EITG05 – Digital Communications

Lecture 6

Receivers continued:
System design criteria, Performance for M-ary signaling

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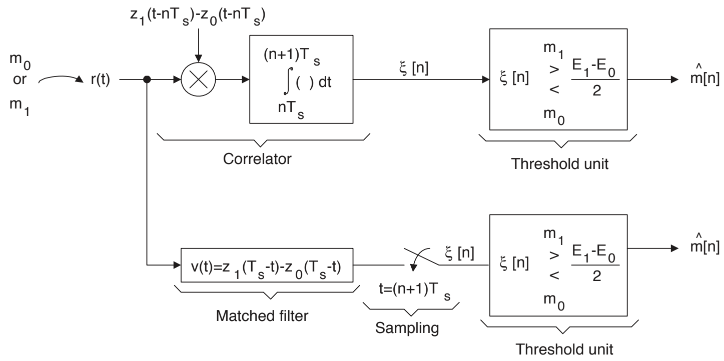
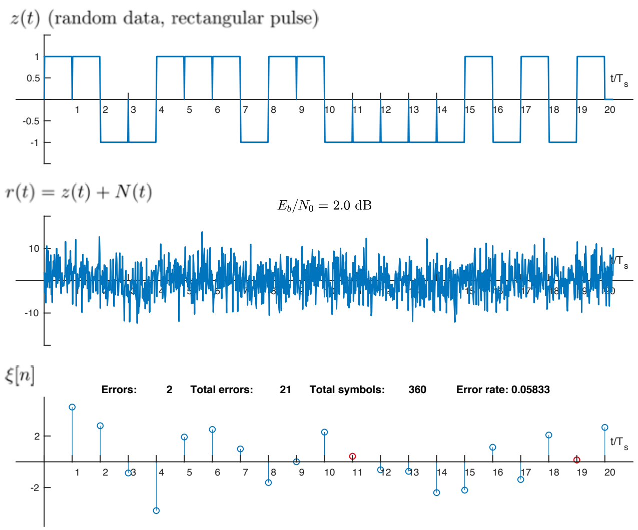
Last week: Analysis Binary Signaling

- Only one correlator or one matched filter is now required:



- Matched filter output needs be sampled at correct time

Example: (see Matlab demo)



An energy efficiency perspective

- Consider the case $P_0 = P_1 = 1/2$
- The average received energy per bit is then

$$\mathcal{E}_b = \frac{1}{2}\int_0^{T_b} z_0^2(t)\, dt + \frac{1}{2}\int_0^{T_b} z_1^2(t)\, dt = \frac{E_0+E_1}{2}$$

- We can then introduce the normalized squared Euclidean distance

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{1}{2\,\mathcal{E}_b}\int_0^{T_b} \left(z_1(t)-z_0(t)\right)^2 dt$$

- With this the bit error probability becomes

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameter $d_{0,1}^2$ is a measure of energy efficiency

Special case 1: antipodal signals

- In case of antipodal signals we have $z_1(t) = -z_0(t)$ and

$$D_{0,1}^2 = \int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2 \, dt = 4 \int_0^{T_b} z_1^2(t) \, dt = 4E$$

- From $E_0 = E_1 = E$ follows

$$\mathcal{E}_b = \frac{E+E}{2} = E$$

and

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{4E}{2E} = 2$$

- The bit error probability for any pair of antipodal signals becomes

$$P_b = Q\left(\sqrt{2\,\frac{\mathcal{E}_b}{N_0}}\right)$$

Special case 2: orthogonal signals

- In case of orthogonal signals we have

$$\int_0^{T_b} z_0(t)\, z_1(t) \, dt = 0$$

and hence (compare page 28)

$$D_{0,1}^2 = \int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2 \, dt = E_0 + E_1$$

- This gives

$$\mathcal{E}_b = \frac{E_0 + E_1}{2}$$

and

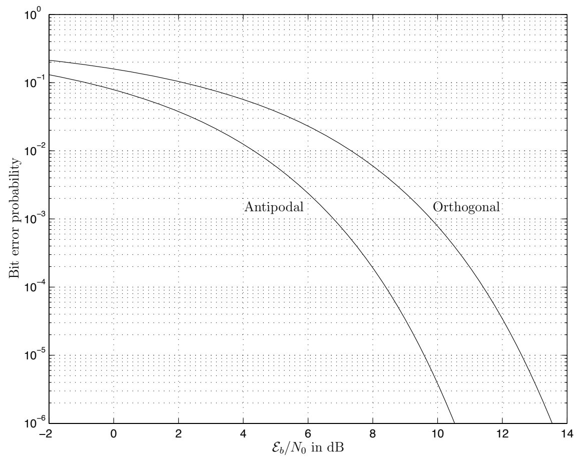
$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{E_0+E_1}{E_0+E_1} = 1$$

