EITG05 – Digital Communications

Lecture 5

Receivers in Digital Communication Systems

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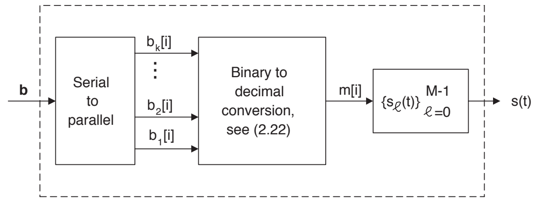
Monday, September 17, 2018

Where are we now?

What we have done so far: (Chapter 2)



- Concepts of digital signaling: bits to analog signals
- Average symbol energy $\overline{E}_s$, Euclidean distance $D_{i,j}$
- Bandwidth of the transmit signal

Chapter 4: Receivers

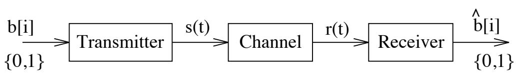
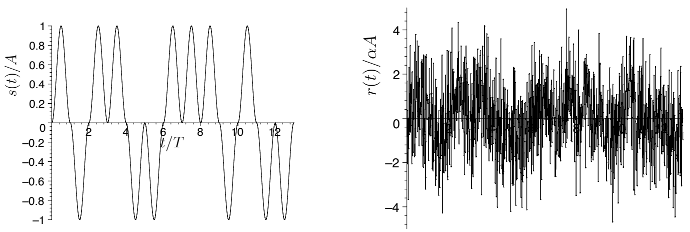
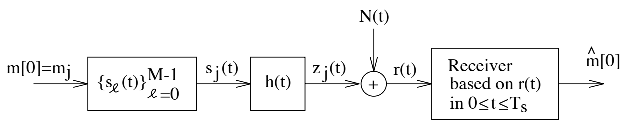


Figure 4.1: A digital communication system.



- How can we estimate the transmitted sequence?
- Is there an optimal way to do this?

The Detection Problem



Assumptions:

- A random (i.i.d.) sequence of messages $m[i]$ is transmitted
- There are $M = 2^k$ possible messages, i.e., k bits per message
- All signal alternatives $z_\ell(t)$, $\ell = 1, \ldots, M$ are known by the receiver
- T_s is chosen such that the signal alternatives $z_\ell(t)$ do not overlap
- $N(t)$ is additive white Gaussian noise (AWGN) with $R_N(f) = N_0/2$

Questions:

- How should decisions be made at the receiver?
- What is the resulting bit error probability P_b?

An optimal decision strategy

- Suppose we want to minimize the symbol error probability P_s
- That means we maximize the probability of a correct decision

$$Pr\{m = \hat{m}(r(t)) \mid r(t)\}$$

where m denotes the transmitted message

- This leads to the following decision rule:

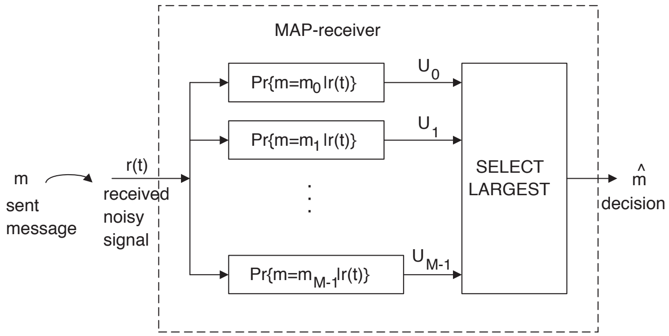
$$\hat{m}(r(t)) = m_\ell \ ,$$

$$\text{where} \quad \ell = \arg\max_i Pr\{m = m_i | r(t)\}$$

- We decide for the message that maximizes the probability above
- A receiver that is based on this decision rule is called maximum-a-posteriori probability (MAP) receiver

Structure of the general MAP receiver

- We know that one of the M messages must be the best
- Hence we can simply test each m_ℓ, $\ell = 0, 1, \ldots, M-1$



