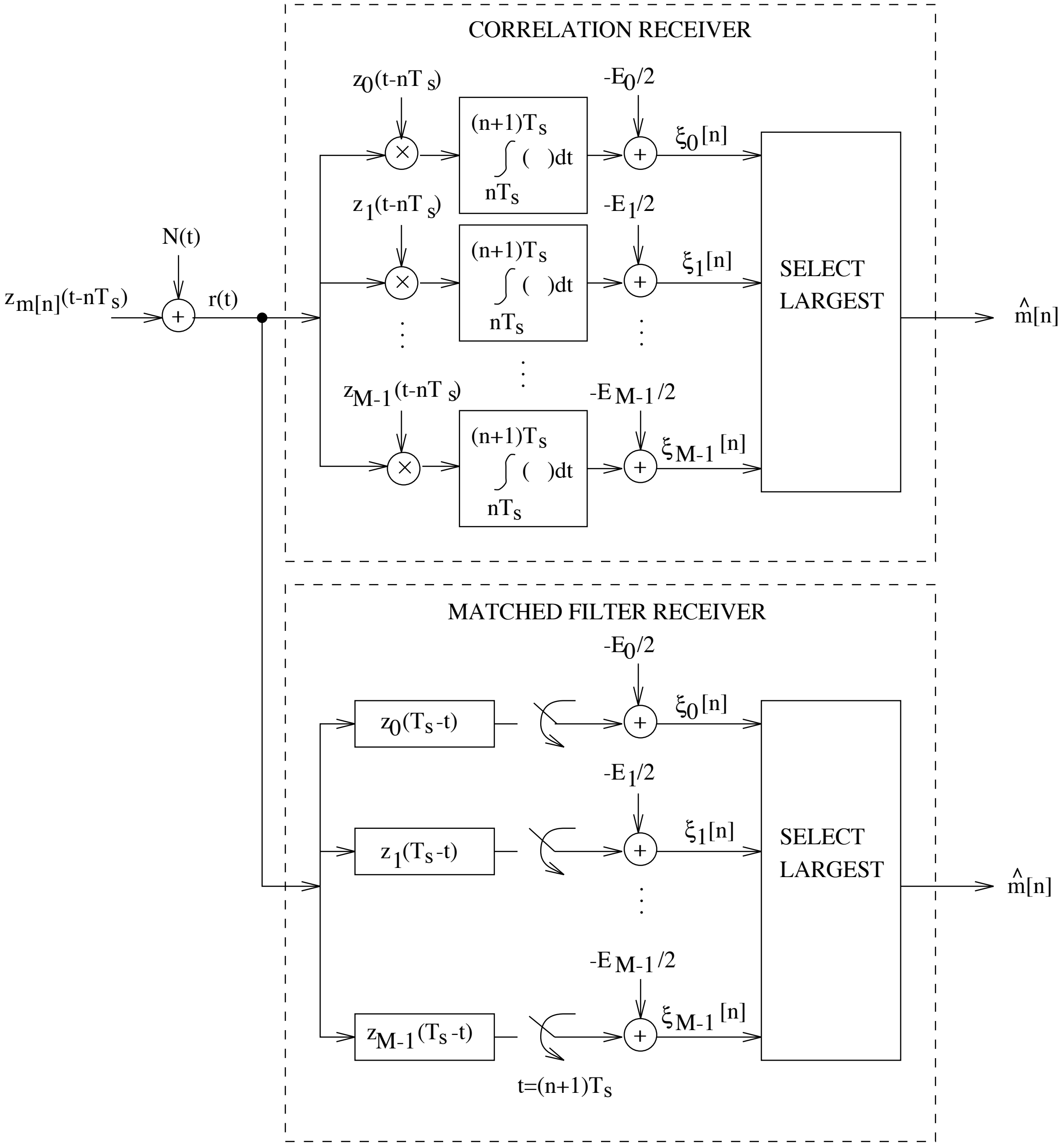
- Let the received signal $r(t)$ enter this matched filter $q(t)$
- The matched filter output, evaluated at time $t = (n+1)T_s$, can be written as

$$r(t) * q(t)\Big|_{t=(n+1)\,T_s} = \int_{n\,T_s}^{(n+1)\,T_s} r(\tau)\, z_i(\tau - n\,T_s)\; d\tau$$

- **Observe:**
  this is exactly the same output value as the correlator produces

$\Rightarrow$ We can replace each correlator with a matched filter which is sampled at times $t = (n+1)\,T_s$

Matched filter vs correlator implementation



Summary: receiver types

- **Minimum Euclidean distance (MED) receiver:**
  decision is based on the signal alternative $z_i(t)$ closest to $r(t)$
- **Correlation receiver:**
  an implementation of the MED receiver based on correlators
- **Matched filter receiver:**
  an implementation of the MED receiver based on matched filters
- **Maximum likelihood (ML) receiver:**
  equivalent to MED receiver under our assumptions: ML = ED
- **Maximum a-posteriori (MAP) receiver:**
  minimizes symbol error probability P_s
  equivalent to ML if $P_i = 1/M$, $i = 0, \ldots, M-1$: ML = ED = MAP

Bit error probability

- Because of the noise the receiver will sometimes make errors
- During a time interval τ we transmit the sequence $\mathbf{b}$ of length

$$B = R_b\,\tau$$

- The detected (estimated) sequence $\hat{\mathbf{b}}$ will contain B_{err} bit errors

$$B_{err} = d_H(\mathbf{b}, \hat{\mathbf{b}}) \leq B$$

- The Hamming distance $d_H(\mathbf{b}, \hat{\mathbf{b}})$ is defined as the number of positions in which the sequences are different

- The bit error probability P_b is defined as

$$P_b = \frac{1}{B}\,\sum_{i=1}^{B} Pr\{\hat{b}[i] \neq b[i]\} = \frac{E\{d_H(\mathbf{b}, \hat{\mathbf{b}})\}}{B}$$

- It measures the average number of bit errors per detected (estimated) information bit

Analysis Binary Signaling

- Binary signaling ($M = 2$, $T_s = T_b$) simplifies the general receiver
- Consider the two decision variables

$$\xi_i[n] = \int_{nT_s}^{(n+1)T_s} r(t)\, z_i(t - nT_s)\; dt - E_i/2\;, \qquad i = 0, 1$$

- The decision $\hat{m}[n]$ is made according to the larger value, i.e.,

$$\xi_1[n] \underset{\hat{m}[n]=m_0}{\overset{\hat{m}[n]=m_1}{\gtrless}} \xi_0[n]$$

- This can be reduced to a single decision variable only

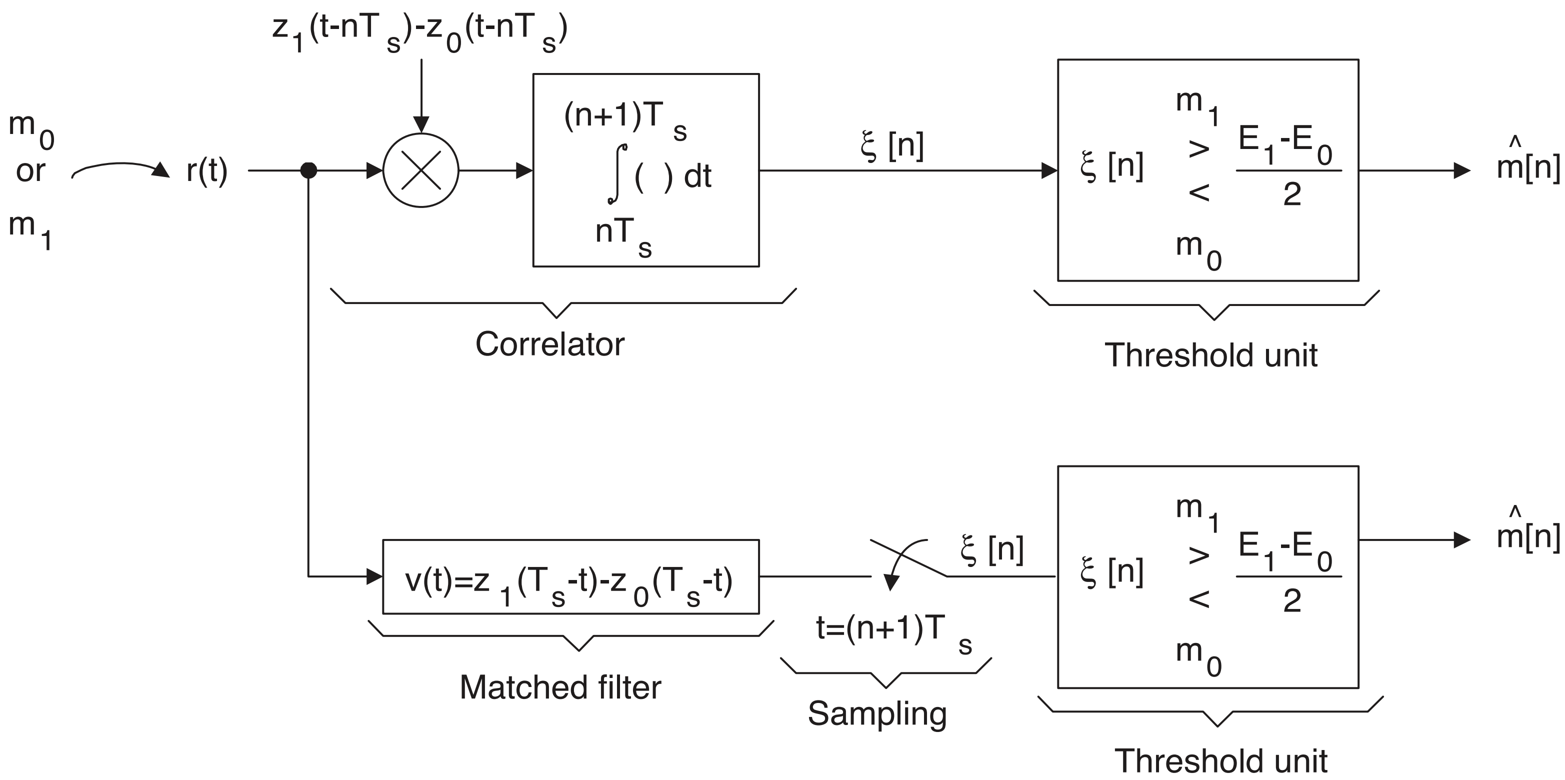
$$\xi[n] = \int_{nT_s}^{(n+1)T_s} r(t)\left(z_1(t - nT_s) - z_0(t - nT_s)\right)\; dt$$

which is compared to a threshold value

$$\xi[n] \underset{\hat{m}[n]=m_0}{\overset{\hat{m}[n]=m_1}{\gtrless}} \frac{E_1 - E_0}{2}$$

Receiver for Binary Signaling

- Only one correlator or one matched filter is now required:



- Matched filter output needs be sampled at correct time

When do we make a wrong decision?

- Assuming $m = m_0$ is sent, the decision variable becomes

$$\xi[n] = \int_0^{T_s} r(t)\big(z_1(t) - z_0(t)\big)\, dt = \int_0^{T_s} \big(z_0(t) + N(t)\big) \cdot \big(z_1(t) - z_0(t)\big)\, dt$$

- We can divide this into a signal component β_0 and a noise component $\mathcal{N}$

$$\xi[n] = \beta_0 + \mathcal{N}$$

$$\beta_0 = \int_0^{T_s} z_0(t)\, \big(z_1(t) - z_0(t)\big)\, dt\ , \quad \mathcal{N} = \int_0^{T_s} N(t)\, \big(z_1(t) - z_0(t)\big)\, dt$$

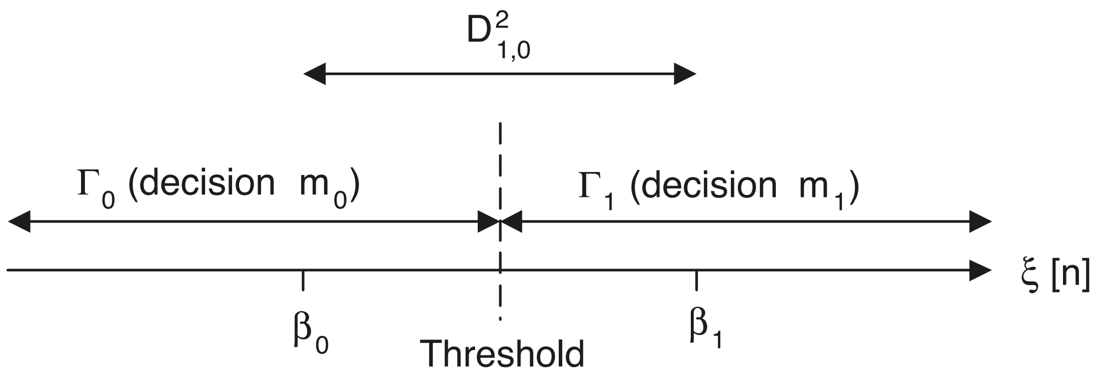
- **Wrong decision:** if $\xi[n] > (E_1 - E_0)/2$ then $\hat{m} = m_1 \neq m_0 = m$

- Analogously, when $m = m_1$ is sent we get

$$\xi[n] = \beta_1 + \mathcal{N}$$

$$\beta_1 = \int_0^{T_s} z_1(t)\, \big(z_1(t) - z_0(t)\big)\, dt$$

Decision regions

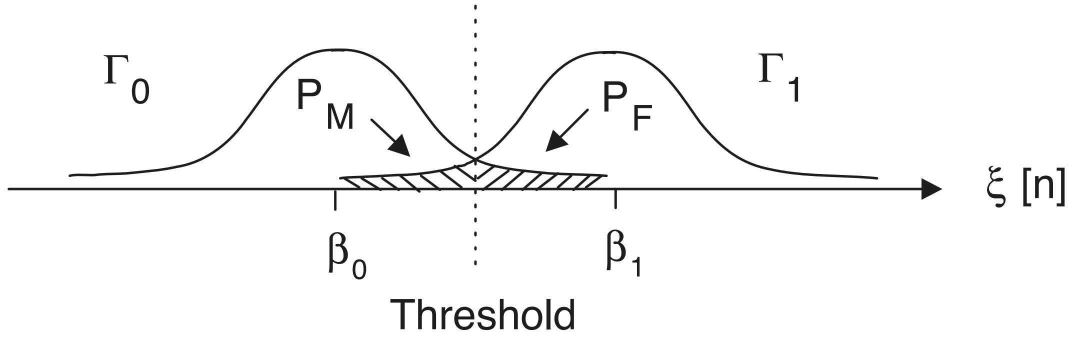


- With
$$\beta_0 + \beta_1 = -\int_0^{T_s} z_0^2(t)\, dt + \int_0^{T_s} z_1^2(t)\, dt = E_1 - E_0$$
the decision threshold lies in the center between β_0 and β_1:
$$\frac{E_1 - E_0}{2} = \frac{\beta_0 + \beta_1}{2}$$

- Furthermore we see that
$$\beta_1 - \beta_0 = \int_0^{T_s} \big(z_1(t) - z_0(t)\big)^2\, dt = D_{1,0}^2 = D_{0,1}^2$$

Probability of a wrong decision

- There exist two ways to make an error:



P_F: false alarm probability $\qquad$ P_M: missed detection probability

- The two probabilities of error can be determined as

$$P_F = Pr\{\hat{m}[n] = m_1 | m = m_0\} = Pr\{\beta_0 + \mathcal{N} > (\beta_0 + \beta_1)/2\}$$

$$P_M = Pr\{\hat{m}[n] = m_0 | m = m_1\} = Pr\{\beta_1 + \mathcal{N} < (\beta_0 + \beta_1)/2\}$$

- We can express these in terms of the $Q(x)$-function:

$$P_F = P_M = Q\left(\frac{\beta_1 - \beta_0}{2\,\sigma}\right)$$

Gaussian Noise

- The noise component $\mathcal{N}$ is a Gaussian random variable with

$$p(\mathcal{N}) = \frac{1}{\sqrt{2\pi\,\sigma^2}}\, e^{-(\mathcal{N}-m)^2/2\sigma^2}$$

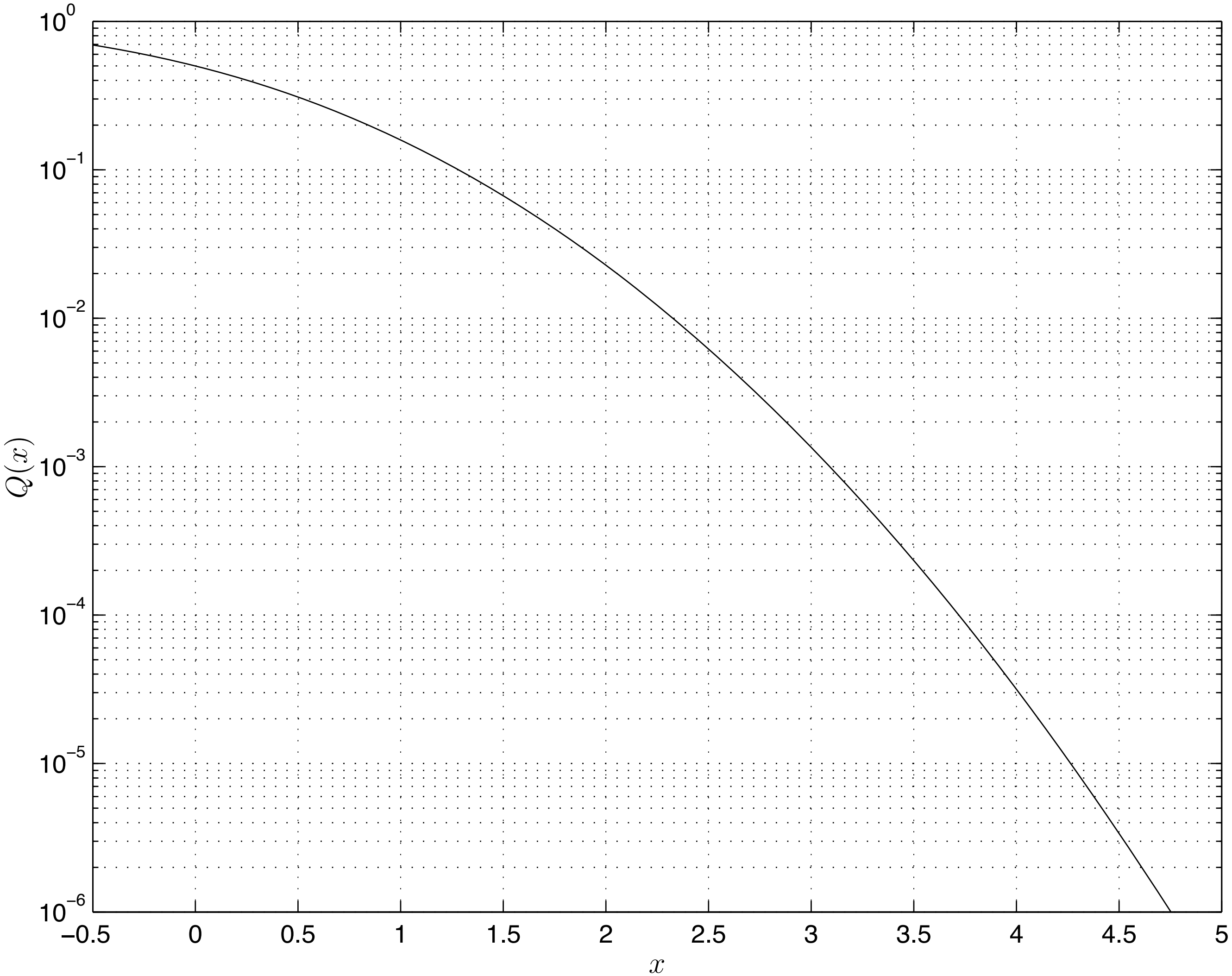
  with mean $m = 0$ and variance $\sigma^2 = N_0/2\, E_v$
- Our bit error probability is related to the probability that the noise value $\mathcal{N}$ is larger than some threshold A

$$Pr\{\mathcal{N} \geq A\} = Pr\left\{\frac{\mathcal{N}-m}{\sigma} \geq \frac{A-m}{\sigma}\right\} = Q\left(\frac{A-m}{\sigma}\right)$$

- The $Q(x)$-function is defined as

$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}}\, e^{-y^2/2}\, dy = \frac{1}{2}\,\mathrm{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

The $Q(x)$-function



The $Q(x)$-function (page 182)

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.0	5.0000e-01	3.0	1.3499e-03	6.0	9.8659e-10	9.0	1.1286e-19
0.1	4.6017e-01	3.1	9.6760e-04	6.1	5.3034e-10	9.1	4.5166e-20
0.2	4.2074e-01	3.2	6.8714e-04	6.2	2.8232e-10	9.2	1.7897e-20
0.3	3.8209e-01	3.3	4.8342e-04	6.3	1.4882e-10	9.3	7.0223e-21
0.4	3.4458e-01	3.4	3.3693e-04	6.4	7.7688e-11	9.4	2.7282e-21
0.5	3.0854e-01	3.5	2.3263e-04	6.5	4.0160e-11	9.5	1.0495e-21
0.6	2.7425e-01	3.6	1.5911e-04	6.6	2.0558e-11	9.6	3.9972e-22
0.7	2.4196e-01	3.7	1.0780e-04	6.7	1.0421e-11	9.7	1.5075e-22
0.8	2.1186e-01	3.8	7.2348e-05	6.8	5.2310e-12	9.8	5.6293e-23
0.9	1.8406e-01	3.9	4.8096e-05	6.9	2.6001e-12	9.9	2.0814e-23
1.0	1.5866e-01	4.0	3.1671e-05	7.0	1.2798e-12	10.0	7.6199e-24
1.1	1.3567e-01	4.1	2.0658e-05	7.1	6.2378e-13		
1.2	1.1507e-01	4.2	1.3346e-05	7.2	3.0106e-13		
1.3	9.6800e-02	4.3	8.5399e-06	7.3	1.4388e-13		
1.4	8.0757e-02	4.4	5.4125e-06	7.4	6.8092e-14		
1.5	6.6807e-02	4.5	3.3977e-06	7.5	3.1909e-14		
1.6	5.4799e-02	4.6	2.1125e-06	7.6	1.4807e-14		
1.7	4.4565e-02	4.7	1.3008e-06	7.7	6.8033e-15		
1.8	3.5930e-02	4.8	7.9333e-07	7.8	3.0954e-15		
1.9	2.8717e-02	4.9	4.7918e-07	7.9	1.3945e-15		
2.0	2.2750e-02	5.0	2.8665e-07	8.0	6.2210e-16		
2.1	1.7864e-02	5.1	1.6983e-07	8.1	2.7480e-16		
2.2	1.3903e-02	5.2	9.9644e-08	8.2	1.2019e-16		
2.3	1.0724e-02	5.3	5.7901e-08	8.3	5.2056e-17		
2.4	8.1975e-03	5.4	3.3320e-08	8.4	2.2324e-17		
2.5	6.2097e-03	5.5	1.8990e-08	8.5	9.4795e-18		
2.6	4.6612e-03	5.6	1.0718e-08	8.6	3.9858e-18		
2.7	3.4670e-03	5.7	5.9904e-09	8.7	1.6594e-18		
2.8	2.5551e-03	5.8	3.3157e-09	8.8	6.8408e-19		
2.9	1.8658e-03	5.9	1.8175e-09	8.9	2.7923e-19		

$Q(1.2816) \approx 10^{-1}$	$Q(5.1993) \approx 10^{-7}$
$Q(2.3263) \approx 10^{-2}$	$Q(5.6120) \approx 10^{-8}$
$Q(3.0902) \approx 10^{-3}$	$Q(5.9978) \approx 10^{-9}$
$Q(3.7190) \approx 10^{-4}$	$Q(6.3613) \approx 10^{-10}$
$Q(4.2649) \approx 10^{-5}$	$Q(6.7060) \approx 10^{-11}$
$Q(4.7534) \approx 10^{-6}$	$Q(7.0345) \approx 10^{-12}$

Bit error probability

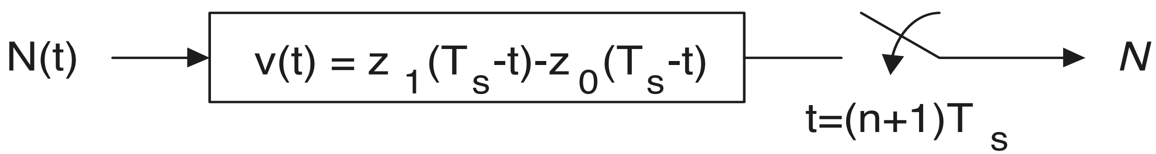
- The bit error probability can be written as

$$P_b = P_0 P_F + P_1 P_M = (P_0 + P_1) P_F = P_F = P_M$$

- With $\beta_1 - \beta_0 = D_{0,1}^2$ and $\sigma^2 = N_0/2 \cdot D_{0,1}^2$ we obtain

$$P_b = Q\left(\frac{\beta_1 - \beta_0}{2\sigma}\right) = Q\left(\frac{D_{0,1}^2}{2\sigma}\right) = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right)$$

- This fundamental result provides the bit error probability P_b of an ML receiver for binary transmission over an AWGN channel
- The additive noise $\mathcal{N}$ is sampled from a filtered noise process



$$\sigma^2 = N_0/2 \cdot E_v = N_0/2 \int_0^{T_s} \big(z_1(t) - z_0(t)\big)^2 \, dt$$

Example

- Let $z_0(t) = 0$ and $z_1(t)$ rectangular with amplitude A and $T = T_b$
- The information bit rate is $R_b = 400$ kbps
- Regarding the noise we know that $A^2/N_0 = 70$ dB

Task: determine the bit error probability P_b

Solution:

- First we find that $D_{0,1}^2 = A^2/R_b$
- Then

$$\frac{D_{0,1}^2}{2N_0} = \frac{A^2}{N_0} \cdot \frac{1}{2R_b} = 12.5$$

- $P_b = Q\left(\sqrt{12.5}\right) = Q(3.536) = 2.3 \cdot 10^{-4}$
- Last step: check Table 3.1 on page 182

An energy efficiency perspective

- Consider the case $P_0 = P_1 = 1/2$
- The average received energy per bit is then

$$\mathcal{E}_b = \frac{1}{2}\int_0^{T_b} z_0^2(t)\,dt + \frac{1}{2}\int_0^{T_b} z_1^2(t)\,dt = \frac{E_0 + E_1}{2}$$

- We can then introduce the normalized squared Euclidean distance

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{1}{2\,\mathcal{E}_b}\int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2\,dt$$

- With this the bit error probability becomes

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameter $d_{0,1}^2$ is a measure of energy efficiency

LUND UNIVERSITY

EITG05 – Digital Communications

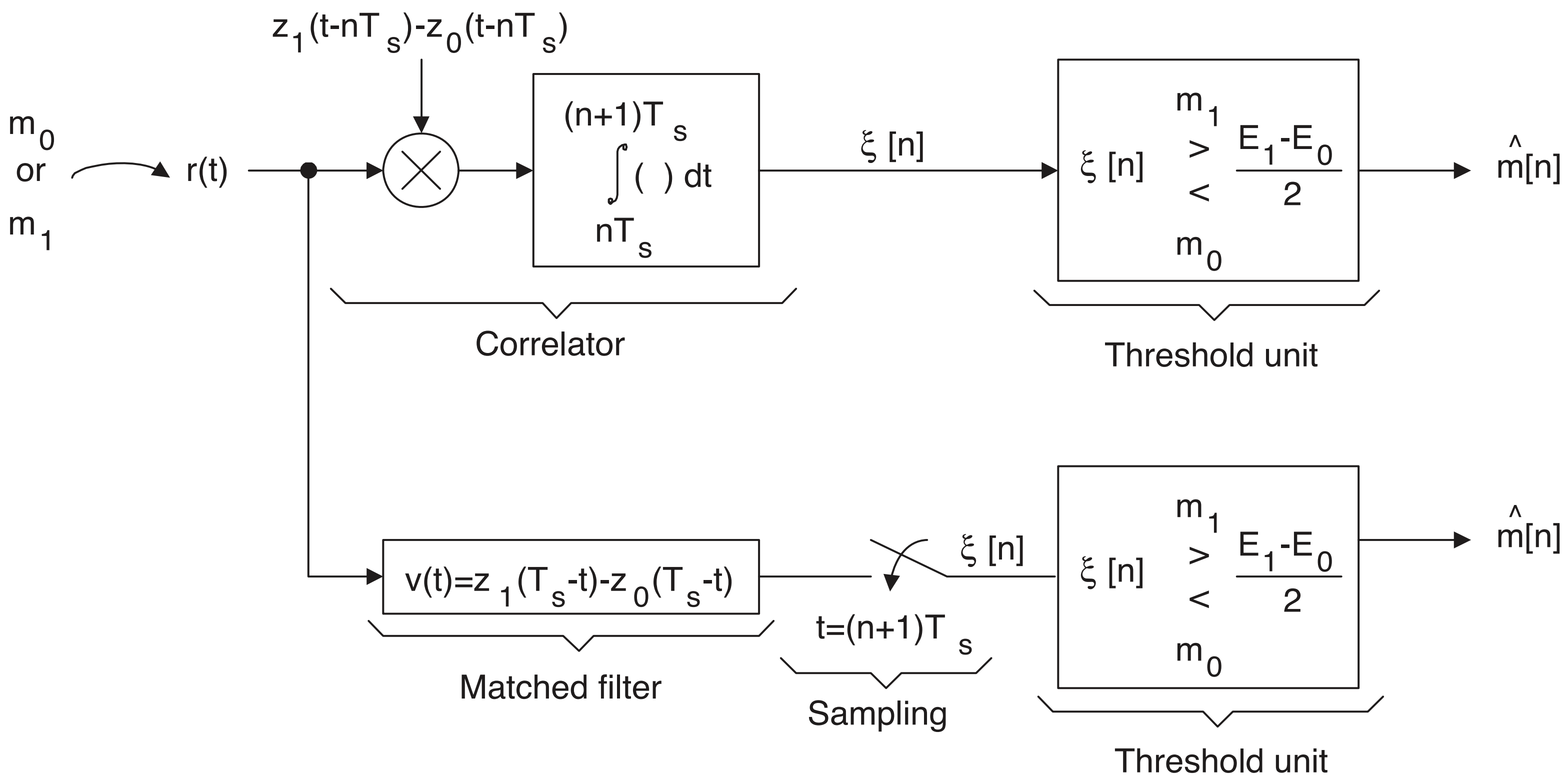
Lecture 6

Receivers continued:
System design criteria, Performance for M-ary signaling

Michael Lentmaier
Monday, September 24, 2018

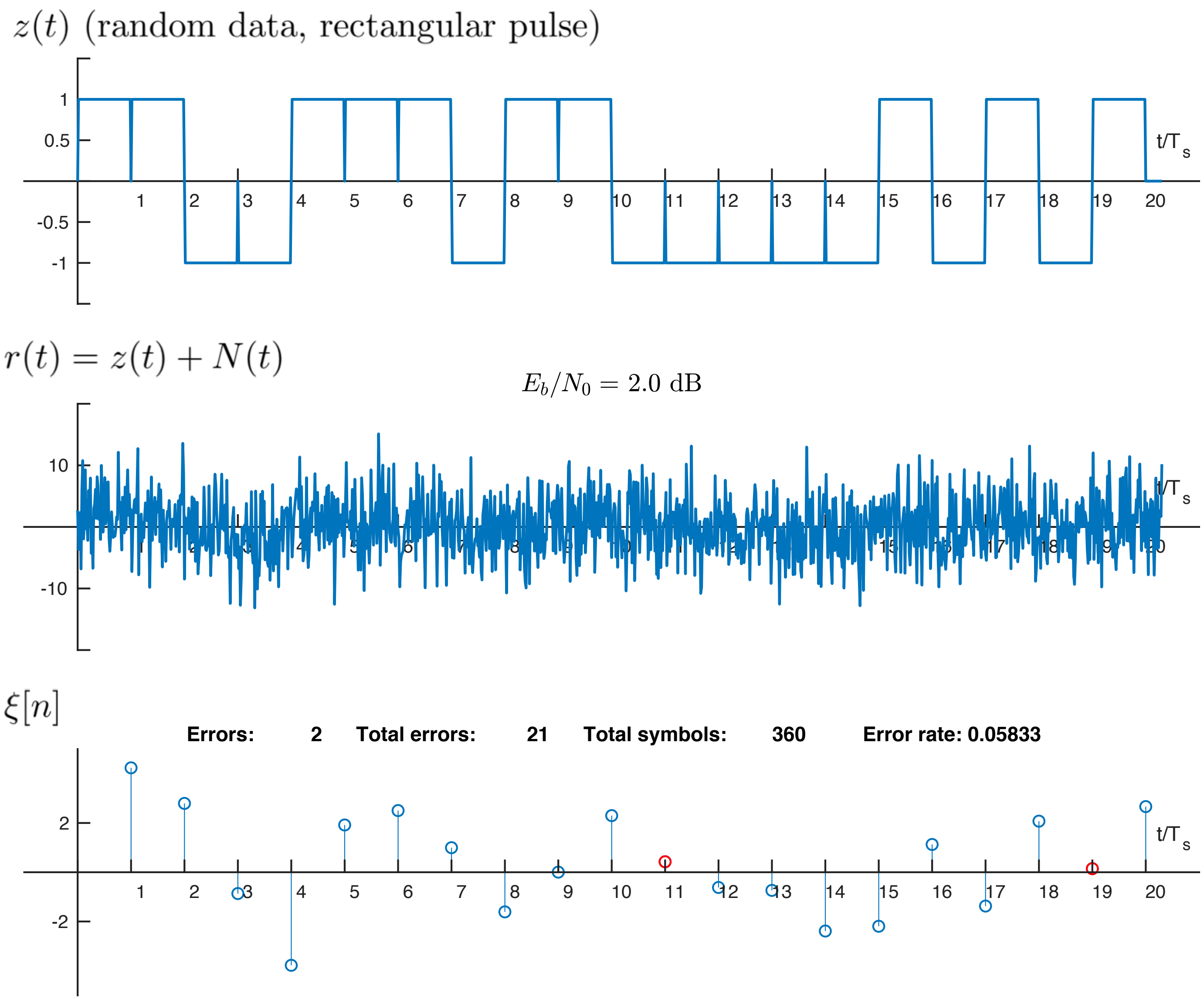
Last week: Analysis Binary Signaling

- Only one correlator or one matched filter is now required:



- Matched filter output needs be sampled at correct time

Example: (see Matlab demo)



An energy efficiency perspective

- Consider the case $P_0 = P_1 = 1/2$
- The average received energy per bit is then

$$\mathcal{E}_b = \frac{1}{2}\int_0^{T_b} z_0^2(t)\, dt \; + \; \frac{1}{2}\int_0^{T_b} z_1^2(t)\, dt = \frac{E_0 + E_1}{2}$$

- We can then introduce the normalized squared Euclidean distance

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{1}{2\,\mathcal{E}_b}\int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2\, dt$$

- With this the bit error probability becomes

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameter $d_{0,1}^2$ is a measure of energy efficiency

Special case 1: antipodal signals

- In case of antipodal signals we have $z_1(t) = -z_0(t)$ and

$$D_{0,1}^2 = \int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2 \, dt = 4 \int_0^{T_b} z_1^2(t) \, dt = 4E$$

- From $E_0 = E_1 = E$ follows

$$\mathcal{E}_b = \frac{E+E}{2} = E$$

and

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{4E}{2E} = 2$$

- The bit error probability for any pair of antipodal signals becomes

$$P_b = Q\left(\sqrt{2\,\frac{\mathcal{E}_b}{N_0}}\right)$$

Special case 2: orthogonal signals

- In case of orthogonal signals we have

$$\int_0^{T_b} z_0(t)\, z_1(t)\; dt = 0$$

and hence (compare page 28)

$$D_{0,1}^2 = \int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2\; dt = E_0 + E_1$$

- This gives

$$\mathcal{E}_b = \frac{E_0 + E_1}{2}$$

and

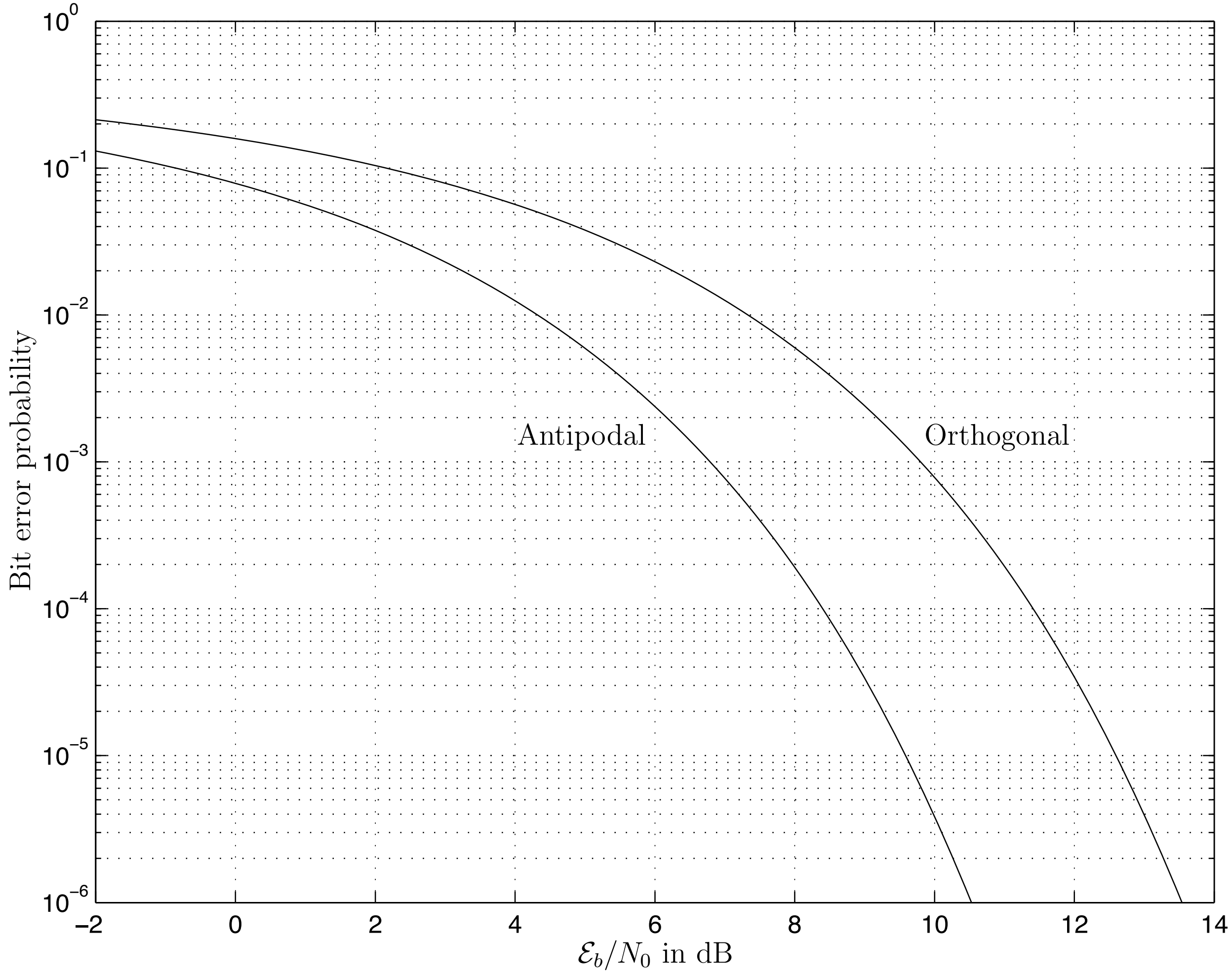
$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{E_0 + E_1}{E_0 + E_1} = 1$$

- The bit error probability for any pair of orthogonal signals is

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

Comparison

Antipodal vs orthogonal signaling:



Larger values of $d^2_{0,1}$ give better energy efficiency

Antipodal vs orthogonal signaling

- There is a constant gap between the two curves
- We can measure the difference in energy efficiency by the ratio

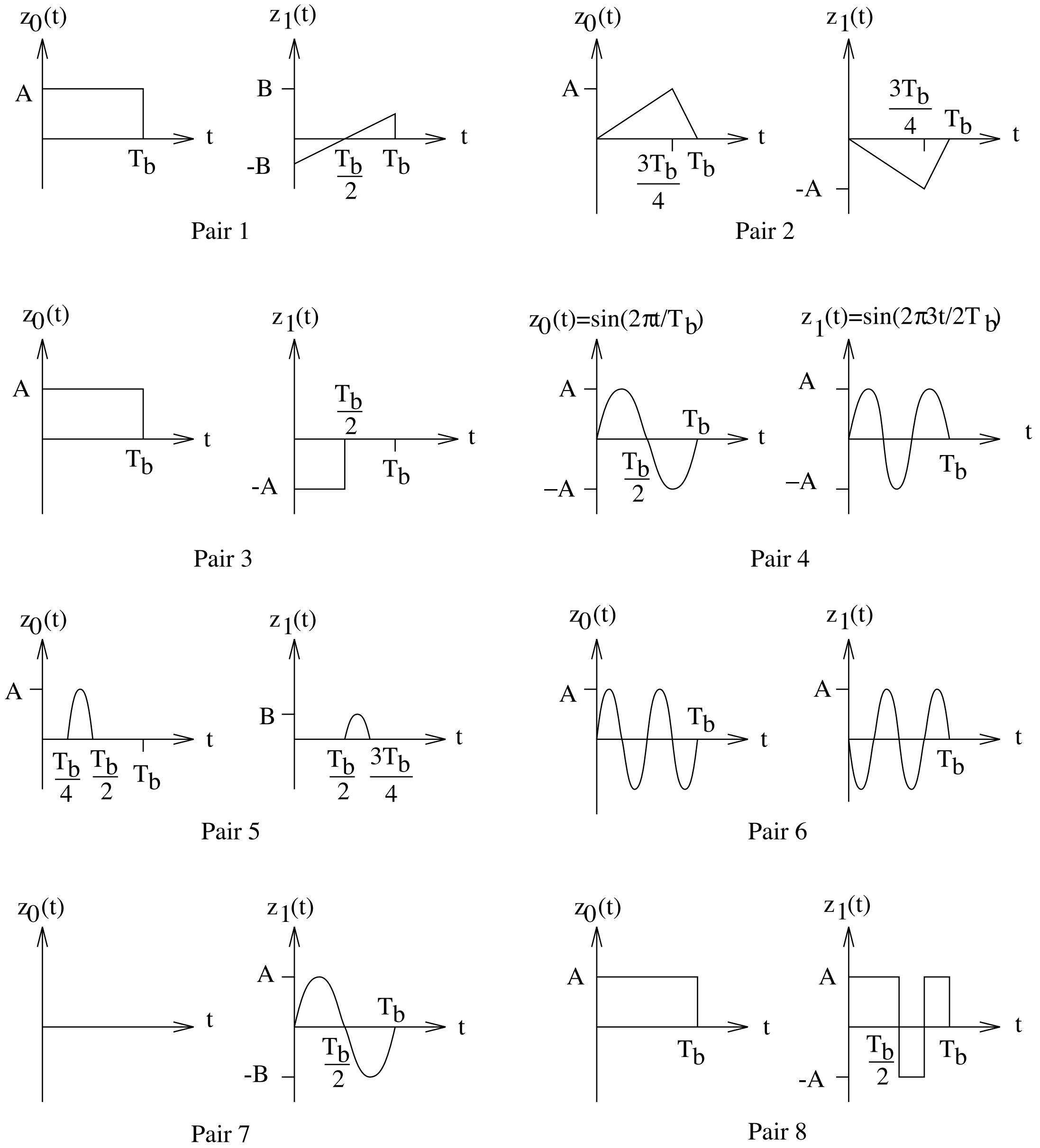
$$\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}} = \frac{d^2_{0,1,ort}}{d^2_{0,1,atp}} = \frac{1}{2}$$

- In terms of dB this corresponds to

$$10\log_{10}\left(\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}}\right) = 10\log_{10}\left(\frac{d^2_{0,1,ort}}{d^2_{0,1,atp}}\right) = -3 \text{ [dB]}$$

$\Rightarrow$ antipodal signaling requires 3 dB less energy for equal P_b

Example 4.11: rank pairs with respect to $d^2_{0,1}$



Can we do better?

