

Collection of Formulas

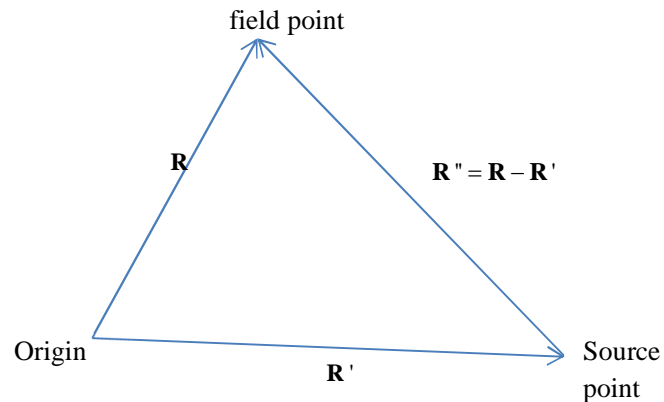
Electromagnetic Fields EITF80

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ELECTROSTATICS



Coulomb's Law

The force \mathbf{F} on a point charge q_1 at point \mathbf{R} , resulting from a point charge q at point \mathbf{R}'

$$\mathbf{F} = \frac{q_1 q}{4\pi\epsilon_0 R^2} \mathbf{a}_R, \text{ where } R = |\mathbf{R}''| = |\mathbf{R} - \mathbf{R}'| \text{ and } \mathbf{a}_R = \frac{\mathbf{R} - \mathbf{R}'}{|\mathbf{R} - \mathbf{R}'|}$$

Electric Field Intensity \mathbf{E} in Vacuum

From point charge q at \mathbf{R}' $\mathbf{E}(\mathbf{R}) = \frac{q}{4\pi\epsilon_0 R^2} \mathbf{a}_R$

From charge distribution $\mathbf{E}(\mathbf{R}) = \int \frac{1}{4\pi\epsilon_0 R^2} \mathbf{a}_R dq(\mathbf{R}')$

$$\text{where } dq(\mathbf{R}') = \begin{cases} \rho dv & \text{for volume charge density } \rho \\ \rho_s ds & \text{for surface charge density } \rho_s \\ \rho_l dl & \text{for line charge density } \rho_l \end{cases}$$

From point dipole $\mathbf{p} = p\mathbf{a}_z$ $\mathbf{E}(\mathbf{R}) = \frac{p}{4\pi\epsilon_0 R^3} (2\cos\theta \mathbf{a}_R + \sin\theta \mathbf{a}_\theta)$

From line charge (of density ρ_l) $\mathbf{E}(\mathbf{R}) = \frac{\rho_l}{2\pi\epsilon_0 r} \mathbf{a}_r$

Force \mathbf{F} on point charge q

$$\mathbf{F} = \begin{cases} q\mathbf{E} & \text{for electrostatics} \\ q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) & \text{for general case} \end{cases}$$

Electric Potential V

From point charge q at \mathbf{R}'
$$V(\mathbf{R}) = \frac{q}{4\pi\epsilon_0 R}$$

From charge distribution
$$V(\mathbf{R}) = \int \frac{1}{4\pi\epsilon_0 R} dq(\mathbf{R}')$$

From point dipole $\mathbf{p} = p\mathbf{a}_z$
$$V(\mathbf{R}) = \frac{\mathbf{p} \cdot \mathbf{a}_R}{4\pi\epsilon_0 R^2} = \frac{p \cos \theta}{4\pi\epsilon_0 R^2}$$

From line charge (of density ρ_l)
$$V(\mathbf{R}) = \frac{\rho_l}{2\pi\epsilon_0} \ln \frac{1}{r}$$

$$\mathbf{E} = -\nabla V \quad \text{for electrostatics}$$

Electric Flux Density \mathbf{D}

Gauss' Law
$$\oint \mathbf{D} \cdot \mathbf{a}_n ds = \int \rho dv$$

where \mathbf{a}_n is the outward unit vector normal to the surface of the volume.

Polarization \mathbf{P}
$$\mathbf{P} = \frac{d\mathbf{p}}{dv}$$

Surface polarization charge density
$$\rho_{ps} = \mathbf{P} \cdot \mathbf{a}_n$$

Volume polarization charge density
$$\rho_p = -\nabla \cdot \mathbf{P}$$

Relationship between \mathbf{P} , \mathbf{E} and \mathbf{D}
$$\begin{cases} \mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} & \text{for general case} \\ \mathbf{D} = \epsilon_r \epsilon_0 \mathbf{E} \end{cases}$$

Boundary Conditions
$$\begin{cases} E_t \text{ is continuous} \\ \rho_s = \mathbf{a}_{n2} \cdot (\mathbf{D}_1 - \mathbf{D}_2) \end{cases}$$

where ρ_s is the surface charge density of free charges, and \mathbf{a}_{n2} is the normal unit vector pointing from region 2 toward region 1.