This receiver minimizes the symbol error probability P_s

A slightly different decision strategy

- The maximum likelihood (ML) receiver is based on a slightly different decision rule:

$$\hat{m}(r(t)) = m_\ell : \quad \ell = \arg\max_i Pr\{r(t) \,|\, m_i \text{ sent}\}$$

- Using the Bayes rule we can write

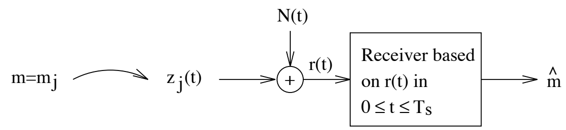
$$Pr\{m = m_i \mid r(t)\} = \frac{Pr\{r(t) \,|\, m_i \text{ sent}\} \cdot P_i}{Pr\{r(t)\}}$$

- The decision rule of the MAP receiver can be formulated as

$$\hat{m}(r(t)) = m_\ell : \quad \ell = \arg\max_i Pr\{r(t) \,|\, m_i \text{ sent}\} \cdot P_i$$

- It follows that the ML receiver is equivalent to the MAP receiver for equally likely messages, $P_i = 1/M, i = 0, 1, \ldots, M-1$.

The Minimum Euclidean Distance Receiver



- For our considered scenario with Gaussian noise: maximizing $Pr\{r(t) \,|\, m_i \text{ sent}\}$ is equivalent to minimizing the squared Euclidean distance $D^2_{r,i}$.
- The received signal is compared with all noise-free signals $z_i(t)$ in terms of the squared Euclidean distance

$$D^2_{r,i} = \int_0^{T_s} \left(r(t) - z_i(t)\right)^2 \, dt$$

- The message is selected according to the decision rule:

$$\hat{m}(r(t)) = m_\ell : \quad \ell = \arg\min_i \ D^2_{r,i}$$

The Minimum Euclidean Distance Receiver

- The squared Euclidean distance is a measure of similarity
- An implementation is often based on correlators with output

$$\int_0^{T_s} r(t)\, z_i(t)\; dt\;, \quad i=0,1,\ldots,M-1$$

- Using

$$D_{r,i}^2 = \int_0^{T_s} \big(r(t)-z_i(t)\big)^2\; dt = E_r - 2\int_0^{T_s} r(t)\, z_i(t)\; dt + E_i$$

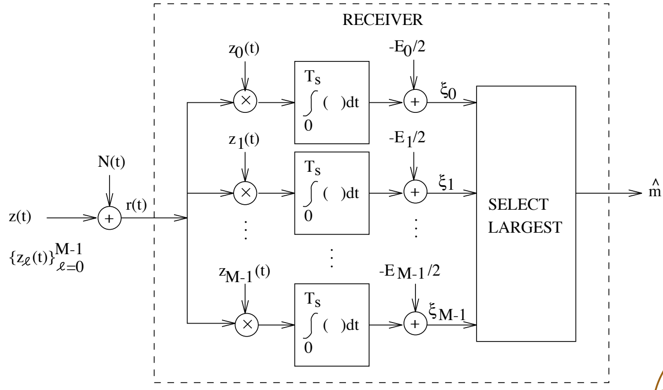
we can write

$$\ell = \arg\min_i\; D_{r,i}^2 = \arg\max_i \int_0^{T_s} r(t)\, z_i(t)\; dt - E_i/2$$

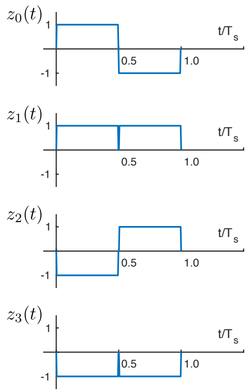
- The received signal is compared with all possible noise-free signal alternatives $z_i(t)$
  The receiver needs to know the channel!