- The bit error probability for any pair of orthogonal signals is

$$P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$$

Comparison

Antipodal vs orthogonal signaling:



Larger values of $d_{0,1}^2$ give better energy efficiency

Antipodal vs orthogonal signaling

- There is a constant gap between the two curves
- We can measure the difference in energy efficiency by the ratio

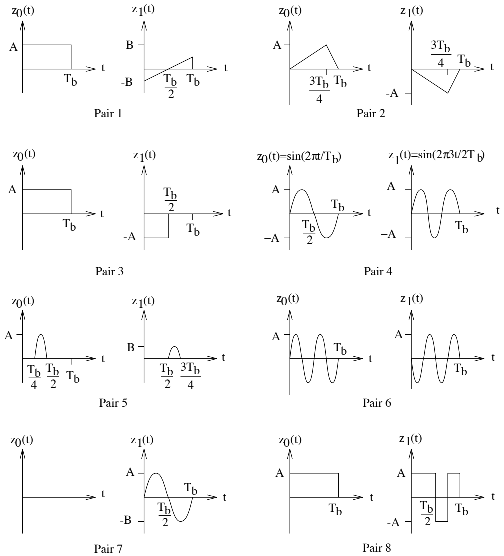
$$\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}} = \frac{d_{0,1,ort}^2}{d_{0,1,atp}^2} = \frac{1}{2}$$

- In terms of dB this corresponds to

$$10\log_{10}\left(\frac{\mathcal{E}_{b,atp}}{\mathcal{E}_{b,ort}}\right) = 10\log_{10}\left(\frac{d_{0,1,ort}^2}{d_{0,1,atp}^2}\right) = -3 \text{ [dB]}$$

$\Rightarrow$ antipodal signaling requires 3 dB less energy for equal P_b

Example 4.11: rank pairs with respect to $d_{0,1}^2$



Can we do better?

- It is possible to show that for two equally likely signal alternatives we always have
$$d_{0,1}^2 \leq 2$$
- Antipodal signaling is hence optimal for binary signaling ($M = 2$)

Remark:

- Channel coding can be used to further increase $d_{0,1}^2$
- Sequences of binary pulses with large separation are designed
- This does not contradict the result from above:
coded binary signals correspond to uncoded signals with $M > 2$

Channel coding can be used for improving energy efficiency
Cost: complexity, latency, (bandwidth)

Relationship between parameters

- The bit error probability can be expressed in different ways

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right) = Q\left(\sqrt{d_{0,1}^2 \frac{\mathcal{E}_b}{N_0}}\right) = Q\left(\sqrt{d_{0,1}^2 \frac{P_z}{R_b N_0}}\right)$$

- Assuming $z_0(t) = \alpha\, s_0(t)$ and $z_1(t) = \alpha\, s_1(t)$ we also get

$$P_b = Q\left(\sqrt{d_{0,1}^2 \frac{\alpha^2 \bar{P}_{sent}}{R_b N_0}}\right) = Q\left(\sqrt{\frac{d_{0,1}^2}{\rho} \cdot \frac{\alpha^2 \bar{P}_{sent}}{N_0 W}}\right)$$

- Recall that $\rho = R_b/W$ is the bandwidth efficiency and $N_0 W$ is the noise power within the bandwidth W

The expression that is most appropriate to use depends on the specific problem to be solved

A "typical" type of problem

- The bit error probability must not exceed a certain level,
$$P_b \leq P_{b,req} = Q(\sqrt{\mathcal{X}})$$
- Example: if $P_{b,req} = 10^{-9}$ then $\mathcal{X} \approx 36$
- **Consequences:**
$$d_{0,1}^2 \frac{\mathcal{E}_b}{N_0} \geq \mathcal{X}$$
$$R_b \leq \frac{d_{0,1}^2}{\mathcal{X}} \cdot \frac{P_z}{N_0}$$
$$R_b \leq \frac{d_{0,1}^2}{\mathcal{X}} \cdot \frac{\alpha^2 \bar{P}_{sent}}{N_0}$$
- Note: the received signal power P_z decreases with communication distance