STEADY ELECTRIC CURRENTS

Current density \mathbf{J} $I = \int \mathbf{J} \cdot \mathbf{a}_n \, ds$

Equation of Continuity (from principle of conservation of charge)

In differential form $\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$

In integral form $\oint \mathbf{J} \cdot \mathbf{a}_n \, ds = -\frac{dQ}{dt}$

Conductivity σ $\mathbf{J} = \sigma \mathbf{E}$

Total Power Dissipation \mathbf{P} $\mathbf{P} = \int \mathbf{J} \cdot \mathbf{E} \, dv$

Boundary Conditions $\left\{ \begin{array}{l} \mathbf{J}_s = \mathbf{a}_{n2} \cdot (\mathbf{J}_1 - \mathbf{J}_2) \quad (\text{for no surface current } \mathbf{J}_s = 0) \\ \mathbf{E}_t \text{ is continuous} \end{array} \right.$

Time Constant $\tau = RC = \frac{\epsilon_r \epsilon_0}{\sigma}$

Analogy between Electrostatics and Steady Electric Currents

$$\mathbf{E}, V \leftrightarrow \mathbf{E}, V$$

$$\mathbf{D} \leftrightarrow \mathbf{J}$$

$$\epsilon_r, \epsilon_0 \leftrightarrow \sigma$$

$$Q \leftrightarrow I$$

$$C \leftrightarrow G$$

MAGNETOSTATICS

Magnetic Flux Density \mathbf{B} in Vacuum

From point dipole $\mathbf{m} = m\mathbf{a}_z$ $\mathbf{B}(\mathbf{R}) = \frac{\mu_0 m}{4\pi R^3} (2\cos\theta\mathbf{a}_R + \sin\theta\mathbf{a}_\theta)$

From current density $\mathbf{J}(\mathbf{R}')$ $\mathbf{B}(\mathbf{R}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{R}') \times \mathbf{a}_R}{R^2} dv'$

From current path $\mathbf{B}(\mathbf{R}) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l}' \times \mathbf{a}_R}{R^2}$

From circular wire loop $\mathbf{B}(x=0, y=0, z) = \frac{\mu_0 I b^2}{2(b^2 + z^2)^{3/2}} \mathbf{a}_z$

From long straight wire path $\mathbf{B}(\mathbf{R}) = \frac{\mu_0 I}{2\pi r} \mathbf{a}_\phi$

Vector Magnetic Potential \mathbf{A} in Vacuum

From point dipole \mathbf{m} $\mathbf{A}(\mathbf{R}) = \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \mathbf{a}_R}{R^2}$

From current density $\mathbf{J}(\mathbf{R}')$ $\mathbf{A}(\mathbf{R}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{R}')}{R} dv'$

From current path $\mathbf{A}(\mathbf{R}) = \frac{\mu_0}{4\pi} \int \frac{I d\mathbf{l}'}{R}$

Magnetic Flux Φ $\Phi = \int \mathbf{B} \cdot \mathbf{a}_n ds = \oint \mathbf{A} \cdot d\mathbf{l}$

Flux Linkage Λ $\Lambda = N\Phi$

Self-inductance L and Mutual-inductance M

$$\Lambda_1 = L_1 I_1 + M I_2$$

$$\Lambda_2 = L_2 I_2 + M I_1$$

Magnetic Field Intensity \mathbf{H}

Ampere's Law

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int \mathbf{J} \cdot \mathbf{a}_n ds = I_{\text{enclosed}}$$

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Relationship between magnetization vector \mathbf{M} , \mathbf{B} and \mathbf{H}

$$\begin{cases} \mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M}) & \text{for general case} \\ \mathbf{B} = \mu_r \mu_0 \mathbf{H} \end{cases}$$

Boundary Conditions

$$\begin{cases} \mathbf{a}_{n2} \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_s \\ B_n \text{ is continuous} \end{cases}$$

where \mathbf{J}_s is the current density of free surface current, and \mathbf{a}_{n2} is the normal unit vector pointing from region 2 toward region 1.