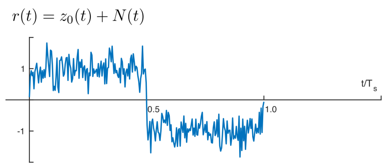
Correlation based implementation

$$\ell = \arg\min_i\; D_{r,i}^2 = \arg\max_i \int_0^{T_s} r(t)\, z_i(t)\; dt - E_i/2$$



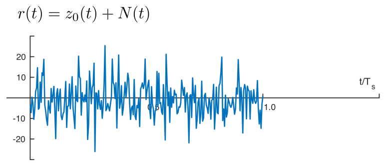
Example: $M=4$





$\xi_0 = 0.4778\,E$
$\xi_1 = -0.5011\,E$
$\xi_2 = -1.4754\,E$
$\xi_3 = -0.4989\,E$

Stronger noise:



$\xi_0 = 0.2187\,E$
$\xi_1 = -1.4575\,E$
$\xi_2 = -1.2447\,E$
$\xi_3 = 0.4575\,E$

$E_0 = E_1 = E_2 = E_3 = E$

$\Rightarrow$ wrong decision: $\hat{m}=3$

Example 4.4: 64-QAM receiver

*Assume that $\{z_\ell(t)\}_{\ell=0}^{M-1}$ is a 64-ary QAM signal constellation. Draw a block-diagram of a minimum Euclidean distance receiver that uses only **two** integrators.*

Solution:
A QAM signal alternative can be written as $z_i(t) = A_i g(t)\cos(\omega_c t) - B_i g(t)\sin(\omega_c t)$, where $g(t)$ is a baseband pulse. The output value from the i:th correlator in Figure 4.8 is,

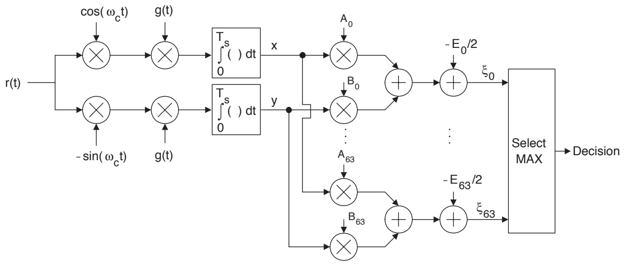
$$\begin{aligned}\int_0^{T_s} r(t)z_i(t)dt &= A_i \underbrace{\int_0^{T_s} r(t)g(t)\cos(\omega_c t)dt}_{x} - B_i \underbrace{\int_0^{T_s} r(t)g(t)\sin(\omega_c t)dt}_{-y} = \\ &= A_i x + B_i y\end{aligned}$$

Observe that x and y do not depend on the index i.
*Hence, a possible implementation of the receiver is to **first** generate x and y, and then calculate the M correlations $A_i x + B_i y$, $i = 0,\ i,\ldots, M-1$. By subtracting the value $E_i/2$ from the i:th correlation, the decision variables $\xi_0,\ldots,\xi_{M-1}$ are finally obtained.*

For M-ary constellations with fixed pulse shape $g(t)$ the implementation can be further simplified

Example 4.4: 64-QAM receiver

The implementation of this receiver is shown below:



The complexity of this receiver is significantly reduced compared to the receiver in Figure 4.8 on page 241! Only two integrators are here used, instead of 64 (= M) in Figure 4.8.

- pulse shape and carrier waveform are recreated at the receiver
  $\Rightarrow$ these parts are very similar to the transmitter
- integration and comparison can be performed separately

A geometric interpretation

- Our receiver computes: (maximum correlation)

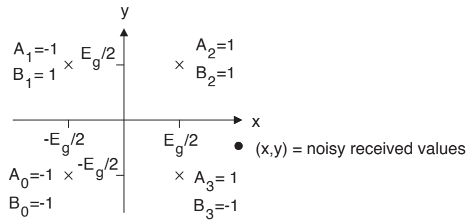
$$\max_i \{x A_i + y B_i - E_g/2\}$$

- Equivalently we can compute: (minimum Euclidean distance)

$$\min_i \left\{ \left(x - \frac{A_i E_g}{2}\right)^2 + \left(y - \frac{B_i E_g}{2}\right)^2 \right\}$$

Ex. QPSK: received point (x,y) is closest to the point of message m_3

x = message points, • = noisy received values (x, y)