- It is possible to show that for two equally likely signal alternatives we always have
$$d_{0,1}^2 \leq 2$$
- Antipodal signaling is hence optimal for binary signaling ($M = 2$)

Remark:

- Channel coding can be used to further increase $d_{0,1}^2$
- Sequences of binary pulses with large separation are designed
- This does not contradict the result from above:
coded binary signals correspond to uncoded signals with $M > 2$

Channel coding can be used for improving energy efficiency
Cost: complexity, latency, (bandwidth)

Relationship between parameters

- The bit error probability can be expressed in different ways

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{P_z}{R_b\,N_0}}\right)$$

- Assuming $z_0(t) = \alpha\, s_0(t)$ and $z_1(t) = \alpha\, s_1(t)$ we also get

$$P_b = Q\left(\sqrt{d_{0,1}^2\,\frac{\alpha^2 \bar{P}_{sent}}{R_b\,N_0}}\right) = Q\left(\sqrt{\frac{d_{0,1}^2}{\rho}\cdot\frac{\alpha^2 \bar{P}_{sent}}{N_0\,W}}\right)$$

- Recall that $\rho = R_b/W$ is the bandwidth efficiency and $N_0\,W$ is the noise power within the bandwidth W

The expression that is most appropriate to use depends on the specific problem to be solved

A "typical" type of problem

- The bit error probability must not exceed a certain level,

$$P_b \leq P_{b,req} = Q(\sqrt{\mathcal{X}})$$

- Example: if $P_{b,req} = 10^{-9}$ then $\mathcal{X} \approx 36$
- **Consequences:**

$$d_{0,1}^2 \frac{\mathcal{E}_b}{N_0} \geq \mathcal{X}$$

$$R_b \leq \frac{d_{0,1}^2}{\mathcal{X}} \cdot \frac{P_z}{N_0}$$

$$R_b \leq \frac{d_{0,1}^2}{\mathcal{X}} \cdot \frac{\alpha^2 \bar{P}_{sent}}{N_0}$$

- Note: the received signal power P_z decreases with communication distance

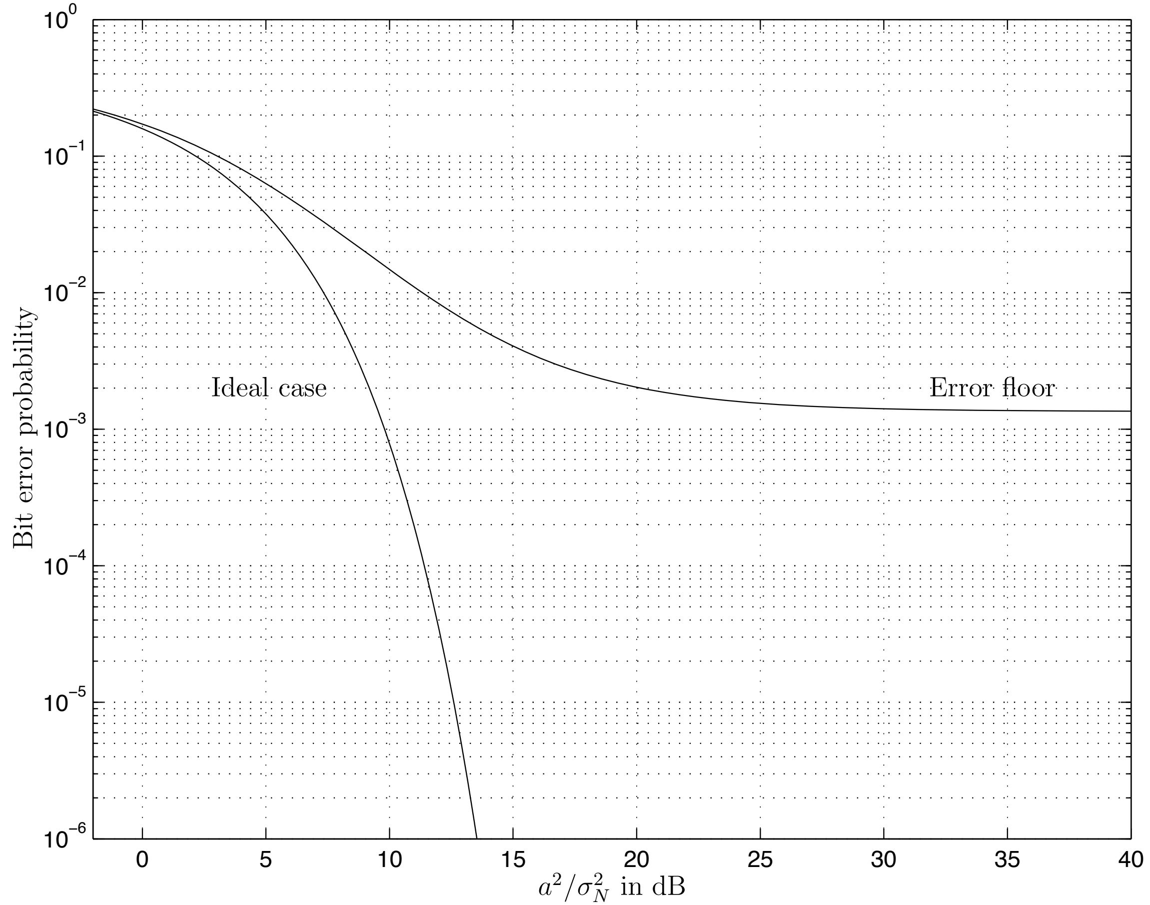
Example 4.12: transmission hidden in noise

In a specific application equally likely binary antipodal signals are used, and the pulse shape is $g_{rc}(t)$ with amplitude A and duration $T \leq T_b$. AWGN with power spectral density $N_0/2$, and the ML receiver is assumed. It is required that the bit error probability must not exceed 10^{-9}. It is also required that the power spectral density satisfies $R(f) \leq N_0/2$ for all frequencies f (the information signal is intentionally "hidden" in the noise). Determine system and signal parameters above such that these two requirements are satisfied.

- $P_b = Q\left(\sqrt{2\,\mathcal{E}_b/N_0}\right) \leq 10^{-9} \Rightarrow \mathcal{E}_b/N_0 \geq 18$
- $R(f) = R_b |G_{rc}(f)|^2$ has maximum at $f = 0$
- $R(0) = R_b\, A^2T^2/4 \leq N_0/2$ (check pulse shape)
- $\mathcal{E}_b/N_0 = 3/8 A^2 T/N_0 \geq 18$
- Hidden in noise: $A^2T/N_0 \leq 2/(R_b T)$
- P_b requirement: $A^2T/N_0 \geq 48$
- Solution:
  choose $T \leq T_b/24$ and $A^2 = 48N_0/T$

Non-ideal receiver conditions

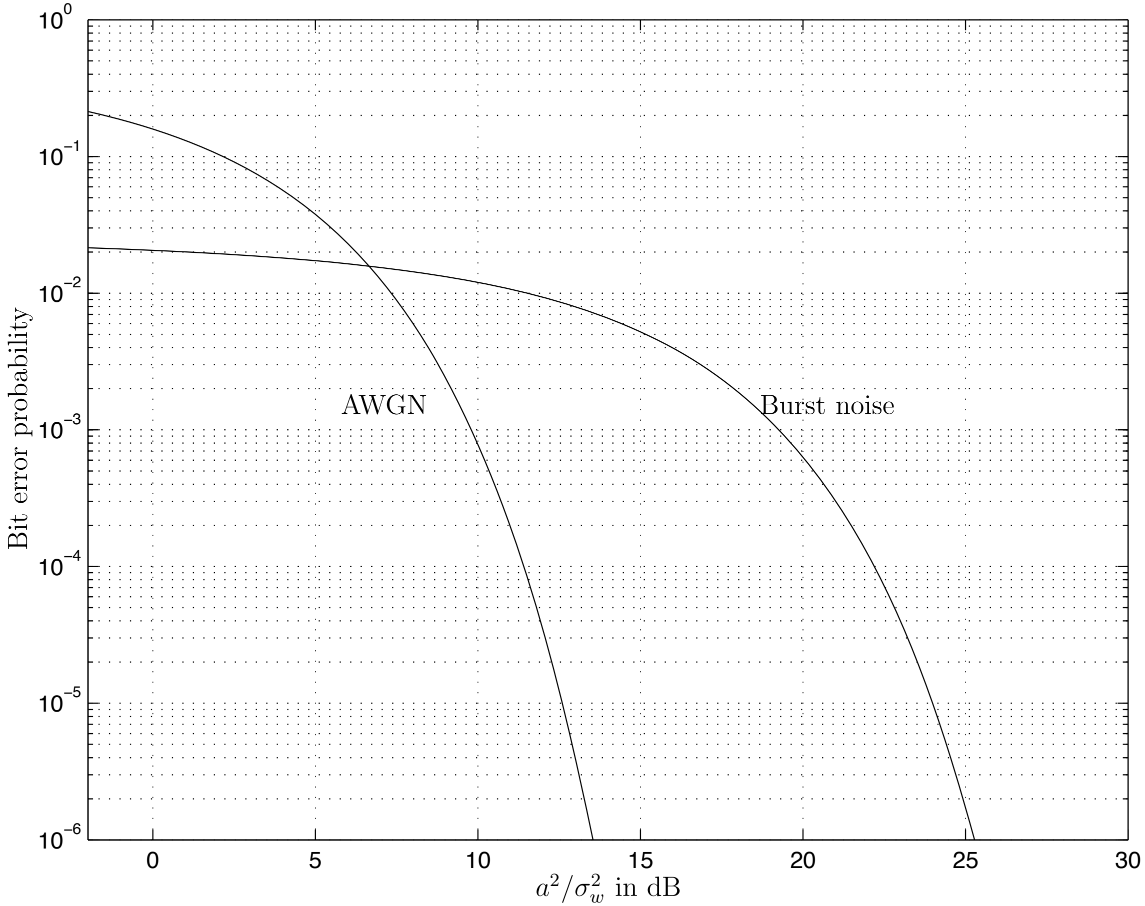
Example 4.15: unexpected additional noise w_x, i.e., $w = w_N + w_x$



Can be analyzed with our methods

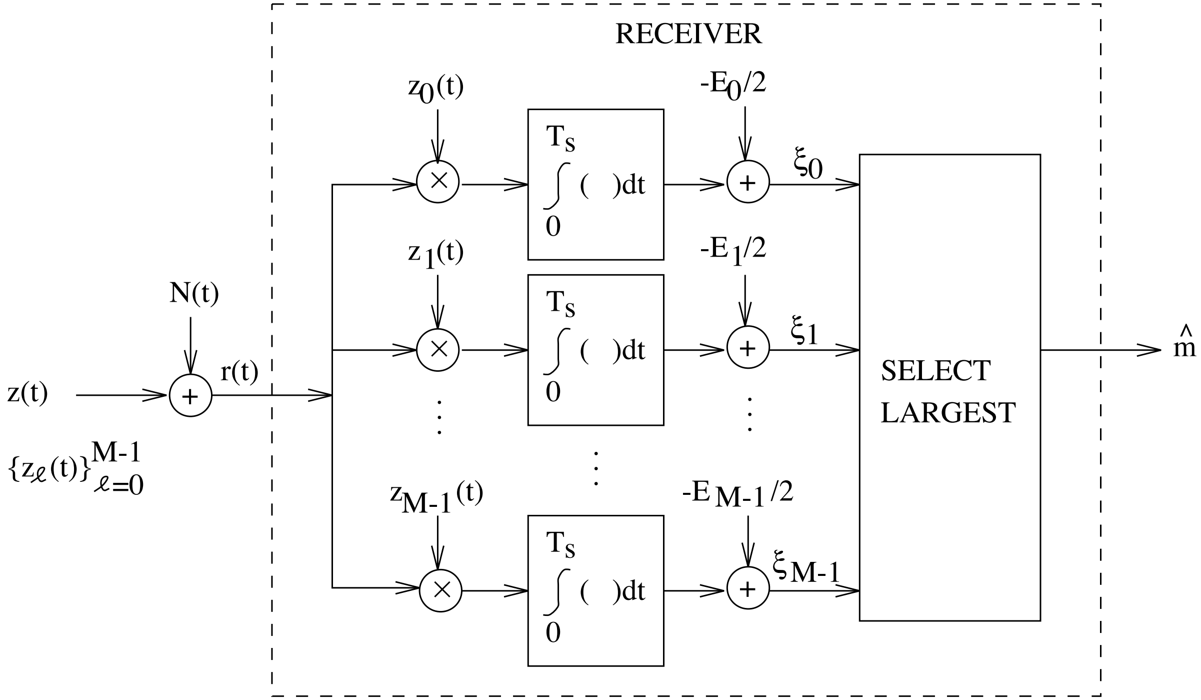
Non-ideal receiver conditions

Example 4.16: hostile bursty interference, active with $p_{on} = 0.05$



Observe: at low power an interference in bursts is more severe than continuous interference

M-ary Signaling



- The receiver computes M decision variables $\xi_0, \xi_1, \ldots, \xi_{M-1}$
- The selected message $\hat{m}$ is based on the largest value

$$\hat{m} = m_\ell \ , \quad \ell = \arg\max_i \xi_i$$

- **Question:** when do we make a wrong decision?

Probability of a wrong decision

- For $M = 2$ we have considered two error probabilities P_F and P_M
- For a given message $m = m_j$, in general there are $M-1$ ways (events) to make a wrong decision,

$$\left\{\xi_i > \xi_j \;\middle|\; m = m_j\right\} \,, \quad i \neq j$$

- The probability of a wrong decision can be upper bounded by

$$Pr\{\hat{m} \neq m_j | m = m_j\} = Pr\left\{ \bigcup_{\substack{i=0 \\ i \neq j}}^{M-1} \xi_i > \xi_j \;\middle|\; m = m_j \right\}$$

$$\leq \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Pr\left\{\xi_i > \xi_j \;\middle|\; m = m_j\right\} \quad \text{(union bound)}$$

- Note: given some events A and B, the union bound states that

$$Pr\{A \cup B\} \leq Pr\{A\} + Pr\{B\} \,,$$

where equality holds if A and B are independent

Symbol error probability

- The symbol error probability can be upper bounded by

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0 \\ i\neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\}$$

- From the binary case $M = 2$ we know that (pick $i = 0$ and $j = 1$)

$$Pr\{\xi_i > \xi_j \mid m = m_j\} = Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

where $D_{i,j}$ is the Euclidean distance between $z_i(t)$ and $z_j(t)$

- We obtain the following main result for M-ary signaling:

$$\max_{\substack{i \\ i\neq j}} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \leq P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0 \\ i\neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

Example: orthogonal signaling

- Consider M orthogonal signals with equal energy E
- Examples: FSK, PPM
- For each pair $z_i(t)$ and $z_j(t)$ we get

$$D_{i,j}^2 = E + E = 2E$$

- From the union bound we obtain

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0 \\ i \neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

$$= (M-1)\, Q\left(\sqrt{\frac{2E}{2N_0}}\right) = (M-1)\, Q\left(\sqrt{\frac{E}{N_0}}\right)$$

- This generalizes the binary case considered previously

Distances $D_{i,j}$ are important

- P_s is determined by the distances $D_{i,j}$ between the signal pairs
- Let us sort these distances

$$D_{min} < D_1 < D_2 < \cdots < D_{max}$$

- Then the upper bound on P_s can be written as

$$P_s \leq c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) + c_1\, Q\left(\sqrt{\frac{D_1^2}{2N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{\frac{D_{max}^2}{2N_0}}\right)$$

- The coefficients are

$$c_\ell = \sum_{j=1}^{M-1} P_j \cdot n_{j,\ell}\ , \quad \ell = 0, 1, 2, \ldots, x$$

- $n_{j,\ell}$: number of signals at distance D_ℓ from signal $z_j(t)$

How many distinct terms do exist for 4-PAM?

A useful approximation of P_s

- The union bound is easy to compute if we know all distances D_ℓ
- At large signal-to-noise ratio (small N_0), i.e., when P_s is small, the first term provides a good approximation

$$P_s \approx c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

- We see that the minimum distance D_{min}^2 and the average number of closest signals c dominate the performance in this case
- **Explanation:**
  the function $Q(x)$ decreases very fast as x increases (faster than exponentially). The other terms become negligible at some point.

$\Rightarrow$ at small P_s (small N_0) we can compare different signal constellations by means of D_{min}^2, similarly to the binary case

Energy efficiency and normalized distances

- Consider the case $P_\ell = 1/M$, $\ell = 0, 1, \ldots, M-1$
- The average received energy per bit is given by

$$\mathcal{E}_b = \frac{1}{k} \sum_{i=0}^{M-1} \frac{1}{M} \int_0^{T_s} z_i^2(t)\, dt \; = \frac{1}{k} \; \frac{E_0 + E_1 + \cdots E_{M-1}}{M}$$

- Using the normalized squared Euclidean distances

$$d_\ell^2 = \frac{D_\ell^2}{2\,\mathcal{E}_b} \ ,$$

the union bound can be written as

$$P_s \leq c\, Q\left(\sqrt{d_{min}^2 \frac{\mathcal{E}_b}{N_0}}\right) + c_1\, Q\left(\sqrt{d_1^2 \frac{\mathcal{E}_b}{N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{d_{max}^2 \frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameters d_ℓ^2 determine the energy efficiency

Approximate P_s for some constellations

- Considering the dominating term in the union bound we obtain

$$P_s \approx c\, Q\left(\sqrt{d^2_{min}\,\frac{\mathcal{E}_b}{N_0}}\right)$$

- This approximation is valid if $\frac{\mathcal{E}_b}{N_0}$ is sufficiently large

	c	$d^2_{\min}$
M-ary PAM	$2(1-1/M)$	$\frac{6\log_2(M)}{M^2-1}$
M-ary PSK $(M>2)$	2	$2\log_2(M)\sin^2(\pi/M)$
M-ary FSK	$M-1$	$\log_2(M)$
M-ary QAM	$4(1-1/\sqrt{M})$	$\frac{3\log_2(M)}{M-1}$

Table 4.1: The coefficient c, and $d^2_{\min}$, for some common signal constellations. Equally likely signal alternatives are assumed. See Subsection 2.4.1.1 for the M-ary PAM case, and Subsection 2.4.5.1 for the M-ary QAM case. M equal energy orthogonal FSK signals are also assumed.

Example 4.19

Assume two signal constellations, denoted A and B respectively, with corresponding parameters $d^2_{\min,A}$ *and* $d^2_{\min,B}$. *From the equality (see e.g. the dominating term in the union bound),*

$$d^2_{\min,A}\mathcal{E}_{b,A}/N_0 = d^2_{\min,B}\mathcal{E}_{b,B}/N_0$$

we find that the difference (in dB) in received energy per information bit is (compare with (2.13) on page 16),

$$10\log_{10}(\mathcal{E}_{b,B}) - 10\log_{10}(\mathcal{E}_{b,A}) = 10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$$

Calculate the value $10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$ *if "A" is binary antipodal PAM, and if "B" is 4-ary PAM. Assume, that the conditions leading to (2.50) are satiesfied.*

- For M-ary PAM we have (Table 4.1 or Table 5.1)

$$d^2_{min} = 6\log_2(M)/(M^2-1) \quad \Rightarrow \quad d^2_{min,A} = 2,\ d^2_{min,B} = 4/5$$

- $10\log_{10} d^2_{min,A}/d^2_{min,B} = 10\log_{10} 5/2 = 3.98$ dB

Binary PAM is 3.98 dB more energy efficient than 4-ary PAM!

Example scenario: M-ary QAM

- We want to ensure that $P_s \leq P_{s,req}$, where for M-ary QAM

$$P_s \leq 4\,Q\left(\sqrt{d^2_{min}\,\frac{\mathcal{E}_b}{N_0}}\right) = 4\,Q\left(\sqrt{\mathcal{X}}\right)\ , \quad d^2_{min} = 3\ \log_2 \frac{M}{M-1}$$

- The pulse shape $g(t)$ is chosen such that

$$\rho = \log_2(M)\ \rho_{BPSK}\ , \quad \text{where } \rho = \frac{R_b}{W} \leq \frac{d^2_{min}}{\mathcal{X}} \cdot \frac{\mathcal{P}_z}{N_0\,W}$$

- Combining these requirements we obtain

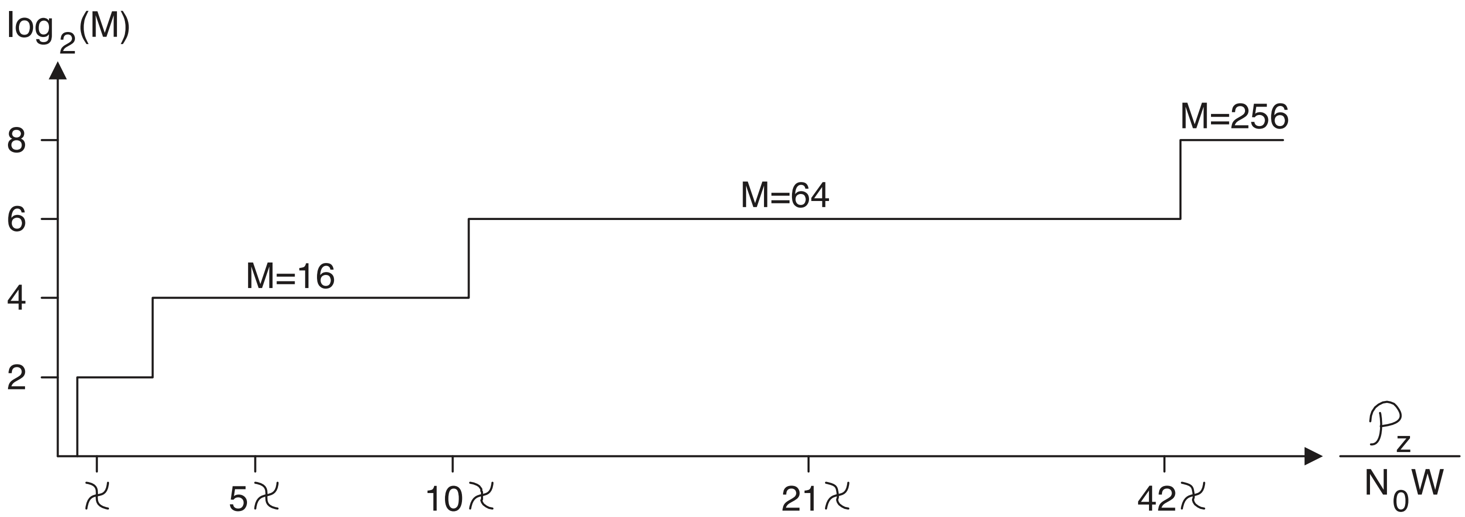
$$M \leq 1 + \frac{3}{\mathcal{X}\,\rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\,W} = 1 + \frac{3}{\mathcal{X}} \cdot \frac{\mathcal{P}_z\,T_s}{N_0}$$

- Hence we want to choose $M = 2^k$ such that (QAM: k even)

$$2^k\ \leq\ 1 + \frac{3}{\mathcal{X}\,\rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\,W}\ <\ 2^{k+2}$$

Example 4.22: adapting M to channel quality

Assume that an M-ary QAM system adapts between 4-ary QAM, 16-ary QAM, 64-ary QAM and 256-ary QAM. Show when a new M is chosen by plotting M (or $\log_2(M)$) versus $\mathcal{P}_z/N_0W$. How large is the bit rate in each case? Assume that $\rho_{BPSK} = 1/2$ [bps/Hz].



Depending on the channel quality we can achieve different bit rates $R_b = W$, $2W$, $3W$, or $4W$[bps]

EITG05 – Digital Communications

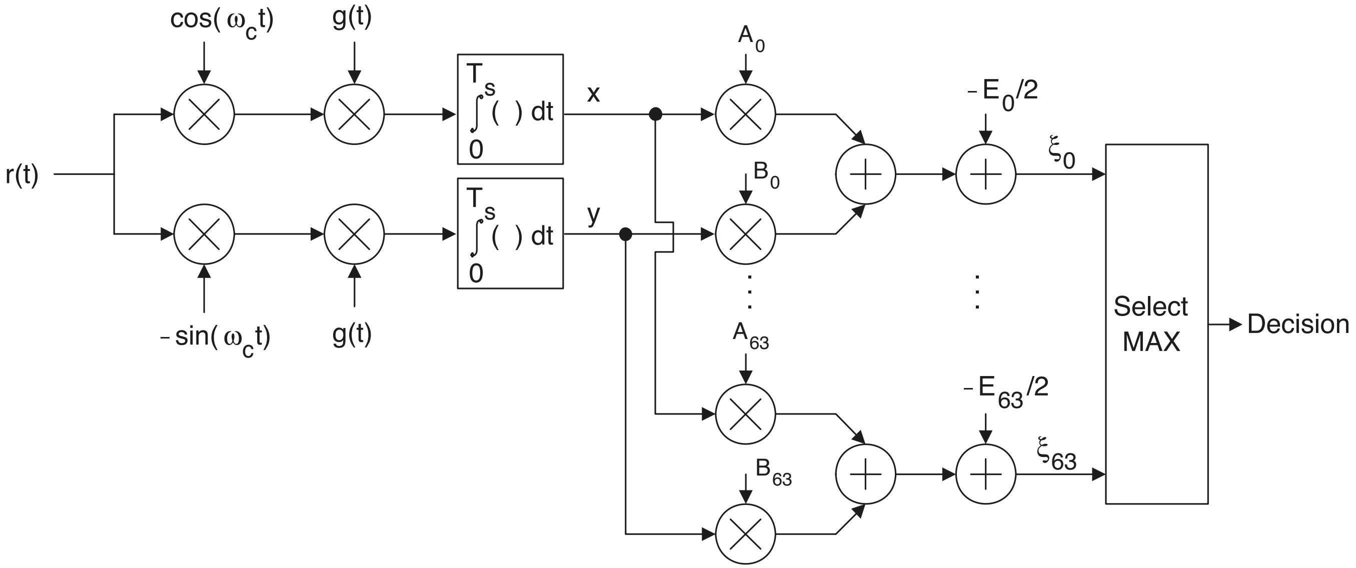
Lecture 7

Receivers continued:
Geometric representation, Capacity,
Multiuser receiver, Non-coherent receiver

Michael Lentmaier
Thursday, September 27, 2018

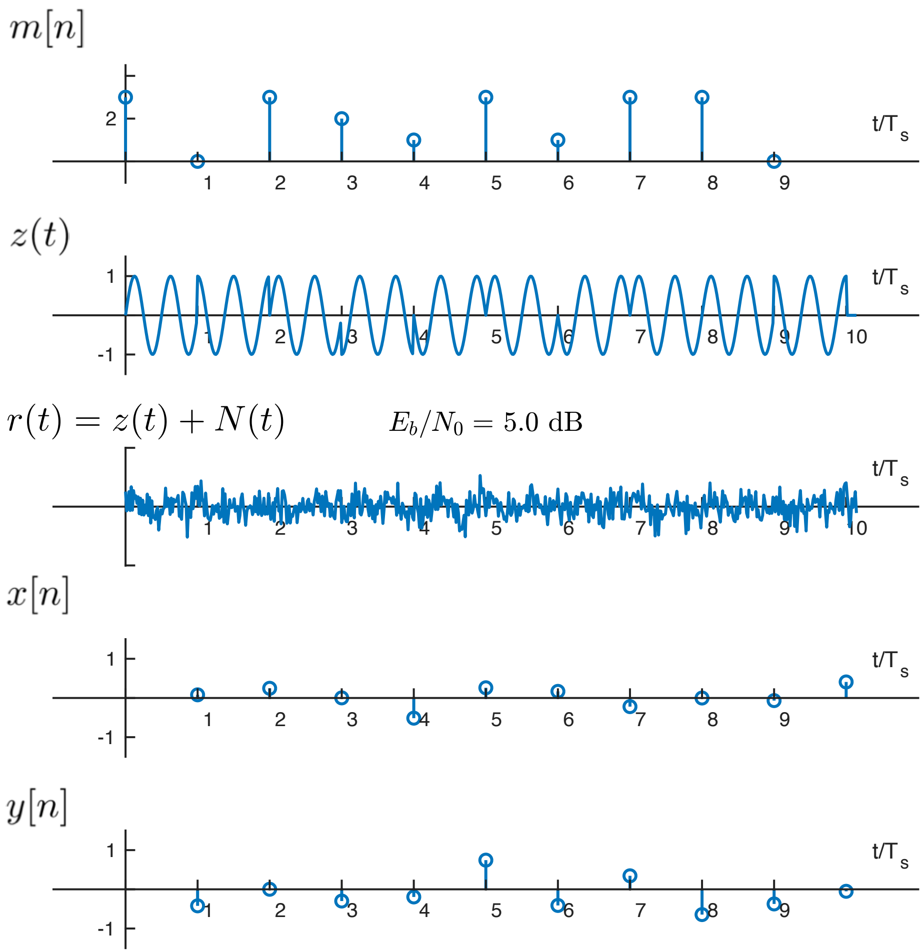
Recall: QAM receiver (Example 4.4)

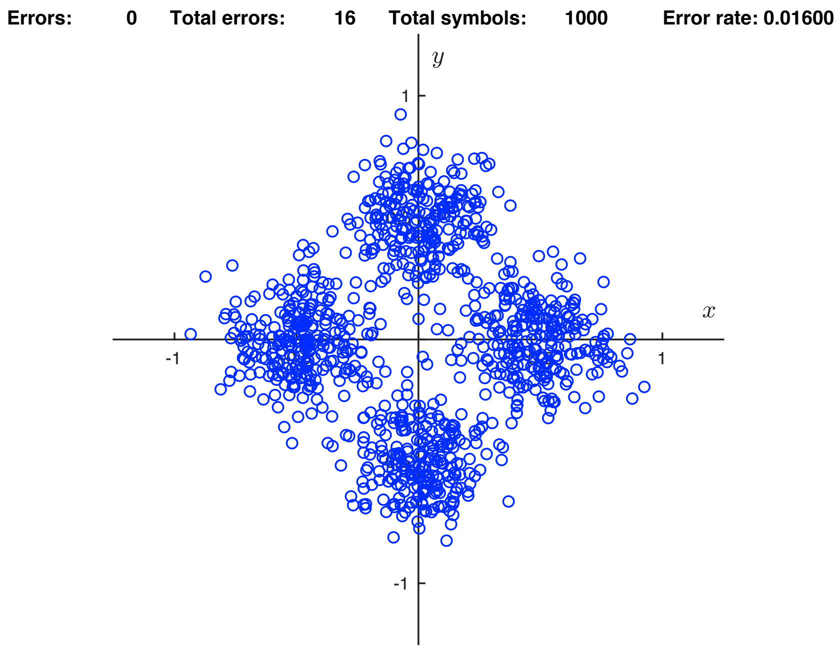
The implementation of this receiver is shown below:



The complexity of this receiver is significantly reduced compared to the receiver in Figure 4.8 on page 241! Only two integrators are here used, instead of 64 (= M) in Figure 4.8.

Example: QPSK (see Matlab demo)





Distances $D_{i,j}$ are important

- P_s is determined by the distances $D_{i,j}$ between the signal pairs
- Let us sort these distances

$$D_{min} < D_1 < D_2 < \cdots < D_{max}$$

- Then the upper bound on P_s can be written as

$$P_s \leq c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) + c_1\, Q\left(\sqrt{\frac{D_1^2}{2N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{\frac{D_{max}^2}{2N_0}}\right)$$

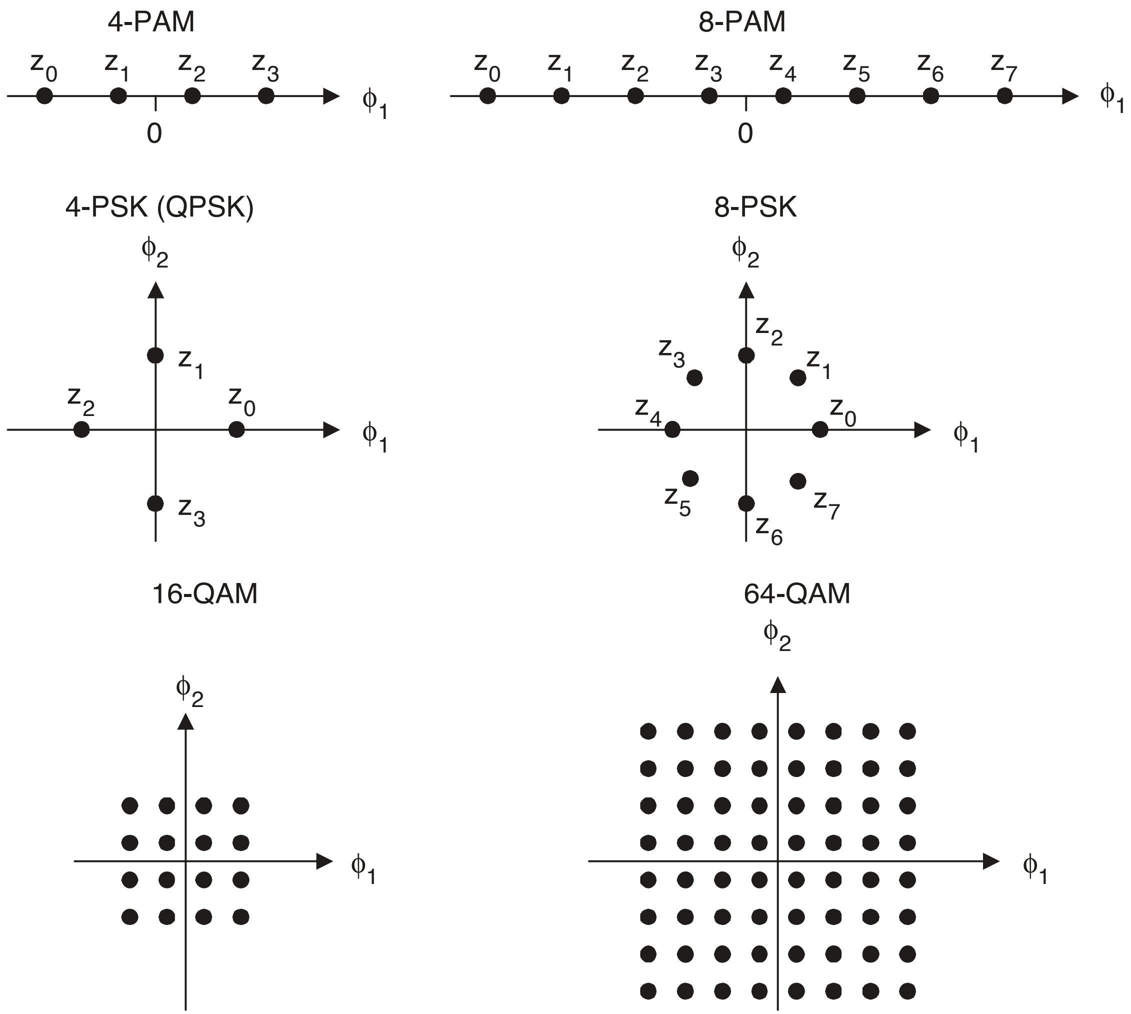
- The coefficients are

$$c_\ell = \sum_{j=1}^{M-1} P_j \cdot n_{j,\ell}\ , \quad \ell = 0, 1, 2, \ldots, x$$

- $n_{j,\ell}$: number of signals at distance D_ℓ from signal $z_j(t)$

How many distinct terms do exist for QPSK?

Signal Space Representation



$$\phi_1(t) = \frac{g(t)}{\sqrt{E_g}}$$

$$\phi_1(t) = \frac{g(t)\cos(2\pi\, f_c\, t)}{\sqrt{E_g/2}}$$

$$\phi_2(t) = \frac{g(t)\sin(2\pi\, f_c\, t)}{\sqrt{E_g/2}}$$

A geometric description

- As we have seen in Chapter 2 we can represent our signal alternatives $z_j(t)$ as vectors (points) in signal space

$$\mathbf{z}_j = \left(z_{j,1}\right) = \left(A_j\sqrt{E_g}\right) \qquad \text{PAM}$$

$$\mathbf{z}_j = \left(z_{j,1} \quad z_{j,2}\right) = \left(A_j\sqrt{\frac{E_g}{2}} \quad B_j\sqrt{\frac{E_g}{2}}\right) \qquad \text{QAM, PSK}$$

- The signal energy can be written as

$$E_j = \int_0^{T_s} z_j^2(t)\, dt = z_{j,1}^2 + z_{j,2}^2$$

- Likewise, the squared Euclidean distance becomes

$$D_{i,j}^2 = \int_0^{T_s} \left(z_i(t) - z_j(t)\right)^2 dt = (z_{i,1} - z_{j,1})^2 + (z_{i,2} - z_{j,2})^2$$

Signal energies and distances have a geometric interpretation

Approximate P_s for some constellations

- Considering the dominating term in the union bound we obtain

$$P_s \approx c\, Q\left(\sqrt{d^2_{min}\,\frac{\mathcal{E}_b}{N_0}}\right)$$

- This approximation is valid if $\frac{\mathcal{E}_b}{N_0}$ is sufficiently large

	c	$d^2_{\min}$
M-ary PAM	$2(1-1/M)$	$\dfrac{6\log_2(M)}{M^2-1}$
M-ary PSK ($M>2$)	2	$2\log_2(M)\sin^2(\pi/M)$
M-ary FSK	$M-1$	$\log_2(M)$
M-ary QAM	$4(1-1/\sqrt{M})$	$\dfrac{3\log_2(M)}{M-1}$

Table 4.1: The coefficient c, and $d^2_{\min}$, for some common signal constellations. Equally likely signal alternatives are assumed. See Subsection 2.4.1.1 for the M-ary PAM case, and Subsection 2.4.5.1 for the M-ary QAM case. M equal energy orthogonal FSK signals are also assumed.

