EITG05 – Digital Communications

Lecture 2

Signal Constellations (p. 31–55)

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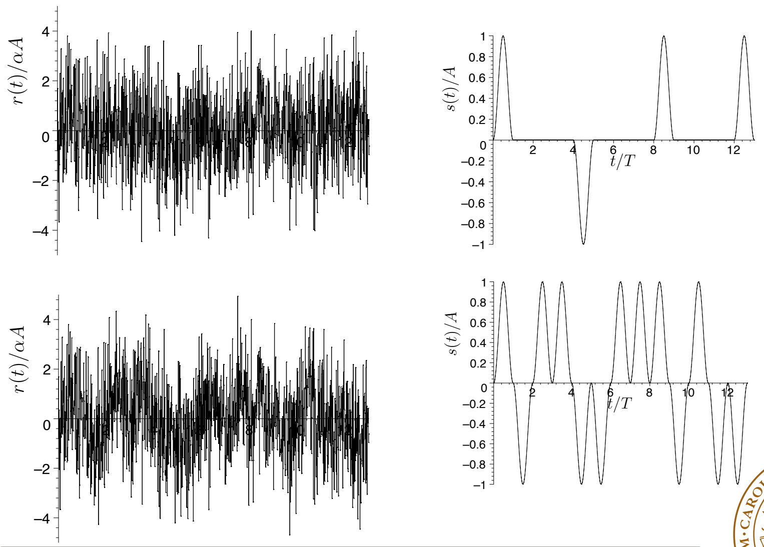
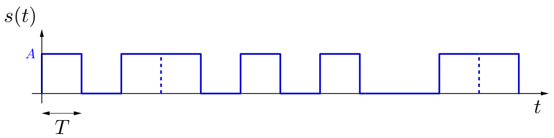
Example: noisy signal at the receiver (p. 13)



Euclidean distance example $M = 2$

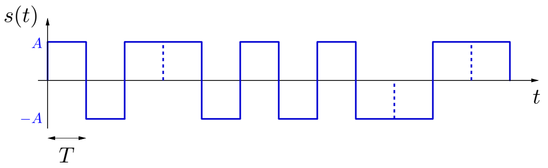
Case 1: on-off signaling



$s_0(t) = A$ and $s_1(t) = 0$ for $0 < t < T_s = T$, which gives $D^2_{0,1} = 2\overline{E}_b$

Observe: on-off signaling is orthogonal

Case 2: antipodal signaling



$s_0(t) = A$ and $s_1(t) = -A$ for $0 < t < T_s = T$, and $D^2_{0,1} = 4\overline{E}_b$

How well can we distinguish two signals?

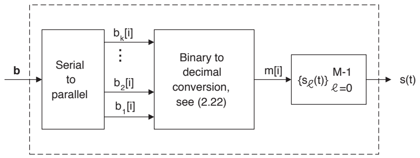
- The squared Euclidean distance between two signals $s_i(t)$ and $s_j(t)$ is defined as

$$D^2_{i,j} = \int_0^{T_s} (s_i(t) - s_j(t))^2\, dt$$
$$= \int_0^{T_s} s_i^2(t) + s_j^2(t) - 2\, s_i(t) s_j(t)\, dt$$
$$= E_i + E_j - 2 \int_0^{T_s} s_i(t) s_j(t)\, dt$$

- The symbol energy E_ℓ of a signal alternative $s_\ell(t)$ is given by

$$E_\ell = \int_0^{T_s} s_\ell^2(t)\, dt < \infty\ , \quad \ell = 0, 1, \ldots, M-1$$

Signal constellations



- In case of M-ary signaling, one of $M = 2^k$ messages $m[i]$ is transmitted by its corresponding signal alternative

$$s_\ell(t) \in \{s_0(t), s_1(t), \dots, s_{M-1}(t)\}$$

- The signal constellation is the set of possible signal alternatives
- The mapping defines which message is assigned to which signal
- When the message equals $m[i] = j$ then $s_j(t - i\,T_s)$ is sent

$$s(t) = s_{m[0]}(t) + s_{m[1]}(t - T_s) + s_{m[2]}(t - 2T_s) + \cdots$$

Question: how should we choose M distinguishable signals?