Matched filter implementation

- A filter with impulse response $q(t)$ is matched to a signal $z_i(t)$ if

$$q(t) = z_i(-t + T_s) = z_i(-(t - T_s))$$

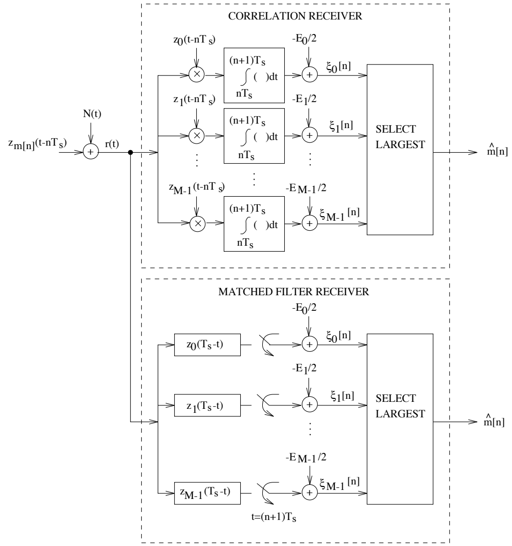
- Let the received signal $r(t)$ enter this matched filter $q(t)$
- The matched filter output, evaluated at time $t = (n+1)T_s$, can be written as

$$r(t) * q(t)\big|_{t=(n+1)T_s} = \int_{nT_s}^{(n+1)T_s} r(\tau)\, z_i(\tau - nT_s)\, d\tau$$

- **Observe:**
  this is exactly the same output value as the correlator produces

$\Rightarrow$ We can replace each correlator with a matched filter which is sampled at times $t = (n+1)T_s$

Matched filter vs correlator implementation



Summary: receiver types

- **Minimum Euclidean distance (MED) receiver:**
  decision is based on the signal alternative $z_i(t)$ closest to $r(t)$
- **Correlation receiver:**
  an implementation of the MED receiver based on correlators
- **Matched filter receiver:**
  an implementation of the MED receiver based on matched filters
- **Maximum likelihood (ML) receiver:**
  equivalent to MED receiver under our assumptions: ML = ED
- **Maximum a-posteriori (MAP) receiver:**
  minimizes symbol error probability P_s
  equivalent to ML if $P_i = 1/M$, $i = 0, \ldots, M-1$: ML = ED = MAP

Bit error probability

- Because of the noise the receiver will sometimes make errors
- During a time interval τ we transmit the sequence $\mathbf{b}$ of length

$$B = R_b \, \tau$$

- The detected (estimated) sequence $\hat{\mathbf{b}}$ will contain B_{err} bit errors

$$B_{err} = d_H(\mathbf{b}, \hat{\mathbf{b}}) \le B$$

- The Hamming distance $d_H(\mathbf{b}, \hat{\mathbf{b}})$ is defined as the number of positions in which the sequences are different
- The bit error probability P_b is defined as

$$P_b = \frac{1}{B} \sum_{i=1}^{B} Pr\{\hat{b}[i] \neq b[i]\} = \frac{E\{d_H(\mathbf{b}, \hat{\mathbf{b}})\}}{B}$$

- It measures the average number of bit errors per detected (estimated) information bit

Analysis Binary Signaling

- Binary signaling ($M = 2$, $T_s = T_b$) simplifies the general receiver
- Consider the two decision variables

$$\xi_i[n] = \int_{nT_s}^{(n+1)T_s} r(t)\, z_i(t - nT_s)\, dt - E_i/2 \,, \quad i = 0, 1$$

- The decision $\hat{m}[n]$ is made according to the larger value, i.e.,

$$\xi_1[n] \underset{\hat{m}[n]=m_0}{\overset{\hat{m}[n]=m_1}{\gtrless}} \xi_0[n]$$

