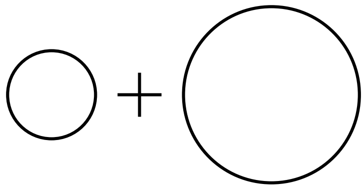
Alternative solution of problem 2.4

This solution will not use Gauss' law. To start off, we remember that the potential obeys superposition and as such, the potential everywhere can be computed as the sum of the contributions of two spheres.



One sphere is held at potential 0 and the other at V_0. Consider,

$$V(\boldsymbol{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathrm{V}} \frac{\rho}{|\boldsymbol{r} - \boldsymbol{r}'|} \,\mathrm{d}v' \tag{1}$$

From this equation we see that charges somewhere in our system is needed in order to have a non-zero potential. In the region we have only two spheres and vacuum, thus any charges must reside on the surfaces of the spheres. For now we assume that the inner sphere, of radius a, have a total charge of Q_1 and consequently a charge distribution of $\sigma_1 = Q_1/4\pi a^2$. Likewise for the larger sphere, $r = 2a$, we have a charge Q_2 and a charge distribution of $\sigma_2 = Q_2/8\pi a^2$. Now let us compute the potential from an arbitrary sphere of radius b and surface charge distribution σ. To do this we consider a ring on the sphere, as illustrated in the figure below.

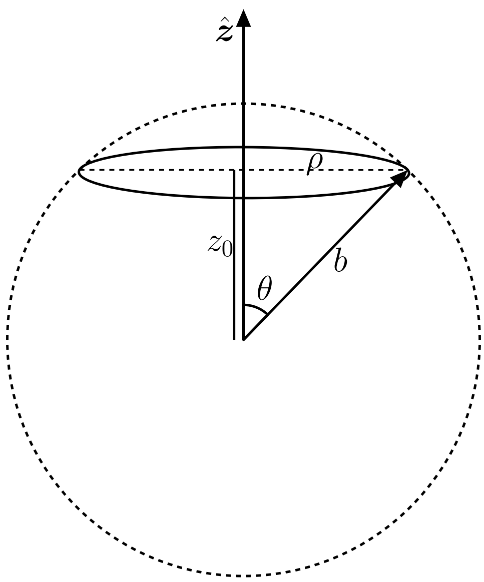


Figure 1: Geometry of a ring situated on a sphere.

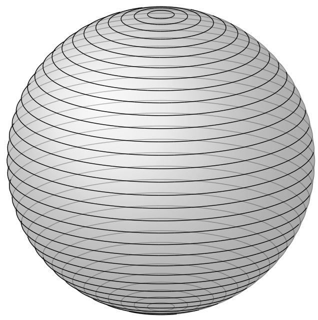
The ring of interest is shown by the solid line and has a radius of ρ and is situated at height z_0. The potential along the symmetry axis of the ring is then given by,

$$V_{\mathrm{ring}}(z) = \frac{1}{4\pi\epsilon_0} \int \frac{\sigma}{|\boldsymbol{r} - \boldsymbol{r}'|} \,\mathrm{d}l \tag{2}$$

Our observation point, $\boldsymbol{r}$, is any point along the z-axis, $\boldsymbol{r} = z\hat{\boldsymbol{z}}$, and the source positions is any point on the circle at height z_0, $\boldsymbol{r}' = \rho\hat{\boldsymbol{r}} + z_0\hat{\boldsymbol{z}}$.

$$V_{\text{ring}}(z) = \frac{1}{4\pi\epsilon_0}\int \frac{\sigma}{|\boldsymbol{r}-\boldsymbol{r}'|}\,\mathrm{d}l = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\sigma}{\sqrt{\rho^2+(z-z_0)^2}}\rho\,\mathrm{d}\phi' = \frac{\sigma}{2\epsilon_0}\frac{\rho}{\sqrt{\rho^2+(z-z_0)^2}} \tag{3}$$