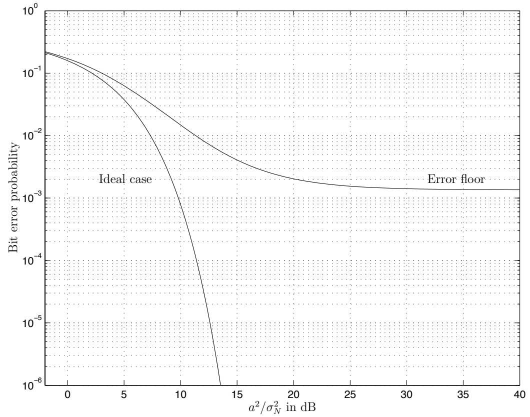
Example 4.12: transmission hidden in noise

In a specific application equally likely binary antipodal signals are used, and the pulse shape is $g_{rc}(t)$ with amplitude A and duration $T \leq T_b$. AWGN with power spectral density $N_0/2$, and the ML receiver is assumed. It is required that the bit error probability must not exceed 10^{-9}. It is also required that the power spectral density satisfies $R(f) \leq N_0/2$ for all frequencies f (the information signal is intentionally "hidden" in the noise). Determine system and signal parameters above such that these two requirements are satisfied.

- $P_b = Q\left(\sqrt{2\,\mathcal{E}_b/N_0}\right) \leq 10^{-9} \Rightarrow \mathcal{E}_b/N_0 \geq 18$
- $R(f) = R_b |G_{rc}(f)|^2$ has maximum at $f = 0$
- $R(0) = R_b\, A^2 T^2/4 \leq N_0/2$ (check pulse shape)
- $\mathcal{E}_b/N_0 = 3/8 A^2 T/N_0 \geq 18$
- Hidden in noise: $A^2 T/N_0 \leq 2/(R_b T)$
- P_b requirement: $A^2 T/N_0 \geq 48$
- Solution:
  choose $T \leq T_b/24$ and $A^2 = 48 N_0/T$

Non-ideal receiver conditions

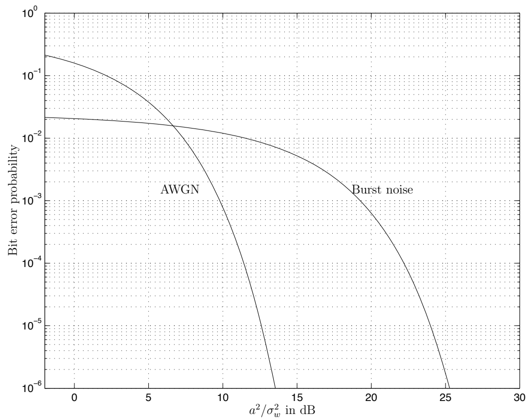
Example 4.15: unexpected additional noise w_x, i.e., $w = w_N + w_x$



Can be analyzed with our methods

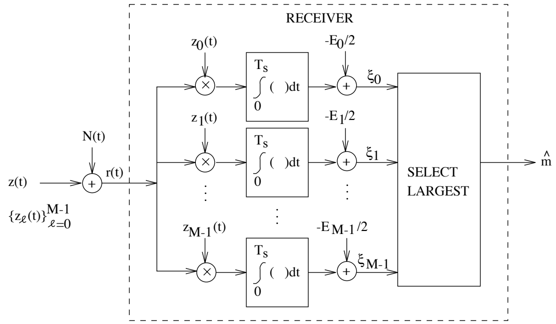
Non-ideal receiver conditions

Example 4.16: hostile bursty interference, active with $p_{on} = 0.05$



Observe: at low power an interference in bursts is more severe than continuous interference

M-ary Signaling



- The receiver computes M decision variables $\xi_0, \xi_1, \ldots, \xi_{M-1}$
- The selected message $\hat{m}$ is based on the largest value

$$\hat{m} = m_\ell \ , \quad \ell = \arg\max_i \xi_i$$

- **Question:** when do we make a wrong decision?