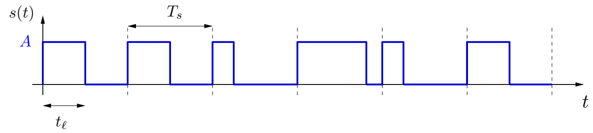
Pulse Width Modulation (PWM)

- In pulse width modulation the message modulates the duration T of a pulse $c(t)$ within the symbol interval T_s

$$s_\ell(t) = c\left(\frac{t}{t_\ell}\right) \;, \quad \ell = 0, 1, \dots, M-1$$

- The duration of the pulse $c(t)$ is equal to $T = 1$
- It follows that $s_\ell(t)$ is zero outside the interval $0 \le t \le t_\ell$
- It is assumed that $t_\ell < T_s$
- Average symbol energy: $\overline{E}_s = E_c\, \bar{t}_\ell$

Example:



Used in control applications, not much for data transmission (e.g., speed of CPU fan, LED intensity)

Pulse Position Modulation (PPM)

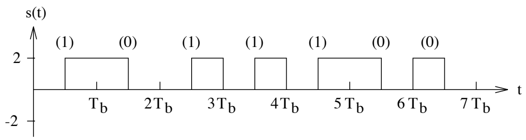
- In pulse position modulation the message modulates the position of a short pulse $c(t)$ within the symbol interval T_s

$$s_\ell(t) = c\left(t - \ell\,\frac{T_s}{M}\right) \;, \quad \ell = 0, 1, \dots, M-1$$

- The duration T of the pulse $c(t)$ has to satisfy $T \le T_s/M$
- The pulses are orthogonal and we get

$$\overline{E}_s = E_c \;, \qquad D^2_{i,j} = E_i + E_j = 2\,E_c$$

Example:



Used for low-power optical links (e.g. IR remote controls)

Pulse Amplitude Modulation (PAM)

- In pulse amplitude modulation the message is mapped into the amplitude only:

$$s_\ell(t) = A_\ell\, g(t) \;, \quad \ell = 0, 1, \dots, M-1$$

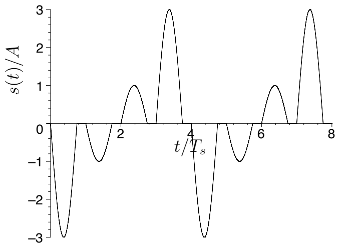
- PAM is a natural generalization of binary on-off signaling and antipodal signaling, which are special cases for $M = 2$

- A common choice are equidistant amplitudes located symmetrically around zero:

$$A_\ell = -M + 1 + 2\ell \;, \quad \ell = 0, 1, \dots, M-1$$

Example of 4-ary PAM

- **Example:** $M=4$, $A_0=-3$, $A_1=-1$, $A_2=+1$, $A_3=+3$



- The same constellation, defined by the amplitudes

$$\{A_\ell\}_{\ell=0}^{M-1} = \{\pm 1, \pm 3, \pm 5, \dots, \pm(M-1)\}$$

could also be used with other mappings

What is the message sequence $m[i]$?

Symbol Energy of PAM

- The symbol energy of a PAM signal is

$$E_\ell = \int_0^{T_s} s_\ell^2(t)\ dt = \int_0^{T_s} A_\ell^2\ g^2(t)\ dt$$

- Using

$$E_g = \int_0^{T_s} g^2(t)\ dt$$

we can write the average symbol energy as

$$\overline{E}_s = E_g \sum_{\ell=0}^{M-1} P_\ell\ A_\ell^2$$

- Often the messages are equally likely, i.e., $P_\ell = \frac{1}{M} = 2^{-k}$, and for the symmetric constellation from above we get

$$\overline{E}_s = E_g \frac{M^2-1}{3}\ .$$

Euclidean distances of PAM signals

- The squared Euclidean distance between two PAM signal alternatives is

$$D_{i,j}^2 = \int_0^{T_s} \left(s_i(t) - s_j(t)\right)^2\ dt = E_g\ (A_i - A_j)^2$$

- With $A_\ell = -M+1+2\ell$ this becomes

$$D_{i,j}^2 = 4E_g\ (i-j)^2$$

Compare this with Example 2.7 on page 28

- We will later see that the minimum Euclidean distance $\min_{i,j} D_{i,j}$ strongly influences the error probability of the receiver
- For this reason, equidistant constellations are often used