- This can be reduced to a single decision variable only

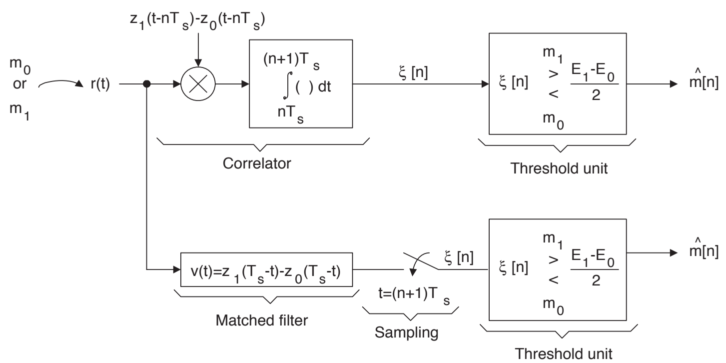
$$\xi[n] = \int_{nT_s}^{(n+1)T_s} r(t) \left(z_1(t - nT_s) - z_0(t - nT_s) \right) dt$$

which is compared to a threshold value

$$\xi[n] \underset{\hat{m}[n]=m_0}{\overset{\hat{m}[n]=m_1}{\gtrless}} \frac{E_1 - E_0}{2}$$

Receiver for Binary Signaling

- Only one correlator or one matched filter is now required:



- Matched filter output needs be sampled at correct time

When do we make a wrong decision?

- Assuming $m = m_0$ is sent, the decision variable becomes

$$\xi[n] = \int_0^{T_s} r(t)\big(z_1(t) - z_0(t)\big)\ dt = \int_0^{T_s} \big(z_0(t) + N(t)\big) \cdot \big(z_1(t) - z_0(t)\big)\ dt$$

- We can divide this into a signal component β_0 and a noise component $\mathcal{N}$

$$\xi[n] = \beta_0 + \mathcal{N}$$

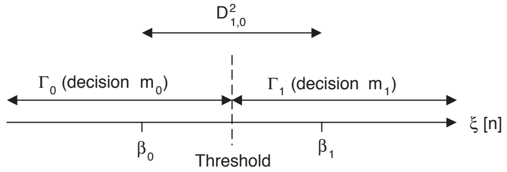
$$\beta_0 = \int_0^{T_s} z_0(t)\,\big(z_1(t) - z_0(t)\big)\ dt\ , \quad \mathcal{N} = \int_0^{T_s} N(t)\,\big(z_1(t) - z_0(t)\big)\ dt$$

- **Wrong decision:** if $\xi[n] > (E_1 - E_0)/2$ then $\hat{m} = m_1 \neq m_0 = m$
- Analogously, when $m = m_1$ is sent we get

$$\xi[n] = \beta_1 + \mathcal{N}$$

$$\beta_1 = \int_0^{T_s} z_1(t)\,\big(z_1(t) - z_0(t)\big)\ dt$$

Decision regions



- With

$$\beta_0 + \beta_1 = -\int_0^{T_s} z_0^2(t)\ dt + \int_0^{T_s} z_1^2(t)\ dt = E_1 - E_0$$

the decision threshold lies in the center between β_0 and β_1:

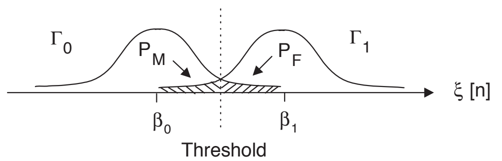
$$\frac{E_1 - E_0}{2} = \frac{\beta_0 + \beta_1}{2}$$

- Furthermore we see that

$$\beta_1 - \beta_0 = \int_0^{T_s} \big(z_1(t) - z_0(t)\big)^2\ dt = D^2_{1,0} = D^2_{0,1}$$

Probability of a wrong decision

- There exist two ways to make an error:



P_F: false alarm probability $\quad$ P_M: missed detection probability

- The two probabilities of error can be determined as

$$P_F = Pr\{\hat{m}[n] = m_1 | m = m_0\} = Pr\{\beta_0 + \mathcal{N} > (\beta_0 + \beta_1)/2\}$$
$$P_M = Pr\{\hat{m}[n] = m_0 | m = m_1\} = Pr\{\beta_1 + \mathcal{N} < (\beta_0 + \beta_1)/2\}$$