Example 4.19

Assume two signal constellations, denoted A and B respectively, with corresponding parameters $d^2_{\min,A}$ and $d^2_{\min,B}$. From the equality (see e.g. the dominating term in the union bound),

$$d^2_{\min,A}\mathcal{E}_{b,A}/N_0 = d^2_{\min,B}\mathcal{E}_{b,B}/N_0$$

we find that the difference (in dB) in received energy per information bit is (compare with (2.13) on page 16),

$$10\log_{10}(\mathcal{E}_{b,B}) - 10\log_{10}(\mathcal{E}_{b,A}) = 10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$$

Calculate the value $10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$ if "A" is binary antipodal PAM, and if "B" is 4-ary PAM. Assume, that the conditions leading to (2.50) are satiesfied.

- For M-ary PAM we have (Table 4.1 or Table 5.1)

$$d^2_{min} = 6\log_2(M)/(M^2-1) \quad \Rightarrow \quad d^2_{min,A} = 2,\ d^2_{min,B} = 4/5$$

- $10\log_{10} d^2_{min,A}/d^2_{min,B} = 10\log_{10} 5/2 = 3.98 \text{ dB}$

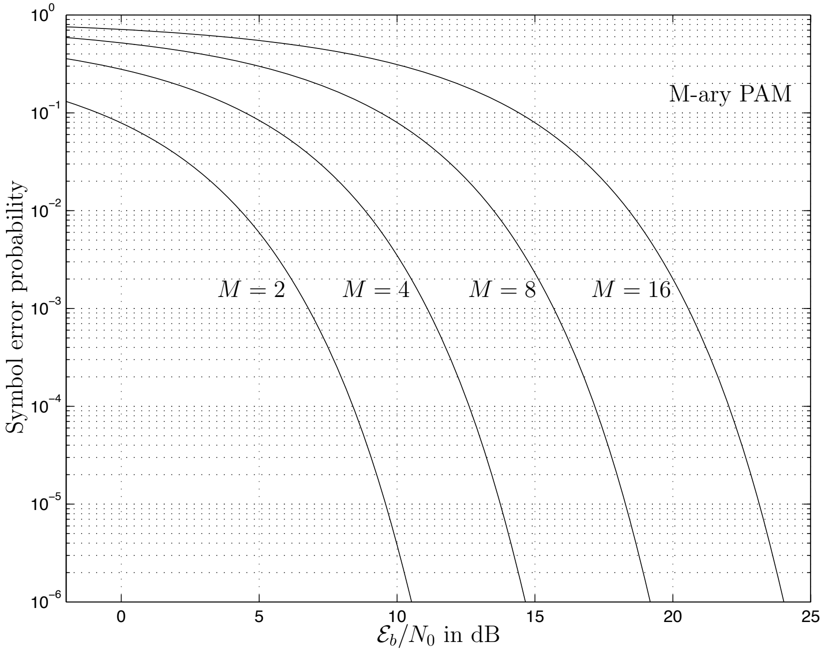
Binary PAM is 3.98 dB more energy efficient than 4-ary PAM!

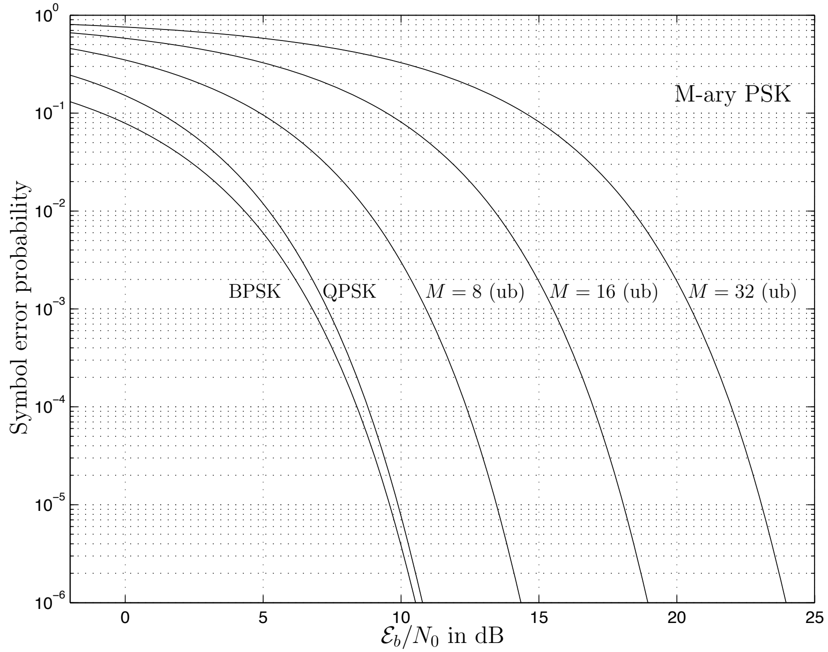
Comparisons

$M = 2$	P_b	$Q\left(\sqrt{d^2_{\min}\,\frac{\mathcal{E}_b}{N_0}}\right)$, (4.55)
	$d^2_{\min}$	$0 \leq d^2_{\min} \leq 2$, (4.57)
	ρ	ρ_{bin} , (2.21)
M-ary PAM	P_s	$2\left(1-\frac{1}{M}\right) Q\left(\sqrt{d^2_{\min}\,\frac{\mathcal{E}_b}{N_0}}\right)$, (5.35)
	$d^2_{\min}$	$\frac{6\log_2(M)}{M^2-1}$, Table 4.1 on page 281, (2.50)
	ρ	$\rho_{2-PAM}\cdot\log_2(M)$, (2.220)
M-ary PSK	P_s	$< 2Q\left(\sqrt{d^2_{\min}\,\frac{\mathcal{E}_b}{N_0}}\right)$, (5.43)
	$d^2_{\min}$	$2\sin^2(\pi/M)\log_2(M)$, Table 4.1, Fig. 5.11
	ρ	$\rho_{BPSK}\cdot\log_2(M)$, (2.229)
M-ary QAM (rect., k even) (QPSK with $M = 4$)	P_s	$4\left(1-\frac{1}{\sqrt{M}}\right) Q\left(\sqrt{d^2_{\min}\,\frac{\mathcal{E}_b}{N_0}}\right) -$ $-4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{d^2_{\min}\,\frac{\mathcal{E}_b}{N_0}}\right)$, (5.50)
	$d^2_{\min}$	$\frac{3\log_2(M)}{M-1}$, Table 4.1, Subsection 2.4.5.1
	ρ	$\rho_{BPSK}\cdot\log_2(M)$, (2.229)
M-ary FSK (orthogonal FSK)	P_s	$\leq (M-1)Q\left(\sqrt{d^2_{\min}\,\frac{\mathcal{E}_b}{N_0}}\right)$, Example 4.18c, Table 4.1
	$d^2_{\min}$	$\log_2(M)$, Table 4.1 on page 281
	ρ	See (2.245)

Table 5.1, p. 361

Symbol error probability comparison





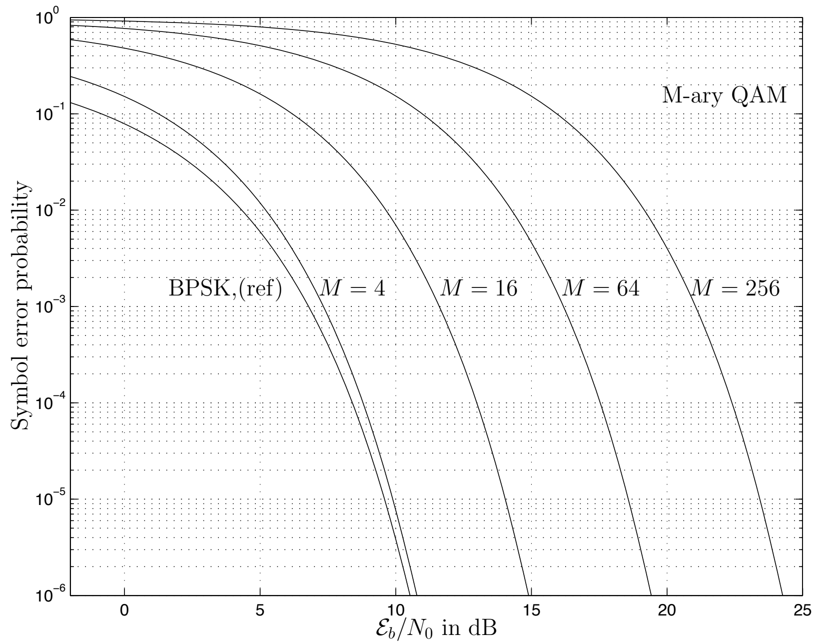
M-ary PAM, $M = 2, 4, 8, 16$

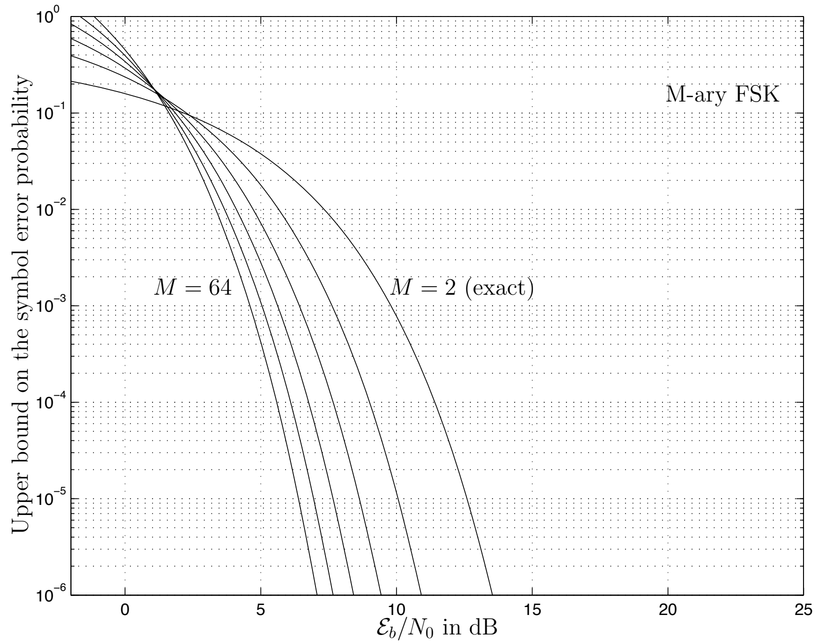
$$d_{min}^2 = 6 \cdot \frac{\log_2 M}{M^2 - 1}$$

M-ary PSK, $M = 2, 4, 8, 16, 32$

$$d_{min}^2 = 2 \sin^2(\pi/M) \ \log_2 M$$

Symbol error probability comparison





M-ary QAM, $M = 4, 16, 64, 256$

$$d_{min}^2 = 3 \cdot \frac{\log_2 M}{M-1}$$

M-ary FSK, $M = 2, 4, 8, 16, 32, 64$

$$d_{min}^2 = \log_2 M$$

Gain in d_{min}^2 compared with binary antipodal

Antipodal	$M = 2$	0[dB]
Orthogonal	$M = 2$	-3.01
M-ary PAM	$M = 2$	0
	$M = 4$	-3.98
	$M = 8$	-8.45
	$M = 16$	-13.27
	$M = 32$	-18.34
	$M = 64$	-23.57
M-ary PSK	$M = 2$	0
	$M = 4$	0
	$M = 8$	-3.57
	$M = 16$	-8.17
	$M = 32$	-13.18
	$M = 64$	-18.40
M-ary QAM	$M = 4$	0
	$M = 16$	-3.98
	$M = 64$	-8.45
	$M = 256$	-13.27
	$M = 1024$	-18.34
	$M = 4096$	-23.57

M-ary FSK	$M = 2$	-3.01
	$M = 4$	0
	$M = 8$	1.76
	$M = 16$	3.01
	$M = 32$	3.98
	$M = 64$	4.77
M -ary bi-orthogonal	$M = 2$	0
	$M = 4$	0
	$M = 8$	1.76
	$M = 16$	3.01
	$M = 32$	3.98
	$M = 64$	4.77

Large values M reduce energy efficiency

Example scenario: M-ary QAM

- We want to ensure that $P_s \leq P_{s,req}$, where for M-ary QAM

$$P_s \leq 4\,Q\left(\sqrt{d_{min}^2 \frac{\mathcal{E}_b}{N_0}}\right) = 4\,Q\left(\sqrt{\mathcal{X}}\right) \;, \quad d_{min}^2 = 3\,\log_2 \frac{M}{M-1}$$

- The pulse shape $g(t)$ is chosen such that

$$\rho = \log_2(M)\ \rho_{BPSK} \;, \quad \text{where } \rho = \frac{R_b}{W} \leq \frac{d_{min}^2}{\mathcal{X}} \cdot \frac{\mathcal{P}_z}{N_0\,W}$$

- Combining these requirements we obtain

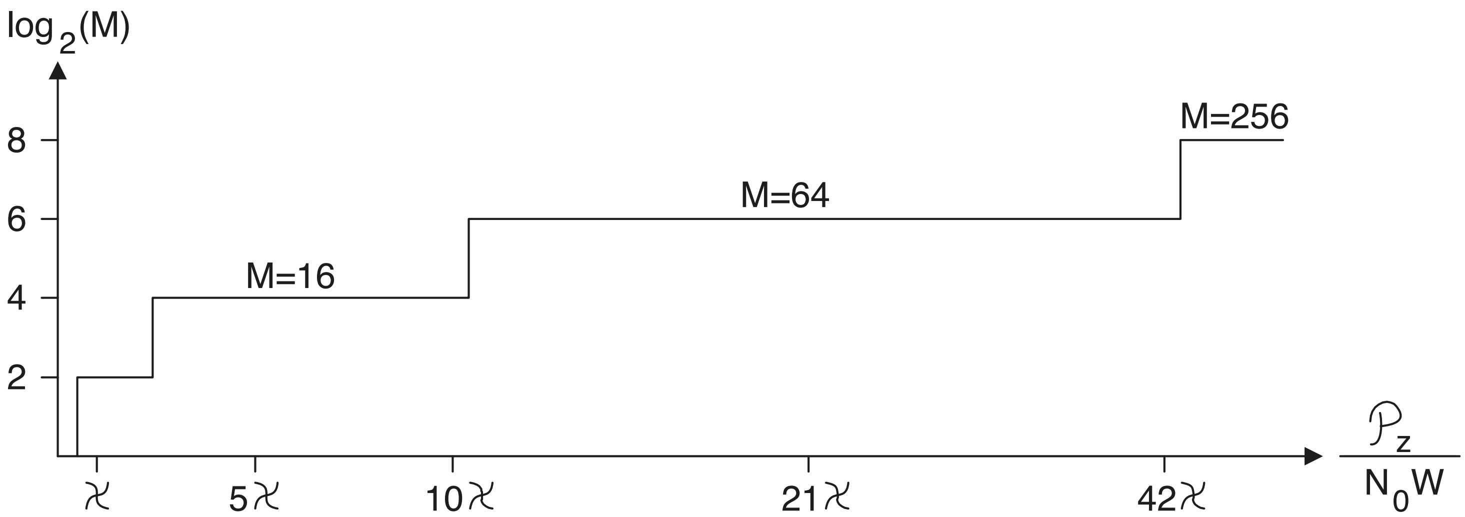
$$M \leq 1 + \frac{3}{\mathcal{X}\,\rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\,W} = 1 + \frac{3}{\mathcal{X}} \cdot \frac{\mathcal{P}_z\,T_s}{N_0}$$

- Hence we want to choose $M = 2^k$ such that (QAM: k even)

$$2^k \;\leq\; 1 + \frac{3}{\mathcal{X}\,\rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\,W} \;<\; 2^{k+2}$$

Example 4.22: adapting M to channel quality

Assume that an M-ary QAM system adapts between 4-ary QAM, 16-ary QAM, 64-ary QAM and 256-ary QAM. Show when a new M is chosen by plotting M (or $\log_2(M)$) versus $\mathcal{P}_z/N_0W$. How large is the bit rate in each case? Assume that $\rho_{BPSK} = 1/2$ [bps/Hz].



Depending on the channel quality we can achieve different bit rates $R_b = W$, $2W$, $3W$, or $4W$[bps]

Bit errors vs symbol errors

- Assume that S symbols are transmitted and S_{err} are in error
- If a symbol $\hat{m} \neq m$ is decided, this causes at least 1 bit error and at most $k = \log_2 M$ bit errors

$$S_{err} \leq B_{err} \leq k\, S_{err}$$

- This leads to the following relationship between P_b and P_s:

$$\frac{P_s}{k} = \frac{E\{S_{err}\}}{S \cdot k} \leq P_b \leq \frac{E\{S_{err} \cdot k\}}{S \cdot k} = P_s$$

- P_s depends on the signal constellation only
- The exact P_b depends on the mapping from bits to messages m_ℓ and hence signal alternatives $s_{m_\ell}(t)$

Example: Which mapping is better for 4-PAM? (and why?)

$$(1) \quad m_0 = 00,\; m_1 = 11,\; m_2 = 01,\; m_3 = 10$$

$$(2) \quad m_0 = 00,\; m_1 = 01,\; m_2 = 11,\; m_3 = 10$$

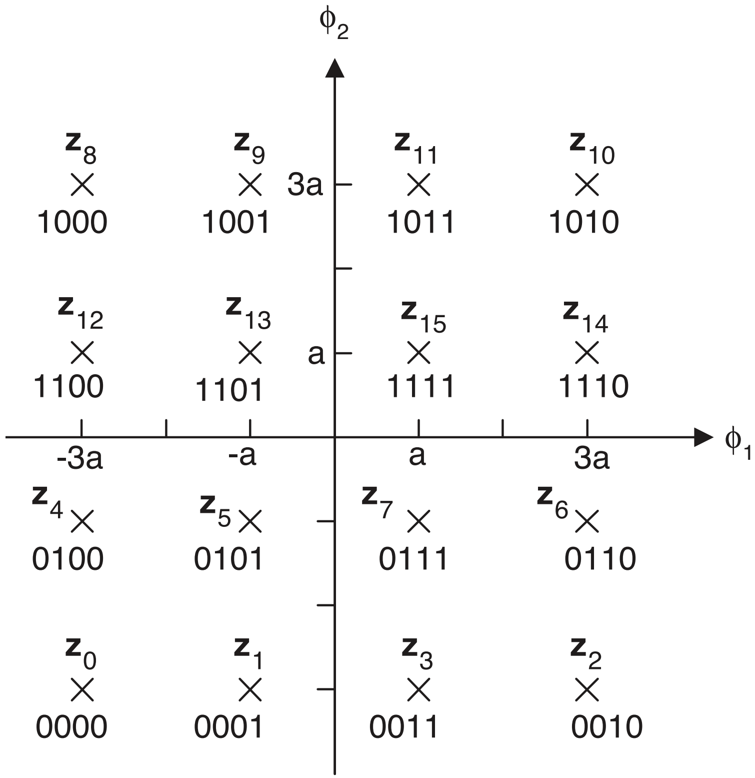
Gray code mappings

- We have seen that for small N_0 we can approximate

$$P_s \approx c\, Q\left(\sqrt{\frac{D^2_{min}}{2\,N_0}}\right)$$

- This motivates the use of Gray code mappings:

Example:
16-QAM



How can we achieve large data rates?

- The bit rate R_b can be increased in different ways
- We can select a signal constellation with large M
  $\Rightarrow$ this typically increases the error probability P_s
  exception: orthogonal signals (FSK): require more bandwidth W
- Achieving equal P_s with larger M is possible by increasing $\mathcal{E}_b/N_0$
  $\Rightarrow$ this reduces the energy efficiency
- We can also increase R_b by increasing the bandwidth W
  $\Rightarrow$ this does not improve the bandwidth efficiency $\rho = R_b/W$

Question:
what is the largest achievable rate R_b for a given error probability P_s, channel quality $\mathcal{E}_b/N_0$ and bandwidth W?

This question was answered by Claude Shannon in 1948:
"A mathematical theory of communication"
Course EITN45: Information Theory (VT2)

A fundamental limit: channel capacity

- Consider a single-path channel ($|H(f)|^2 = \alpha^2$) with finite bandwidth W and additive white Gaussian noise (AWGN) $N(t)$
- The capacity for this channel is given by

$$\mathcal{C} = W \log_2 \left(1 + \frac{\mathcal{P}_z}{N_0 W} \right) \ \text{[bps]}$$

- Shannon showed that reliable communication requires that

$$R_b \leq \mathcal{C}$$

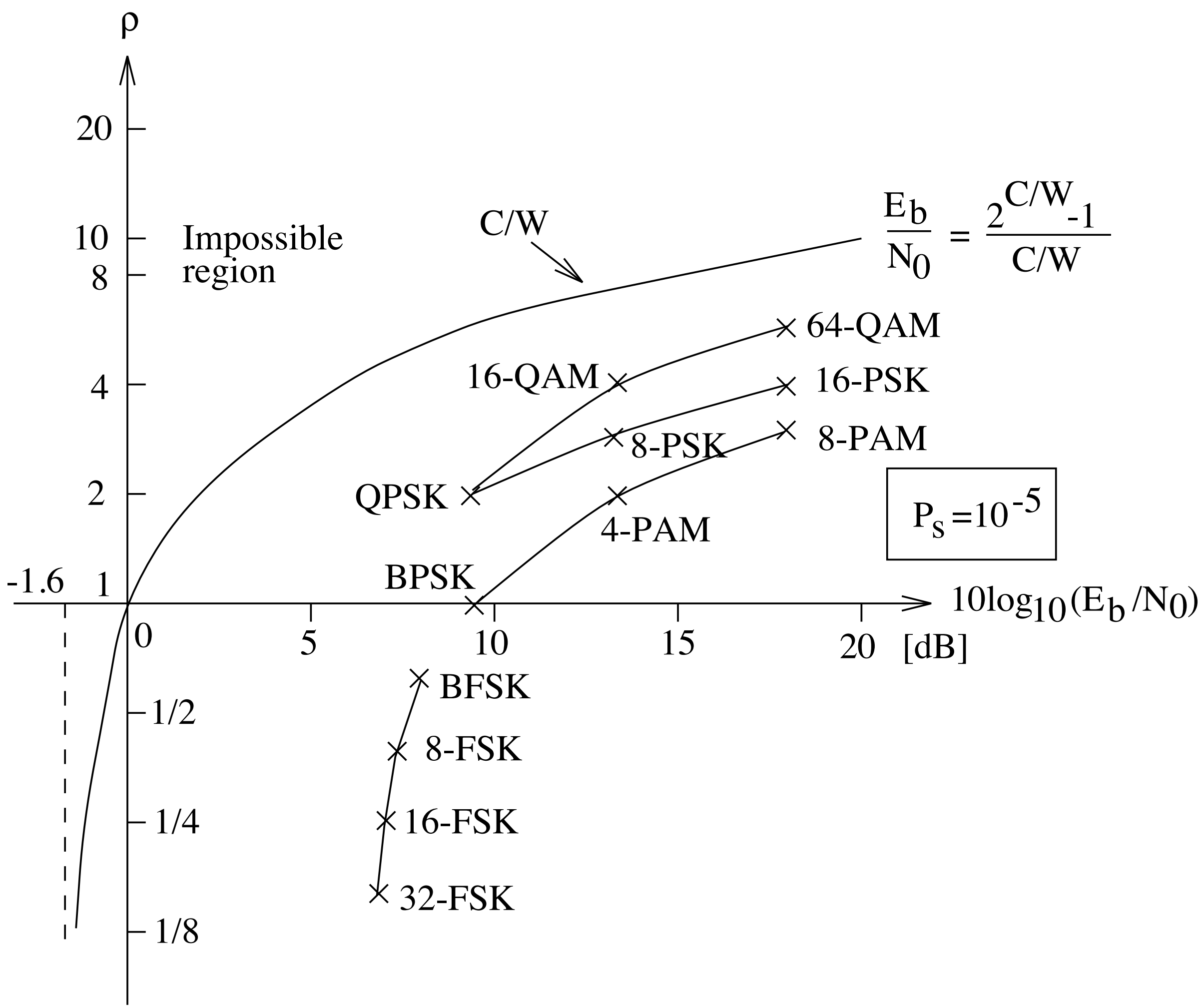
- **Observe:** the capacity formula does not include P_s (why?)
- Shannon also showed that if $R_b < \mathcal{C}$, then the probability of error P_s can be made arbitrarily small

$$P_s \to 0$$

if messages are coded in blocks of length $N \to \infty$

Bandwidth efficiency and gap to capacity

(p. 369)



- $\rho \leq C/W$: reliable communication is impossible above
- this limit can be approached with channel coding

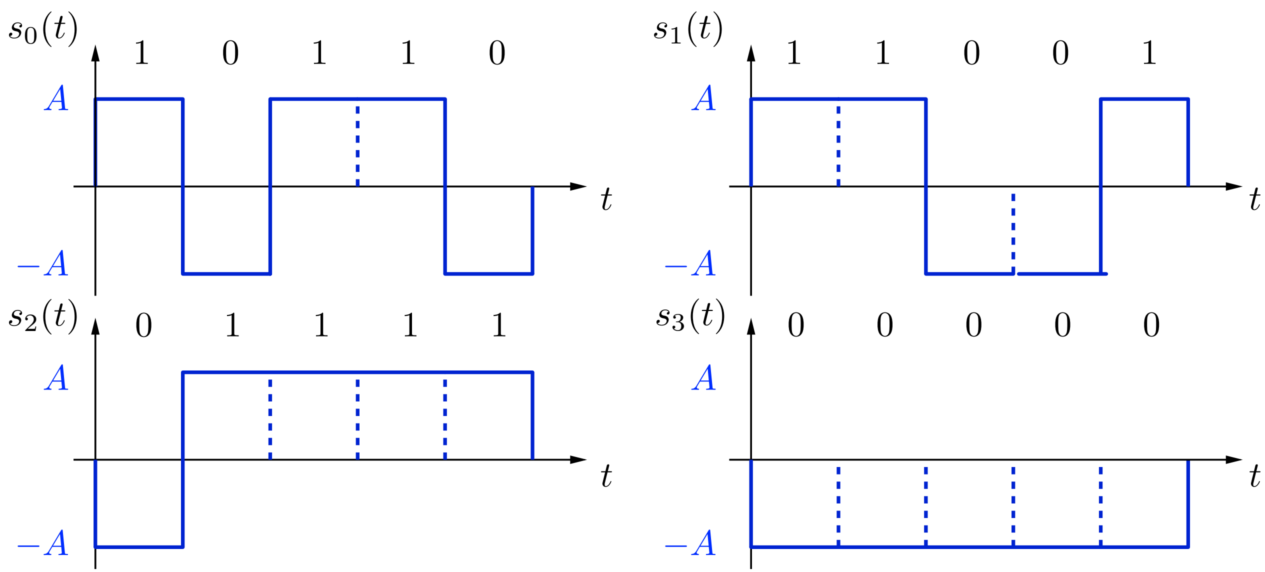
How does channel coding work?

- We have seen that a large minimum distance d^2_{min} between signals is required to improve the energy efficiency
- For binary signaling ($M = 2$) we have seen that $d^2_{min} \leq 2$

Idea of coding:

- generate M binary sequences of length N
- use binary antipodal signaling to create M signals $s_\ell(t)$

Example: $N = 5$, $M = 4$, $g_{rec}(t)$ pulse with $T = T_s/N$ (what is D^2_{min}?)



Increasing d_{min}^2 with coding

- In our example we have

$$D_{min}^2 = 4A^2 T \cdot 3 = 4 E_g 3 = 12 E_g$$

- Normalizing by the average energy $\mathcal{E}_b = N E_g / k$ this gives

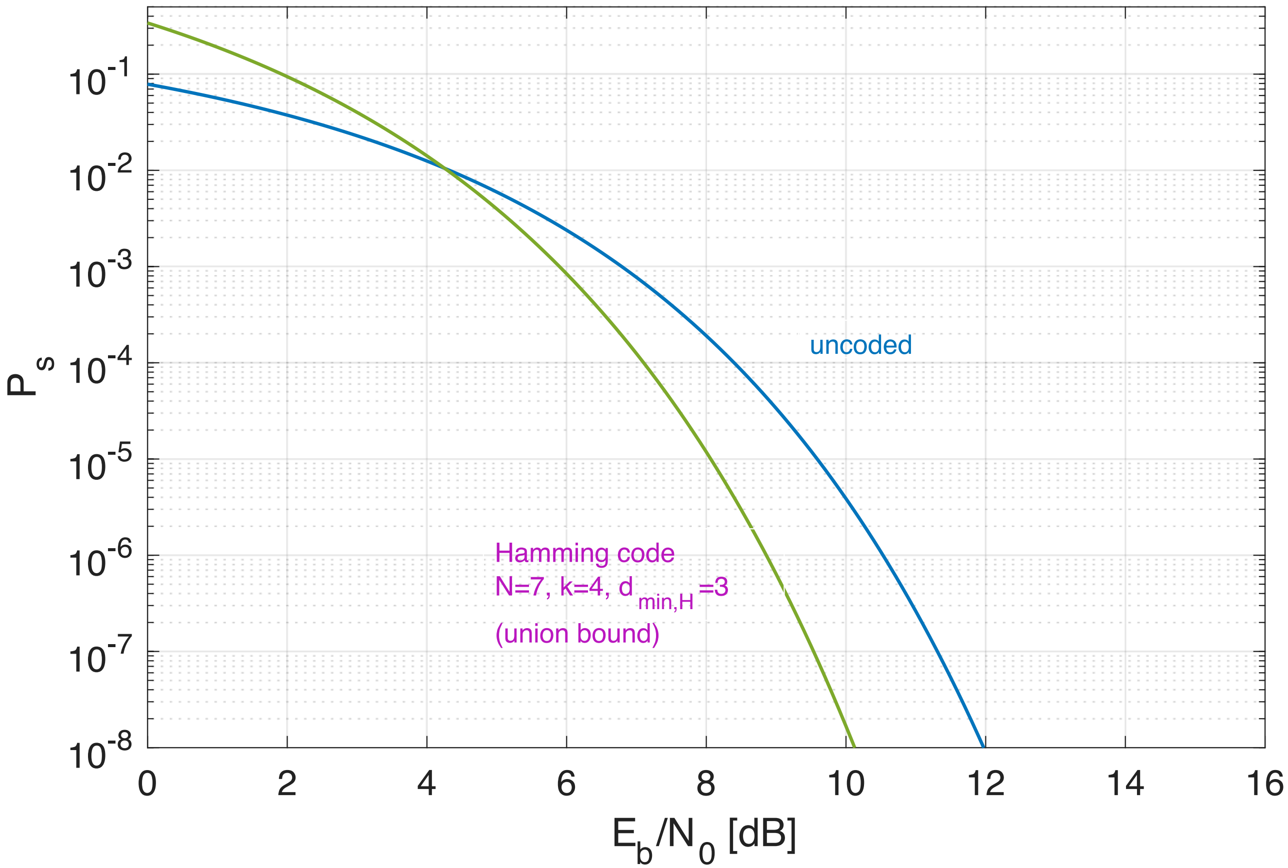
$$d_{min}^2 = \frac{D_{min}^2}{2\mathcal{E}_b} = \frac{12 E_g}{2N/k E_g} = 6 \cdot \frac{k}{N} = \frac{12}{5} = 2.4$$

- Let $d_{min,H}$ denote the minimum Hamming distance between the binary code sequences $\Rightarrow$ in our example: $d_{min,H} = 3$
- Then we can write

$$d_{min}^2 = 2 \frac{k}{N} d_{min,H}$$

  where $R = k/N$ is called the code rate
- Larger $d_{min,H}$ values can be achieved with larger N

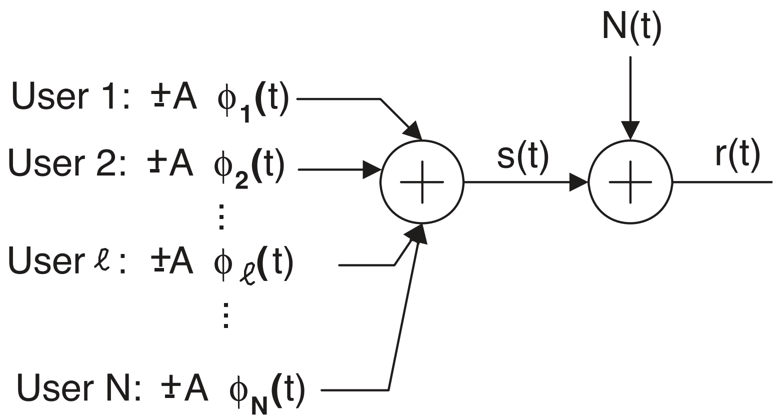
Example: symbol error probability



- Hamming code, $N = 7$, $k = 4$, $d_{min,H} = 3 \Rightarrow d^2_{min} = 3.43$
- How can we construct good codes?

Multiuser Communication

(p. 395/396)



A simple model:

- N users transmit at same time with orthonormal waveforms $\phi_\ell(t)$
- Binary antipodal signaling is used in this example, such that

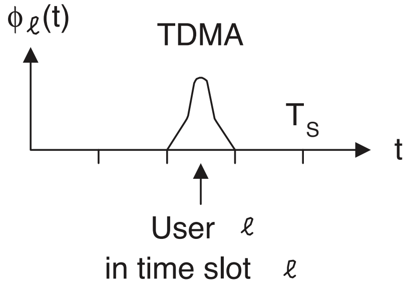
$$s(t) = \sum_{n=1}^{N} A_n \, \phi_n(t) \ , \quad A_n \in \pm A$$

- The orthonormal waveforms satisfy

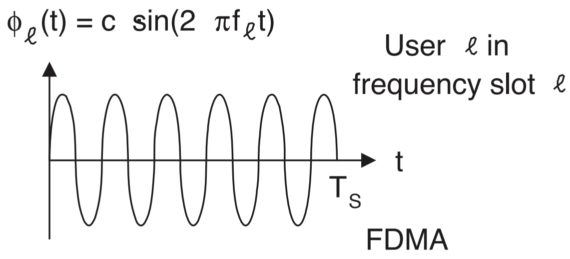
$$\int_0^{T_s} \phi_i(t) \, \phi_j(t) \, dt = \begin{cases} 0 & \text{if } i \neq j \ , \\ 1 & \text{if } i = j \end{cases}$$

Multiuser Communication

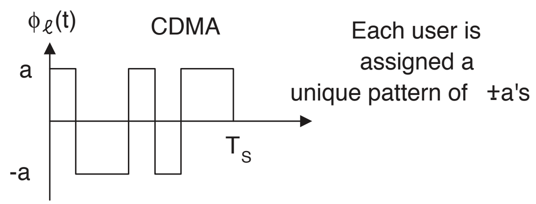
- The separation of users can be achieved in different ways
- **TDMA:** (time-division multiple access)



- **FDMA / OFDMA:** (frequency-division multiple access)

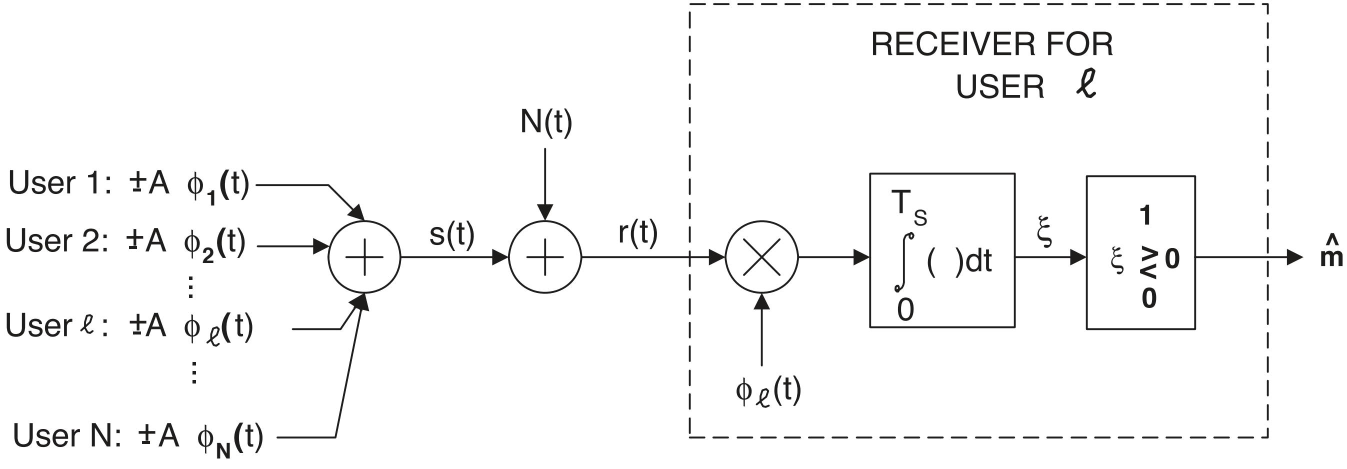


- **CDMA:** (code-division multiple access)



- **MC-CDMA:** (multi-carrier CDMA) combined OFDM/CDMA

Receiver for Multiuser Communication



- This permits a simple receiver structure for each user ℓ
- The decision variable becomes

$$\xi = \int_0^{T_s} \phi_\ell(t)\, r(t)\; dt = \int_0^{T_s} \phi_\ell(t) \left(\sum_{n=1}^{N} A_n\, \phi_n(t) + N(t) \right) \; dt$$

$$= A_\ell + \int_0^{T_s} \phi_\ell(t)\, N(t)\; dt = A_\ell + \mathcal{N}$$

$\Rightarrow$ receiver is only disturbed by noise and not by other users!

Non-coherent receivers

- With phase-shift keying (PSK) the message $m[n]$ at time nT_s is put into the phase θ_n of the transmit signal

$$s(t) = g(t)\,\sqrt{2E}\,\cos(2\pi f_c t + \theta_n)\,, \quad nT_s \leq t \leq (n+1)T_s$$

- The channel introduces some attenuation α, some additive noise $N(t)$ and also some phase offset ν into the received signal

$$r(t) = \alpha\, g(t)\,\sqrt{2E}\,\cos(2\pi f_c t + \theta_n + \nu) + N(t)$$

- **Challenge:** the optimal receiver needs to know α and ν
- In some applications an accurate estimation of ν is infeasible (cost, complexity, size)
- **Non-coherent receivers:**
  receiver structures that can work well without knowledge of the exact phase offset

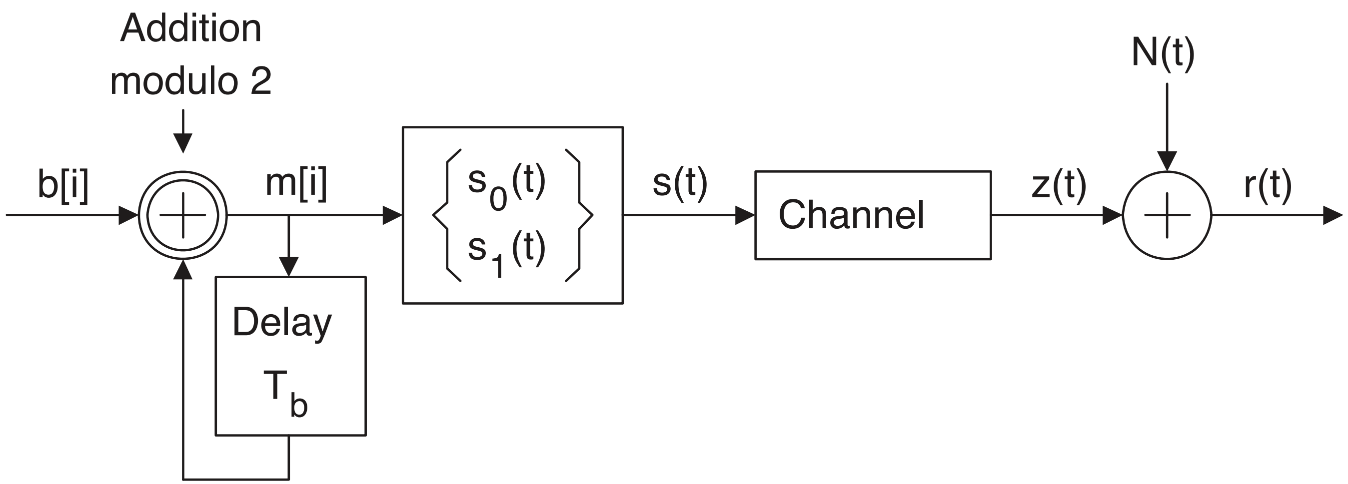
How can we modify our PSK transmission accordingly?