Probability of a wrong decision

- For $M = 2$ we have considered two error probabilities P_F and P_M
- For a given message $m = m_j$, in general there are $M-1$ ways (events) to make a wrong decision,

$$\{\xi_i > \xi_j \mid m = m_j\} , \quad i \neq j$$

- The probability of a wrong decision can be upper bounded by

$$Pr\{\hat{m} \neq m_j | m = m_j\} = Pr\left\{ \bigcup_{\substack{i=0\\ i\neq j}}^{M-1} \xi_i > \xi_j \;\middle|\; m = m_j \right\}$$

$$\leq \sum_{\substack{i=0\\ i\neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\} \quad \text{(union bound)}$$

- Note: given some events A and B, the union bound states that

$$Pr\{A \cup B\} \leq Pr\{A\} + Pr\{B\} ,$$

where equality holds if A and B are independent

Symbol error probability

- The symbol error probability can be upper bounded by

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0\\ i\neq j}}^{M-1} Pr\{\xi_i > \xi_j \mid m = m_j\}$$

- From the binary case $M = 2$ we know that (pick $i = 0$ and $j = 1$)

$$Pr\{\xi_i > \xi_j \mid m = m_j\} = Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

where $D_{i,j}$ is the Euclidean distance between $z_i(t)$ and $z_j(t)$

- We obtain the following main result for M-ary signaling:

$$\max_{\substack{i\\ i\neq j}} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right) \leq P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0\\ i\neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

Example: orthogonal signaling

- Consider M orthogonal signals with equal energy E
- Examples: FSK, PPM
- For each pair $z_i(t)$ and $z_j(t)$ we get

$$D_{i,j}^2 = E + E = 2E$$

- From the union bound we obtain

$$P_s \leq \sum_{j=0}^{M-1} P_j \sum_{\substack{i=0\\ i\neq j}}^{M-1} Q\left(\sqrt{\frac{D_{i,j}^2}{2N_0}}\right)$$

$$= (M-1)\, Q\left(\sqrt{\frac{2E}{2N_0}}\right) = (M-1)\, Q\left(\sqrt{\frac{E}{N_0}}\right)$$

- This generalizes the binary case from the previous lecture

Distances $D_{i,j}$ are important

- P_s is determined by the distances $D_{i,j}$ between the signal pairs
- Let us sort these distances

$$D_{min} < D_1 < D_2 < \cdots < D_{max}$$

- Then the upper bound on P_s can be written as

$$P_s \leq c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right) + c_1\, Q\left(\sqrt{\frac{D_1^2}{2N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{\frac{D_{max}^2}{2N_0}}\right)$$

- The coefficients are

$$c_\ell = \sum_{j=1}^{M-1} P_j \cdot n_{j,\ell} , \quad \ell = 0, 1, 2, \ldots, x$$

- $n_{j,\ell}$: number of signals at distance D_ℓ from signal $z_j(t)$

How many distinct terms do exist for 4-PAM?

A useful approximation of P_s

- The union bound is easy to compute if we know all distances D_ℓ
- At large signal-to-noise ratio (small N_0), i.e., when P_s is small, the first term provides a good approximation

$$P_s \approx c\, Q\left(\sqrt{\frac{D_{min}^2}{2N_0}}\right)$$

- We see that the minimum distance D_{min}^2 and the average number of closest signals c dominate the performance in this case
- **Explanation:**
the function $Q(x)$ decreases very fast as x increases (faster than exponentially). The other terms become negligible at some point.

$\Rightarrow$ at small P_s (small N_0) we can compare different signal constellations by means of D_{min}^2, similarly to the binary case

Energy efficiency and normalized distances

- Consider the case $P_\ell = 1/M$, $\ell = 0, 1, \ldots, M-1$
- The average received energy per bit is given by

$$\mathcal{E}_b = \frac{1}{k}\sum_{i=0}^{M-1}\frac{1}{M}\int_0^{T_s} z_i^2(t)\, dt \;=\frac{1}{k}\,\frac{E_0+E_1+\cdots E_{M-1}}{M}$$

- Using the normalized squared Euclidean distances

$$d_\ell^2 = \frac{D_\ell^2}{2\,\mathcal{E}_b}\ ,$$

the union bound can be written as

$$P_s \le c\, Q\left(\sqrt{d_{min}^2\frac{\mathcal{E}_b}{N_0}}\right) + c_1\, Q\left(\sqrt{d_1^2\frac{\mathcal{E}_b}{N_0}}\right) + \cdots + c_x\, Q\left(\sqrt{d_{max}^2\frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameters d_ℓ^2 determine the energy efficiency