- We can express these in terms of the $Q(x)$-function:

$$P_F = P_M = Q\left(\frac{\beta_1 - \beta_0}{2\sigma}\right)$$

Gaussian Noise

- The noise component $\mathcal{N}$ is a Gaussian random variable with

$$p(\mathcal{N}) = \frac{1}{\sqrt{2\pi\ \sigma^2}}\ e^{-(\mathcal{N}-m)^2/2\sigma^2}$$

with mean $m = 0$ and variance $\sigma^2 = N_0/2\ E_v$

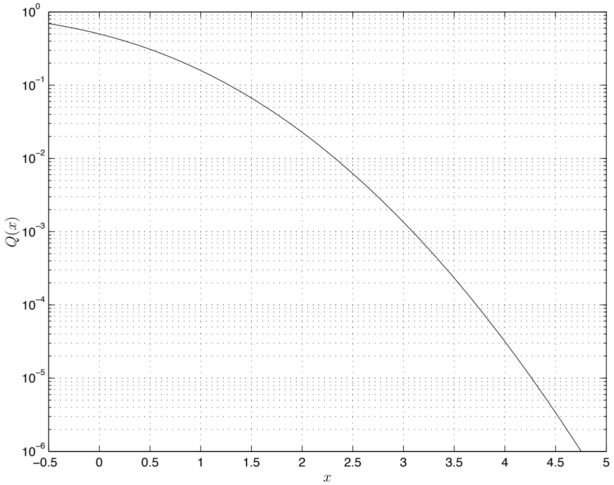
- Our bit error probability is related to the probability that the noise value $\mathcal{N}$ is larger than some threshold A

$$Pr\{\mathcal{N} \geq A\} = Pr\left\{\frac{\mathcal{N} - m}{\sigma} \geq \frac{A - m}{\sigma}\right\} = Q\left(\frac{A - m}{\sigma}\right)$$

- The $Q(x)$-function is defined as

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}}\ e^{-y^2/2}\ dy = \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

The $Q(x)$-function



The $Q(x)$-function (page 182)

x	$Q(x)$	x	$Q(x)$	x	$Q(x)$	x	$Q(x)$
0.0	5.0000e-01	3.0	1.3499e-03	6.0	9.8659e-10	9.0	1.1286e-19
0.1	4.6017e-01	3.1	9.6760e-04	6.1	5.3034e-10	9.1	4.5166e-20
0.2	4.2074e-01	3.2	6.8714e-04	6.2	2.8232e-10	9.2	1.7897e-20
0.3	3.8209e-01	3.3	4.8342e-04	6.3	1.4882e-10	9.3	7.0223e-21
0.4	3.4458e-01	3.4	3.3693e-04	6.4	7.7688e-11	9.4	2.7282e-21
0.5	3.0854e-01	3.5	2.3263e-04	6.5	4.0160e-11	9.5	1.0495e-21
0.6	2.7425e-01	3.6	1.5911e-04	6.6	2.0558e-11	9.6	3.9972e-22
0.7	2.4196e-01	3.7	1.0780e-04	6.7	1.0421e-11	9.7	1.5075e-22
0.8	2.1186e-01	3.8	7.2348e-05	6.8	5.2310e-12	9.8	5.6293e-23
0.9	1.8406e-01	3.9	4.8096e-05	6.9	2.6001e-12	9.9	2.0814e-23
1.0	1.5866e-01	4.0	3.1671e-05	7.0	1.2798e-12	10.0	7.6199e-24
1.1	1.3567e-01	4.1	2.0658e-05	7.1	6.2378e-13		
1.2	1.1507e-01	4.2	1.3346e-05	7.2	3.0106e-13		
1.3	9.6800e-02	4.3	8.5399e-06	7.3	1.4388e-13		
1.4	8.0757e-02	4.4	5.4125e-06	7.4	6.8092e-14		
1.5	6.6807e-02	4.5	3.3977e-06	7.5	3.1909e-14		
1.6	5.4799e-02	4.6	2.1125e-06	7.6	1.4807e-14		
1.7	4.4565e-02	4.7	1.3008e-06	7.7	6.8033e-15		
1.8	3.5930e-02	4.8	7.9333e-07	7.8	3.0954e-15		
1.9	2.8717e-02	4.9	4.7918e-07	7.9	1.3945e-15		
2.0	2.2750e-02	5.0	2.8665e-07	8.0	6.2210e-16		
2.1	1.7864e-02	5.1	1.6983e-07	8.1	2.7480e-16		
2.2	1.3903e-02	5.2	9.9644e-08	8.2	1.2019e-16		
2.3	1.0724e-02	5.3	5.7901e-08	8.3	5.2056e-17		
2.4	8.1975e-03	5.4	3.3320e-08	8.4	2.2324e-17		
2.5	6.2097e-03	5.5	1.8990e-08	8.5	9.4795e-18		
2.6	4.6612e-03	5.6	1.0718e-08	8.6	3.9858e-18		
2.7	3.4670e-03	5.7	5.9904e-09	8.7	1.6594e-18		
2.8	2.5551e-03	5.8	3.3157e-09	8.8	6.8408e-19		
2.9	1.8658e-03	5.9	1.8175e-09	8.9	2.7923e-19		