Differential Phase Shift Keying

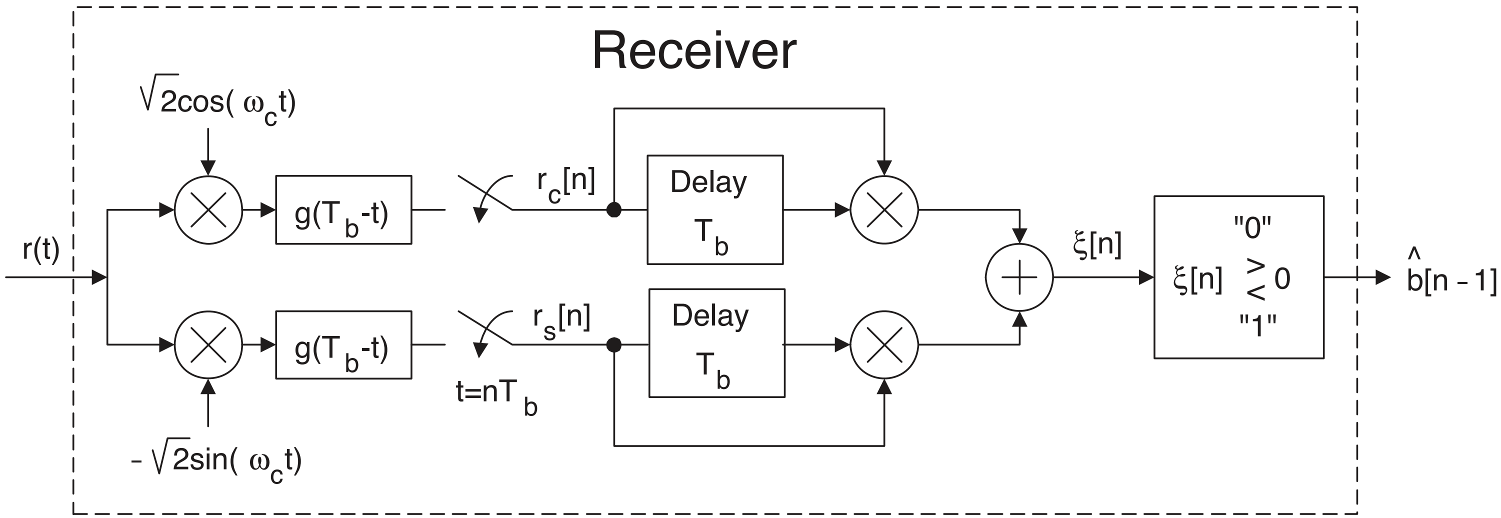
- With differential PSK, the message $m[n] = m_\ell$ is mapped to the phase according to

$$\theta_n = \theta_{n-1} + \frac{2\pi\ell}{M} \qquad \ell = 0, \ldots, M-1$$

- The transmitted phase θ_n depends on both θ_{n-1} and $m[n]$
- This differential encoding introduces memory and the transmitted signal alternatives become dependent
- **Example 5.25:** binary DPSK



Differential Phase Shift Keying ($M = 2$)



- The receiver uses no phase offset v in the carrier waveforms
- Without noise, the decision variable is

$$\begin{aligned}\xi[n] &= r_c[n]\, r_c[n-1] + r_s[n]\, r_s[n-1] \\ &= A\cos(\theta_{n-1}+v)\, A\cos(\theta_{n-2}+v) + A\sin(\theta_{n-1}+v)\, A\sin(\theta_{n-2}+v) \\ &= A^2\cos(\theta_{n-1}-\theta_{n-2}) \quad \Rightarrow \text{ independent of } v\end{aligned}$$

- Note: non-coherent reception increases variance of noise

EITG05 – Digital Communications

Lecture 8

Chapter 3: Carrier modulation techniques

Bandpass signals, digital and analog modulation

Michael Lentmaier
Monday, October 1, 2018

From last lecture: Non-coherent receivers

- With phase-shift keying (PSK) the message $m[n]$ at time nT_s is put into the phase θ_n of the transmit signal

$$s(t) = g(t)\,\sqrt{2E}\,\cos(2\pi f_c t + \theta_n)\ , \quad nT_s \leq t \leq (n+1)T_s$$

- The channel introduces some attenuation α, some additive noise $N(t)$ and also some phase offset ν into the received signal

$$r(t) = \alpha\; g(t)\,\sqrt{2E}\,\cos(2\pi f_c t + \theta_n + \nu) + N(t)$$

- **Challenge:** the optimal receiver needs to know α and ν
- In some applications an accurate estimation of ν is infeasible (cost, complexity, size)
- **Non-coherent receivers:**
  receiver structures that can work well without knowledge of the exact phase offset

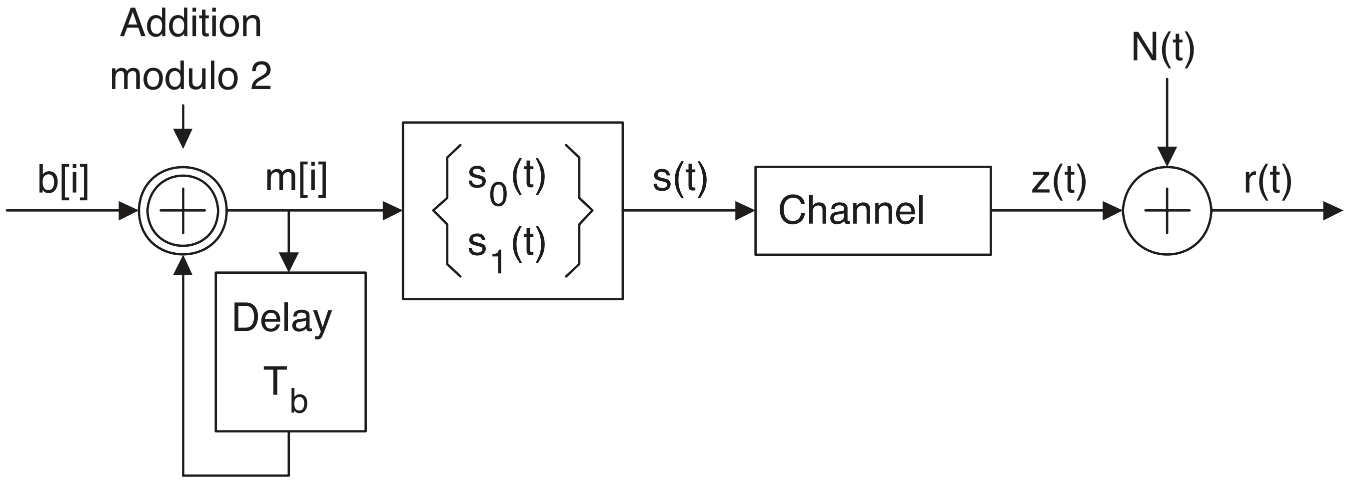
How can we modify our PSK transmission accordingly?

Differential Phase Shift Keying

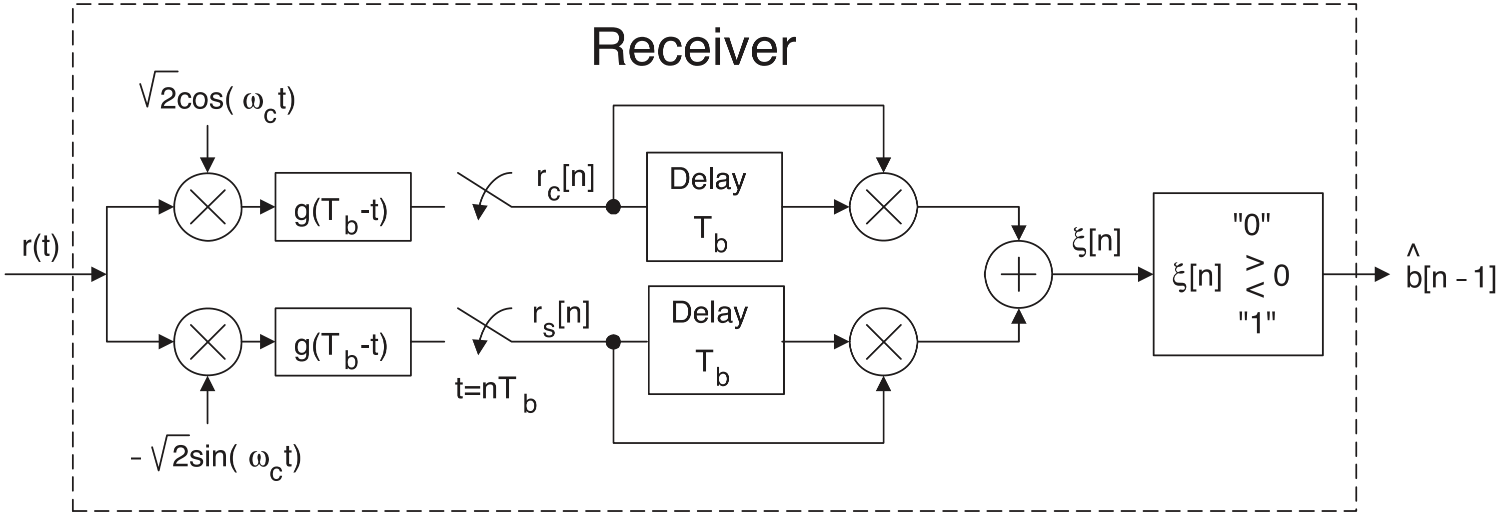
- With differential PSK, the message $m[n] = m_\ell$ is mapped to the phase according to

$$\theta_n = \theta_{n-1} + \frac{2\pi\ell}{M} \qquad \ell = 0, \ldots, M-1$$

- The transmitted phase θ_n depends on both θ_{n-1} and $m[n]$
- This differential encoding introduces memory and the transmitted signal alternatives become dependent
- **Example 5.25:** binary DPSK



Differential Phase Shift Keying ($M = 2$)



- The receiver uses no phase offset ν in the carrier waveforms
- Without noise, the decision variable is

$$
\begin{aligned}
\xi[n] &= r_c[n]\, r_c[n-1] + r_s[n]\, r_s[n-1] \\
&= A\cos(\theta_{n-1}+\nu)\; A\cos(\theta_{n-2}+\nu) + A\sin(\theta_{n-1}+\nu)\; A\sin(\theta_{n-2}+\nu) \\
&= A^2\cos(\theta_{n-1}-\theta_{n-2}) \quad \Rightarrow \text{ independent of } \nu
\end{aligned}
$$

- Note: non-coherent reception increases variance of noise

Chapter 3: Carrier modulation techniques

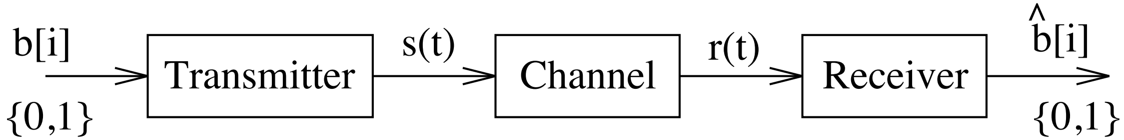
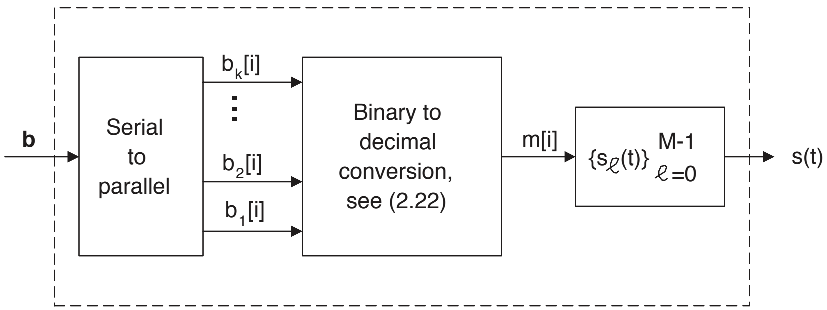


Figure 4.1: A digital communication system.

What we have done so far:

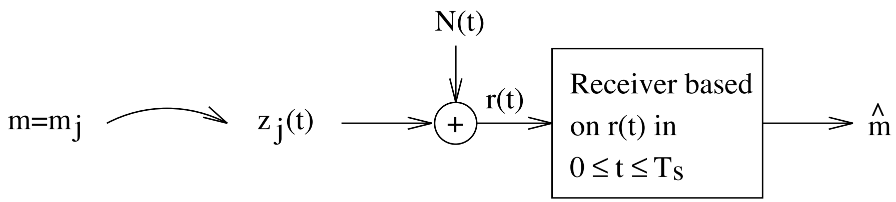
Chapter 2:

From $b[i]$ and $m[i]$ to signals $s_\ell(t)$



Chapter 4:

From signals $z_j(t) + N(t)$ to $\hat{m}[i]$ and $\hat{b}[i]$



Now more on:

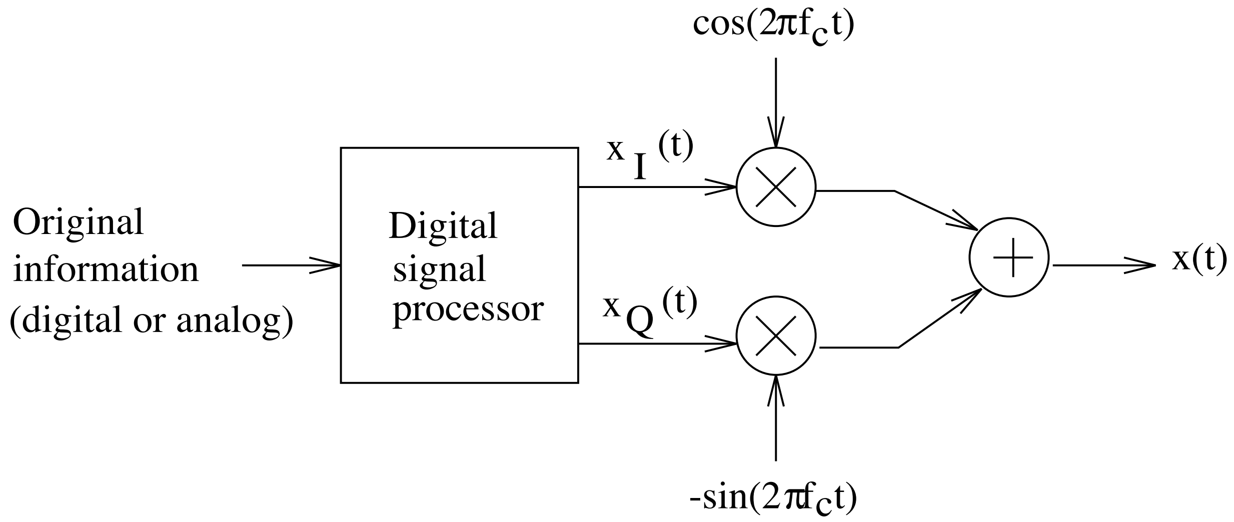
- properties of bandpass signals
- the channel: from $s(t)$ over $z(t)$ to $r(t)$
- efficient receivers for bandpass signals

Bandpass Signals

- A general bandpass signal can always be written as

$$x(t) = x_I(t)\ \cos(2\pi f_c\, t)\ -\ x_Q(t)\ \sin(2\pi f_c\, t)\ , \qquad -\infty \le t \le \infty$$

- $x_I(t)$: inphase component $\qquad x_Q(t)$: quadrature component
- Corresponding transmitter structure:



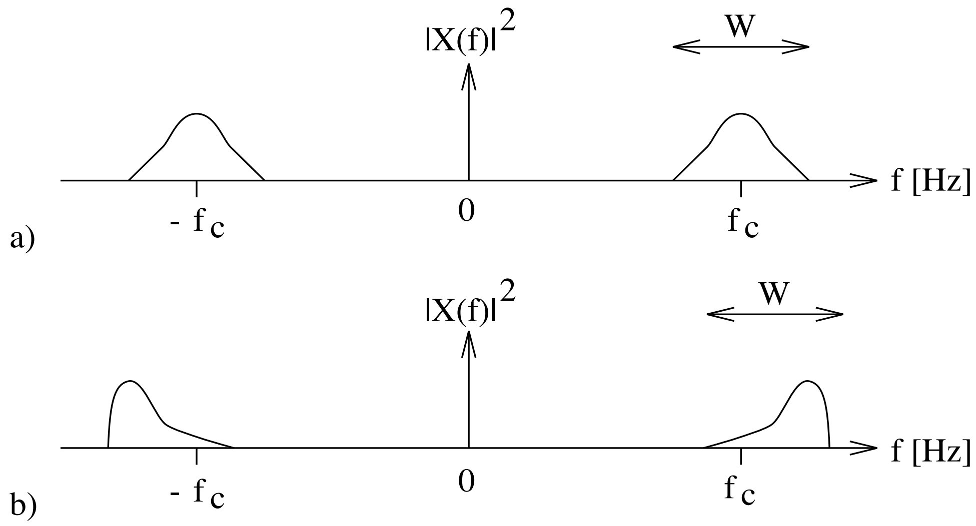
- The information is contained in the signals $x_I(t)$ and $x_Q(t)$ (for both analog or digital modulation)
- Not only wireless systems use carrier modulation

Spectrum of bandpass signals

- Computing the Fourier transform of $x(t)$ we get

$$X(f) = \frac{X_I(f+f_c) - j\,X_Q(f+f_c)}{2} + \frac{X_I(f-f_c) + j\,X_Q(f-f_c)}{2}$$

- Normally, $X_I(t)$ and $X_Q(t)$ have baseband characteristic, and f_c is much larger than their bandwidth
- The spectrum can be symmetric or non-symmetric around f_c



- **Remember:** real signals $x(t)$ always have even $|X(f)|$

DSB-SC Carrier Modulation

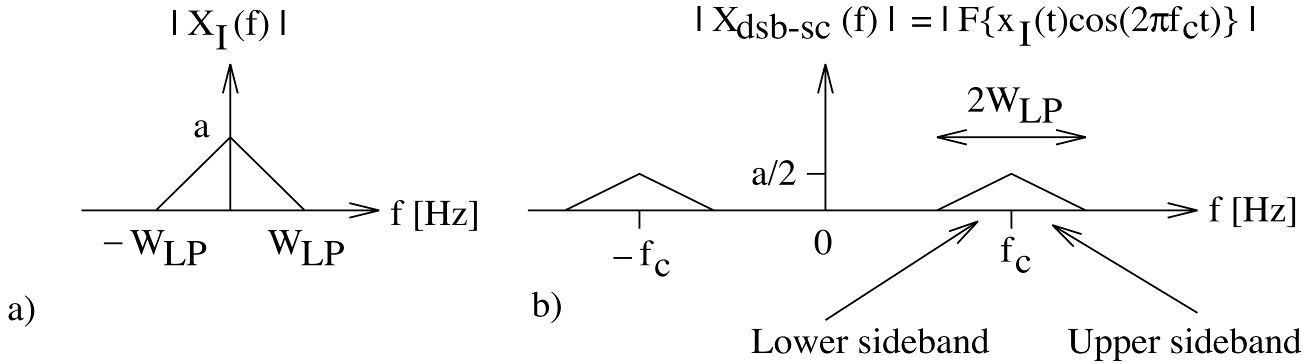
- Double sideband-suppressed (DSB-SC) carrier modulation is a special case of our general model
- In this case only $x_I(t)$ contains information and $x_Q(t) = 0$, i.e.,

$$x_{dsb-sc}(t) = x_I(t)\cos(2\pi f_c t)$$

- The Fourier transform then simplifies to

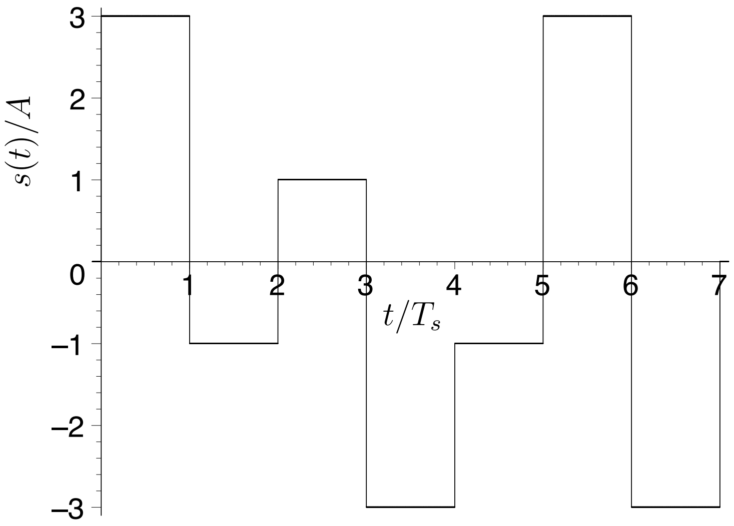
$$X(f) = \frac{X_I(f+f_c)}{2} + \frac{X_I(f-f_c)}{2}$$

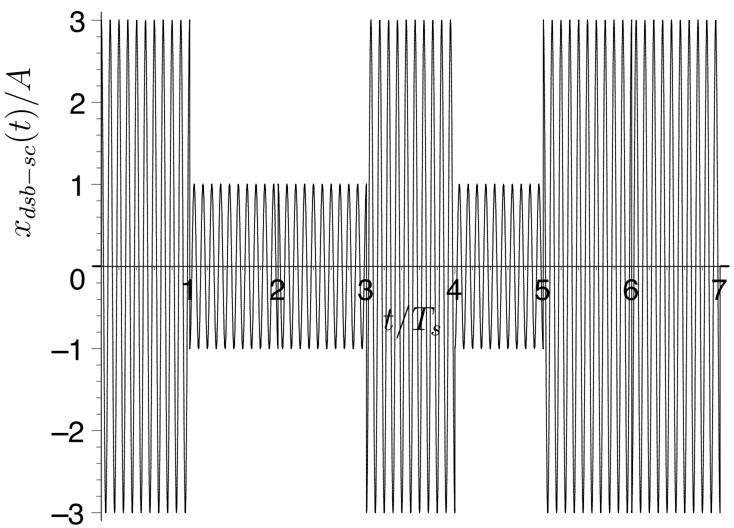
- $X_I(f)$ is symmetric around $f = 0$ $\Rightarrow$ $X_I(f)$ is symmetric around f_c



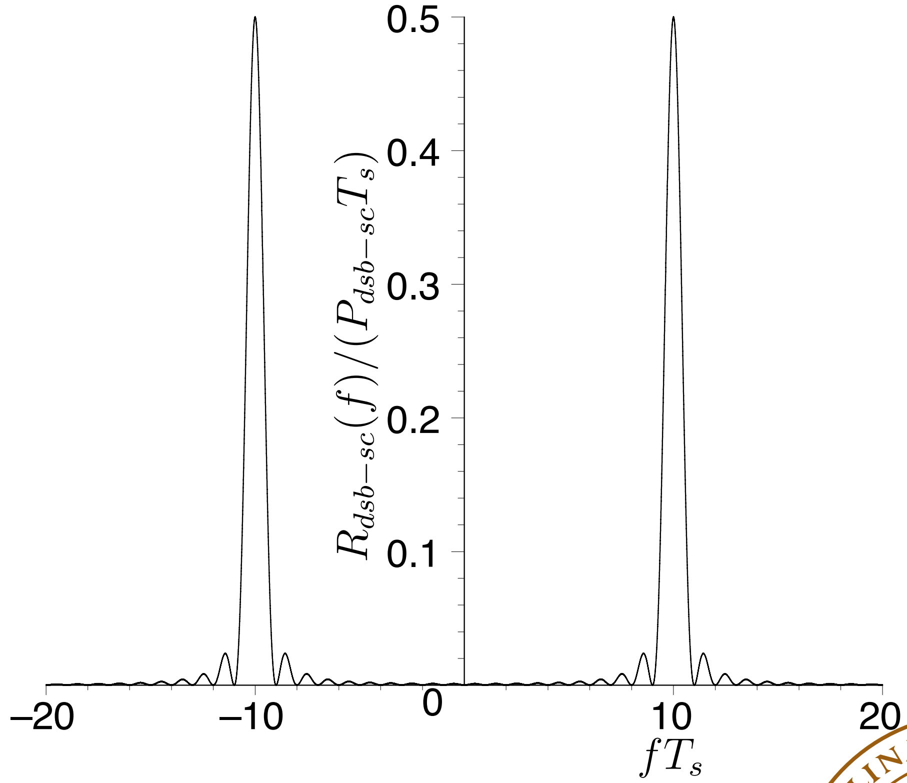
Where does the name come from?

Example 3.1: 4-ary PAM



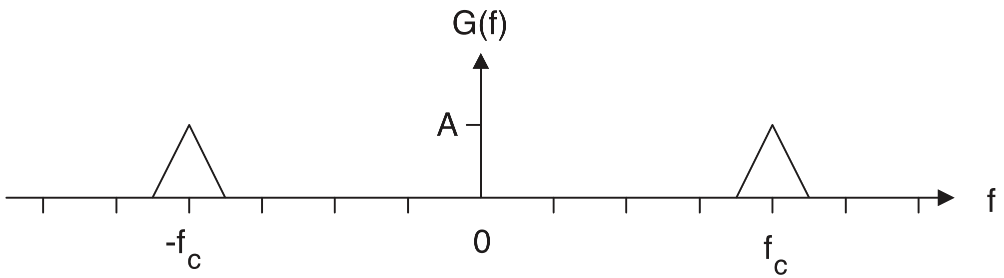


$$x_I(t) = s(t) = \sum_{n=-\infty}^{\infty} A_{m[n]}\, g_{rec}(t - nT_s)$$



How can we revert the frequency shift to f_c?

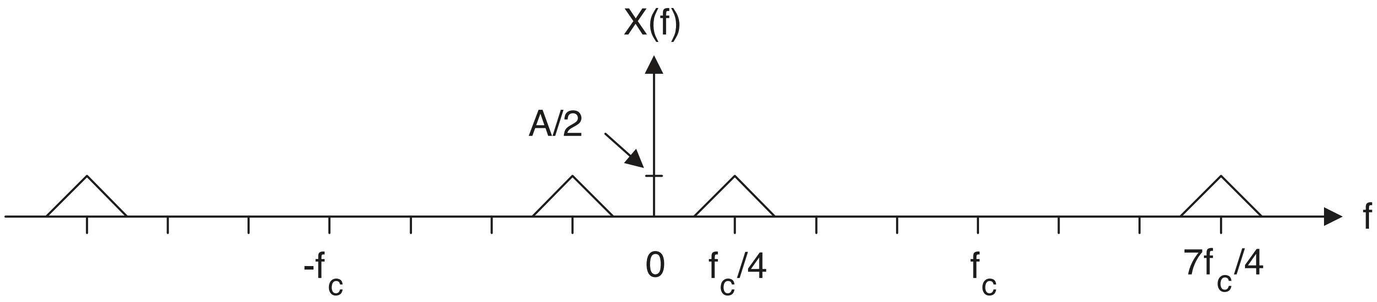
Hint: check Example 2.19 (p. 68)



Find the frequency content of

$$x(t) = g(t)\cos(2\pi f_0 t) \ , \quad f_0 = 3f_c/4$$

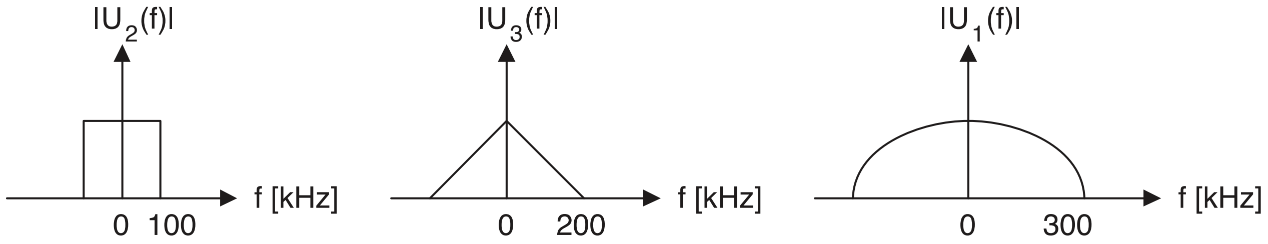
Solution:
If we apply (2.157) using $G(f)$ above, we obtain the frequency content in $x(t)$ as

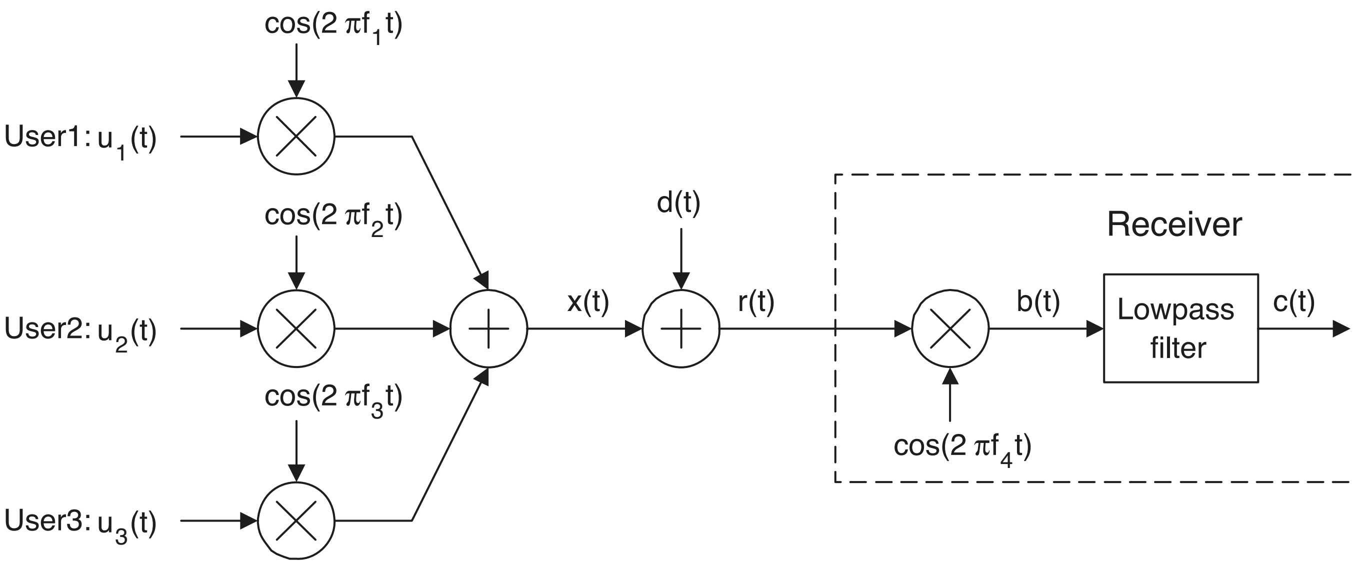


How should we choose f_0 to get the baseband signal back?

Problem 3.9

In the three-user (digital) communication system below, the frequency content in the user information signals $u_1(t)$, $u_2(t)$ and $u_3(t)$ are,





It is known that the individual carrier frequencies are: $f_1 = 3.5$ MHz, $f_2 = 4.0$ MHz, $f_3 = 3$ MHz. The disturbance $d(t)$ is $d(t) = \cos(2\pi 2 f_d t)$ where $f_d = 1.7$ MHz.
Only frequencies up to 100 kHz pass the lowpass filter.

Envelope and Phase

- A frequency shift corresponds to a multiplication with $e^{j2\pi f_c t}$
- For connecting this to the cosine and sine function we use

$$e^{j2\pi f_c t} = \cos(2\pi f_c t) \; + \; j\sin(2\pi f_c t)$$

- The general bandpass signal can then be written in terms of a frequency shifted version of a complex signal $x_I(t) + jx_Q(t)$

$$\begin{aligned} x(t) &= x_I(t)\cos(2\pi f_c t) \; - \; x_Q(t)\sin(2\pi f_c t) \\ &= Re\left\{\left(x_I(t) + jx_Q(t)\right)e^{j2\pi f_c t}\right\} \end{aligned}$$

- Expressing $x_I(t) + jx_Q(t)$ in terms of magnitude and phase we get

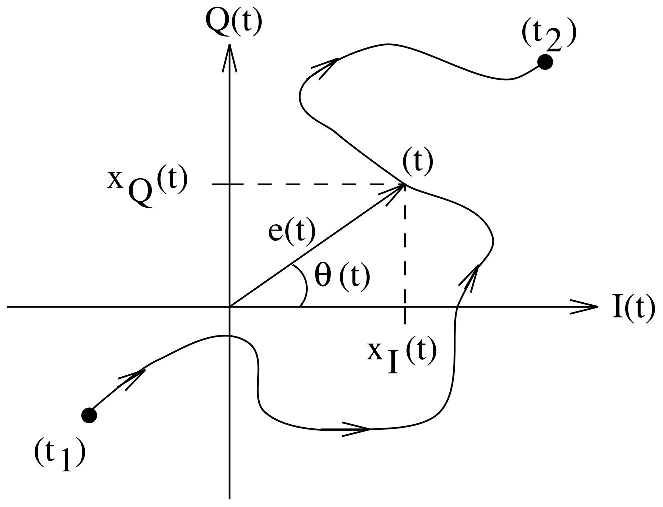
$$x(t) = e(t)\cos(2\pi f_c t + \theta(t)) \; , \quad -\infty \leq t \leq \infty$$

with

$$\begin{aligned} e(t) &= \sqrt{x_I^2(t) + x_Q^2(t)} \geq 0 \\ x_I(t) &= e(t)\cos(\theta(t)) \\ x_Q(t) &= e(t)\sin(\theta(t)) \end{aligned}$$

I-Q Diagram

- In the representation
$$x(t) = x_I(t)\cos(2\pi f_c t) \; - \; x_Q(t)\sin(2\pi f_c t)$$
the information is contained in the inphase component $x_I(t)$ and quadrature component $x_Q(t)$
- In the representation
$$x(t) = e(t)\cos(2\pi f_c t + \theta(t)) \; , \quad -\infty \le t \le \infty$$
the information is contained in the envelope $e(t)$ and instantaneous phase $\theta(t)$



connection: I-Q diagram

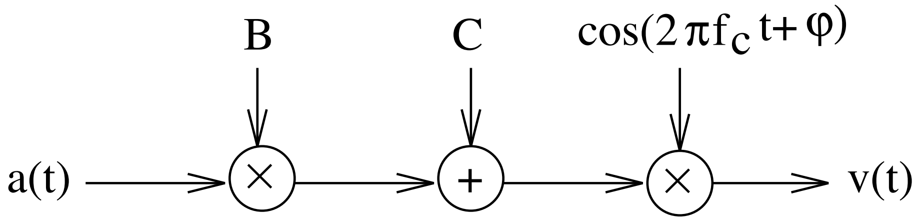
Analog Information Transmission

- Suppose that the information signal is an analog waveform $a(t)$
  Examples: music, speech, video
- If we use digital modulation, the waveform $a(t)$ is first converted to a binary sequence $b[i]$, which then is mapped to signals $s_\ell(t)$
- In case of analog modulation, the waveform $a(t)$ is used directly to modulate the carrier signal
- Let $v(t)$ denote the bandpass signal of an analog transmitter

$$\begin{aligned} v(t) &= v_I(t)\cos(2\pi f_c t) \;-\; v_Q(t)\sin(2\pi f_c t)\;, \quad -\infty \leq t \leq \infty \\ &= e(t)\cos\left(2\pi f_c t + \theta(t)\right) \end{aligned}$$

- **Amplitude modulation (AM):**
  the waveform $a(t)$ modulates the envelope $e(t)$ only
- **Frequency modulation (FM):**
  here $a(t)$ modulates the instantaneous phase $\theta(t)$ only

Amplitude Modulation (AM)



- The AM signal is the sum of a DSB-SC signal and carrier wave

$$\begin{aligned} v(t) &= \big(a(t)\,B + C\big)\,\cos(2\pi f_c\,t + \varphi) \\ &= a(t)\,B\cos(2\pi f_c\,t+\varphi) \;+\; C\cos(2\pi f_c\,t+\varphi) \end{aligned}$$

- Let us introduce the modulation index

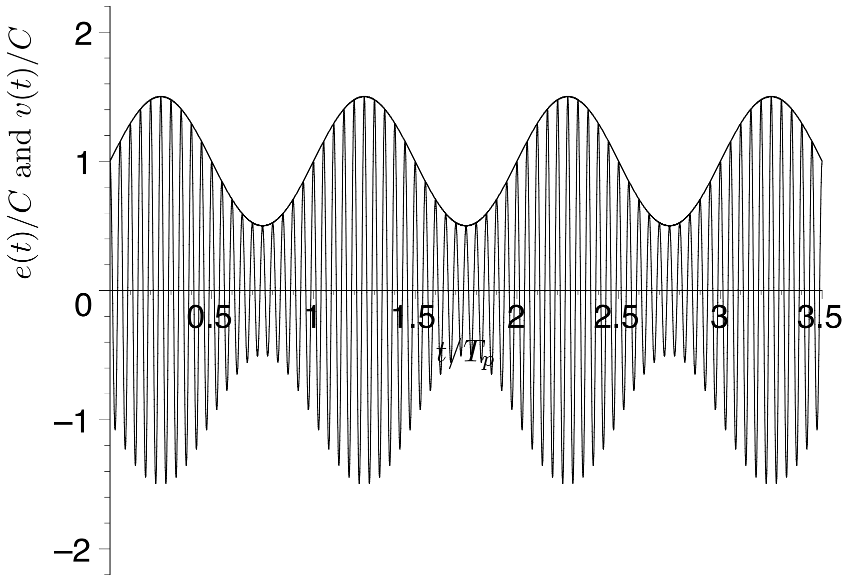
$$m = \frac{B\,a_{max}}{C} \leq 1\,, \quad \text{where} \;\; a_{max} = \max|a(t)|$$

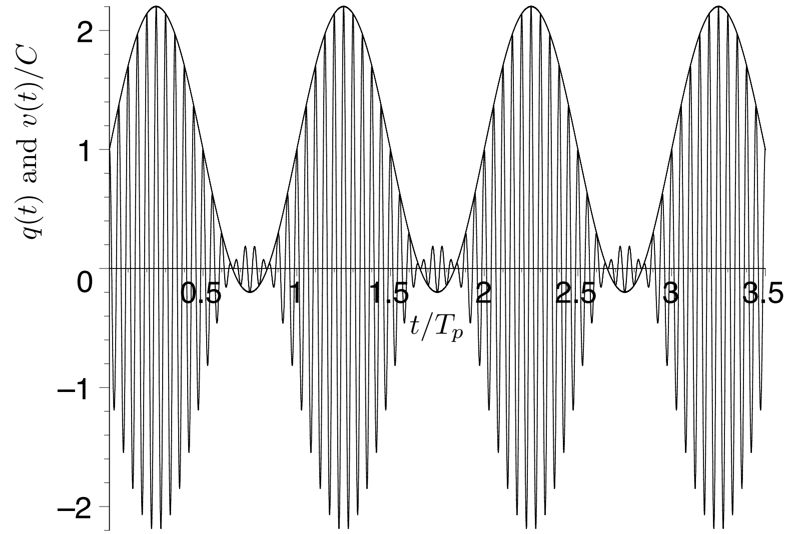
- Using the normalized signal $a_n(t) = a(t)/a_{max}$ we can write

$$v(t) = \big(1 + m\,a_n(t)\big)\,C\cos(2\pi f_c\,t+\varphi)$$

Example: AM signal

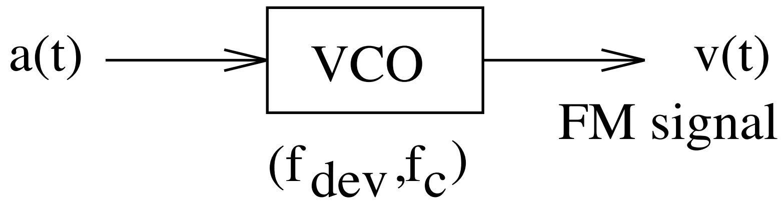
$$e(t)/C = 1 + m\,a(t)\ ,\ a_n(t) = \sin(2\pi f_p t)\ ,\ f_p = 1/Tp$$





- $m = 0.5 < 1$**:**
  the information signal $a_n(t)$ is contained in the envelope $e(t)$
- $m = 1.2 > 1$**:** (right picture)
  overmodulation: the baseband signal $q(t) = (1 + 1.2\,a_n(t))$
  is no longer equal to $e(t)$

Frequency Modulation (FM)



- With FM modulation, the transmitted signal

$$v(t) = \sqrt{2P}\cos(2\pi f_c t + \theta(t))$$

  is generated by a voltage controlled oscillator (VCO)
- The information carrying signal $a(t)$ is related to the phase $\theta(t)$ by

$$\frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_{dev} \cdot a(t)$$

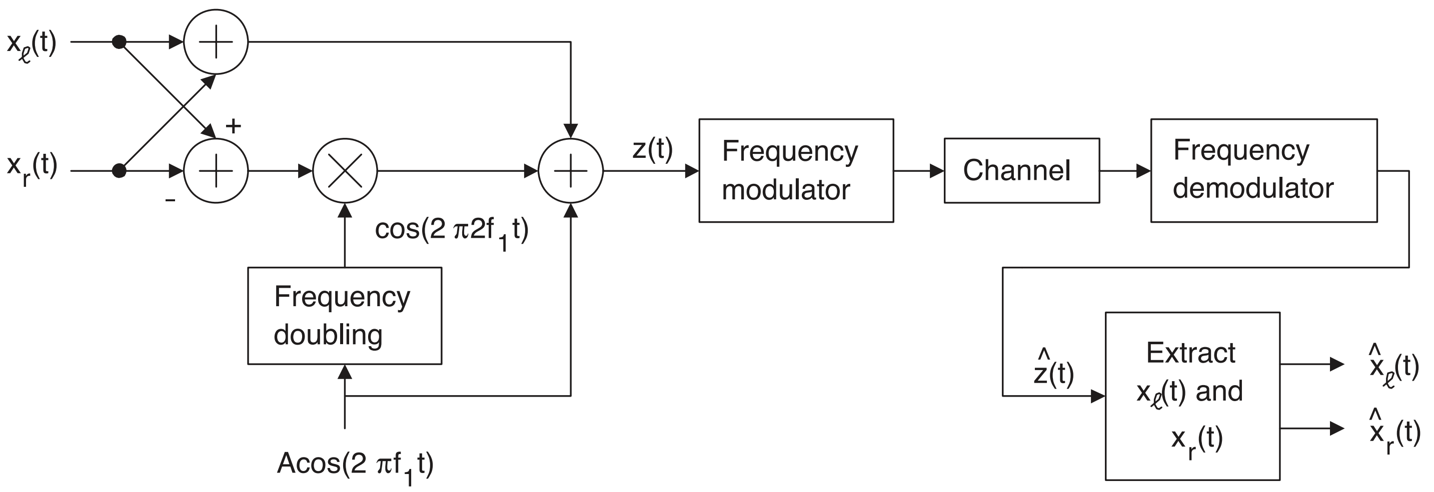
- The signal $a(t)$ hence modulates the instantaneous frequency

$$f_{ins}(t) = f_c + \frac{1}{2\pi}\frac{d\theta(t)}{dt} = f_c + f_{dev}\, a(t)$$

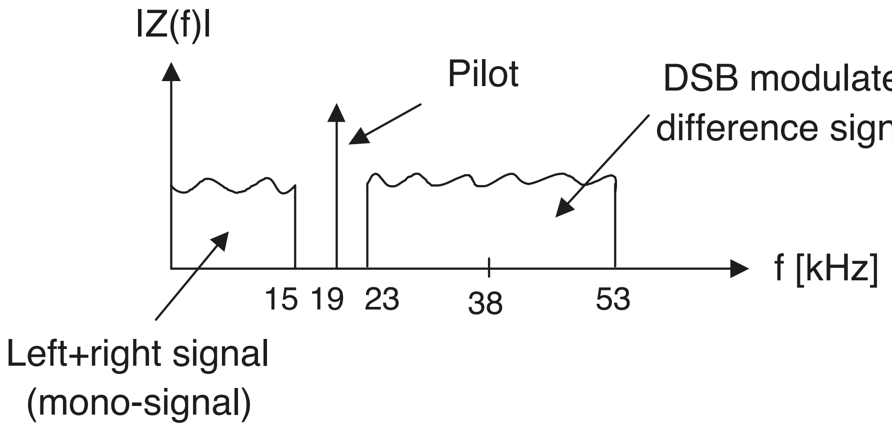
- FM modulation is a non-linear operation, hard to analyze

Example 3.13: FM stereo

A possible block-diagram of conventional analog FM stereo is shown below.