Reluctance

$$R = \frac{l}{\mu_r \mu_0 S}$$

Magnetic Force

$$d\mathbf{F}_m = I d\mathbf{l} \times \mathbf{B}$$

Magnetic Moment \mathbf{m} for Current Loop

$$\mathbf{m} = \int I \mathbf{a}_n ds$$

Torque \mathbf{T}_m on Magnetic Dipole \mathbf{m}

$$\mathbf{T}_m = \mathbf{m} \times \mathbf{B}$$

Force on Magnetic Dipole \mathbf{m}

$$\mathbf{F} = (\mathbf{m} \cdot \nabla) \mathbf{B} + \mathbf{m} \times (\nabla \times \mathbf{B})$$

ELECTROMAGNETIC FIELD

Faraday's Law of Electromagnetic Induction

$$v = RI = -\frac{d\Phi}{dt}$$

Induced emf v

$$v = \oint (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}$$

$$v = -\frac{d\Lambda}{dt} \text{ for coils}$$

Faraday's Law of Electromagnetic Induction

In differential form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

In integral form

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{a}_n ds$$

Maxwell's Equations

$$\begin{cases} \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \\ \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} = \rho \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

Electromagnetic Constants

$$\mu_0 = 4\pi \cdot 10^{-7} \text{ H/m}, \quad \epsilon_0 \approx \frac{10^{-9}}{36\pi} \text{ F/m}, \quad c \approx 3 \cdot 10^8 \text{ m/s}$$

$$\frac{1}{\mu_0 \epsilon_0} = c^2, \quad \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}, \quad \eta_0 \approx 120\pi \Omega \approx 377 \Omega$$

Potentials

Relationship between vector magnetic potential \mathbf{A} and magnetic flux density \mathbf{B}

$$\mathbf{B} = \nabla \times \mathbf{A}$$

Relationship between scalar electric potential V , vector magnetic potential \mathbf{A} and electric field intensity \mathbf{E}

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

Poynting Vector

$$P(\mathbf{R}, t) = \mathbf{E}(\mathbf{R}, t) \times \mathbf{H}(\mathbf{R}, t)$$

$$P_{\text{av}}(\mathbf{R}) = \frac{1}{2} \text{Re} \left[\mathbf{E}(\mathbf{R}) \times \mathbf{H}^*(\mathbf{R}) \right] \text{ (time harmonic signal)}$$

TIME-HARMONIC FIELD

Plane Wave

$$\mathbf{E} = \hat{E} \cos(\omega t - \mathbf{k} \cdot \mathbf{R} + \phi) \mathbf{a}_E \quad \text{instantaneous value}$$

$$\mathbf{E} = E_0 e^{-jk \cdot \mathbf{R}} \mathbf{a}_E \quad \text{complex form}$$

$$E_0 = \hat{E} e^{j\phi} \quad \text{peak value}$$

$$E_0 = \frac{\hat{E}}{\sqrt{2}} e^{j\phi} \quad \text{root-mean-square (rms) value}$$

Group Velocity

$$v = \frac{1}{\sqrt{\mu_r \mu_0 \epsilon_r \epsilon_0}}, \quad v = \frac{\omega}{k}, \quad k = |\mathbf{k}|$$

Characteristic Impedance

$$\eta = \sqrt{\frac{\mu_r \mu_0}{\epsilon_r \epsilon_0}}$$

Right Hand Rule

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H, \quad E = \eta H, \quad \mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_B, \quad E = vB$$

VECTOR IDENTITIES

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

Divergence Theorem

$$\int_{\text{Volume}} \nabla \cdot \mathbf{A} \, dv = \oint_{\text{Surface}} \mathbf{A} \cdot \mathbf{a}_n \, ds$$

Stoke's Theorem

$$\int_{\text{Surface}} (\nabla \times \mathbf{A}) \cdot \mathbf{a}_n \, ds = \oint_{\text{Curve}} \mathbf{A} \cdot d\mathbf{l}$$

Null Identities

$$\nabla \times (\nabla V) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

COORDINATE SYSTEM

Cartesian Coordinates (x, y, z)

Position vector $\mathbf{R} = x \mathbf{a}_x + y \mathbf{a}_y + z \mathbf{a}_z$

Line element $d\mathbf{l} = dx \mathbf{a}_x + dy \mathbf{a}_y + dz \mathbf{a}_z$

Volume element $dv = dx dy dz$

Surface elements $ds_x = dy dz$

$$ds_y = dx dz$$

$$ds_z = dx dy$$

Differential operators $\nabla V = \mathbf{a}_x \frac{\partial V}{\partial x} + \mathbf{a}_y \frac{\partial V}{\partial y} + \mathbf{a}_z \frac{\partial V}{\partial z}$

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_x \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) + \mathbf{a}_y \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) + \mathbf{a}_z \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates (r, ϕ, z)

Position vector $\mathbf{R} = r \mathbf{a}_r + z \mathbf{a}_z$

Unit vectors $\mathbf{a}_r = \cos \phi \mathbf{a}_x + \sin \phi \mathbf{a}_y$

$$\mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$$

Line element $d\mathbf{l} = dr \mathbf{a}_r + r d\phi \mathbf{a}_\phi + dz \mathbf{a}_z$