$Q(1.2816) \approx 10^{-1}$	$Q(5.1993) \approx 10^{-7}$
$Q(2.3263) \approx 10^{-2}$	$Q(5.6120) \approx 10^{-8}$
$Q(3.0902) \approx 10^{-3}$	$Q(5.9978) \approx 10^{-9}$
$Q(3.7190) \approx 10^{-4}$	$Q(6.3613) \approx 10^{-10}$
$Q(4.2649) \approx 10^{-5}$	$Q(6.7060) \approx 10^{-11}$
$Q(4.7534) \approx 10^{-6}$	$Q(7.0345) \approx 10^{-12}$

Bit error probability

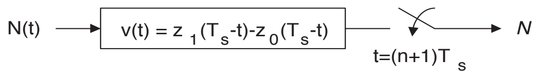
- The bit error probability can be written as

$$P_b = P_0 P_F + P_1 P_M = (P_0 + P_1) P_F = P_F = P_M$$

- With $\beta_1 - \beta_0 = D_{0,1}^2$ and $\sigma^2 = N_0/2 \cdot D_{0,1}^2$ we obtain

$$P_b = Q\left(\frac{\beta_1 - \beta_0}{2\sigma}\right) = Q\left(\frac{D_{0,1}^2}{2\sigma}\right) = Q\left(\sqrt{\frac{D_{0,1}^2}{2N_0}}\right)$$

- This fundamental result provides the bit error probability P_b of an ML receiver for binary transmission over an AWGN channel
- The additive noise $\mathcal{N}$ is sampled from a filtered noise process



$$\sigma^2 = N_0/2 \cdot E_v = N_0/2 \int_0^{T_s} \left(z_1(t) - z_0(t)\right)^2 \, dt$$

Example

- Let $z_0(t) = 0$ and $z_1(t)$ rectangular with amplitude A and $T = T_b$
- The information bit rate is $R_b = 400$ kbps
- Regarding the noise we know that $A^2/N_0 = 70$ dB

Task: determine the bit error probability P_b

Solution:

- First we find that $D_{0,1}^2 = A^2/R_b$
- Then

$$\frac{D_{0,1}^2}{2N_0} = \frac{A^2}{N_0} \cdot \frac{1}{2R_b} = 12.5$$

- $P_b = Q\left(\sqrt{12.5}\right) = Q(3.536) = 2.3 \cdot 10^{-4}$
- Last step: check Table 3.1 on page 182

An energy efficiency perspective

- Consider the case $P_0 = P_1 = 1/2$
- The average received energy per bit is then

$$\mathcal{E}_b = \frac{1}{2}\int_0^{T_b} z_0^2(t)\ dt\ +\ \frac{1}{2}\int_0^{T_b} z_1^2(t)\ dt = \frac{E_0 + E_1}{2}$$

- We can then introduce the normalized squared Euclidean distance

$$d_{0,1}^2 = \frac{D_{0,1}^2}{2\,\mathcal{E}_b} = \frac{1}{2\,\mathcal{E}_b}\int_0^{T_b} \big(z_1(t) - z_0(t)\big)^2\ dt$$

- With this the bit error probability becomes

$$P_b = Q\left(\sqrt{\frac{D_{0,1}^2}{2\,N_0}}\right) = Q\left(\sqrt{d_{0,1}^2\,\frac{\mathcal{E}_b}{N_0}}\right)$$

- The parameter $d_{0,1}^2$ is a measure of energy efficiency