$x_\ell(t)$ and $x_r(t)$ denotes the left and the right audio-channel, respectively, and they are both bandlimited to 15 [kHz]. *The frequency $f_1 = 19$* [kHz] *(often referred to as a so-called pilot-tone).*



Digital Information Transmission

- In Chapter 2 the signal alternatives $s_\ell(t)$ could have arbitrary shape within the signaling interval $0 \leq t \leq T_s$
- The bandpass signal for digital modulation then has the form

$$\begin{aligned} x(t) &= x_I(t)\cos(2\pi f_c t) \; - \; x_Q(t)\sin(2\pi f_c t) \\ &= \left(\sum_{n=-\infty}^{\infty} s_{m[n],I}(t-nT_s) \right) \cos(2\pi f_c t) \\ &\quad - \left(\sum_{n=-\infty}^{\infty} s_{m[n],Q}(t-nT_s) \right) \sin(2\pi f_c t) \end{aligned}$$

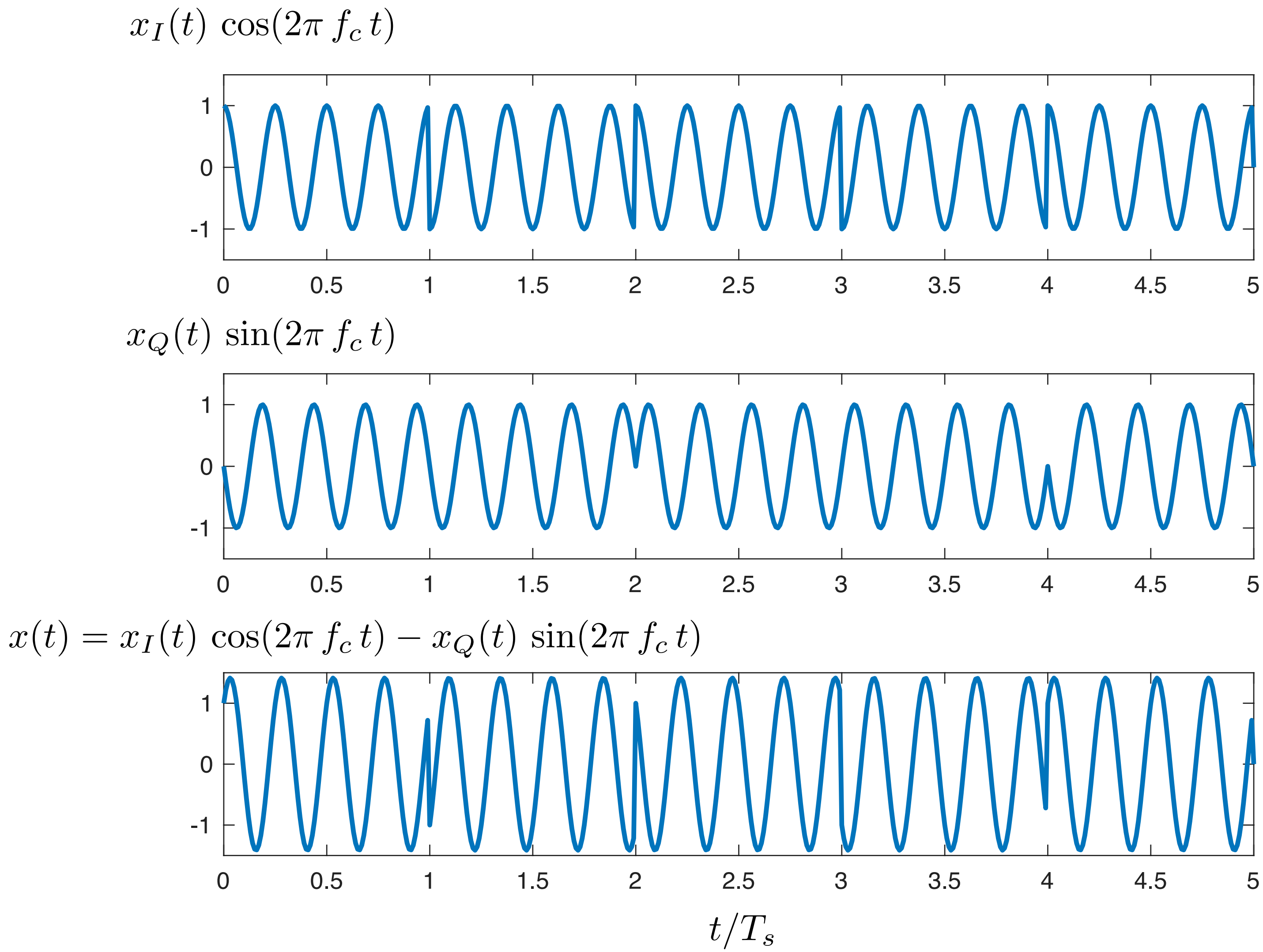
- In case of M-ary QAM we have

$$x_I(t) = \sum_{n=-\infty}^{\infty} A_{m[n]}\, g(t-nT_s) \; , \quad x_Q(t) = \sum_{n=-\infty}^{\infty} B_{m[n]}\, g(t-nT_s)$$

- Also M-ary FSK signals have bandpass characteristics

A simple Matlab example

How does a QPSK signal look like? Here is an example:



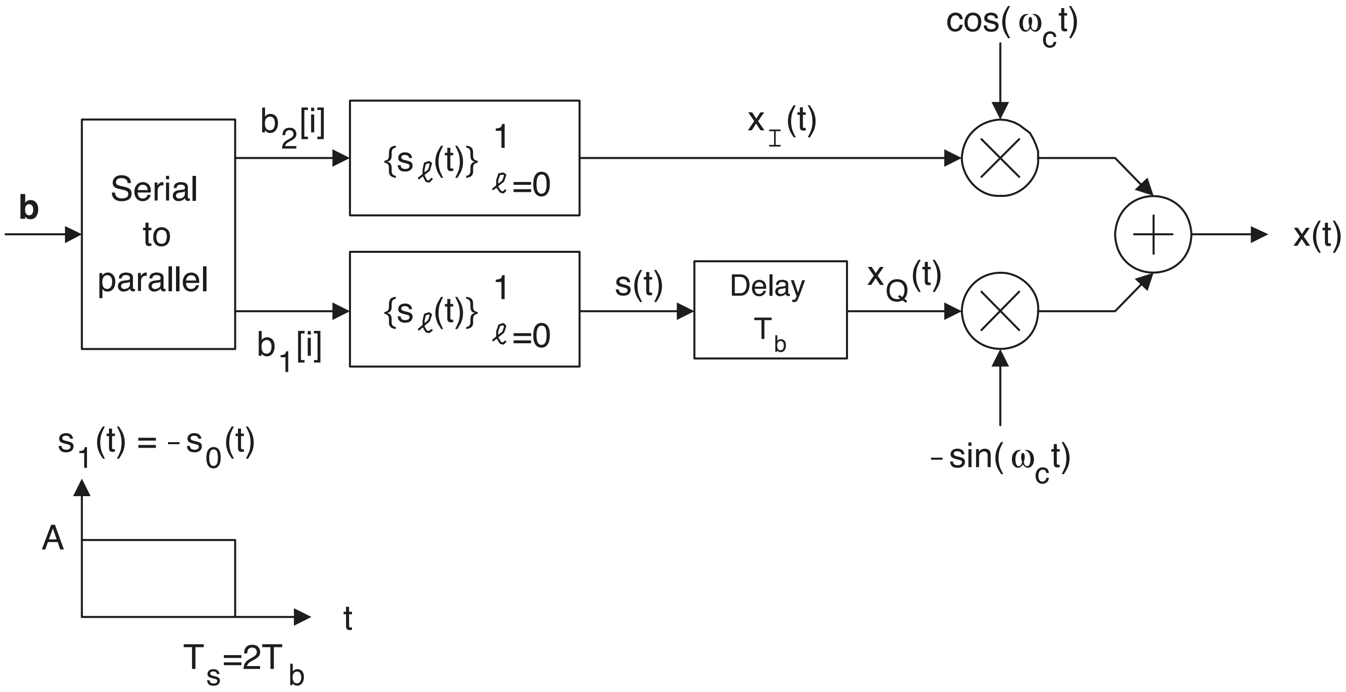
And how it was done:

```
% Example: QPSK signal

t=0:0.01:5;
fc=4;
pRec=ones(1,(length(t)-1)/5);
sI=zeros(1,length(t)); sQ=zeros(1,length(t));

dataI=[1 -1 1 -1 1];
indPulse=1:(length(t)-1)/5;
for i=1:length(dataI),
  sI(indPulse)=dataI(i)*pRec;
  indPulse=indPulse+length(indPulse);
end;

dataQ=[-1 -1 1 1 -1];
indPulse=1:(length(t)-1)/5;
for i=1:length(dataQ),
  sQ(indPulse)=dataQ(i)*pRec;
  indPulse=indPulse+length(indPulse);
end;

sCarI=cos(2*pi*t*fc); sCarQ=sin(2*pi*t*fc);

figure(1);
subplot(3,1,1); plot(t,sI.*sCarI);
set(gca,'YLim',[-1.5 1.5]); xlabel('fT_s');

subplot(3,1,2); plot(t,sQ.*sCarQ);
set(gca,'YLim',[-1.5 1.5]); xlabel('fT_s');

subplot(3,1,3); plot(t,sI.*sCarI - sQ.*sCarQ);
set(gca,'YLim',[-1.5 1.5]); xlabel('fT_s');
```

script | Ln 32 Col 30

Example 3.5: offset QPSK

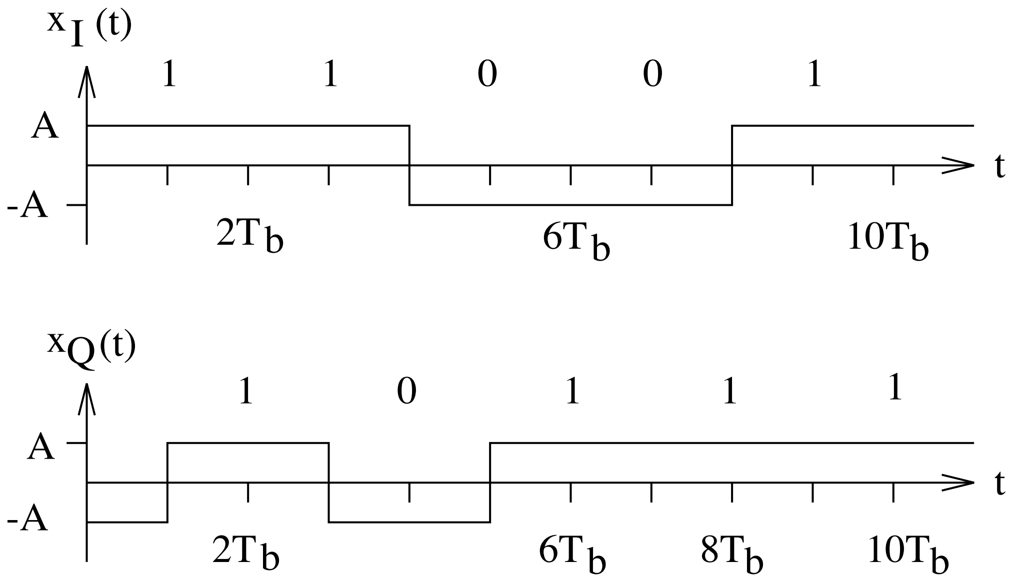
Below, two information carrying baseband signals $x_I(t)$ and $s(t)$ are first generated. Binary antipodal signaling with a rectangular pulse shape is used for both $x_I(t)$ and $s(t)$. The signal $x_Q(t)$ is a delayed version of $s(t)$, $x_Q(t) = s(t - T_b)$.

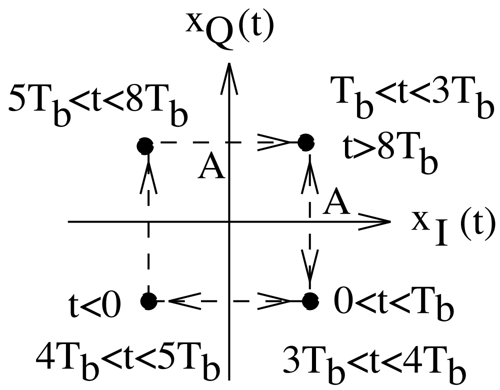


*The information bit rate (in **b**) is $R_b = 1/T_b$. Hence, the signaling rate in the quadrature components is $R_s = R_b/2$.*

QPSK signal with delayed transmission of $x_Q(t)$

Example 3.5: offset QPSK

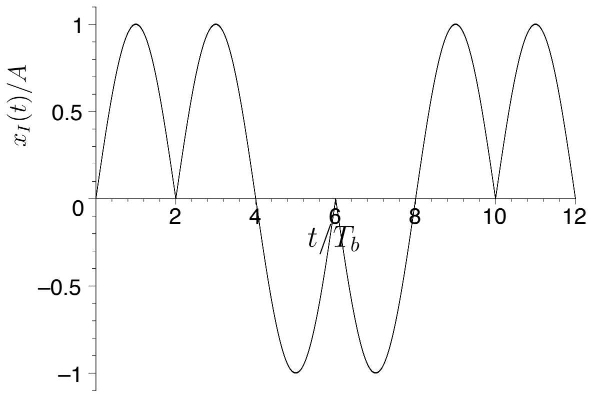


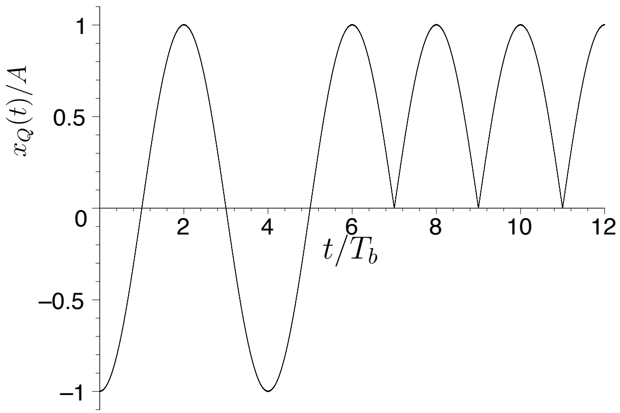


- **Special feature:**
  $x_I(t)$ and $x_Q(t)$ can never change at the same time
- it follows that the envelope does not pass the origin, i.e., $e(t) > 0$
- the variation of instantaneous power $\mathcal{P}(t) = e^2(t)/2$ is small, which allows more efficient power amplifiers

Example 3.6: constant envelope signaling

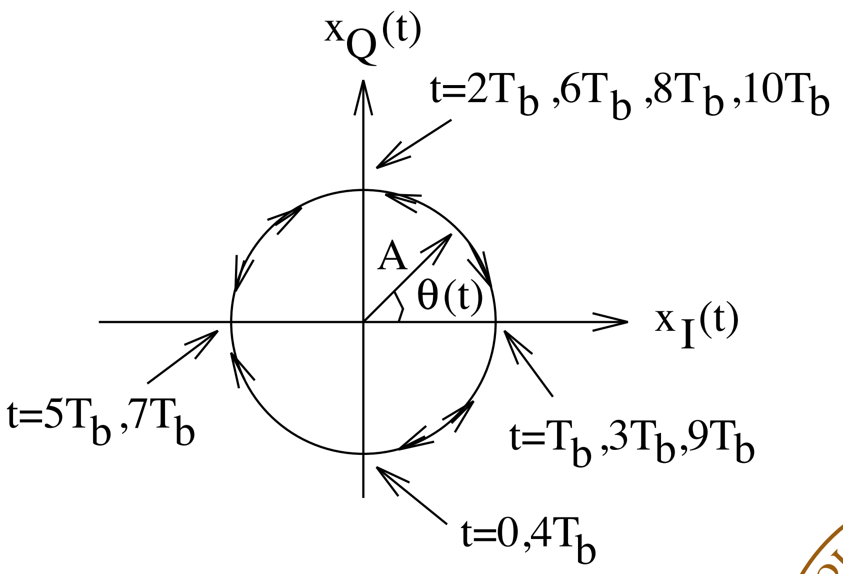
Change pulse shape:
half cycle sinusoidal $g_{hcs(t)}$
instead of $g_{rec}(t)$





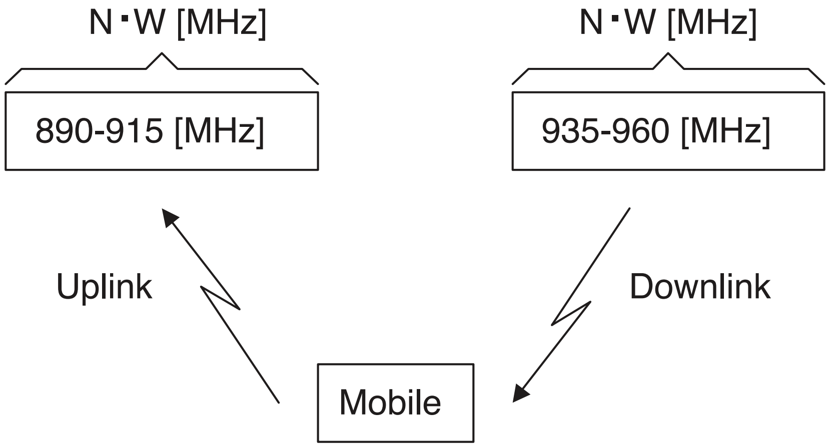
The squared envelope becomes

$$
\begin{aligned}
e^2(t) &= x_I^2(t) + x_Q^2(t) \\
&= A^2 \sin^2(\pi t/(2T_b)) + A^2 \cos^2(\pi t/(2T_b)) \\
&= A^2 \qquad \Rightarrow \text{constant envelope } e(t) = A
\end{aligned}
$$



Continuous phase modulation (CPM) is used in GSM

Example 3.7: GSM



Each sub-band of W [Hz] carries information from X users, which are time-multiplexed using X time-slots. The total number of speech-channels (or data-channels) in the uplink (and in the downlink) is $N \cdot X$.

A specific user is allocated one of the N sub-bands, and one of the X time-slots. A time-slot has duration 576.92 [μs], and 148 binary symbols are transmitted within this time, see the figure below.

Start	Coded bits	Flag	Training sequence	Flag	Coded bits	Stop	ISI
3	57	1	26	1	57	3	margin

148 (Start through Stop)

30.46 µs (ISI margin)

576.92 µs (total)

From 2G to 4G

- **GSM:** (Global System for Mobile Communications)
  based on combined time-division multiple access (TDMA) and frequency division multiple access (FDMA)
- **UMTS:** (Universal Mobile Telecommunications Service)
  based on wideband code division multiple access (W-CDMA)
  each user has an individual code, no TDMA or FDMA
- **LTE (advanced):** (Long Term Evolution)
  orthogonal frequency-division multiple access (OFDMA)

Multiple access:
refers to how different active users are separated

EITG05 – Digital Communications

Lecture 9

Chap. 3: N-ray channel model, noise,
Receivers for bandpass signals

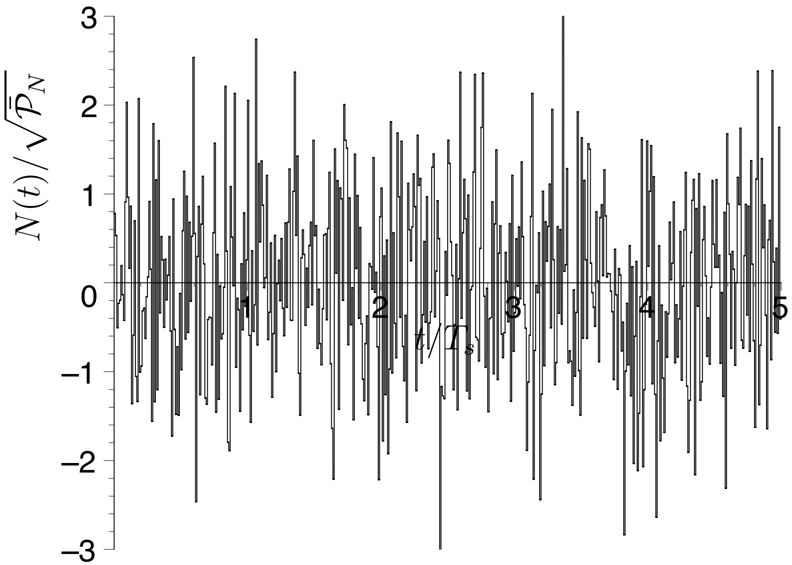
Chap. 4: Filtered channel receiver

Michael Lentmaier
Thursday, October 4, 2018

Channel Noise

- In almost all applications the received signal $r(t)$ is disturbed by some additive noise $N(t)$:

$$r(t) = z(t) + N(t)$$



- Since the received noise disturbs that transmitted signal, we need to characterize its influence on the performance in terms of bit error rate or achievable information bit rate

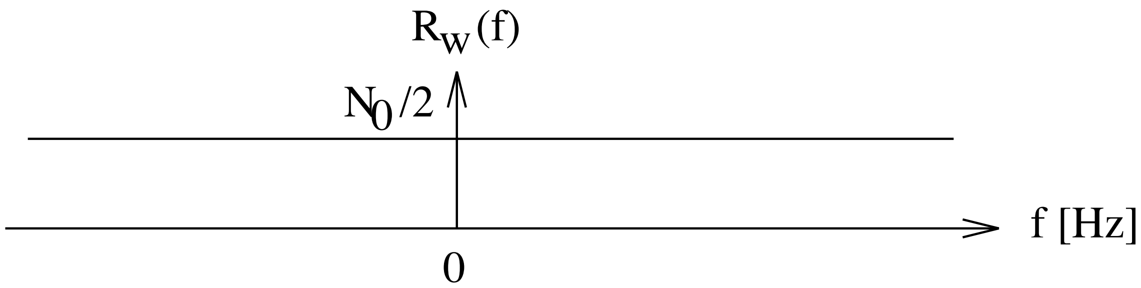
White Gaussian Noise

- White Gaussian noise $w(t)$ is a common model for background noise, such as created by electronic equipment
- The samples of $w(t)$ have a zero-mean Gaussian distribution
- Any two distinct samples of $w(t)$ are uncorrelated

$$r_w(\tau) = E\{w(t+\tau)\,w(t)\} = \frac{N_0}{2}\,\delta(\tau)$$

- This leads to a constant power spectral density

$$R_w(f) = \int_{-\infty}^{\infty} r_w(\tau)\, e^{-j2\pi f\tau}\, d\tau = \frac{N_0}{2}\;, \qquad -\infty \le f \le \infty$$



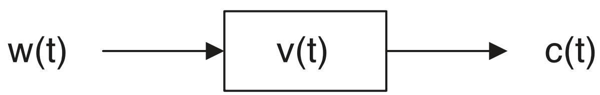
All frequencies are disturbed equally strongly

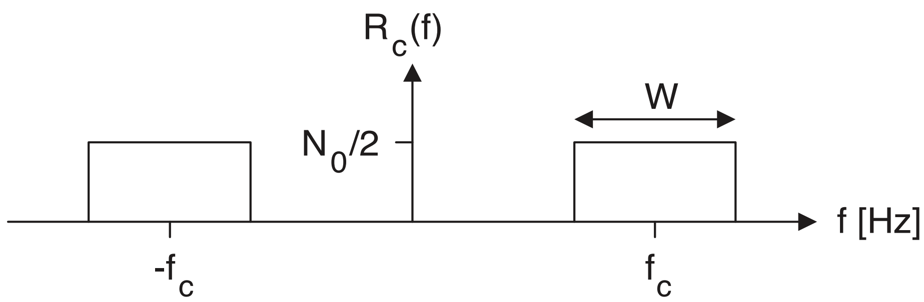
Filtered Gaussian Noise

- In reality we usually deal with filtered noise of limited bandwidth, so-called colored noise
- Assuming that white Gaussian noise $w(t)$ passes a filter $v(t)$ we obtain colored noise $c(t)$ with power spectral density

$$R_c(f) = R_w(f)\,|V(f)|^2 = \frac{N_0}{2}\,|V(f)|^2$$

- For an ideal bandpass filter $v(t)$ with bandwidth W the spectrum is shown below:





Filtered Gaussian Noise

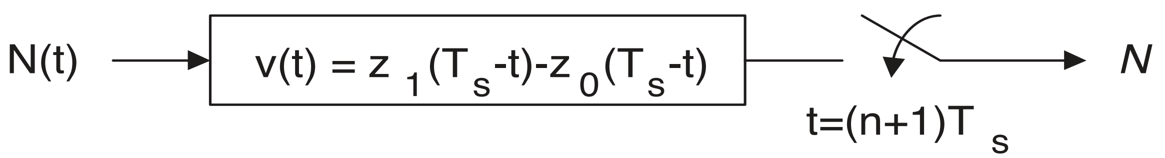
- Since $R(f)$ is constant within the bandwidth W, such a process $c(t)$ is usually referred to as "white" bandpass process
- Let the noise process $c(t)$ be sampled at some time $t = t_0$. Then the sample value $c(t_0)$ is a Gaussian random variable with

$$p(c) = \frac{1}{\sqrt{2\pi\,\sigma^2}}\; e^{-(c-m)^2/2\sigma^2}$$

with mean $m = 0$ and variance $\sigma^2 = N_0/2\; E_v = N_0\, W = \mathcal{P}_c$

Example: matched filter output (recall Chapter 4)

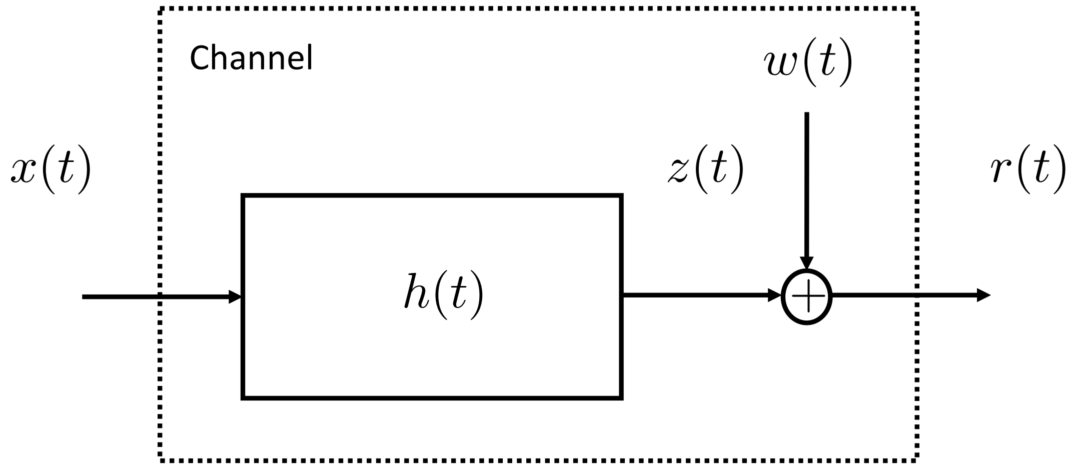
The additive noise $\mathcal{N}$ is sampled from a filtered noise process



$$\sigma^2 = N_0/2 \cdot E_v = N_0/2 \int_0^{T_s} \left(z_1(t) - z_0(t)\right)^2 dt$$

Linear-Filter Channels

- The channel is often modeled as time-invariant filter with noise



- $h(t)$ is the channel impulse response and $w(t)$ the additive noise
- The received signal becomes

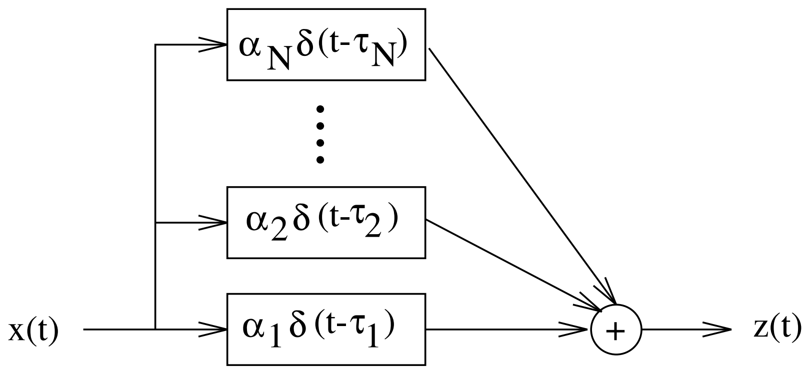
$$r(t) = x(t) * h(t) + w(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\; d\tau + w(t)$$

- The simplest case is an attenuated noisy channel:

$$h(t) = \alpha\,\delta(t) \qquad \Rightarrow r(t) = \alpha\, s(t) + w(t)$$

N-ray Channel Model

- In many applications (wired and wireless) the transmitted signal $x(t)$ reaches the receiver along several different paths
- Such multi-path propagation motivates the N-ray channel model



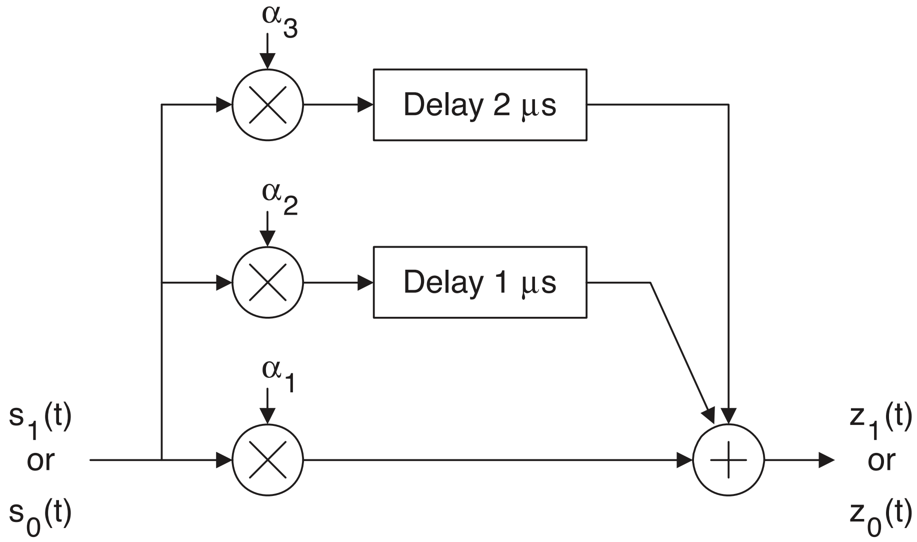
- The output signal becomes

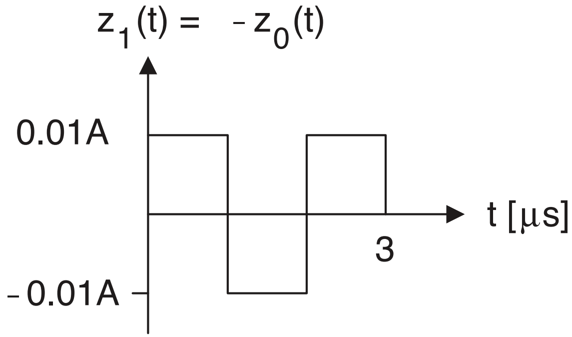
$$z(t) = \sum_{i=1}^{N} \alpha_i \, x(t-\tau_i) = x(t) * h(t)$$

- The impulse response $h(t)$ and its Fourier transform are given by

$$h(t) = \sum_{i=1}^{N} \alpha_i \, \delta(t-\tau_i) \; , \quad H(f) = \sum_{i=1}^{N} \alpha_i \, e^{-j2\pi f \, \tau_i}$$

Example 3.19: multipath propagation





$$s_1(t) = -s_0(t) = \begin{cases} A & , \quad 0 \le t \le 10^{-6} \\ 0 & , \quad otherwise \end{cases}$$

$$\alpha_1 = 0.01, \alpha_2 = -0.01, \alpha_3 = 0.01$$

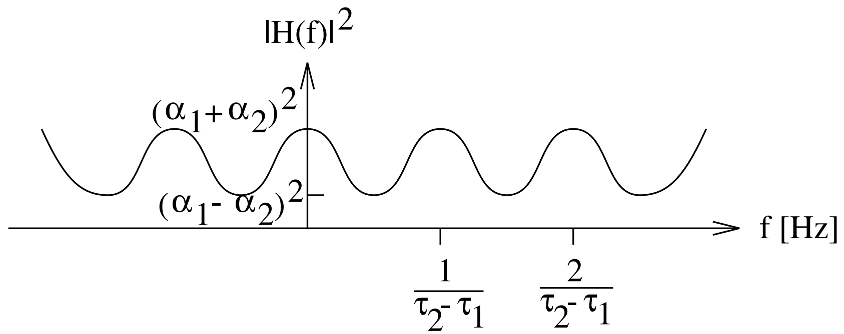
- The channel (= filter) increases the length of the signals
- Signals exceed their time interval and will overlap if T_s is not increased accordingly $\Rightarrow$ inter-symbol interference (ISI)

Example 3.20

EXAMPLE 3.20
Calculate and sketch $|H(f)|^2$ for the 2-ray channel model.