Bandpass Signals

- In many applications we want to transmit signals at high frequencies, centered around a carrier frequency f_c
- A typical bandpass signal has the form

$$s(t) = A(t) \cdot \cos\left(2\pi\, f(t)\ t\ +\ \varphi(t)\right)$$

- The general idea of carrier modulation techniques is to map the messages $m[i]$ to the different signal parameters:
  - **PAM**: amplitude $A(t)$
  - **PSK**: phase $\varphi(t)$
  - **FSK**: frequency $f(t)$
  - **QAM**: amplitude $A(t)$ and phase $\varphi(t)$
  - **OFDM**: amplitude $A(t)$, phase $\varphi(t)$, and frequency $f(t)$

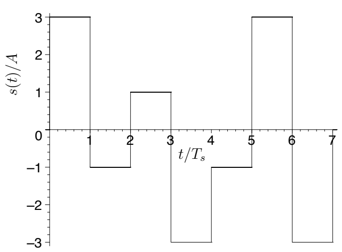
Remark:
analog modulation (AM or FM) changes the parameters by means of a continuous input signal

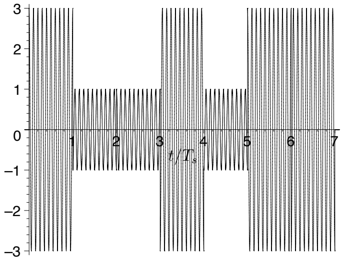
Bandpass M-ary PAM

- To modulate the pulse amplitude, we can multiply the original PAM signal $s(t)$ with a sinusoidal signal

$$s_{bp}(t) = s(t) \cdot \cos(2\pi f_c\, t) = \sum_{i=0}^{\infty} A_{m[i]}\; g(t - i\,T_s) \cdot \cos(2\pi f_c\, t)$$

Example:





Phase Shift Keying (PSK)

- We have seen that with PAM signaling the message modulates the amplitude A_ℓ of the signal $s_\ell(t)$
- The idea of phase shift keying signaling is to modulate instead the phase v_ℓ of $s_\ell(t)$

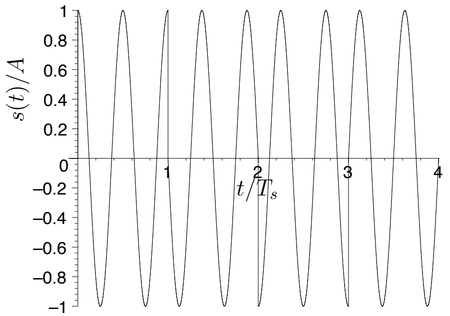
$$s_\ell(t) = g(t)\;\cos(2\pi f_c\, t \;+\; v_\ell)\;, \quad \ell = 0, 1, \dots, M-1\;,$$

- $M = 2$**:** binary PSK (BPSK) with $v_0 = 0$ and $v_1 = \pi$ is equivalent to binary PAM with $A_0 = +1$ and $A_1 = -1$
- $M = 4$**:** 4-ary PSK is also called quadrature PSK (QPSK)
- If we choose

$$f_c = n\, R_s$$

  for some positive integer n, then n full cycles of the carrier wave are contained within a symbol interval T_s

Example of QPSK



$$f_c = 2\,R_s\;, \quad v_0 = 0,\; v_1 = \pi/2,\; v_2 = \pi,\; \text{and}\; v_3 = 3\pi/2$$

What is the message sequence $m[i]$?

Symmetric M-ary PSK

- Normally, the phase alternatives are located symmetrically on a circle

$$v_\ell = \frac{2\pi\,\ell}{M} \;+\; v_{const}\;, \quad \ell = 0, 1, \dots, M-1\;,$$

  where v_{const} is a contant phase offset value

- If $P_\ell = \frac{1}{M}$, and $f_c \gg R_s$, then the average symbol energy is

$$\overline{E}_s = \frac{E_g}{2}$$

  and

$$D^2_{i,j} = E_g\;(1 - \cos(v_i - v_j))$$

- PSK has a constant symbol energy

Frequency Shift Keying (FSK)

- Instead of amplitude and phase, the message can modulate the frequency f_ℓ

$$s_\ell(t) = A\ \cos(2\pi\, f_\ell\ t\ +\ \nu)\ , \quad \ell = 0,1,\ldots,M-1$$

- Amplitude A and phase ν are constants
- In many applications the frequency alternatives f_ℓ are chosen such that the signals are orthogonal, i.e.,