Volume element $dv = r dr d\phi dz$

Surface elements $ds_r = r d\phi dz$

$$ds_\phi = dr dz$$

$$ds_z = r dr d\phi$$

Differential operators $\nabla V = \mathbf{a}_r \frac{\partial V}{\partial r} + \mathbf{a}_\phi \frac{1}{r} \frac{\partial V}{\partial \phi} + \mathbf{a}_z \frac{\partial V}{\partial z}$

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times \mathbf{A} = \mathbf{a}_r \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \mathbf{a}_\phi \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \mathbf{a}_z \frac{1}{r} \left[\frac{\partial}{\partial r} (r A_\phi) - \frac{\partial A_r}{\partial \phi} \right]$$

$$\nabla^2 V = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial V}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$$

Spherical Coordinates (R, θ, ϕ)

Position vector $\mathbf{R} = R \mathbf{a}_R$

Unit vectors $\mathbf{a}_R = \sin \theta \cos \phi \mathbf{a}_x + \sin \theta \sin \phi \mathbf{a}_y + \cos \theta \mathbf{a}_z$

$$\mathbf{a}_\theta = \cos \theta \cos \phi \mathbf{a}_x + \cos \theta \sin \phi \mathbf{a}_y - \sin \theta \mathbf{a}_z$$

$$\mathbf{a}_\phi = -\sin \phi \mathbf{a}_x + \cos \phi \mathbf{a}_y$$

Line element $d\mathbf{l} = dR \mathbf{a}_R + R d\theta \mathbf{a}_\theta + R \sin \theta d\phi \mathbf{a}_\phi$

Volume element $dv = R^2 \sin \theta dR d\theta d\phi$

Surface elements $ds_R = R^2 \sin \theta d\theta d\phi$

$$ds_\theta = R \sin \theta dR d\phi$$

$$ds_\phi = R dR d\theta$$

Differential operators $\nabla V = \mathbf{a}_R \frac{\partial V}{\partial R} + \mathbf{a}_\theta \frac{1}{R} \frac{\partial V}{\partial \theta} + \mathbf{a}_\phi \frac{1}{R \sin \theta} \frac{\partial V}{\partial \phi}$

$$\nabla \cdot \mathbf{A} = \frac{1}{R^2} \frac{\partial}{\partial R} (R^2 A_R) + \frac{1}{R \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{R \sin \theta} \frac{\partial A_\phi}{\partial \phi}$$

$$\begin{aligned} \nabla \times \mathbf{A} = & \mathbf{a}_R \frac{1}{R \sin \theta} \left(\frac{\partial}{\partial \theta} (A_\phi \sin \theta) - \frac{\partial A_\theta}{\partial \phi} \right) + \mathbf{a}_\theta \frac{1}{R} \left[\frac{1}{\sin \theta} \frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R} (R A_\phi) \right] \\ & + \mathbf{a}_\phi \frac{1}{R} \left(\frac{\partial}{\partial R} (R A_\theta) - \frac{\partial A_R}{\partial \theta} \right) \end{aligned}$$

$$\nabla^2 V = \frac{1}{R^2} \frac{\partial}{\partial R} \left(R^2 \frac{\partial V}{\partial R} \right) + \frac{1}{R^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{R^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2}$$

INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, n \neq -1$$

$$\int \frac{1}{x} dx = \ln x$$

$$\int \sqrt{x^2 + a^2} dx = \frac{1}{2} \left[x\sqrt{x^2 + a^2} + a^2 \ln(x + \sqrt{x^2 + a^2}) \right]$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

$$\int \frac{1}{(x^2 + a^2)^{3/2}} dx = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a}$$

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \arctan \frac{x}{a}$$

$$\int \frac{1}{\cos^2 x} dx = \tan x$$

$$\int \frac{1}{\sin x} dx = \ln \left| \tan \frac{x}{2} \right|$$

$$\int \ln x dx = x \ln x - x$$

TRIGONOMETRIC IDENTITIES

$$\cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos^2 \alpha = \frac{1 + \cos 2\alpha}{2}$$

$$\sin^2 \alpha = \frac{1 - \cos 2\alpha}{2}$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2}$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{\alpha + \beta}{2} \sin \frac{\alpha - \beta}{2}$$

$$\begin{cases} a \sin t + b \cos t = \sqrt{a^2 + b^2} \sin(t + \varphi) \\ \sin \varphi = \frac{b}{\sqrt{a^2 + b^2}}, \cos \varphi = \frac{a}{\sqrt{a^2 + b^2}} \end{cases}$$