Solution:
From (3.128) we obtain,

$$
\begin{aligned}
H(f) &= \alpha_1 e^{-j2\pi f\tau_1} + \alpha_2 e^{-j2\pi f\tau_2} = \\
&= e^{-j2\pi f\tau_1}\left(\alpha_1 + \alpha_2 e^{-j2\pi f(\tau_2-\tau_1)}\right) \\
|H(f)|^2 &= \left(\alpha_1 + \alpha_2 e^{-j2\pi f(\tau_2-\tau_1)}\right)\left(\alpha_1 + \alpha_2 e^{+j2\pi f(\tau_2-\tau_1)}\right) = \\
&= \alpha_1^2 + \alpha_2^2 + \alpha_1\alpha_2\left(e^{j2\pi f(\tau_2-\tau_1)} + e^{-j2\pi f(\tau_2-\tau_1)}\right) = \\
&= \alpha_1^2 + \alpha_2^2 + 2\alpha_1\alpha_2\cos(2\pi f(\tau_2-\tau_1))
\end{aligned}
$$



Channel fading: some frequencies are attenuated strongly

Features of Multipath Channels

Challenges:

- the receiver needs to know the channel
- training sequences need be transmitted for channel estimation
- the impulse response can change over time
- the line-of-sight (LOS) component is sometimes not received

Opportunities:

- with multiple paths we can collect more signal energy
- receiver can work without direct LOS component
- channel knowledge, once we have it, can give useful information:
  Examples: distance, angle of arrival, speed (Doppler)
- positioning/navigation is often based on channel estimation

If you want to know more:

EITN85: Wireless Communication Channels, VT 1

Receiver for linear filter channel model

- For a simple channel with a direct transmission path only

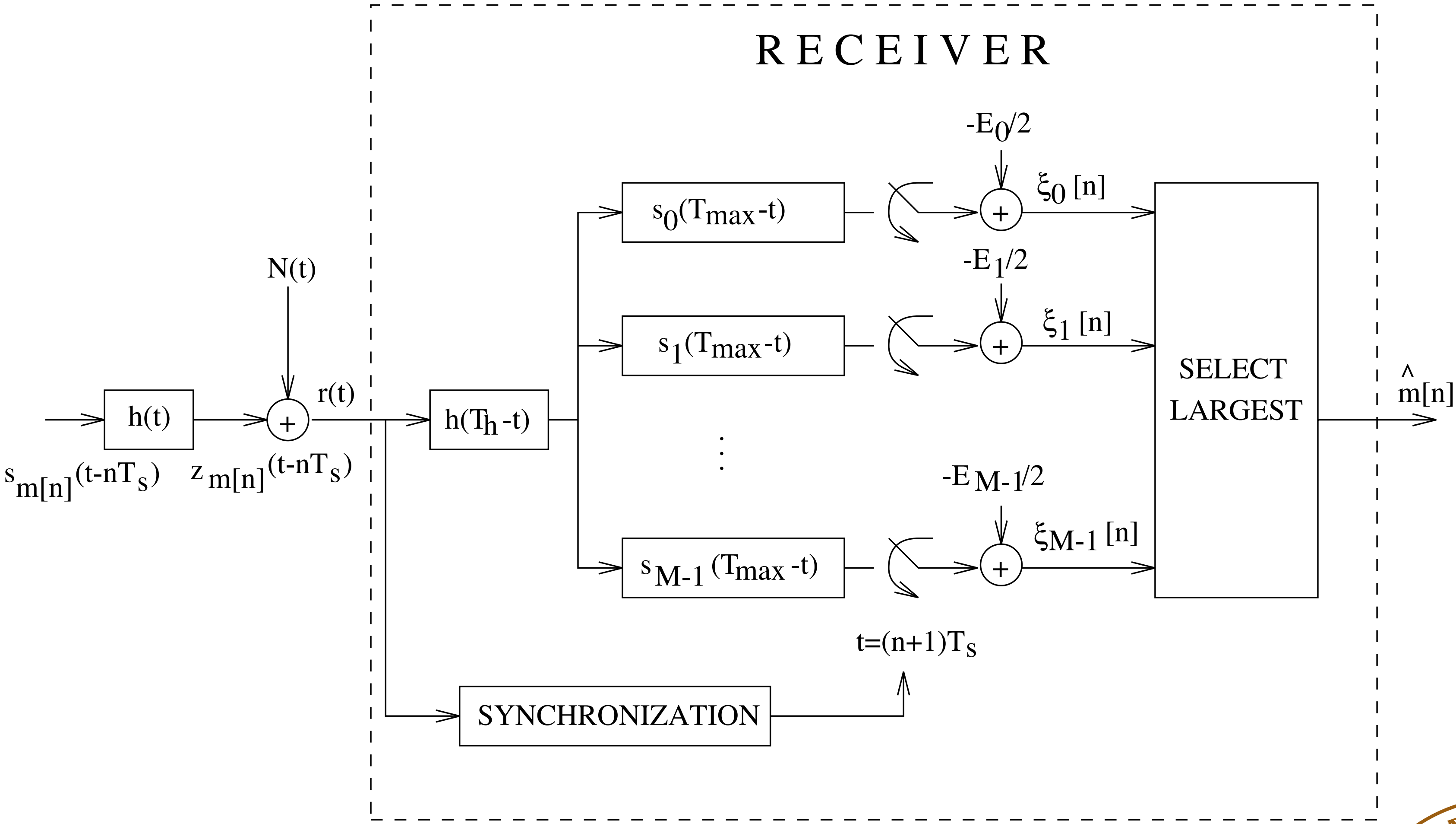
$$h(t) = \alpha\, \delta(t) \quad \Rightarrow \; z_\ell(t) = \alpha\, s_\ell(t)$$

- In case of multipath propagation the channel filter can change the shape and duration of the signals $z_\ell(t)$
- It can be shown that the matched filter of the overall system can be replaced with a cascade of two separate matched filters

$$z_\ell(T_s - t) \quad \Leftrightarrow \quad h(T_h - t)\ ,\ s_\ell(T_{max} - t)\ , \quad T_s = T_{max} + T_h$$

- The channel matching filter $h(T_h - t)$ simplifies the implementation of the receiver

ML receiver with channel matching filter



Example: three-ray channel

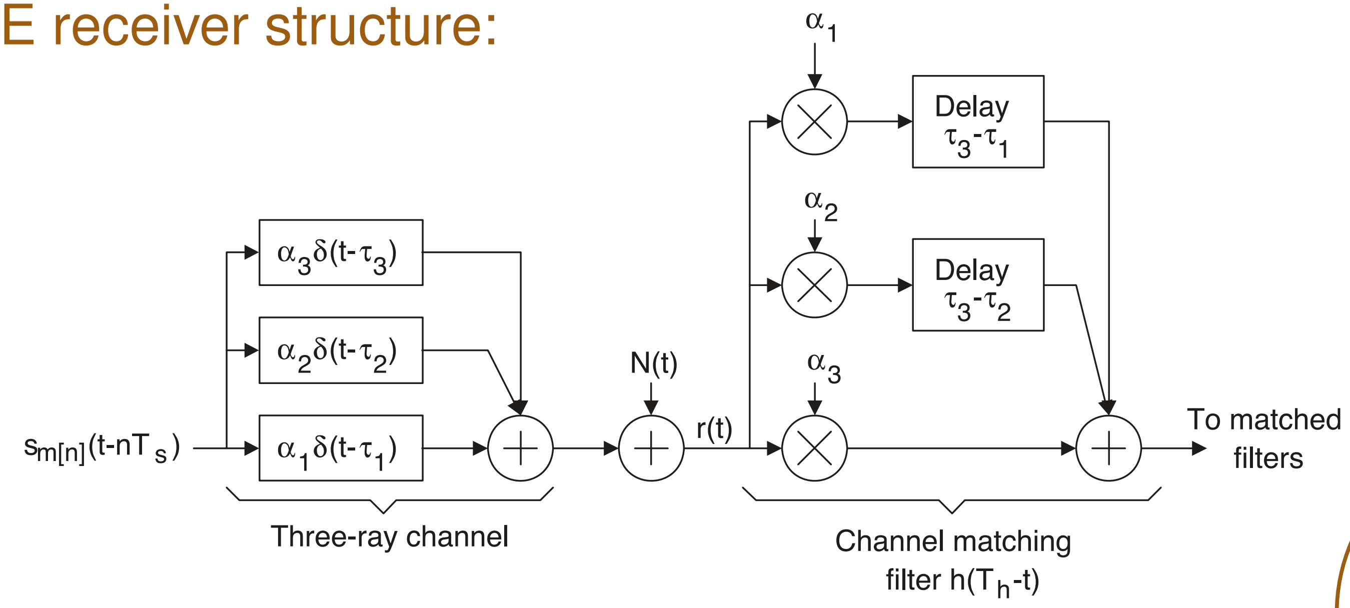
- Consider a channel with three signal paths

$$h(t) = \alpha_1\,\delta(t-\tau_1) + \alpha_2\,\delta(t-\tau_2) + \alpha_3\,\delta(t-\tau_3)$$

- Assuming $\tau_1 < \tau_2 < \tau_3$ we have $T_h = \tau_3$
- The channel matching filter becomes

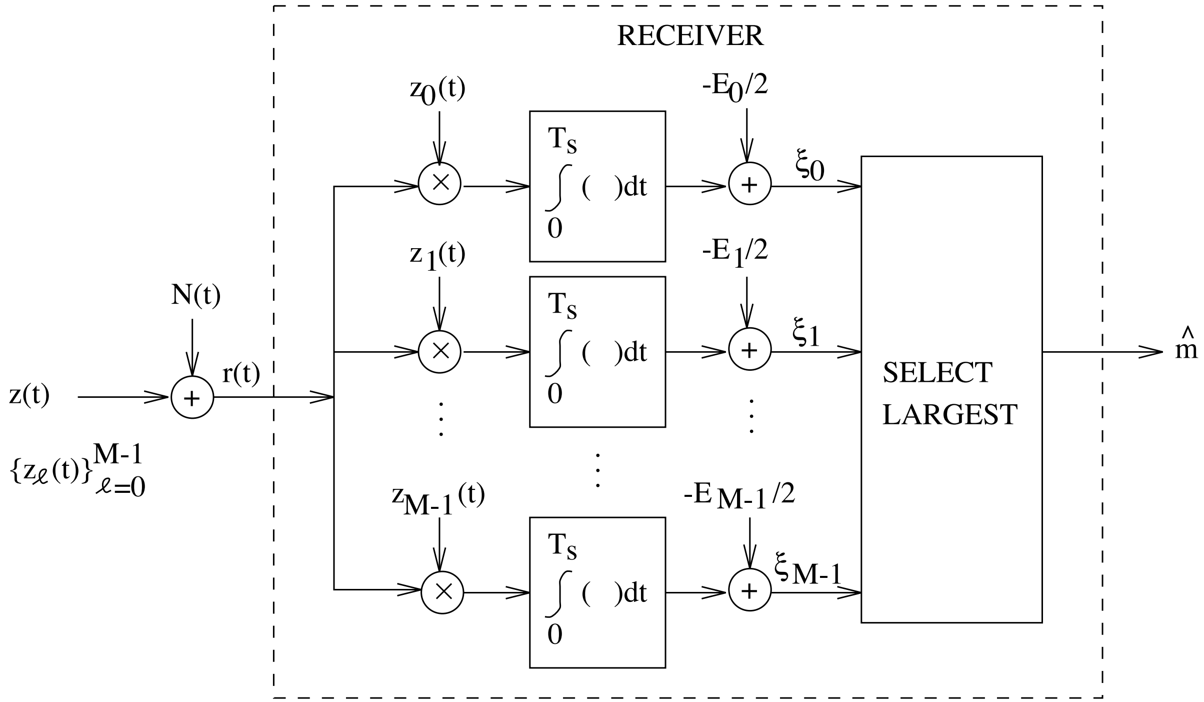
$$\begin{aligned} h(T_h - t) &= h(\tau_3 - t) \\ &= \alpha_3\;\delta(t) + \alpha_2\,\delta(t-(\tau_3-\tau_2)) + \alpha_1\,\delta(t-(\tau_3-\tau_1)) \end{aligned}$$

RAKE receiver structure:



Recall: receiver for M-ary signaling

- Consider the general receiver structure from Chapter 4:

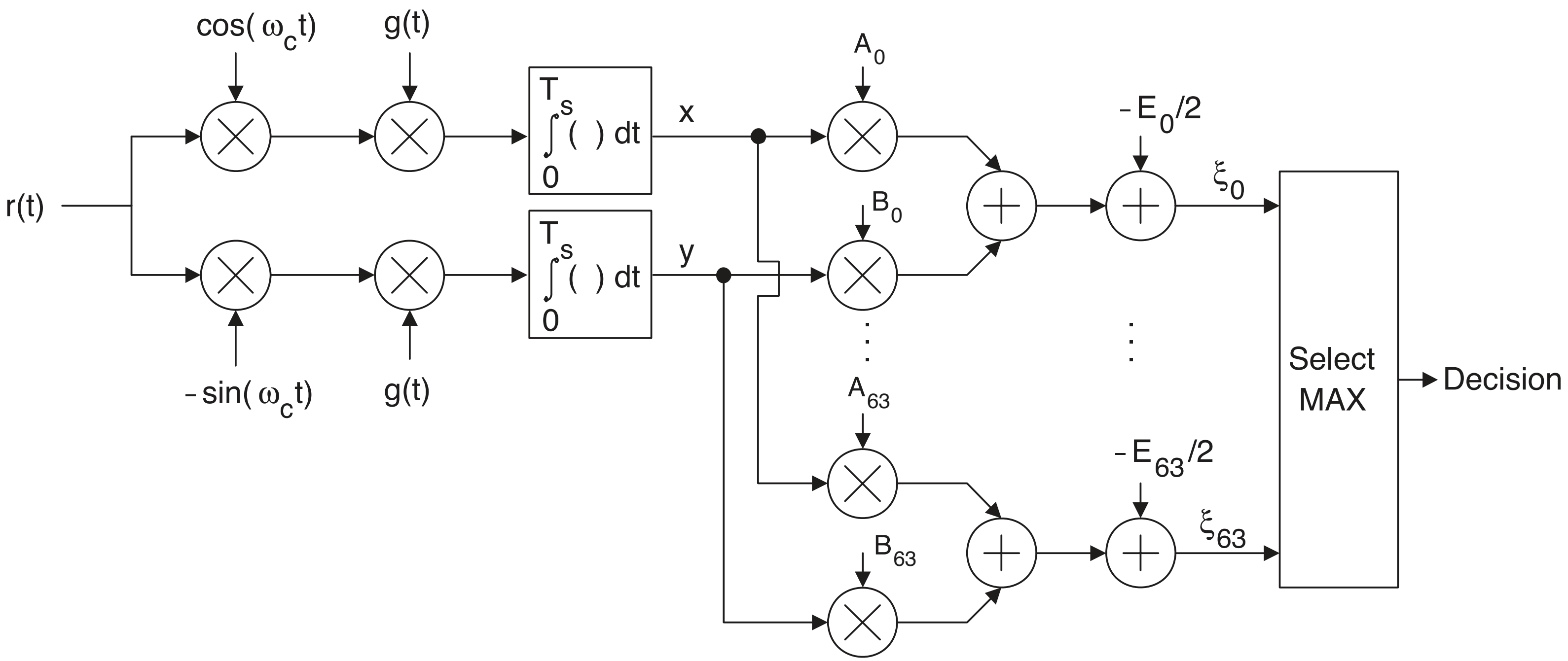


- Decision variables are computed by correlators or matched filters
- Each possible signal alternative is recreated in the receiver
- **Question:** can we apply this to bandpass signals? Yes!

But: recreating signals at large frequencies f_c is a challenge

Example: QAM Signaling

- Recall the simplified receiver considered in Example 4.4:

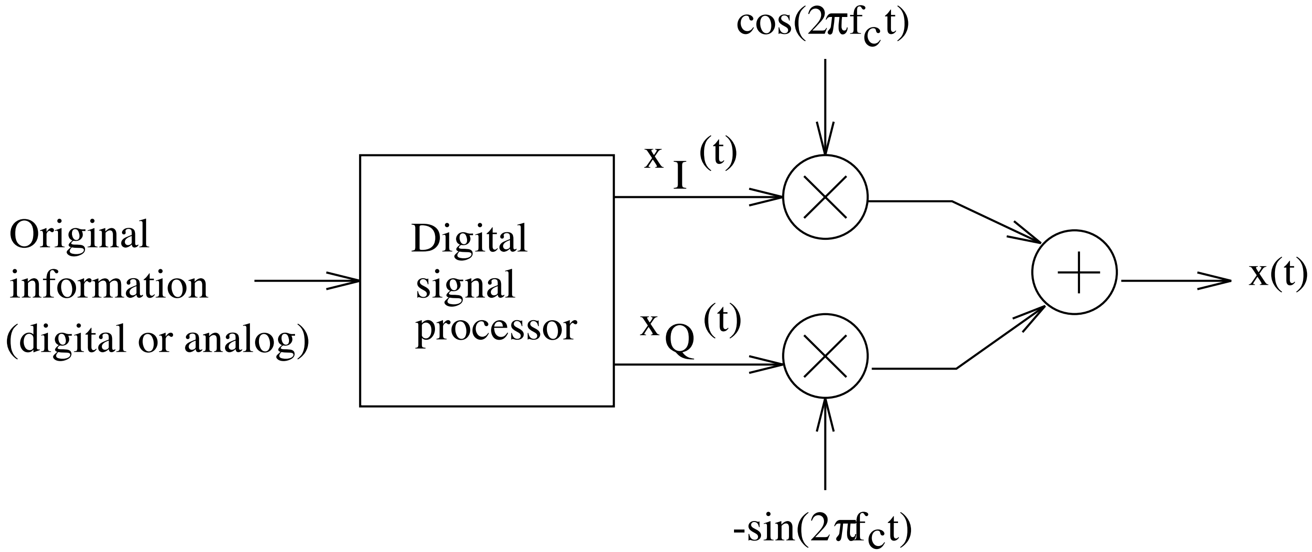


- Only two correlator branches are required instead of M
- Separation of carrier waveforms from baseband pulse possible

Our aim: a general baseband representation of the receiver

Transmission of bandpass signals

- Recall from last lecture:



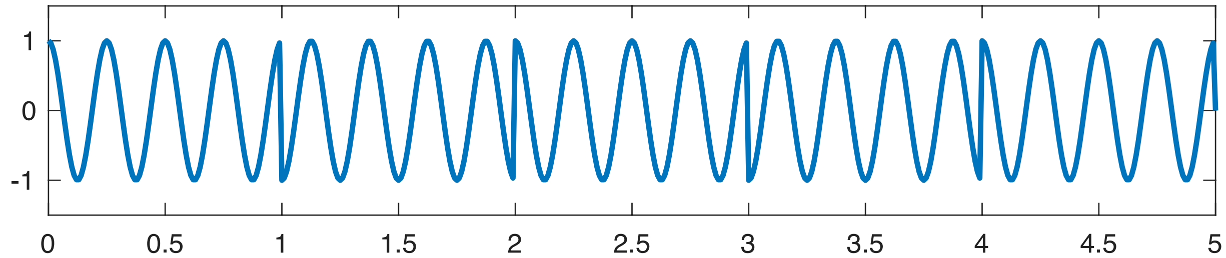
- A general bandpass signal can always be written as

$$x(t) = x_I(t)\ \cos(2\pi f_c\, t)\ -\ x_Q(t)\ \sin(2\pi f_c\, t)\ , \quad -\infty \leq t \leq \infty$$

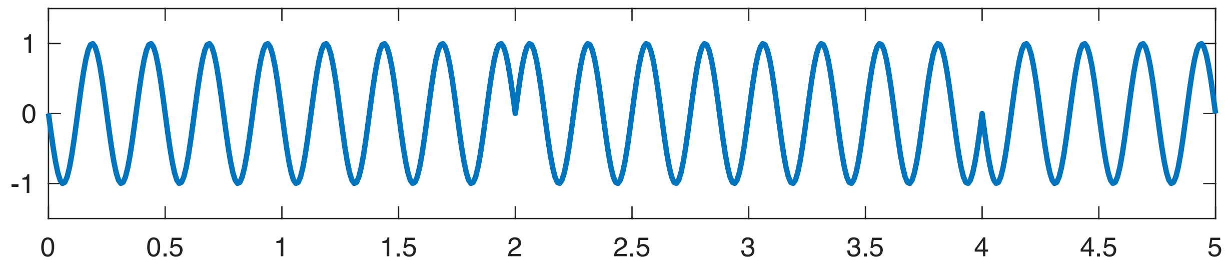
- $x_I(t)$: inphase component $\qquad$ $x_Q(t)$: quadrature component

QPSK Example

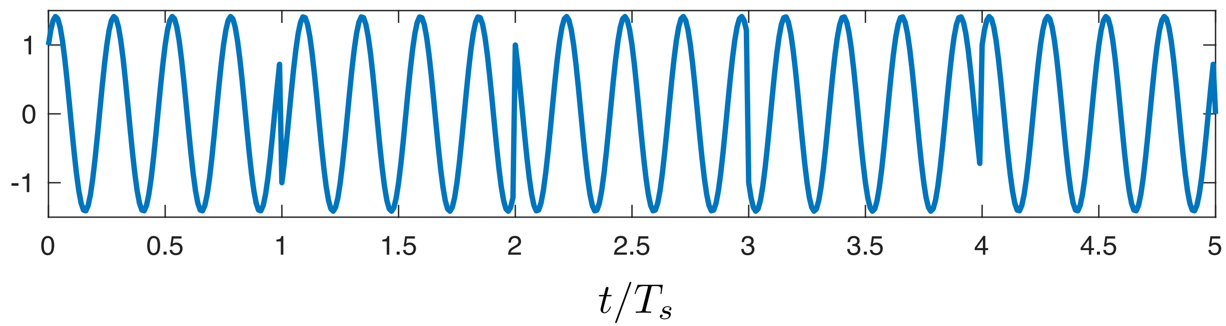
$x_I(t)\ \cos(2\pi\, f_c\, t)$



$x_Q(t)\ \sin(2\pi\, f_c\, t)$



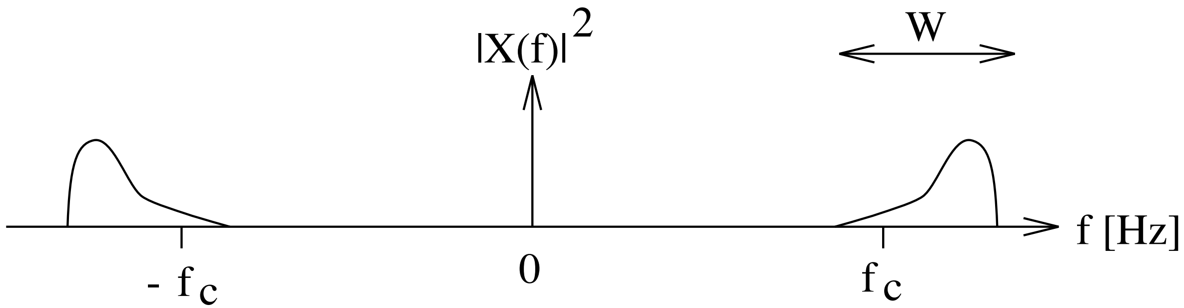
$x(t) = x_I(t)\ \cos(2\pi\, f_c\, t) - x_Q(t)\ \sin(2\pi\, f_c\, t)$



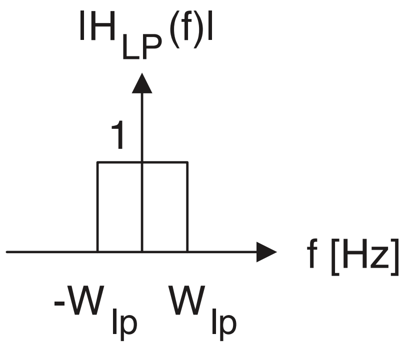
What are $x_I(t)$ and $x_Q(t)$ in this case?

Receivers for bandpass signals

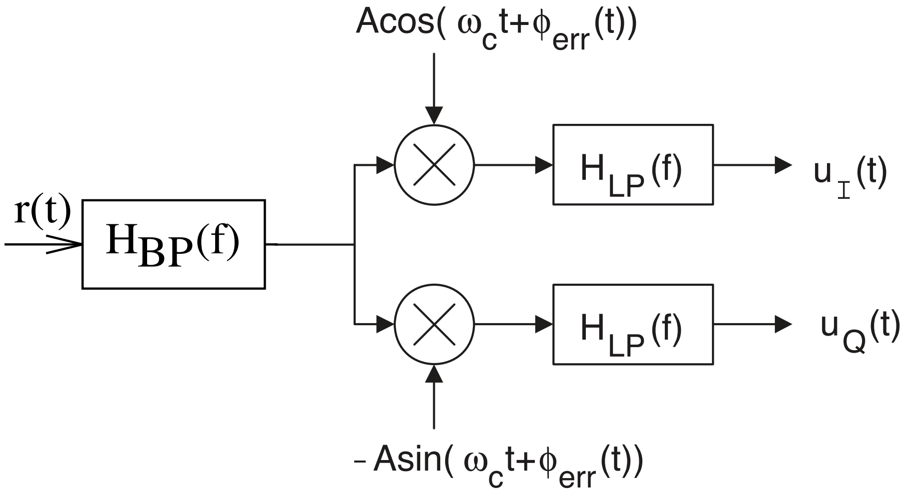
- **Our goal:** reproduce components $x_I(t)$ and $x_Q(t)$ at the receiver
- In the transmitted bandpass signal $x(t)$ these components were shifted to the carrier frequency f_c



- **Idea:** shifting the signal back to the baseband by multiplying with the carrier waveform again (see Ex. 2.19 and Problem 3.9)
- A lowpass filter $H_{LP}(f)$ is then applied in the baseband to remove undesired other signals or copies from the carrier multiplication



Homodyne receiver frontend



- Receiver is not synchronized to transmitter: phase errors $\phi_{err}(t)$
- Assume first $r(t) = x_I(t)\cos(2\pi f_c t)$ $\quad (x_Q(t) = 0$ and no noise)

$$
\begin{aligned}
u_I(t) &= \left[x_I(t)\cos(2\pi f_c t)\cdot A\cos\left(2\pi f_c t + \phi_{err}(t)\right)\right]_{LP} \\
&= \left[\frac{x_I(t)}{2} A\left(\cos(\phi_{err}(t)) + \cos(2\pi 2 f_c t + \phi_{err}(t))\right)\right]_{LP} \\
&= \frac{x_I(t)}{2} A\cos(\phi_{err}(t))
\end{aligned}
$$

- Likewise $$u_Q(t) = -\frac{x_I(t)}{2} A\sin(\phi_{err}(t))$$

The impact of phase errors

- Assuming $r(t) = x_I(t)\cos(2\pi f_c t)$ we have found that

$$u_I(t) = \frac{x_I(t)}{2} A \cos(\phi_{err}(t)) \; , \quad u_Q(t) = -\frac{x_I(t)}{2} A \sin(\phi_{err}(t))$$

- **Ideal case:** $\phi_{err}(t) = 0$

$$u_I(t) = x_I(t)/2 \cdot A \quad \text{and} \quad u_Q(t) = 0$$

$\Rightarrow$ the inphase branch is independent of the quadrature branch

- **Phase errors:** $\phi_{err}(t) \neq 0$

$$u_I(t) < x_I(t)/2 \cdot A \quad \text{and} \quad u_Q(t) \neq 0 \;\; \text{(crosstalk)}$$

- If $\phi_{err}(t)$ changes randomly (jitter) the average $u_I(t)$ can vanish
- Ignoring the effect of phase errors can lead to bad performance

Question: what can we then do about phase errors?

Coherent receivers

- Assume now that we can estimate $\phi_{err}(t)$
- The signal $x_I(t)$ is contained in both $u_I(t)$ and $u_Q(t)$

$$u_I(t) = \frac{x_I(t)}{2} A\cos(\phi_{err}(t))\;, \quad u_Q(t) = -\frac{x_I(t)}{2} A\sin(\phi_{err}(t))$$

- **Coherent reception:**

  by combining both components the signal can be recovered by

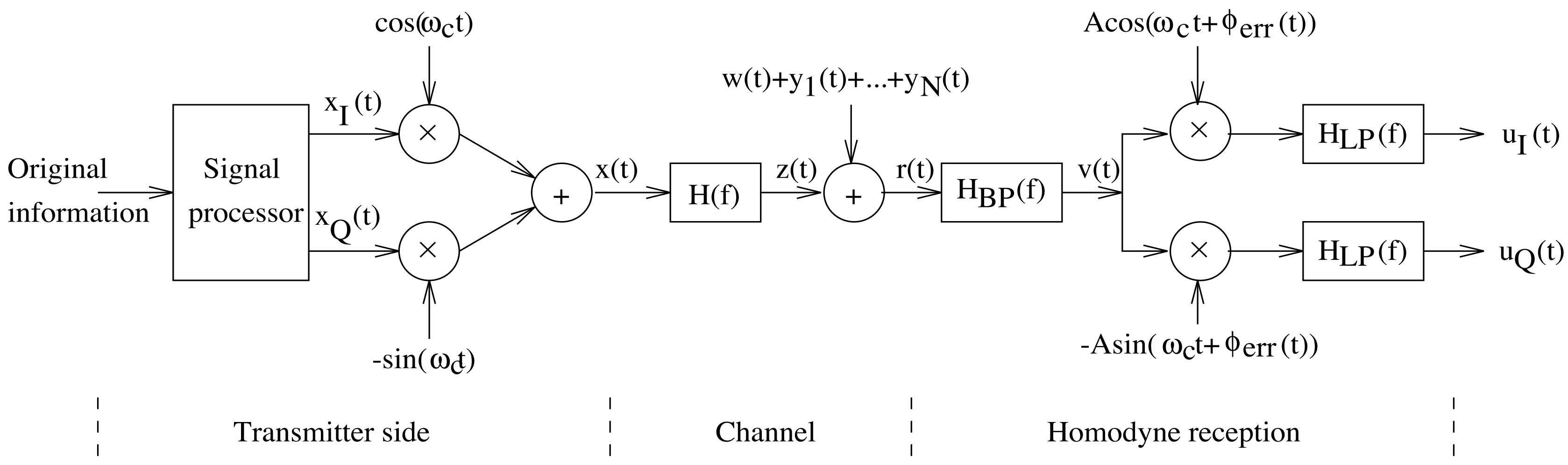
$$\hat{u}_I(t) = u_I(t)\cdot\cos(\phi_{err}(t)) - u_Q(t)\cdot\sin(\phi_{err}(t))$$

$$= \frac{x_I(t)}{2} A\cos^2(\phi_{err}(t)) + \frac{x_I(t)}{2} A\sin^2(\phi_{err}(t)) = \frac{x_I(t)}{2} A$$

- Observe: same result as in the ideal case $\phi_{err}(t) = 0$

Compare: non-coherent DPSK receiver (last lecture, p. 400-403) can be used if phase estimation is not possible

Overall transmission model



- The signal $y(t)$ is given by

$$y(t) = z(t) + w(t) = x(t) \, * \, h(t) + w(t)$$

- It can be written as

$$y(t) = y_I(t)\cos(2\pi f_c t) - y_Q(t)\sin(2\pi f_c t)$$

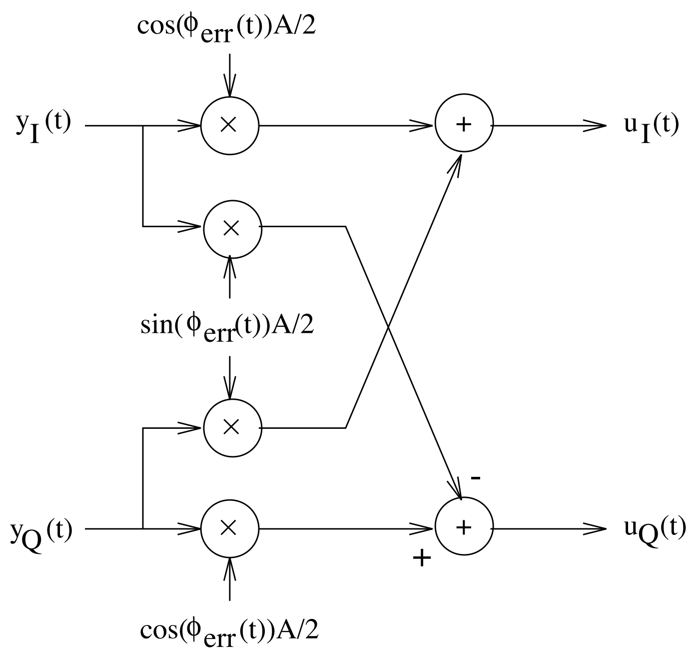
Can we express $u_I(t)$ and $u_Q(t)$ in terms of $x_I(t)$ and $x_Q(t)$?

Inphase and quadrature relationship

- With the complete signal $r(t)$ entering the receiver the output signals become

$$
\begin{aligned}
u_I(t) &= \big[y(t)A\cos(2\pi f_c t + \phi_{err}(t))\big]_{LP} \\
&= \frac{y_I(t)}{2}A\cos(\phi_{err}(t)) \\
&\quad + \frac{y_Q(t)}{2}A\sin(\phi_{err}(t))
\end{aligned}
$$

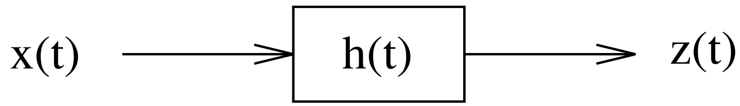
$$
\begin{aligned}
u_Q(t) &= \big[-y(t)A\sin(2\pi f_c t + \phi_{err}(t))\big]_{LP} \\
&= \frac{y_Q(t)}{2}A\cos(\phi_{err}(t)) \\
&\quad - \frac{y_I(t)}{2}A\sin(\phi_{err}(t))
\end{aligned}
$$



Including the channel filter

- Before we can relate $y(t) = z(t) + w(t)$ to $x(t)$ we need to consider the effect of the channel

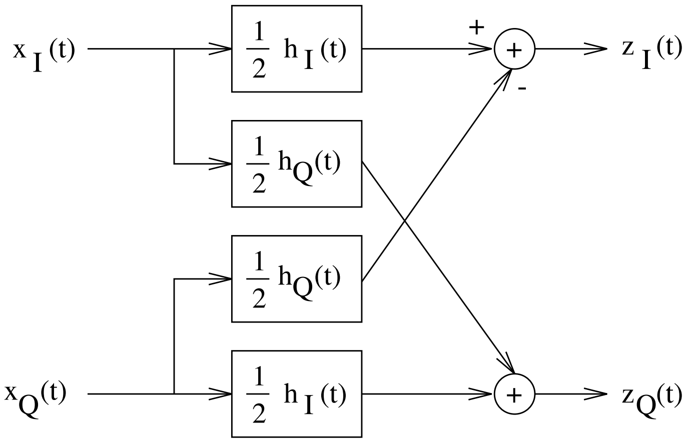
$$z(t) = x(t) \ast h(t)$$



- We assume that the impulse response $h(t)$ can be represented as a bandpass signal

$$h(t) = h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t)$$

- With some calculations the signals can be written as (p. 159-160)

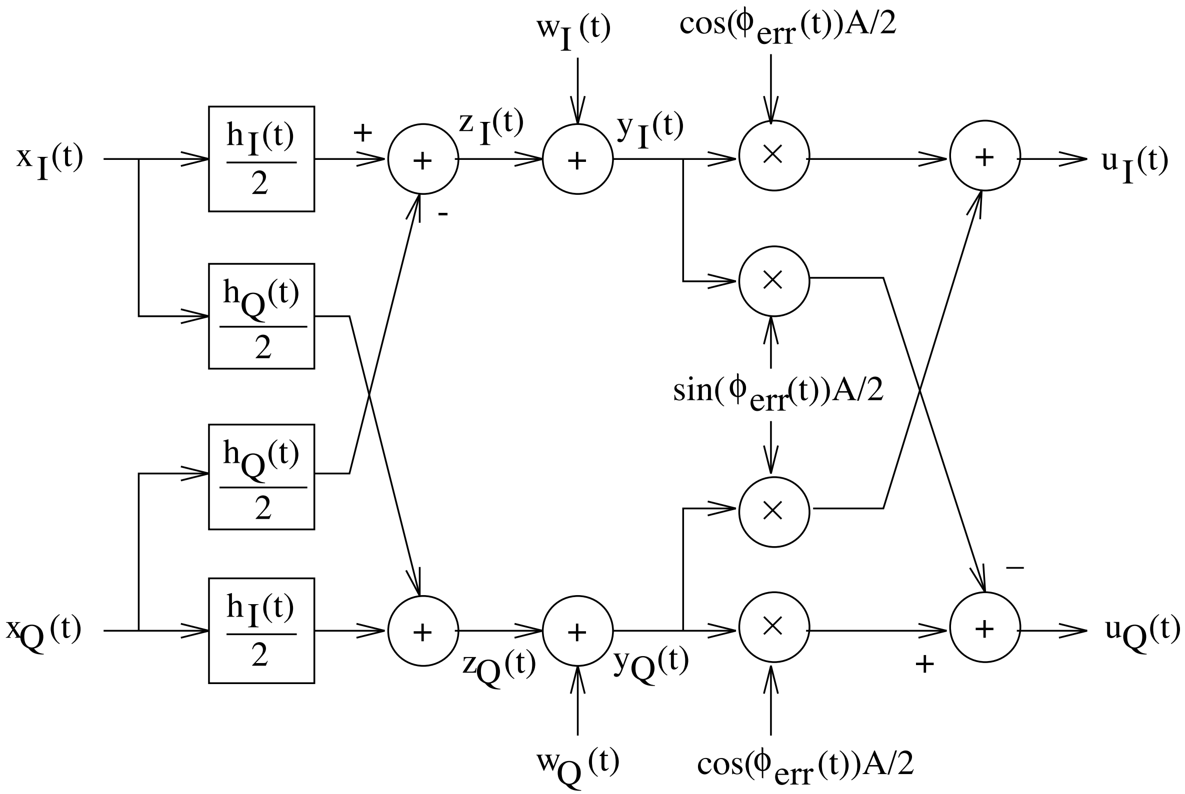


$$z_I(t) = \frac{1}{2}\Big(x_I(t) \ast h_I(t) - x_Q(t) \ast h_Q(t)\Big)$$

$$z_Q(t) = \frac{1}{2}\Big(x_I(t) \ast h_Q(t) + x_Q(t) \ast h_I(t)\Big)$$

Equivalent baseband model

- Combining the channel with the receiver frontend we obtain



- Observe that all the involved signals are in the baseband
- The same is true for channel filter, noise and phase error

Digital signal processing can be applied easily in baseband

What happened with the carrier waveforms?

EITG05 – Digital Communications

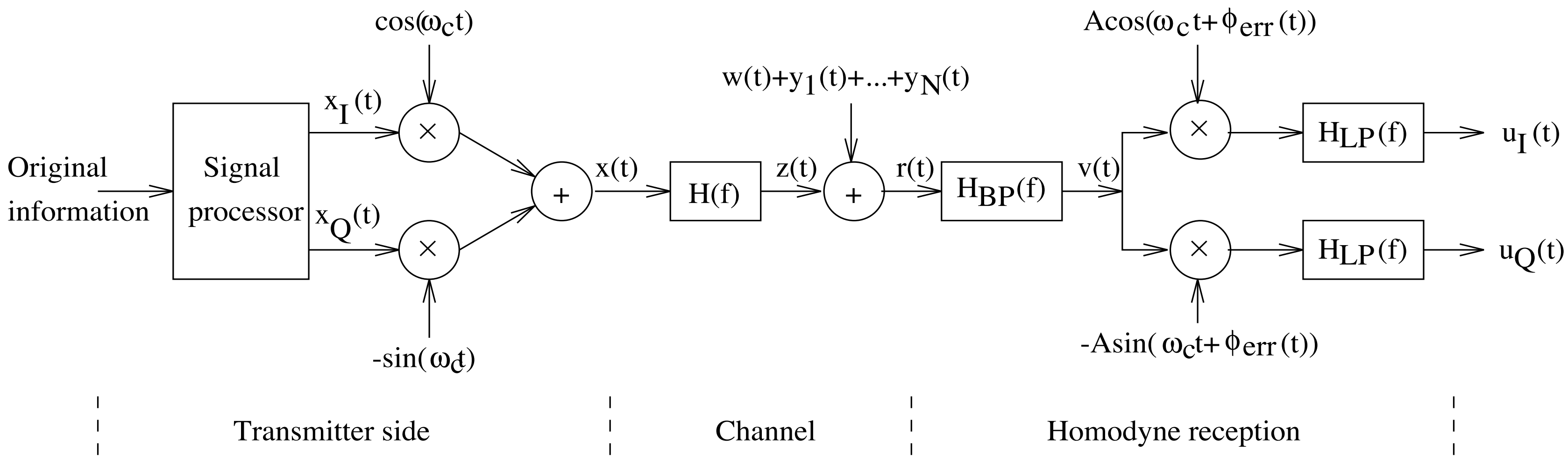
Lecture 10

Equivalent baseband model, Compact description

Chapter 6: Intersymbol interference
ISI, Increasing the signaling rate

Michael Lentmaier
Monday, October 8, 2018

Overall transmission model



- The signal $y(t)$ is given by

$$y(t) = z(t) + w(t) = x(t) \, * \, h(t) + w(t)$$

- It can be written as

$$y(t) = y_I(t)\cos(2\pi f_c t) - y_Q(t)\sin(2\pi f_c t)$$

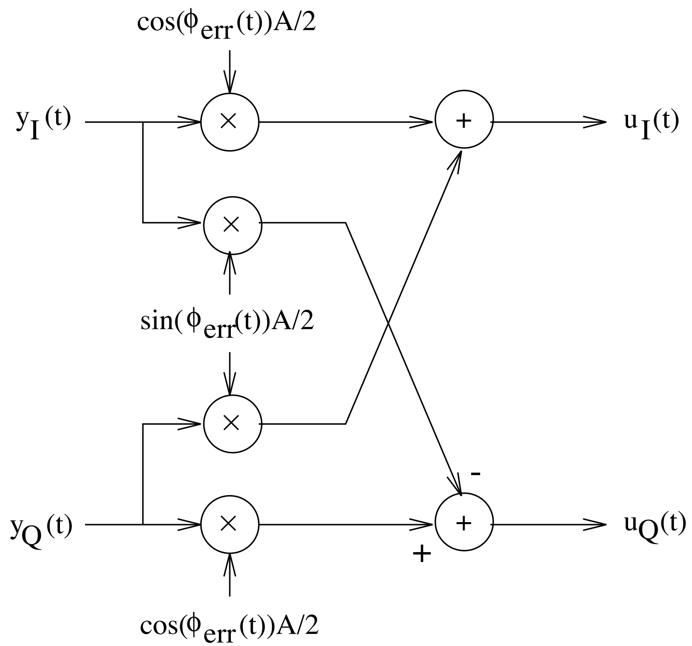
Can we express $u_I(t)$ and $u_Q(t)$ in terms of $x_I(t)$ and $x_Q(t)$?