Now a full sphere can be seen as the sum of many rings. By integrating over θ in spherical coordinates we can obtain the potential for the full sphere. During this integration the radius of the circles will vary according to $\rho = b\sin\theta$ and the position in z as $z_0 = b\cos\theta$, see Fig. 1. Along the z-axis we have that the potential of the full sphere becomes,



$$V_{\text{sphere}}(z) = \int_0^{\pi} V_{\text{ring}}(z)\sin\theta\,\mathrm{d}\theta = \frac{\sigma}{2\epsilon_0}\int_0^{\pi}\frac{b^2\sin\theta}{\sqrt{b^2\sin^2\theta+(z-b\cos\theta)^2}}\,\mathrm{d}\theta \tag{4}$$

Consider the variable substitution $u = \cos\theta$ ($\mathrm{d}u = -\sin\theta\,\mathrm{d}\theta$).

$$V_{\text{sphere}}(z) = \frac{\sigma}{2\epsilon_0}\int_{-1}^{1}\frac{b^2}{\sqrt{b^2(1-u^2)+(z-bu)^2}}\,\mathrm{d}u = \frac{\sigma}{2\epsilon_0}\int_{-1}^{1}\frac{b^2}{\sqrt{b^2-2zbu+z^2}}\,\mathrm{d}u \tag{5}$$

Now another variable substitution, $s = b^2 - 2buz + z^2$ ($\mathrm{d}s = -2bz\,\mathrm{d}u$).

$$V_{\text{sphere}}(z) = \frac{b\sigma}{4z\epsilon_0}\int_{\alpha}^{\beta}\frac{1}{\sqrt{s}}\,\mathrm{d}s \tag{6}$$

where $\alpha = b^2 - 2bz + z^2$ and $\beta = b^2 + 2bz + z^2$. Solving the integral, we obtain,

$$V_{\text{sphere}}(z) = \frac{\sigma b}{2\epsilon_0 z}\left(|b+z| - |b-z|\right) \tag{7}$$

Alternatively written as,

$$V_{\text{sphere}}(z) = \begin{cases} \dfrac{\sigma b}{\epsilon_0}, & z \le b \\[2ex] \dfrac{\sigma b^2}{\epsilon_0 z}, & z > b \end{cases} \tag{8}$$

Due to the symmetries of the sphere, this is valid for any chosen axis not just for z. Thus we can replace z with r to get the potential at any position.

$$V_{\text{sphere}}(r) = \begin{cases} \dfrac{Q}{4\pi\epsilon_0 b}, & r \le b \\[2ex] \dfrac{Q}{4\pi\epsilon_0 r}, & r > b \end{cases} \tag{9}$$

Now we can use superposition to obtain the potential in this specific problem.

$$V_{\text{tot}}(r) = V_1(r) + V_2(r) = \begin{cases} \dfrac{Q_1}{4\pi\epsilon_0 a} + \dfrac{Q_2}{8\pi\epsilon_0 a}, & r \leq a \\ \dfrac{Q_1}{4\pi\epsilon_0 r} + \dfrac{Q_2}{8\pi\epsilon_0 a}, & a < r \leq 2a \\ \dfrac{Q_1}{4\pi\epsilon_0 r} + \dfrac{Q_2}{4\pi\epsilon_0 r}, & r > 2a \end{cases} \tag{10}$$

The potential is known at two positions $V_{\text{tot}}(a) = V_0$ and $V_{\text{tot}}(2a) = 0$,

$$V_{\text{tot}}(2a) = 0 \quad \Rightarrow \quad Q_1 + Q_2 = 0 \quad \Rightarrow \quad Q_1 = -Q_2 \tag{11}$$

$$V_{\text{tot}}(a) = V_0 \quad \Rightarrow \quad \frac{Q_1}{4\pi\epsilon_0 a} - \frac{Q_1}{8\pi\epsilon_0 a} = V_0 \quad \Rightarrow \quad Q_1 = 8\pi\epsilon_0 a V_0 \tag{12}$$

Putting it all together, we get the result at last,

$$V_{\text{tot}}(r) = \begin{cases} V_0, & r \leq a \\ V_0\left(\dfrac{2a}{r} - 1\right), & a < r \leq 2a \\ 0, & r > 2a \end{cases} \tag{13}$$