Approximate P_s for some constellations

- Considering the dominating term in the union bound we obtain

$$P_s \approx c\, Q\left(\sqrt{d_{min}^2\frac{\mathcal{E}_b}{N_0}}\right)$$

- This approximation is valid if $\frac{\mathcal{E}_b}{N_0}$ is sufficiently large

	c	$d^2_{\min}$
M-ary PAM	$2(1-1/M)$	$\frac{6\log_2(M)}{M^2-1}$
M-ary PSK ($M>2$)	2	$2\log_2(M)\sin^2(\pi/M)$
M-ary FSK	$M-1$	$\log_2(M)$
M-ary QAM	$4(1-1/\sqrt{M})$	$\frac{3\log_2(M)}{M-1}$

Table 4.1: The coefficient c, and $d^2_{\min}$, for some common signal constellations. Equally likely signal alternatives are assumed. See Subsection 2.4.1.1 for the M-ary PAM case, and Subsection 2.4.5.1 for the M-ary QAM case. M equal energy orthogonal FSK signals are also assumed.

Example 4.19

Assume two signal constellations, denoted A and B respectively, with corresponding parameters $d^2_{\min,A}$ and $d^2_{\min,B}$. From the equality (see e.g. the dominating term in the union bound),

$$d^2_{\min,A}\mathcal{E}_{b,A}/N_0 = d^2_{\min,B}\mathcal{E}_{b,B}/N_0$$

we find that the difference (in dB) in received energy per information bit is (compare with (2.13) on page 16),

$$10\log_{10}(\mathcal{E}_{b,B}) - 10\log_{10}(\mathcal{E}_{b,A}) = 10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$$

Calculate the value $10\log_{10}\left(\frac{d^2_{\min,A}}{d^2_{\min,B}}\right)$ if "A" is binary antipodal PAM, and if "B" is 4-ary PAM. Assume, that the conditions leading to (2.50) are satiesfied.

- For M-ary PAM we have (Table 4.1 or Table 5.1)

$$d_{min}^2 = 6\log_2(M)/(M^2-1) \quad \Rightarrow\ d_{min,A}^2 = 2,\ d_{min,B}^2 = 4/5$$

- $10\log_{10} d_{min,A}^2/d_{min,B}^2 = 10\log_{10} 5/2 = 3.98$ dB

Binary PAM is 3.98 dB more energy efficient than 4-ary PAM!

Example scenario: M-ary QAM

- We want to ensure that $P_s \leq P_{s,req}$, where for M-ary QAM

$$P_s \leq 4\, Q\left(\sqrt{d^2_{min}\,\frac{\mathcal{E}_b}{N_0}}\right) = 4\, Q\left(\sqrt{\mathcal{X}}\right) \;, \quad d^2_{min} = 3\,\log_2 \frac{M}{M-1}$$

- The pulse shape $g(t)$ is chosen such that

$$\rho = \log_2(M)\ \rho_{BPSK} \;, \quad \text{where } \rho = \frac{R_b}{W} \leq \frac{d^2_{min}}{\mathcal{X}} \cdot \frac{\mathcal{P}_z}{N_0\,W}$$

- Combining these requirements we obtain

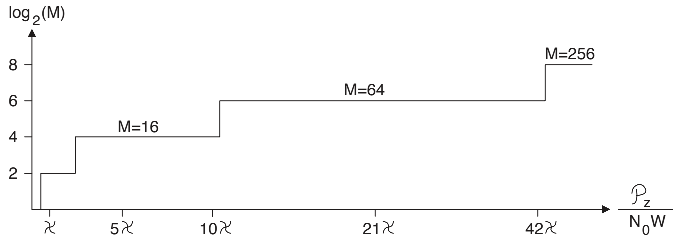
$$M \leq 1 + \frac{3}{\mathcal{X}\,\rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\,W} = 1 + \frac{3}{\mathcal{X}} \cdot \frac{\mathcal{P}_z\,T_s}{N_0}$$

- Hence we want to choose $M = 2^k$ such that (QAM: k even)

$$2^k \;\leq\; 1 + \frac{3}{\mathcal{X}\,\rho_{BPSK}} \cdot \frac{\mathcal{P}_z}{N_0\,W} \;<\; 2^{k+2}$$

Example 4.22: adapting M to channel quality

Assume that an M-ary QAM system adapts between 4-ary QAM, 16-ary QAM, 64-ary QAM and 256-ary QAM. Show when a new M is chosen by plotting M (or $\log_2(M)$) versus $\mathcal{P}_z/N_0W$. How large is the bit rate in each case? Assume that $\rho_{BPSK} = 1/2$ [bps/Hz].



Depending on the channel quality we can achieve different bit rates $R_b = W$, $2W$, $3W$, or $4W$[bps]