Inphase and quadrature relationship

- With the complete signal $r(t)$ entering the receiver the output signals become

$$u_I(t) = \left[y(t) A \cos\left(2\pi f_c t + \phi_{err}(t)\right) \right]_{LP} = \frac{y_I(t)}{2} A \cos(\phi_{err}(t)) + \frac{y_Q(t)}{2} A \sin(\phi_{err}(t))$$

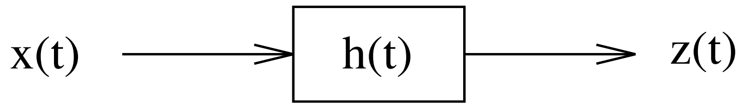
$$u_Q(t) = \left[-y(t) A \sin\left(2\pi f_c t + \phi_{err}(t)\right) \right]_{LP} = \frac{y_Q(t)}{2} A \cos(\phi_{err}(t)) - \frac{y_I(t)}{2} A \sin(\phi_{err}(t))$$



Including the channel filter

- Before we can relate $y(t) = z(t) + w(t)$ to $x(t)$ we need to consider the effect of the channel

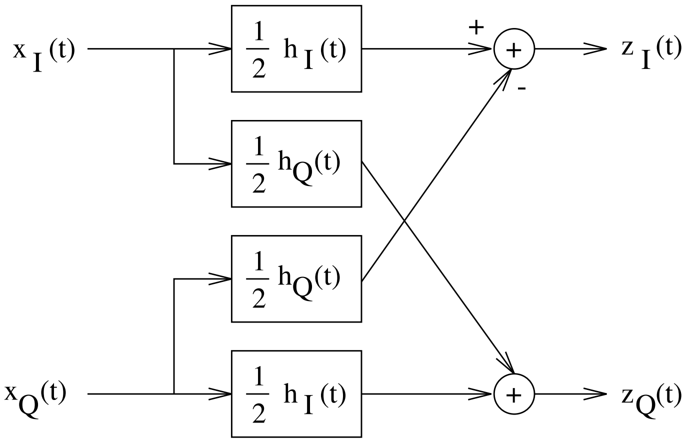
$$z(t) = x(t) \, * \, h(t)$$



- We assume that the impulse response $h(t)$ can be represented as a bandpass signal

$$h(t) = h_I(t)\cos(2\pi f_c t) - h_Q(t)\sin(2\pi f_c t)$$

- With some calculations the signals can be written as (p. 159-160)

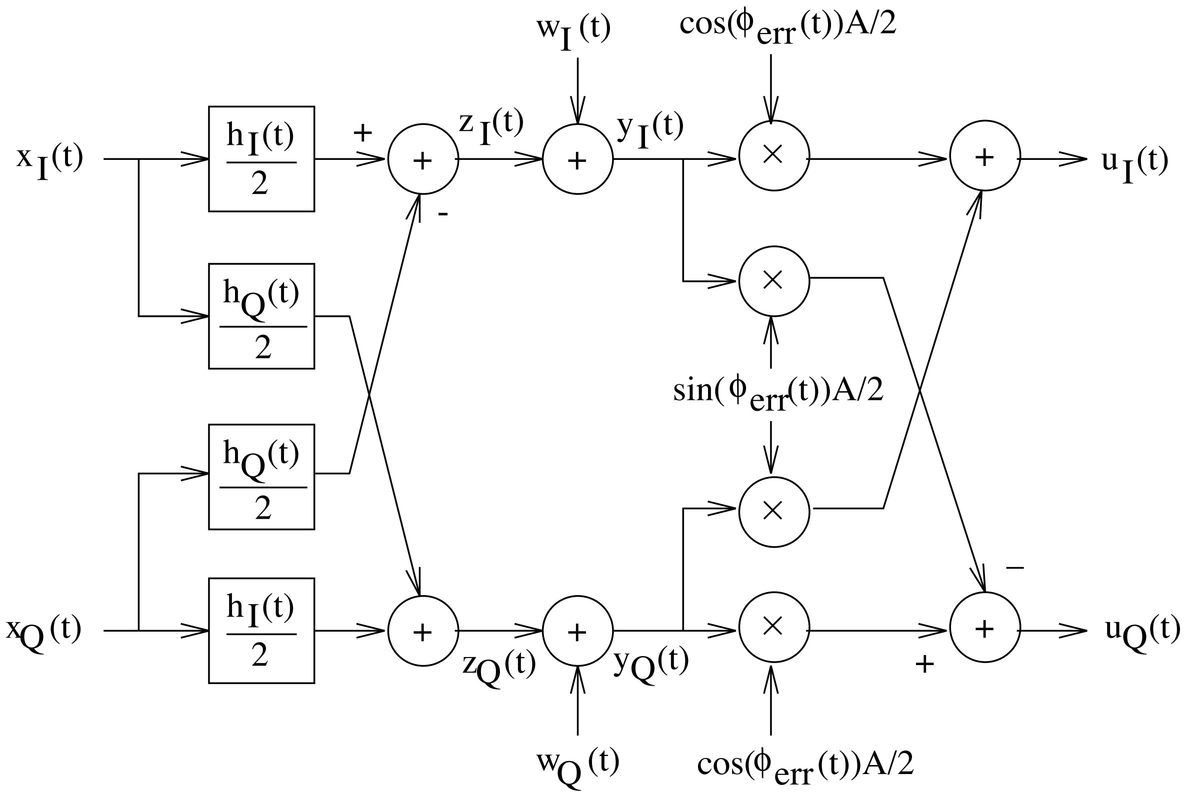


$$z_I(t) = \frac{1}{2}\big(x_I(t) \, * \, h_I(t) - x_Q(t) \, * \, h_Q(t)\big)$$

$$z_Q(t) = \frac{1}{2}\big(x_I(t) \, * \, h_Q(t) + x_Q(t) \, * \, h_I(t)\big)$$

Equivalent baseband model

- Combining the channel with the receiver frontend we obtain



- Observe that all the involved signals are in the baseband
- The same is true for channel filter, noise and phase error

Digital signal processing can be applied easily in baseband

What happened with the carrier waveforms?

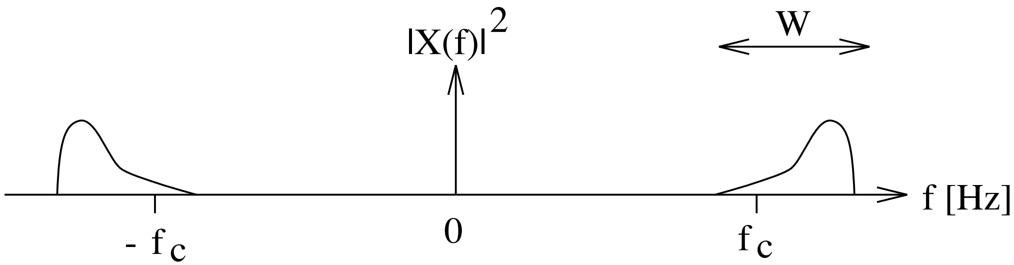
A compact description

- A more compact description is possible by combining $x_I(t)$ and $x_Q(t)$ to an equivalent baseband signal

$$\tilde{x}(t) = x_I(t) + j\,x_Q(t)$$

- The transmitted signal can then be described as

$$x(t) = Re\left\{(x_I(t) + j\,x_Q(t))\,e^{+j2\pi f_c t}\right\} = Re\left\{\tilde{x}(t)\,e^{+j2\pi f_c t}\right\}$$



- With $Re\{a\} = (a + a^*)/2$ we can write

$$x(t) = \frac{\tilde{x}(t)}{2} \cdot e^{+j2\pi f_c t} + \frac{\tilde{x}^*(t)}{2} \cdot e^{-j2\pi f_c t}$$

A compact description

- Let us first ignore the effect of the channel: $w(t) = 0$, $h(t) = \delta(t)$
- The receiver can invert the frequency shift operation by

$$\tilde{u}(t) = \left[x(t) \cdot A\, e^{-j(2\pi f_c t + \phi_{err}(t))} \right]_{LP}$$

- Using the expression for $x(t)$ from the previous slide we get

$$\tilde{u}(t) = \left[\frac{A}{2} \left(\tilde{x}(t) e^{+j2\pi f_c t} + \tilde{x}^*(t) e^{-j2\pi f_c t} \right) \cdot e^{-j(2\pi f_c t + \phi_{err}(t))} \right]_{LP}$$

$$= \frac{\tilde{x}(t)}{2} A \cdot e^{-j\phi_{err}(t)} = u_I(t) + j u_Q(t)$$

- Observe that this expression is equivalent to our earlier result

$$\tilde{u}(t) = \left(\frac{x_I(t)}{2} A \cos(\phi_{err}(t)) + \frac{x_Q(t)}{2} A \sin(\phi_{err}(t)) \right)$$

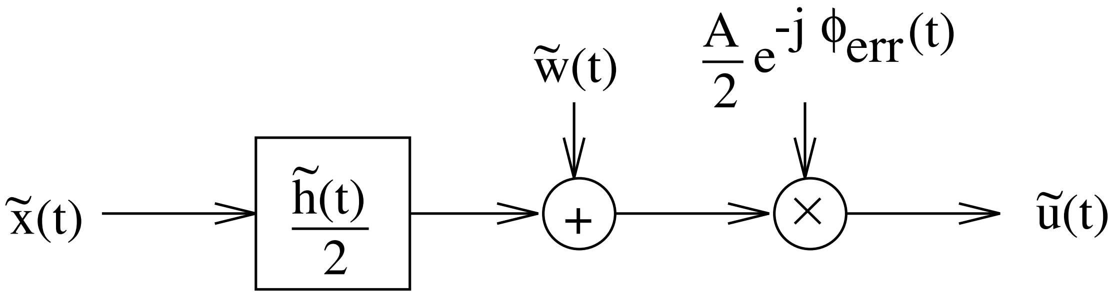
$$+ j \left(\frac{x_Q(t)}{2} A \cos(\phi_{err}(t)) - \frac{x_I(t)}{2} A \sin(\phi_{err}(t)) \right)$$

Compact equivalent baseband model

- The effect of the channel filter becomes

$$\tilde{z}(t) = z_I(t) + j z_Q(t) = \tilde{x}(t) * \frac{\tilde{h}(t)}{2}$$

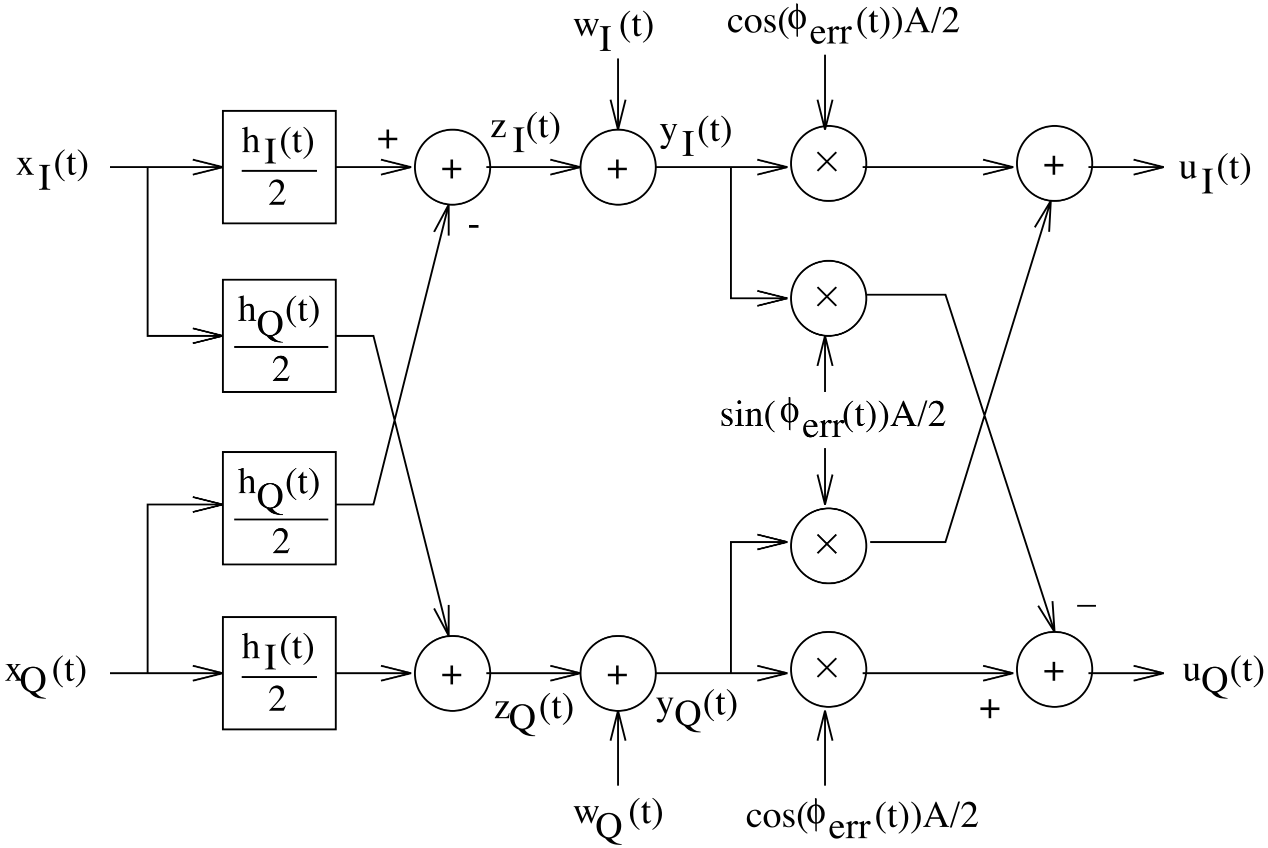
- Combining these parts and the noise we obtain the simple model

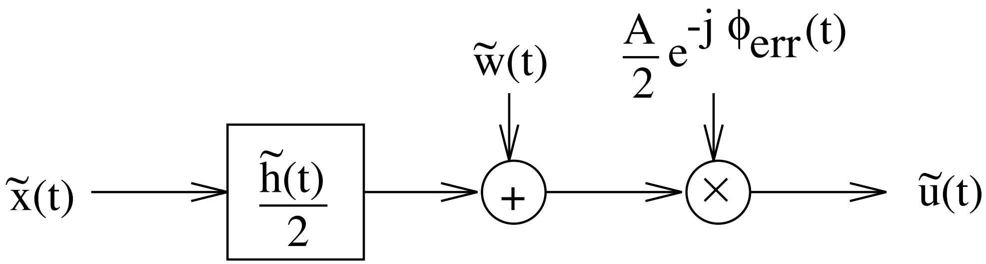


$$\tilde{u}(t) = \left[\left(\tilde{x}(t) * \frac{\tilde{h}(t)}{2} \right) + \tilde{w}(t) \right] \cdot e^{-j\phi_{err}(t)} \cdot \frac{A}{2} \, , \qquad \tilde{w}(t) = w_I(t) + j w_Q(t)$$

- Complex signal notation simplifies expressions significantly

The two equivalent baseband models





M-ary QAM signaling

- Considering M-ary QAM signals we get

$$x_I(t) = \sum_{n=-\infty}^{\infty} A_{m[n]}\, g(t - nT_s)\ , \quad x_Q(t) = \sum_{n=-\infty}^{\infty} B_{m[n]}\, g(t - nT_s)$$

- Let us now introduce

$$\tilde{A}_m[n] = A_{m[n]} + j B_{m[n]}$$

- Then our complex baseband signal $\tilde{x}(t)$ can be written as

$$\tilde{x}(t) = x_I(t) + j\, x_Q(t) = \sum_{n=-\infty}^{\infty} \tilde{A}_{m[n]}\, g(t - nT_s)$$

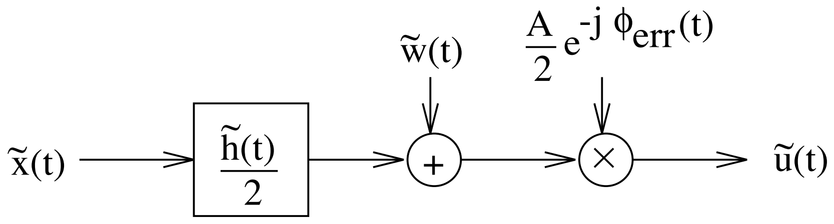
- **Example:** (on the board)

  Consider 4-QAM transmission of $\mathbf{b} = 1\ 0\ 1\ 1\ 1\ 0\ 0\ 1$
  Determine $A_{m[n]}$, $B_{m[n]}$ and $\tilde{A}_{m[n]}$

How can we design the receiver for QAM signals?

Matched filter receiver

- At the receiver we see the complex baseband signal $\tilde{u}(t)$

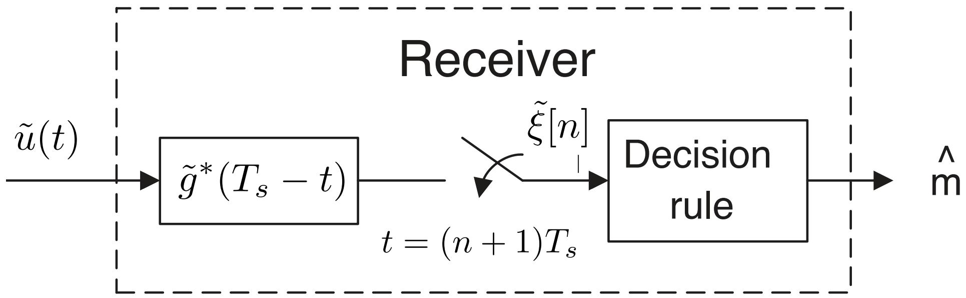


- If we know the channel we can design a matched filter for

$$\tilde{z}(t) = \tilde{x}(t) * \frac{\tilde{h}(t)}{2} \quad \Rightarrow \quad \tilde{v}(t) = \tilde{z}^*(T_s - t)$$

- It is often convenient to match $\tilde{v}(t)$ to the pulse $g(t)$ instead

$$\tilde{v}(t) = g^*(T_s - t) \quad \Rightarrow \quad \tilde{\xi}[n] = \big[\tilde{u}(t) * g^*(T_s - t)\big]_{t=(n+1)T_s}$$



Decision rule

- Consider now $\tilde{h}(t) = \delta(t)$ and $\tilde{w}(t) = 0$
- The ideal values of the decision variable are then given by

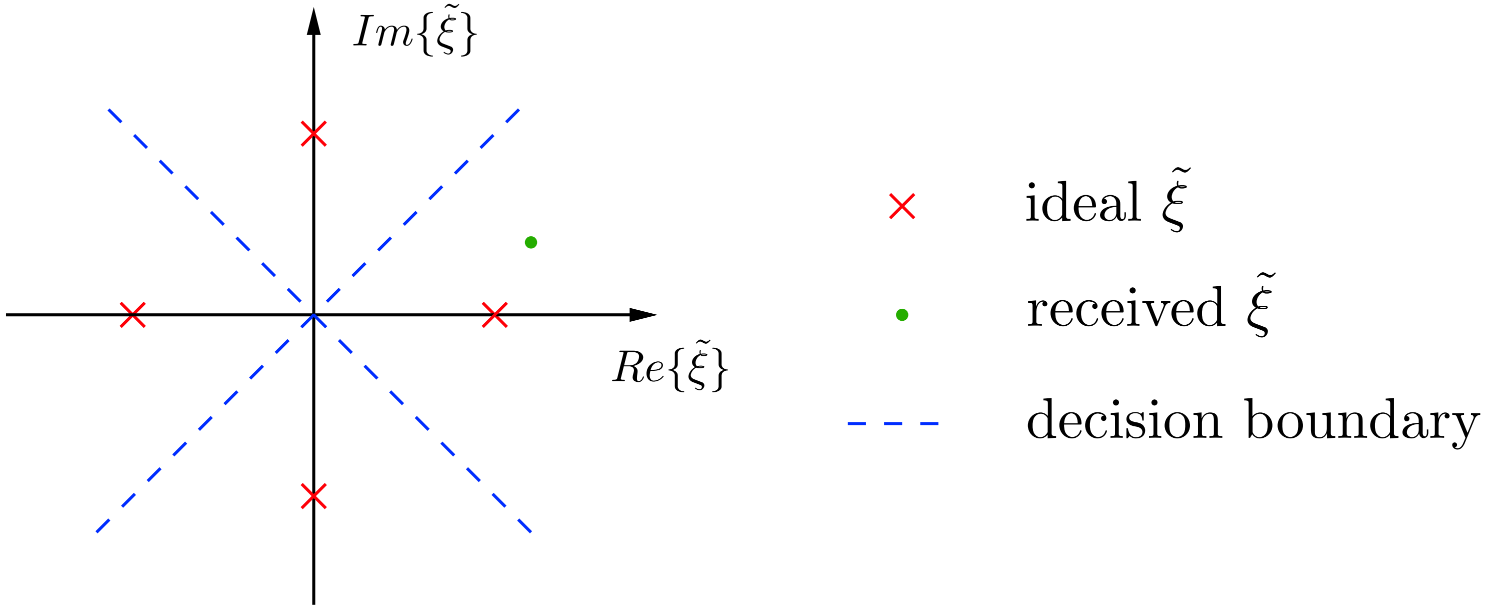
$$
\begin{aligned}
\tilde{\xi}_{m[n]} &= \big[\tilde{u}(t) \; * \; g^*(T_s - t)\big]_{t=(n+1)T_s} \\
&= \left[\left(\tilde{A}_{m[n]} g(t - nT_s) \cdot e^{-j\phi_{err}(t)} \cdot \frac{A}{2}\right) * \; g^*(T_s - t)\right]_{t=(n+1)T_s} \\
&= \tilde{A}_{m[n]} e^{-j\phi_{err}(t)} \cdot \frac{A}{2} \left[g(t - nT_s) \; * \; g^*(T_s - t)\right]_{t=(n+1)T_s} \\
&= \tilde{A}_{m[n]} e^{-j\phi_{err}((n+1)T_s)} \cdot \frac{A}{2} \, E_g
\end{aligned}
$$

- Due to noise $w(t) \neq 0$ and non-ideal channel $\tilde{h}(t)$ the decision variables at the receiver will differ from these ideal values
- The Euclidean distance receiver will base its decision on the ideal value $\tilde{\xi}_{m[n]}$ which is closest to the received value $\tilde{\xi}[i]$

Example: 4-PSK

- Assuming $\phi_{err}(t) = 0$ we obtain the ideal decision variables

$$\tilde{\xi}_{m[n]} = \tilde{A}_{m[n]} \cdot \frac{A}{2} \, E_g = (A_{m[n]} + jB_{m[n]}) \cdot \frac{A}{2} \, E_g$$

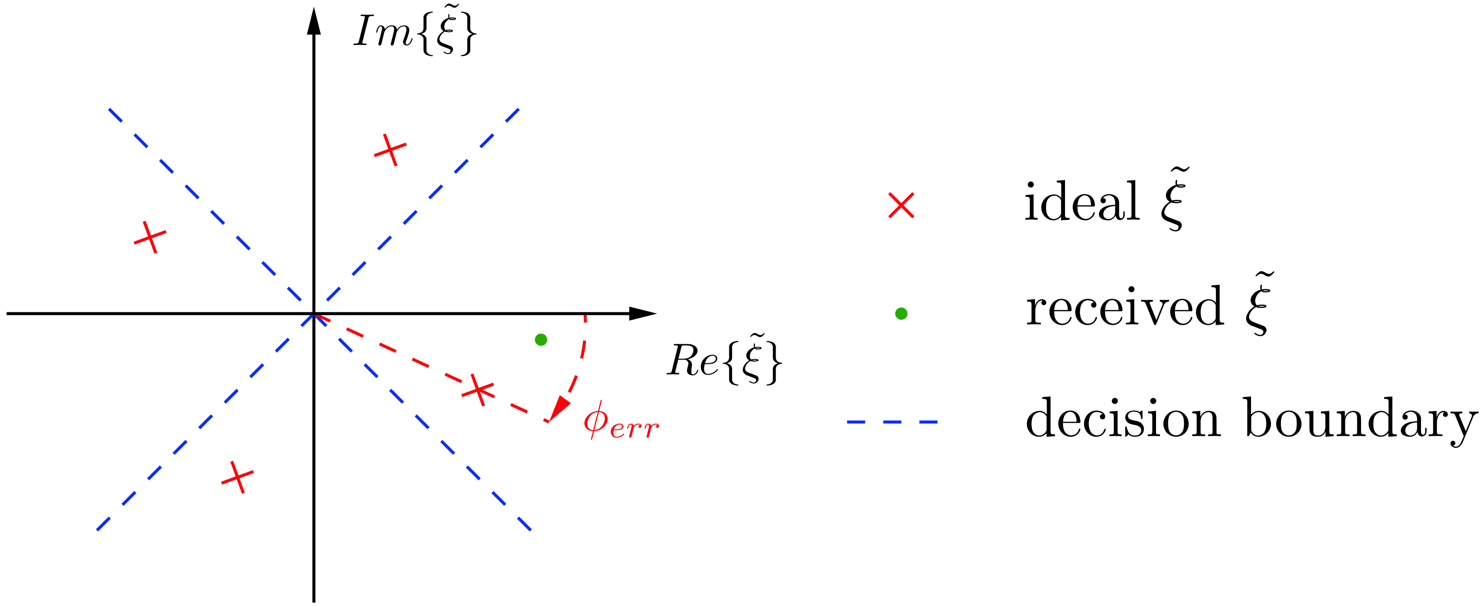


- Based on the received value $\tilde{\xi}[n]$ we decide for

$$\hat{m}[n] : \quad \tilde{A}_{\hat{m}[n]} = (1 + j \cdot 0)$$

Example: 4-PSK with phase offset

- Consider now a constant phase offset of $\phi_{err}(t) = \phi_{err} = 25^\circ$
- As a result the values $\tilde{\xi}_{m[n]}$ and $\tilde{\xi}[n]$ are rotated accordingly



How can we compensate for ϕ_{err}?

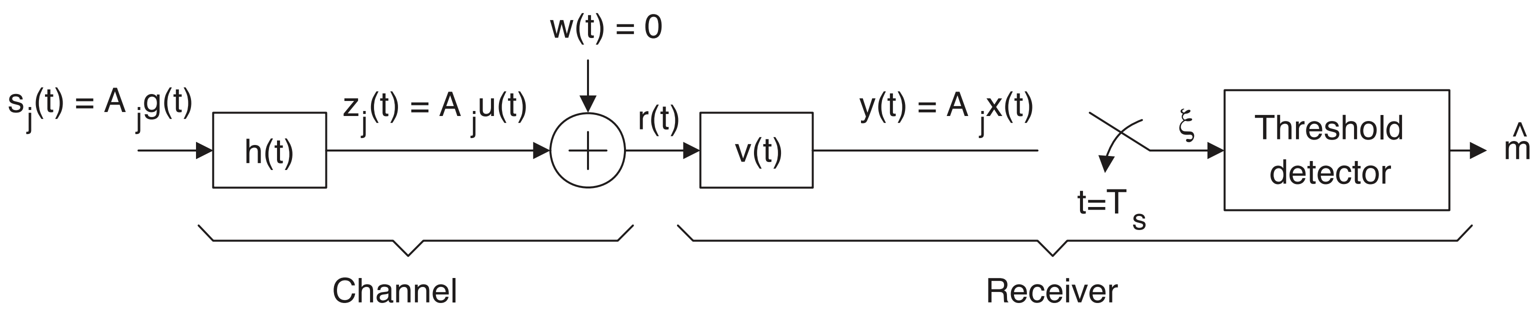
1. we can rotate the decision boundaries by the same amount
2. or we can rotate back $\tilde{\xi}[n]$ by multiplying with $e^{+j\phi_{err}}$

Summary: M-ary QAM transmission

- We can describe the transmitted messages $\tilde{A}_{\hat{m}[n]}$ and the decision variables $\tilde{\xi}[n]$ at the receiver as complex variables
- The effect of the noise $\tilde{w}(t)$ and the channel filter $\tilde{h}(t)$ on $\tilde{\xi}[n]$ can be described by the equivalent baseband model
- The transmitter and receiver frontends can be separated from the (digital) baseband processing
- Assumptions:
  - the pulse shape $g(t)$ satisfies the ISI-free condition
  - the carrier frequency f_c is much larger than the bandwidth of $g(t)$
- Under these conditions the design of the baseband receiver and its error probability analysis can be applied as in Chapter 4

Intersymbol Interference (ISI)

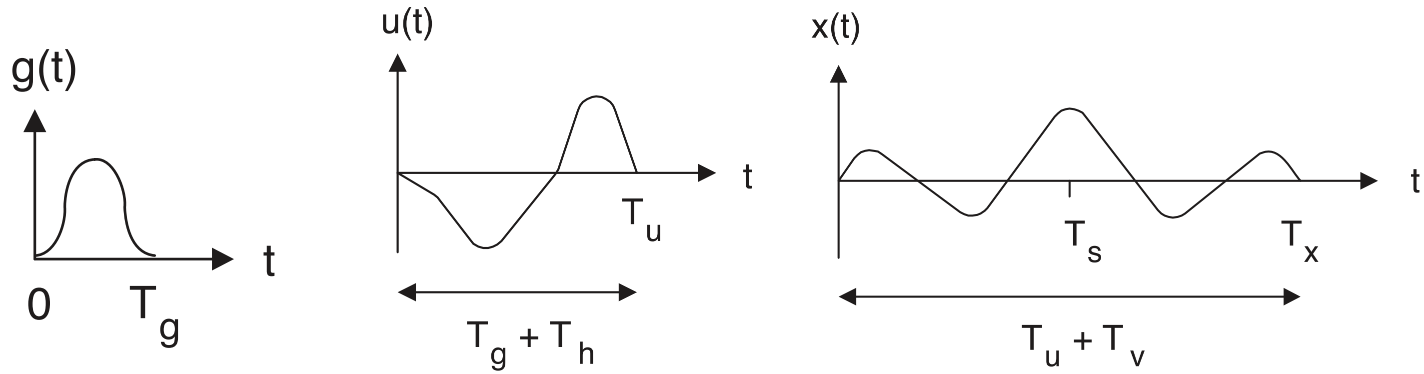
- Consider transmission of a single M-ary PAM signal alternative



- In the noise-free case $(w(t) = 0)$ the signal $x(t)$ can be written as

$$x(t) = u(t) \ * \ v(t) = g(t) \ * \ h(t) \ * \ v(t)$$

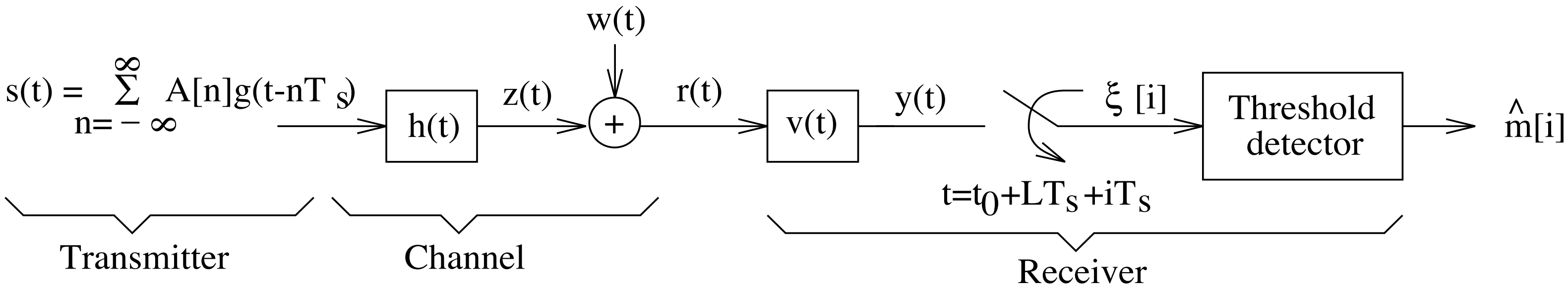
Example:



What happens if $T_u = T_g + T_h \geq T_s$? $\Rightarrow$ ISI occurs

Intersymbol Interference (ISI)

- For $R_s = 1/T_s < 1/T_u$ we can use the ML receiver from Chapter 4
- **Question:** can we use such a receiver for larger rates $R_s \geq 1/T_u$?
- Consider the following receiver structure (compare to last slide)



- Note that $z(t)$ now is a superposition of overlapping pulses $u(t)$
- The signal $y(t)$ after the receiver filter $v(t)$ is

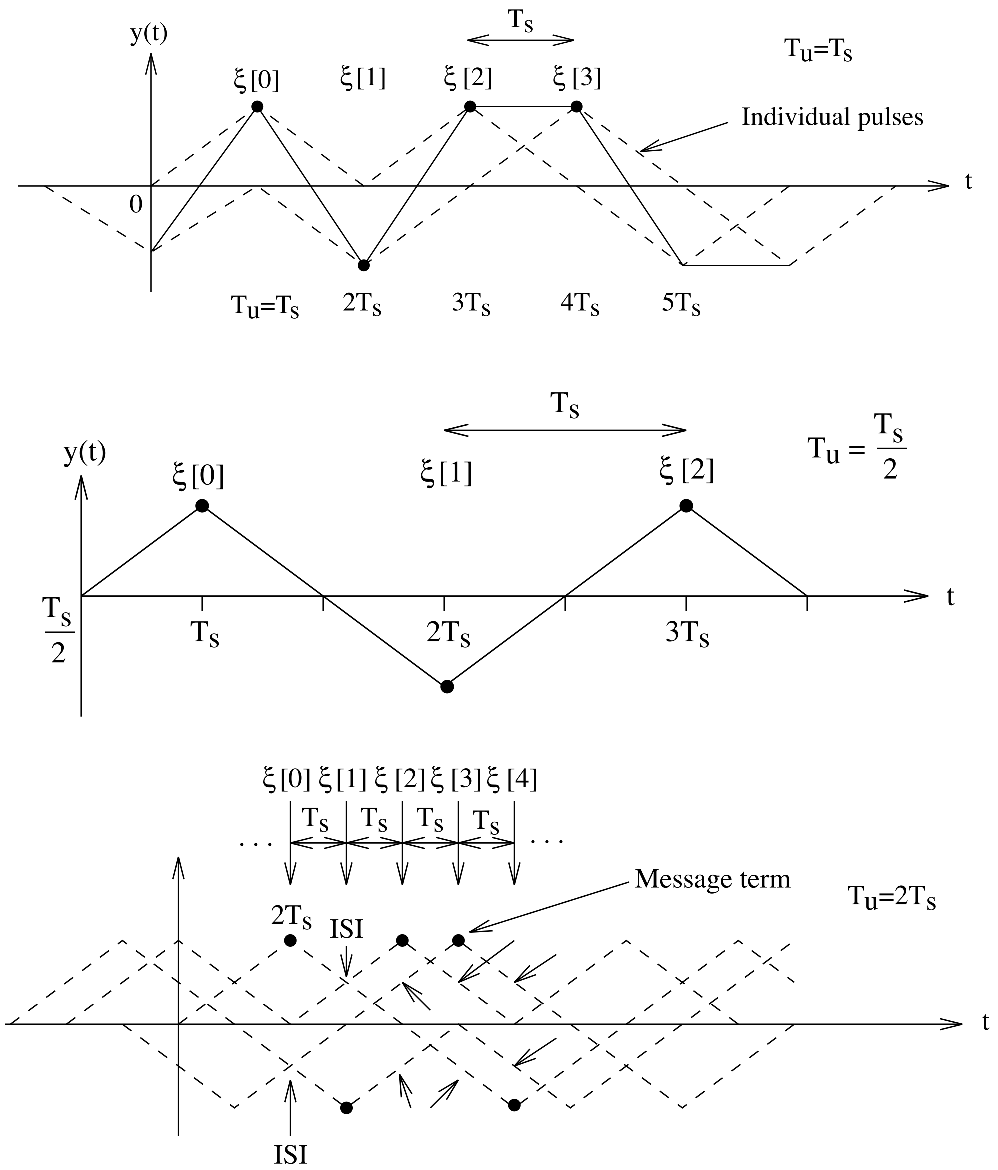
$$y(t) = \sum_{n=-\infty}^{\infty} A[n]\, x(t - nT_s) + w_c(t) \ ,$$

where $w_c(t)$ is a filtered Gaussian process

- The decision variable is obtained after sampling

$$\xi[i] = y(\mathcal{T} + iT_s) \ , \quad \mathcal{T} = t_0 + LT_s \ , \text{ where } LT_s \geq T_u$$

Illustration of ISI in the receiver



Discrete time model for ISI

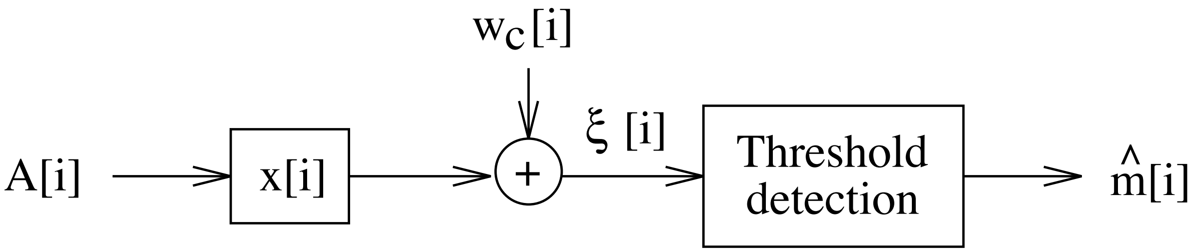
- According to our model the decision variable can be written as

$$\xi[i] = y(\mathcal{T} + iT_s) = \sum_{n=-\infty}^{\infty} A[n]\,x(\mathcal{T} + iT_s - nT_s) + w_c(\mathcal{T} + iT_s)$$

- Let us introduce the discrete sequences

$$x[i] = x(\mathcal{T} + iT_s)\ , \quad w_c[i] = w_c(\mathcal{T} + iT_s)$$

- This leads to the following discrete-time model of our system

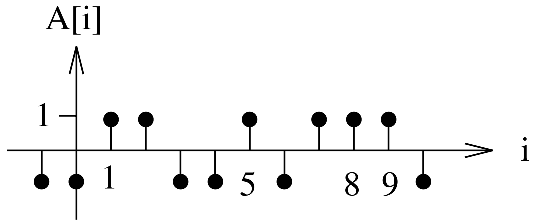


$$\xi[i] = \sum_{n=-\infty}^{\infty} A[n]\,x[i-n] + w_c[i] = A[i] * x[i] + w_c[i]$$

Remark: the discrete-time impulse response $x[i]$ represents pulse shape $g(t)$, channel filter $h(t)$, and receiver filter $v(t)$

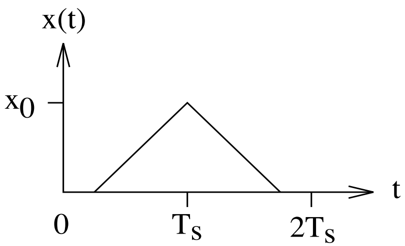
Example 6.1

The transmitted sequence of amplitudes $A[i]$ is given as,

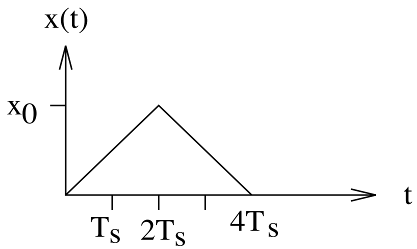


Calculate, and plot, the sequence of decision variables $\xi[i]$ in Figure 6.2, for $0 \leq i \leq 8$, in the noiseless case (i.e. $w(t) = 0$) if $t_0 = 0$ and if the output pulse $x(t)$ is:

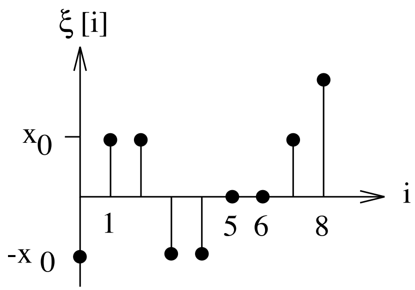
i) L=1 and x(t) as below.



ii) L=2 and x(t) as below.