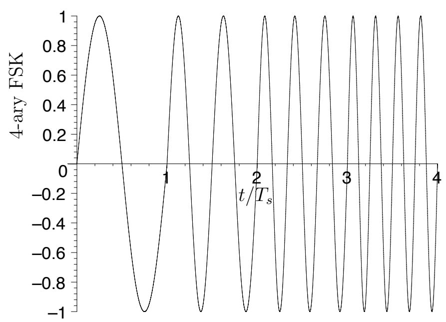
$$\int_0^{T_s} s_i(t)\ s_j(t)\ dt = 0\ , \quad i \neq j$$

- If $\nu = 0$ or $\nu = -\pi/2$ (often used), then we can choose

$$f_\ell = n_0 \frac{R_s}{2} + \ell\ I\ \frac{R_s}{2} \stackrel{\text{def}}{=} f_0 + \ell\, f_\Delta\ , \quad \ell = 0,1,\ldots,M-1\ ,$$

where n_0 and I are positive integers

Example of 4-ary FSK



$$\nu = -\frac{\pi}{2}\ , \quad f_0 = R_s,\ f_1 = 2R_s,\ f_2 = 3R_s,\ \text{and}\ f_3 = 4R_s$$

What is the message sequence $m[i]$?

Quadrature Amplitude Modulation (QAM)

- With QAM signaling the message modulates the amplitudes of two orthogonal signals (inphase and quadrature component)

$$s_\ell(t) = A_\ell\ g(t)\ \cos(2\pi\, f_c\ t) - B_\ell\ g(t)\ \sin(2\pi\, f_c\ t)\ , \quad \ell = 0,1,\ldots,M-1$$

- We can interpret $s_\ell(t)$ as the sum of two bandpass PAM signals
- **Motivation:** We can transmit two signals independently using the same carrier frequency and bandwidth

With QAM we can change both amplitude and phase

Quadrature Amplitude Modulation (QAM)

$$s_\ell(t) = A_\ell\ g(t)\ \cos(2\pi\, f_c\ t) - B_\ell\ g(t)\ \sin(2\pi\, f_c\ t)$$

- The signal $s_\ell(t)$ can also be expressed as

$$s_\ell(t) = g(t)\sqrt{A_\ell^2 + B_\ell^2}\ \cos(2\pi\, f_c\ t\ +\ \nu_\ell)$$

- It follows that QAM is a generalization of PSK: selecting $A_\ell^2 + B_\ell^2 = 1$ we can put the information into ν_ℓ and get

$$A_\ell = \cos(\nu_\ell)\ , \quad B_\ell = \sin(\nu_\ell)$$

Energy and Distance of M-ary QAM

- Choosing $f_c \gg R_s$ it can be shown that

$$E_\ell = \left(A_\ell^2 + B_\ell^2\right) \frac{E_g}{2}$$

$$D_{i,j}^2 = \left((A_i - A_j)^2 + (B_i - B_j)^2\right) \frac{E_g}{2}$$

- A common choice are equidistant amplitudes located symmetrically around zero: (two $\sqrt{M}$-ary PAM with $k/2$ bits each)

$$\{A_\ell\}_{\ell=0}^{\sqrt{M}-1} = \{B_\ell\}_{\ell=0}^{\sqrt{M}-1} = \left\{\pm 1, \pm 3, \pm 5, \dots, \pm\left(\sqrt{M}-1\right)\right\}$$

- For equally likely messages $P_\ell = \frac{1}{M}$, this results in the average energy

$$\overline{E}_s = \sum_{\ell=0}^{M-1} \frac{1}{M} E_\ell = \frac{2(M-1)}{3} \frac{E_g}{2}$$

Geometric interpretation

- It is possible to describe QAM signals as two-dimensional vectors in a so-called signal space
- For this the signal

$$s_\ell(t) = A_\ell \; g(t) \; \cos(2\pi f_c \; t) - B_\ell \; g(t) \; \sin(2\pi f_c \; t)$$