- i) $\xi[i] = x_0 A[i]$ ii) $\xi[i] = \frac{x_0}{2} A[i+1] + x_0 A[i] + \frac{x_0}{2} A[i-1]$



EITG05 – Digital Communications

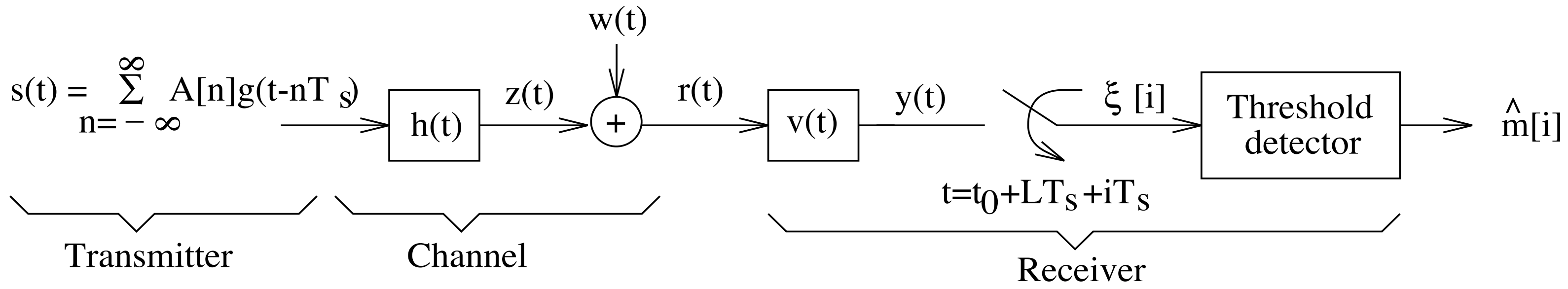
Lecture 11

Intersymbol Interference
Nyquist condition, Spectral raised cosine, Equalizers

Michael Lentmaier
Thursday, October 11, 2018

Intersymbol Interference (ISI)

- For $R_s = 1/T_s < 1/T_u$ we can use the ML receiver from Chapter 4
- **Question:** can we use such a receiver for larger rates $R_s \geq 1/T_u$?
- Consider the following receiver structure (compare to last slide)



- Note that $z(t)$ now is a superposition of overlapping pulses $u(t)$
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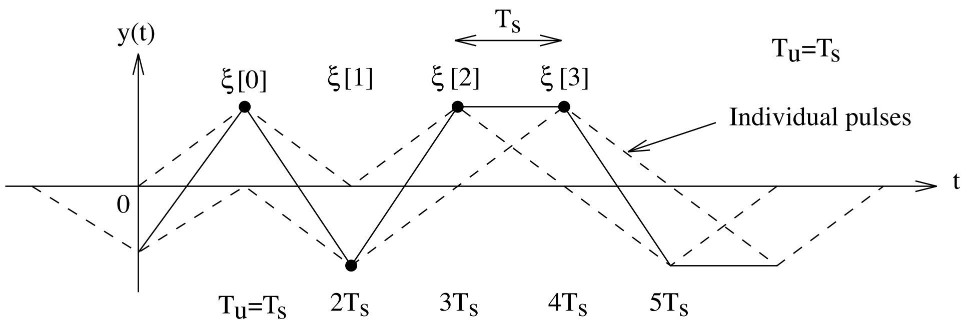
$$y(t) = \sum_{n=-\infty}^{\infty} A[n]\, x(t - nT_s) + w_c(t) \ ,$$

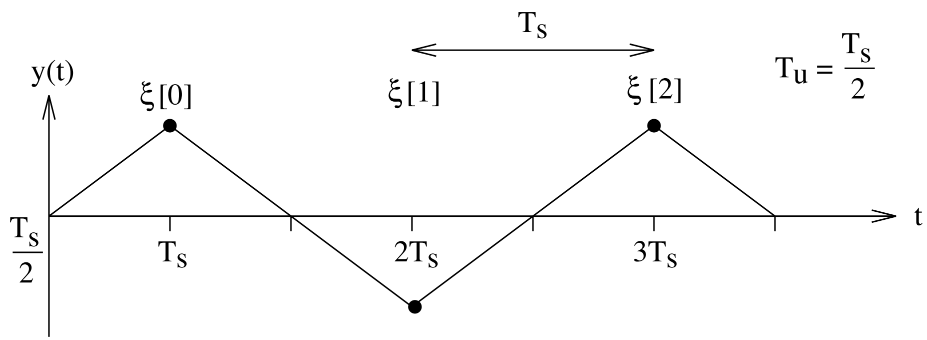
where $w_c(t)$ is a filtered Gaussian process

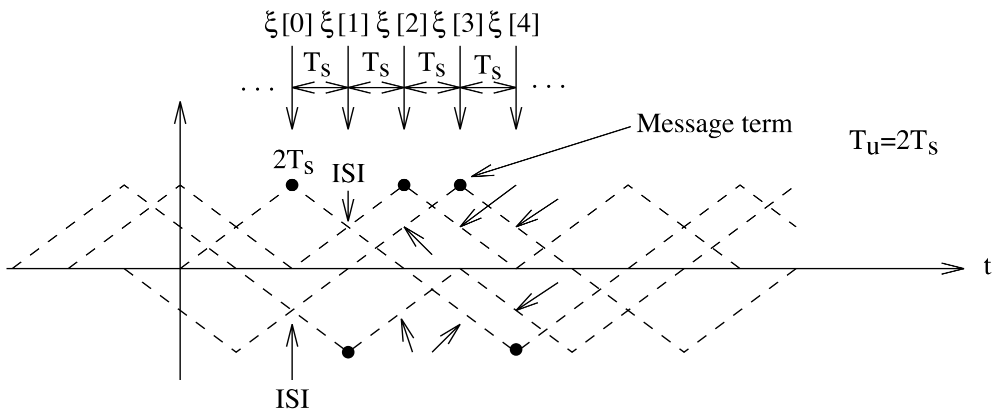
- The decision variable is obtained after sampling

$$\xi[i] = y(\mathcal{T} + iT_s) \ , \quad \mathcal{T} = t_0 + LT_s \ , \text{ where } LT_s \geq T_u$$

Illustration of ISI in the receiver







Discrete time model for ISI

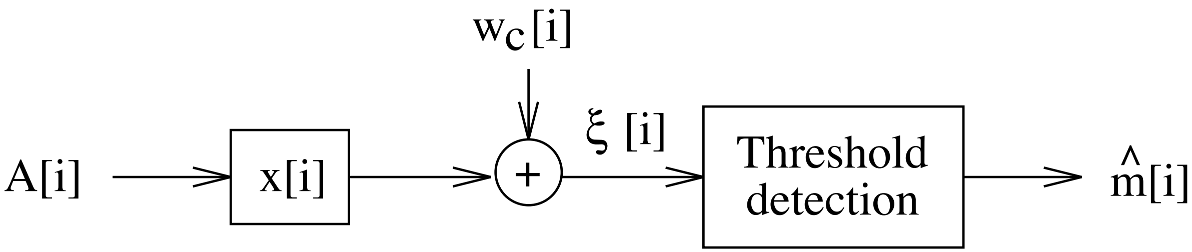
- According to our model the decision variable can be written as

$$\xi[i] = y(\mathcal{T} + iT_s) = \sum_{n=-\infty}^{\infty} A[n]\, x(\mathcal{T} + iT_s - nT_s) + w_c(\mathcal{T} + iT_s)$$

- Let us introduce the discrete sequences

$$x[i] = x(\mathcal{T} + iT_s)\ , \quad w_c[i] = w_c(\mathcal{T} + iT_s)$$

- This leads to the following discrete-time model of our system

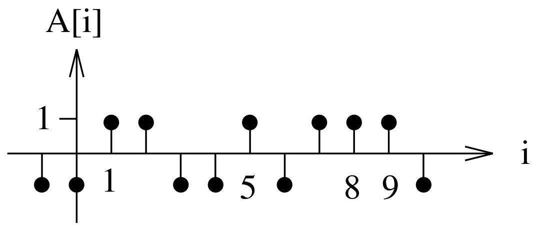


$$\xi[i] = \sum_{n=-\infty}^{\infty} A[n]\, x[i-n] + w_c[i] = A[i] * x[i] + w_c[i]$$

Remark: the discrete-time impulse response $x[i]$ represents pulse shape $g(t)$, channel filter $h(t)$, and receiver filter $v(t)$

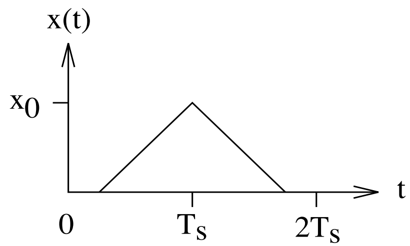
Example 6.1

The transmitted sequence of amplitudes $A[i]$ *is given as,*

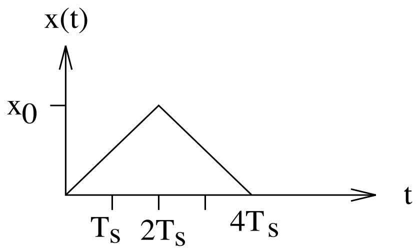


Calculate, and plot, the sequence of decision variables $\xi[i]$ *in Figure 6.2, for* $0 \leq i \leq 8$, *in the noiseless case (i.e.* $w(t) = 0$*) if* $t_0 = 0$ *and if the output pulse* $x(t)$ *is:*

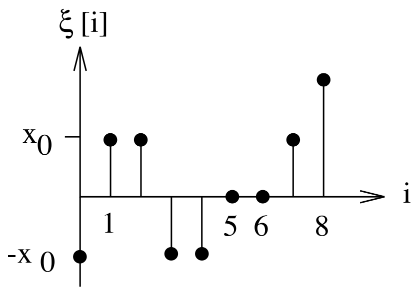
i) L=1 and x(t) as below.



ii) L=2 and x(t) as below.



- i) $\xi[i] = x_0 A[i]$ ii) $\xi[i] = \frac{x_0}{2} A[i+1] + x_0 A[i] + \frac{x_0}{2} A[i-1]$

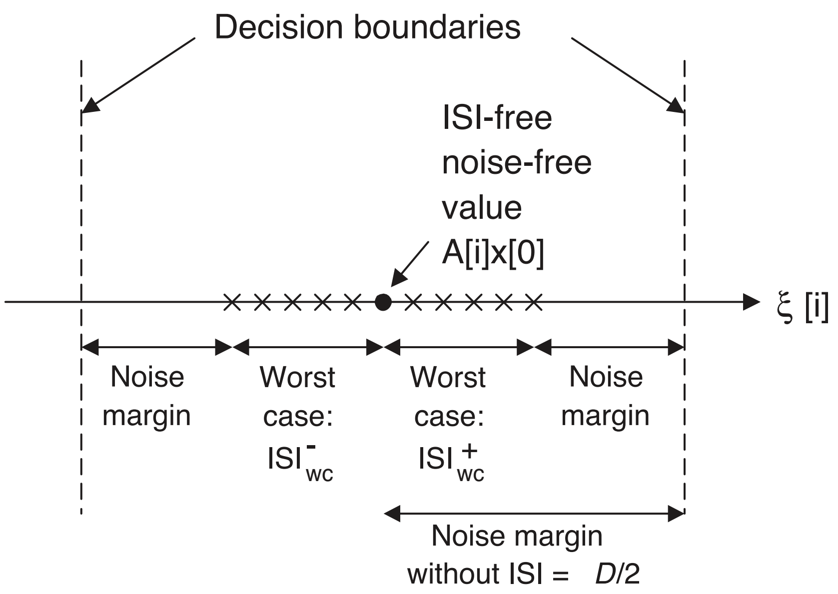


How much ISI can we tolerate?

- We can divide the decision variable $\xi[i]$ into a desired term (message) and an undesired term (interference plus noise)

$$\xi[i] = \underbrace{A[i]\,x[0]}_{\text{message}} + \underbrace{\sum_{\substack{n=-\infty \\ n\neq i}}^{\infty} A[n]\,x[i-n]}_{\text{ISI}} + \underbrace{w_c[i]}_{\text{noise}}$$

- The influence of ISI depends on its relative strength



Worst case ISI

- The ISI term can be written as

$$ISI = \sum_{\substack{n=-\infty \\ n\neq i}}^{\infty} A[n]\,x[i-n] = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} A[i-n]\,x[n]$$

- **Question:** when does this term become largest?
- For symmetric M-ary PAM we have $\max|A[i]| = M-1$ and get

$$ISI_{wc}^{+} = \max(ISI) = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \max\left(A[i-n]\,x[n]\right) = (M-1) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |x[n]|$$

- Similarly, the worst case minimal ISI becomes

$$ISI_{wc}^{-} = \min(ISI) = -(M-1) \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |x[n]|$$

Observe: the worst case ISI occurs for a information sequence $A[i]$ consisting of a particular pattern of $\pm(M-1)$ values

Condition for ISI free reception

- Let us assume that $x[i]$ satisfies the following condition:

$$x[i] = x(\mathcal{T} + i\,T_s) = x_0\,\delta[i] = \begin{cases} x_0 & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

- Then

$$\xi[i] = \sum_{n=-\infty}^{\infty} A[n]\,x[i-n] + w_c[i] = A[i]\,x[0] \; + w_c[i]$$

- Otherwise there always will exist some non-zero ISI term
- For this reason we are interested in signals

$$x(t) = g(t) \, * \, h(t) \, * \, v(t)$$

for which the above condition is satisfied

Which parts of $x(t)$ can we influence?

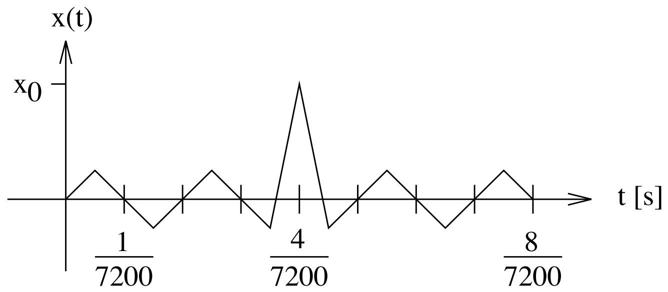
Symbol rates for ISI free reception

- Suppose that the ISI free condition is satisfied for symbol rate R_s^*
- Then it will be satisfied for rates

$$R_s = \frac{R_s^*}{\ell}\ , \quad \ell = 1, 2, 3, \dots$$

Example 6.6:

Consider the overall pulse shape $x(t)$ below, and $\mathcal{T} = 4/7200$.



Assume the bitrate 14400 [b/s] and 16-ary PAM signaling. Does ISI occur in the receiver?

Representation in frequency domain

- The discrete sequence $x[i]$ can be obtained by sampling a non-causal pulse $x_{nc}(t)$ at times $i\,T_s$,

$$x[i] = x_{nc}(i\,T_s)\ , \qquad \text{where } x_{nc}(t) = x(\mathcal{T} + t)\ ,$$

- The Fourier transform $\mathcal{X}(\nu)$ of $x[i]$ can then be expressed in terms of the Fourier transform $X_{nc}(f)$ of the signal $x_{nc}(t)$:

$$\mathcal{X}(\nu) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j2\pi\,\nu\,n} = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} X_{nc}\left(\frac{\nu - n}{T_s}\right)\ ,$$

where

$$X_{nc}(f) = \int_{-\infty}^{\infty} x_{nc}(t) e^{-j2\pi f\,t}\ dt = G(f)\,H(f)\,V(f)\,e^{+j2\pi f\,\mathcal{T}}$$

Observe: the spectrum of the sampled sequence $x[i]$ consists of the periodically repeated spectrum of the continuous signal

Nyquist condition in frequency domain

- Let us now formulate the ISI free condition in frequency domain:

$$x[i] = x_0\,\delta[i] \quad \Rightarrow \mathcal{X}(\nu) = \mathcal{F}\{x[i]\} = x_0 \quad \forall\,\nu$$

- Choosing $\nu = f\,T_s$ this leads to the equivalent Nyquist condition

$$\frac{\mathcal{X}(f\,T_s)}{R_s} = \sum_{n=-\infty}^{\infty} X_{nc}(f - n\,R_s) = \frac{x_0}{R_s}\;, \quad R_s = \frac{1}{T_s}$$

- Let W_{lp} denote the baseband bandwidth of $x_{nc}(t)$,

$$X_{nc}(f) = 0, \quad |f| > W_{lp}$$

- Then ISI always will be present if the symbol rate satisfies

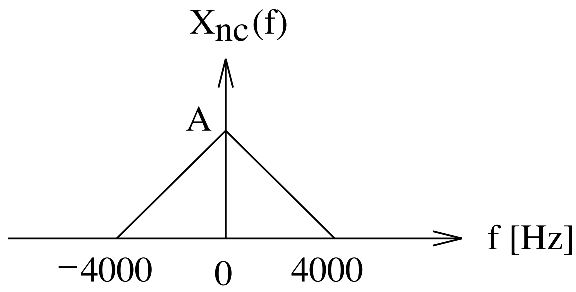
$$R_s > 2\,W_{lp}$$

(non-overlapping spectrum cannot add up to a constant)

- If we have $R_s \leq 2\,W_{lp}$:
ISI-free reception is possible if $X_{nc}(f)$ has a proper shape

Example 6.7

Assume that $X_{nc}(f)$ *is given below.*



a) Sketch the left hand side of (6.33), $\sum_{n=-\infty}^{\infty} X_{nc}(f - nR_s)$*, if* $R_s = 12000$ *symbols per second.*

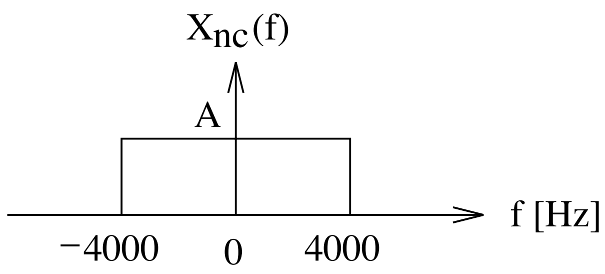
b) Does ISI occur in the receiver?

What happens if $R_s = 8000$?

And $R_s = 4000$?

Example 6.8

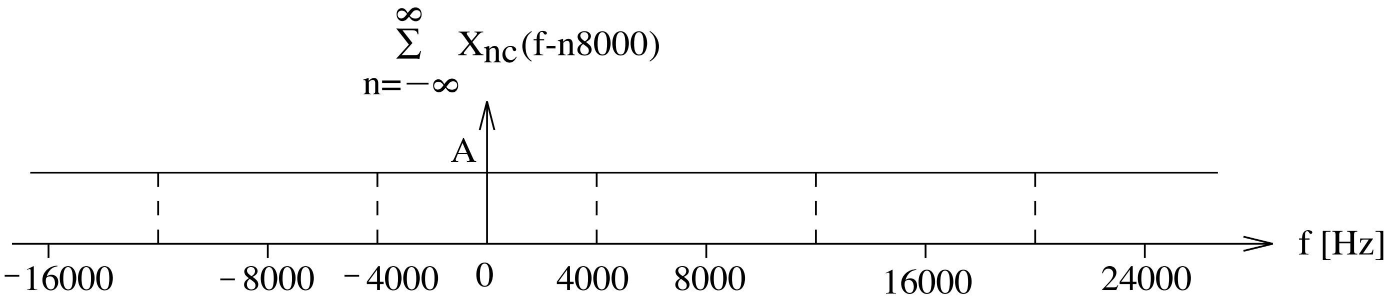
Assume that $X_{nc}(f)$ *is,*



$A = x_0 T_s$.

Show that there is no ISI if the symbol rate is $R_s = 8000$ *[symbol/s].*

Solution:



Since $\sum_{n=-\infty}^{\infty} X_{nc}(f - n8000) = x_0/R_s$, *for all* f, *there is no ISI in the receiver.*

Ideal Nyquist pulse

- The maximum possible signaling rate for ISI-free reception is

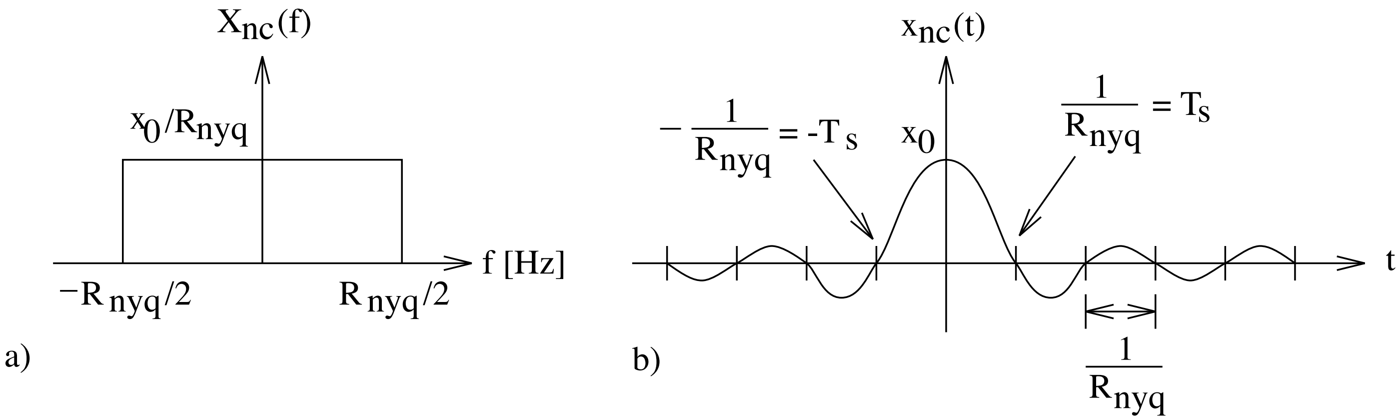
$$R_{nyq} = R_s = \frac{1}{T_s} = 2\,W_{lp} \quad \text{(Nyquist rate)}$$

- With ideal Nyquist signaling, the bandwidth efficiency is

$$\rho_{nyq} = \frac{R_b}{W_{lp}} = \frac{R_{nyq}\log_2(M)}{R_{nyq}/2} = 2\log_2 M = 2k \ [\text{bps/Hz}]$$

- The ideal Nyquist pulse must have rectangular spectrum

$$X_{nc}(f) = \begin{cases} x_0/R_{nyq}\ , & \text{if } |f| \le R_{nyq}/2 \\ 0\ , & \text{else} \end{cases} \quad \Rightarrow x_{nc}(t) = x_0 \frac{\sin(\pi R_{nyq}\, t)}{\pi R_{nyq}\, t}$$



Some comments on bandwidth

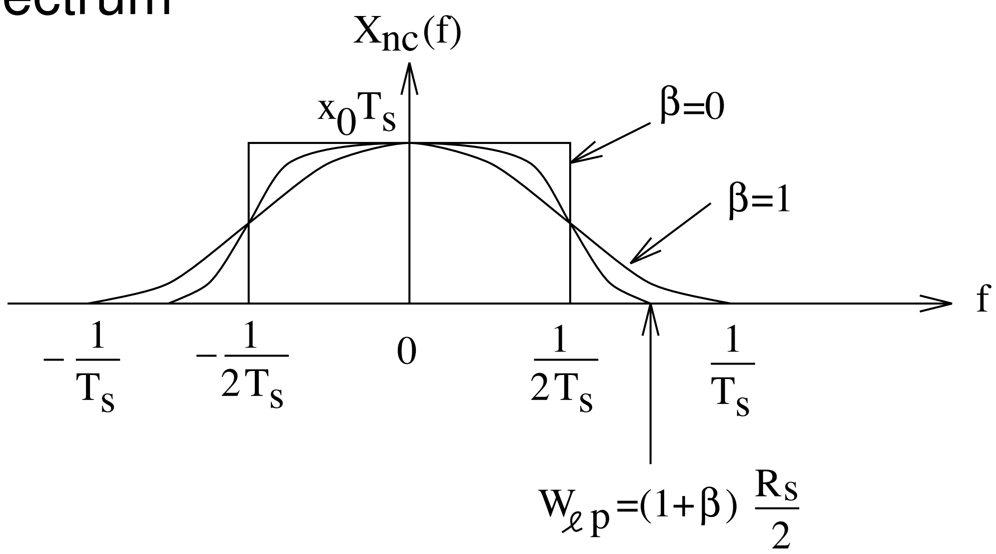
- **Remember:** in Chapter 2 we have seen that strictly band-limited signals always have to be unlimited in time
- In practice we have to find compromises, which was leading to different definitions of bandwidth for time-limited signals

Pulse shape	W_{lobe}	% power in W_{lobe}	W_{90}	W_{99}	$W_{99.9}$	Asymptotic decay
rec	2/T	90.3	1.70/T	20.6/T	204/T	f^{-2}
tri	4/T	99.7	1.70/T	2.60/T	6.24/T	f^{-4}
hcs	3/T	99.5	1.56/T	2.36/T	5.48/T	f^{-4}
rc	4/T	99.95	1.90/T	2.82/T	3.46/T	f^{-6}
Nyquist	R_s	100	$0.9R_s$	$0.99R_s$	$0.999R_s$	ideal

- We can see that time-limited signals need at least about twice the Nyquist bandwidth
- For OFDM with many sub-carriers N this is negligible (why?)
- For single-carrier systems, some close-to-Nyquist pulses are typically used in practice

Spectral Raised Cosine Pulses

- The spectral raised cosine pulse shape is defined by the following spectrum



- The name refers to the way the shape is composed

$$X_{nc}(f) = \begin{cases} x_0\, T_s \;, & 0 \leq |f| \leq \frac{1-\beta}{2T_s} \\ \frac{x_0\, T_s}{2}\left[1+\cos\left(\frac{\pi|f|T_s}{\beta} - \frac{\pi}{2}\cdot\frac{1-\beta}{\beta}\right)\right] \;, & \frac{1-\beta}{2T_s} \leq |f| \leq W_{lp} \\ 0 & |f| > W_{lp} \end{cases}$$

where $W_{lp} = \dfrac{1+\beta}{2T_s} = (1+\beta)\dfrac{R_s}{2} \;, \quad 0 \leq \beta \leq 1$

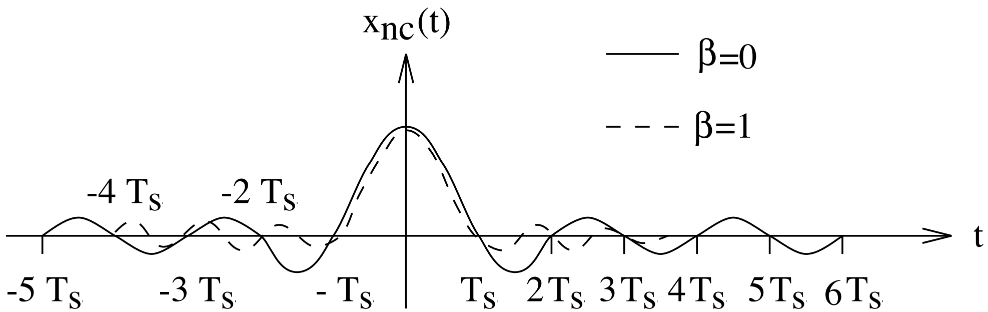
Spectral Raised Cosine Pulses

- The parameter β, $0 \leq \beta \leq 1$, is called the rolloff factor and can be used to smoothly control the bandwidth efficiency

$$\rho_{src} = \frac{R_b}{W_{lp}} = \frac{R_s \log_2 M}{(1+\beta)R_s/2} = \frac{2\log_2 M}{1+\beta} = \frac{2k}{1+\beta}$$

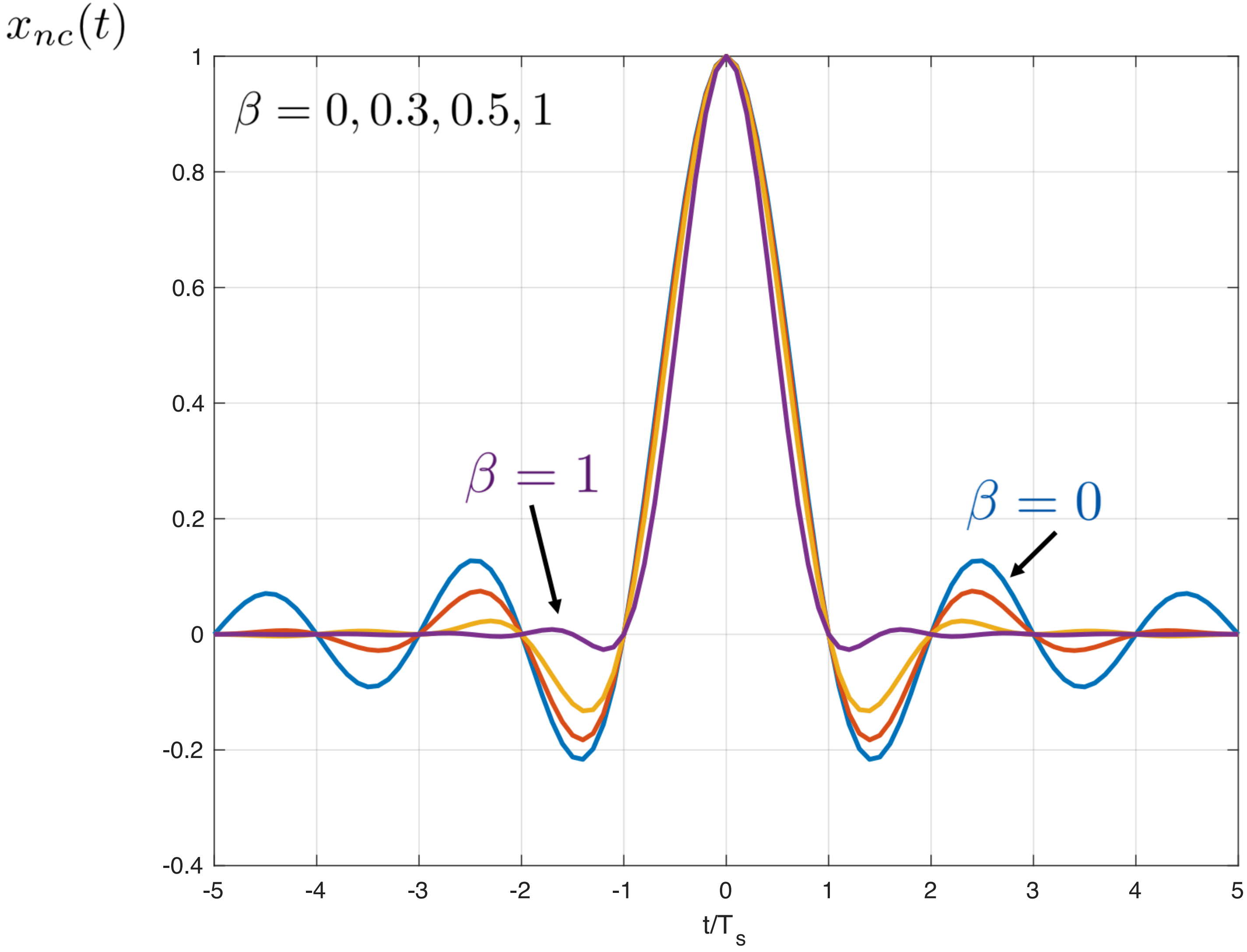
- In time domain the signal can be expressed as

$$x_{nc}(t) = x_0 \frac{\sin(\pi t/T_s)}{\pi t/T_s} \cdot \frac{\cos(\pi\beta t/T_s)}{1-(2\beta t/T_s)^2}\,, \qquad -\infty \leq t \leq \infty$$



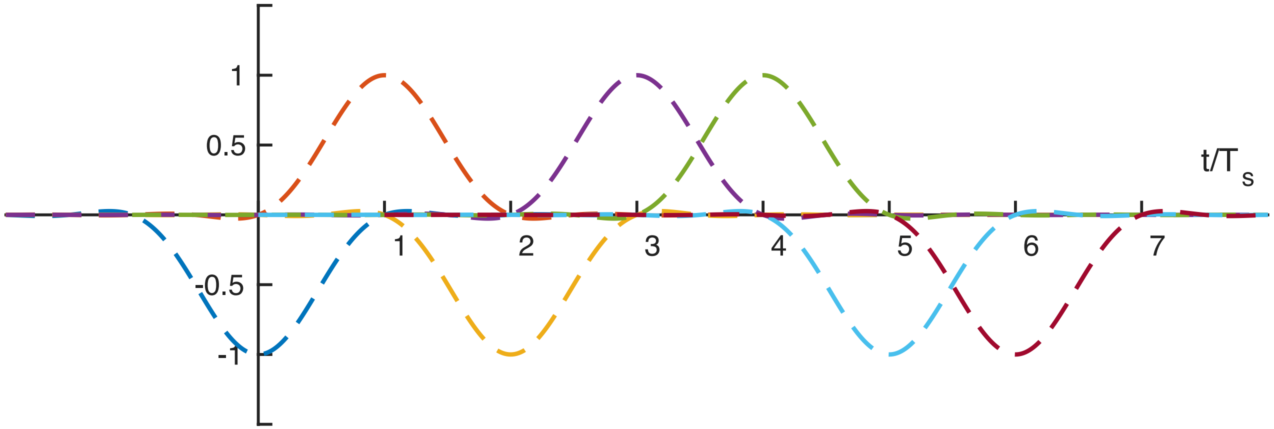
- Larger rolloff factors $\beta \Rightarrow$ faster amplitude decay of $x_{nc}(t)$

Spectral Raised Cosine Pulses

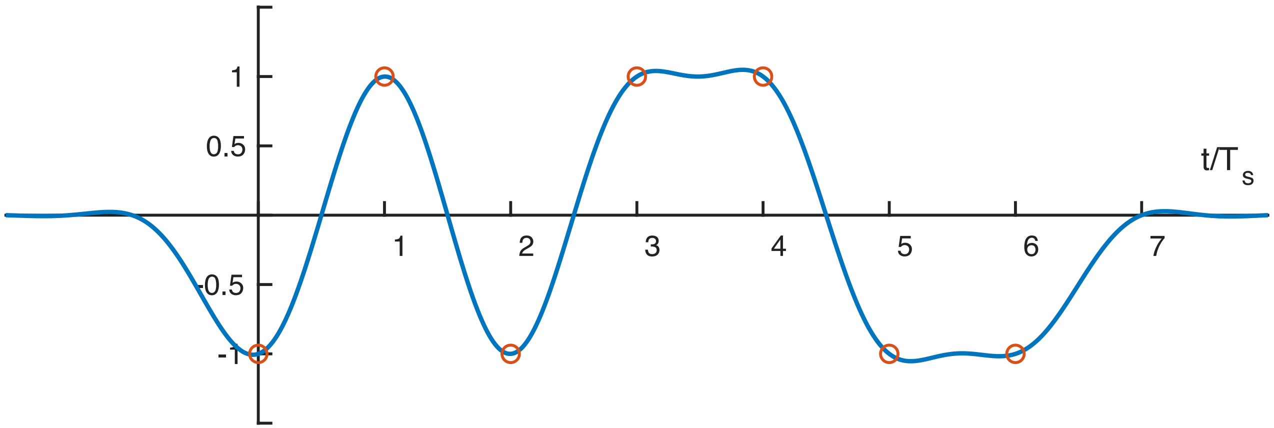


Signaling with overlapping pulses: $\beta = 1$

$A[n]\, x(t - nTs)$

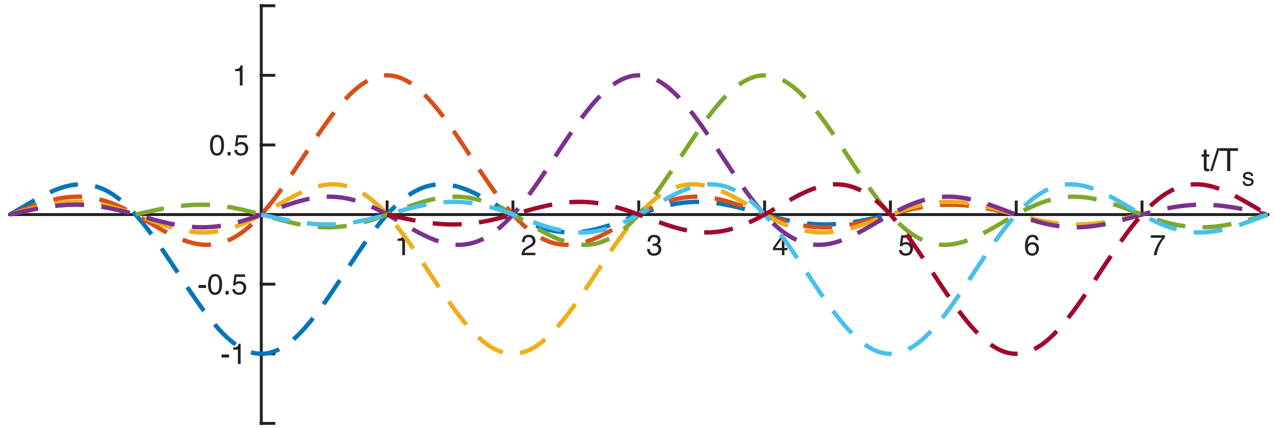


$y(t)$

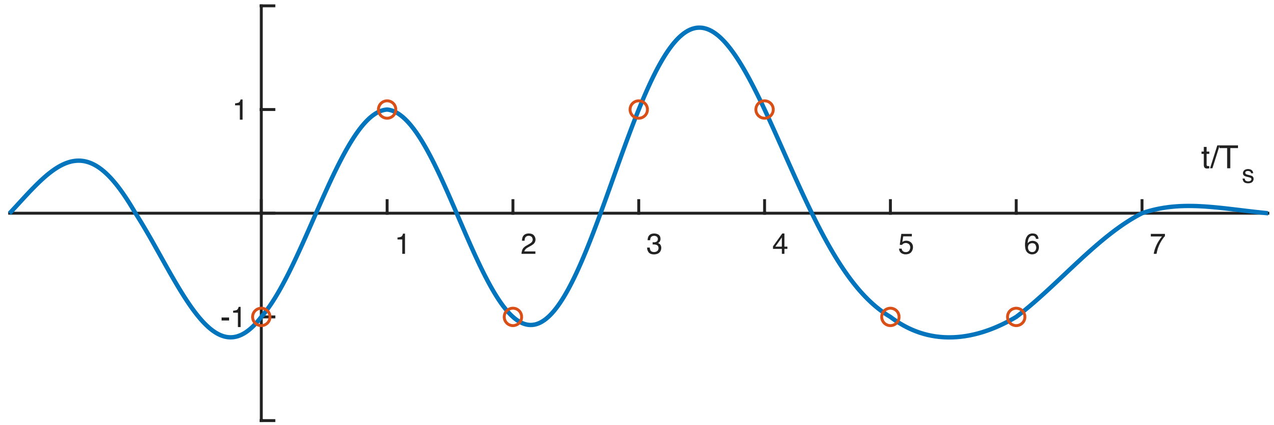


Signaling with overlapping pulses: $\beta = 0$

$A[n]\,x(t - nTs)$



$y(t)$



Spectral Root Raised Cosine Pulse

- When analyzing the Nyquist condition we have considered the output signal of the receiver filter $v(t)$, i.e.,

$$x_{nc}(t) = g(t) \, * \, h(t) \, * \, v(t) = u(t) \, * \, v(t)$$

- The matched filter for our receiver structure with delay $\mathcal{T} = LT_s$ should be equal to

$$v(t) = u(LT_s - t)$$

- As a consequence, we need to choose pulse shape $g(t)$ and receiver filter $v(t)$ in such a way that

$$|V(f)| = \sqrt{X_{nc}^{rc}(f)} \quad \text{and} \quad |G(f)H(f)| = \sqrt{X_{nc}^{rc}(f)}$$

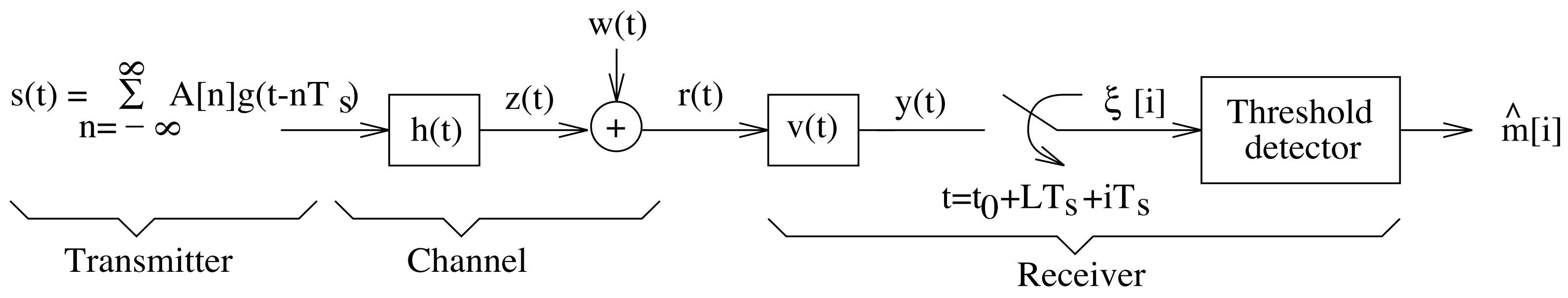
in order to ensure a raised cosine spectrum for

$$X_{nc}(f) = |G(f)H(f)|^2 = |V(f)|^2 = X_{nc}^{rc}(f)$$

- Hence $v(t)$ is a pulse with root-raised cosine spectrum

Introduction to equalizers

- We have considered the receiver structure



- When ISI occurs this receiver is suboptimal and is no longer equivalent to the ML rule (sequence estimation, Viterbi algorithm)
- **Equalization:**
  instead of tolerating the ISI in the above structure, an equalizer can be used for removing (or reducing) the effect of ISI
- Linear equalizer: zero-forcing, MMSE
  can be implemented by linear filters, low complexity
- Decision feedback equalizer:
  non-linear device with feedback, aims at subtracting the estimated ISI from the signal

Introduction to equalizers