is written as

$$s_\ell(t) = s_{\ell,1} \; \phi_1(t) + s_{\ell,2} \; \phi_2(t)$$

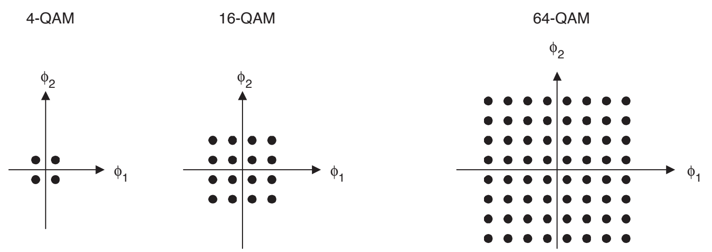
- Here $s_{\ell,1} = A_\ell \sqrt{E_g/2}$ and $s_{\ell,2} = B_\ell \sqrt{E_g/2}$ are the coordinates
- The functions $\phi_1(t)$ and $\phi_2(t)$ form an orthonormal basis of a vector space that spans all possible transmit signals:

$$\phi_1(t) = \frac{g(t) \; \cos(2\pi f_c \; t)}{\sqrt{E_g/2}} \; , \quad \phi_2(t) = -\frac{g(t) \; \sin(2\pi f_c \; t)}{\sqrt{E_g/2}}$$

This looks abstract, but can be very useful!

Signal space representation of QAM

- Now we can describe each signal alternative $s_\ell(t)$ as a point with coordinates $(s_{\ell,1}, s_{\ell,2})$ within a constellation diagram

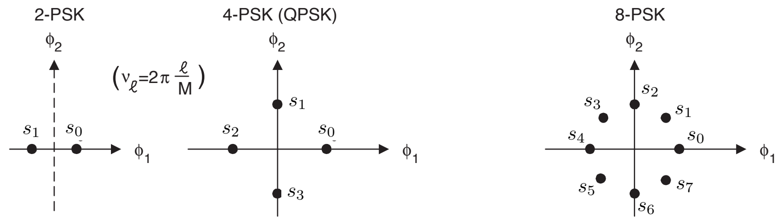


$$s_{\ell,1} = A_\ell \sqrt{E_g/2} \; , \quad s_{\ell,2} = B_\ell \sqrt{E_g/2}$$

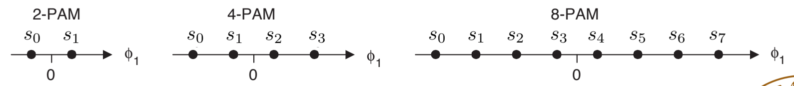
- The signal energy E_ℓ and the Euclidean distance $D_{i,j}^2$ can be determined in the signal space

Signal space representation of PSK and PAM

- PSK and PAM can be seen as a special cases of QAM:



$$s_{\ell,1} = \cos(\nu_\ell) \sqrt{E_g/2} \; , \quad s_{\ell,2} = \sin(\nu_\ell) \sqrt{E_g/2}$$



$$s_{\ell,1} = (-M + 1 + 2\,\ell) \; \sqrt{E_g}$$

Multitone Signaling: OFDM

- With FSK signaling, orthogonal signal alternatives are transmitted at different frequencies
- Disadvantage: only one frequency can be used at the same time
- **Orthogonal Frequency Division Multiplexing (OFDM)**: use QAM at N orthogonal frequencies and transmit the sum
- OFDM is widely used in modern communication systems: WLAN, LTE, DAB (radio), DVB (TV), DSL

Example:

$N = 4096$
64-ary QAM at each frequency (carrier)

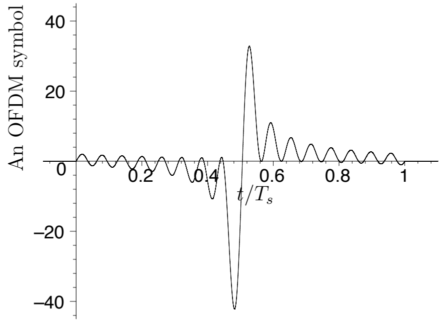
Then an OFDM signal carries $4096 \cdot 6 = 24\,576$ bits

How does a typical OFDM signal look like?

How can such a system be realized in practice?
$\Rightarrow$ OFDM will be explained in detail in the advanced course

Example of an OFDM symbol

$N = 16$, 16-ary QAM in each subcarrier (p. 52)



$$x(t) = \sum_{n=0}^{N-1} \left(a_I[n]\; g(t)\; \cos(2\pi f_n\, t) - a_Q[n]\; g(t)\; \sin(2\pi f_n\, t) \right) , \quad 0 \le t \le T_s$$

In this example the symbol $x(t)$ carries $16 \cdot 4 = 64$ bits