EITF75 Systems and Signals

Lecture 9
Sampling and reconstruction

Fredrik Rusek

EITF75 Systems and Signals

An important part of digital signal processing:
Process time-continuous signals digitally

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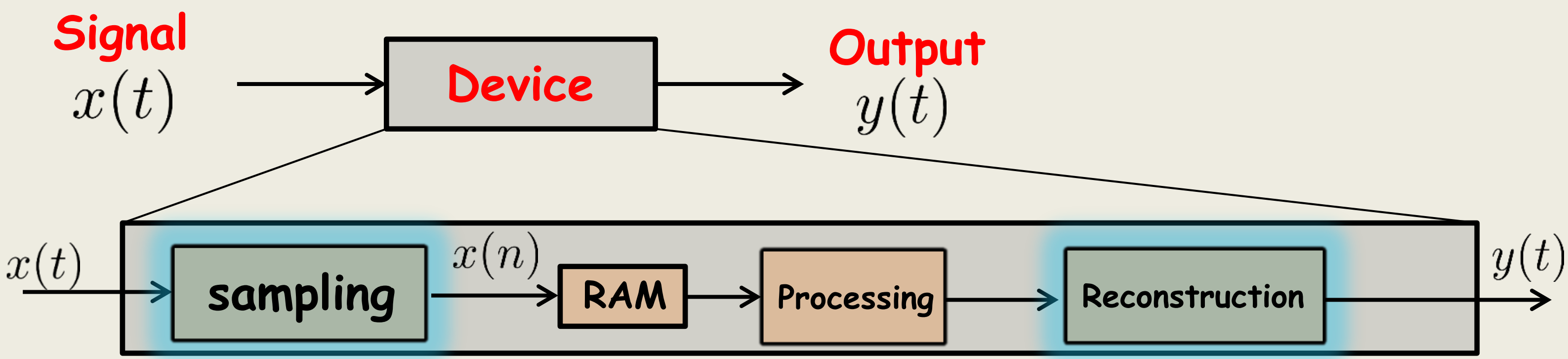

An important part of digital signal processing:
Process time-continuous signals digitally

To do so, we need

1. Study effects of sampling (A/D)
2. Study ways to implement (D/A)
3. Understand when 1&2 are optimally done

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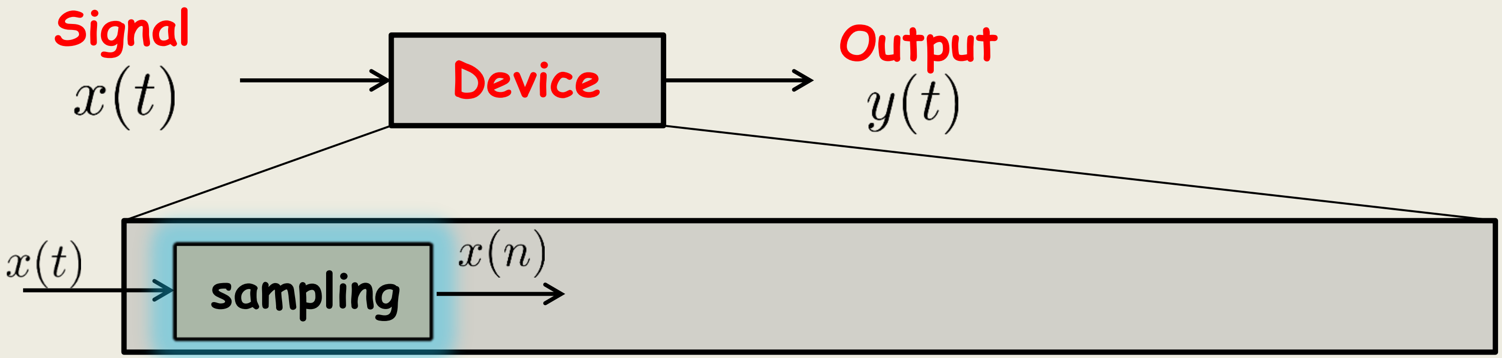

When is the sampling unit optimal?

(optimal means lossless)

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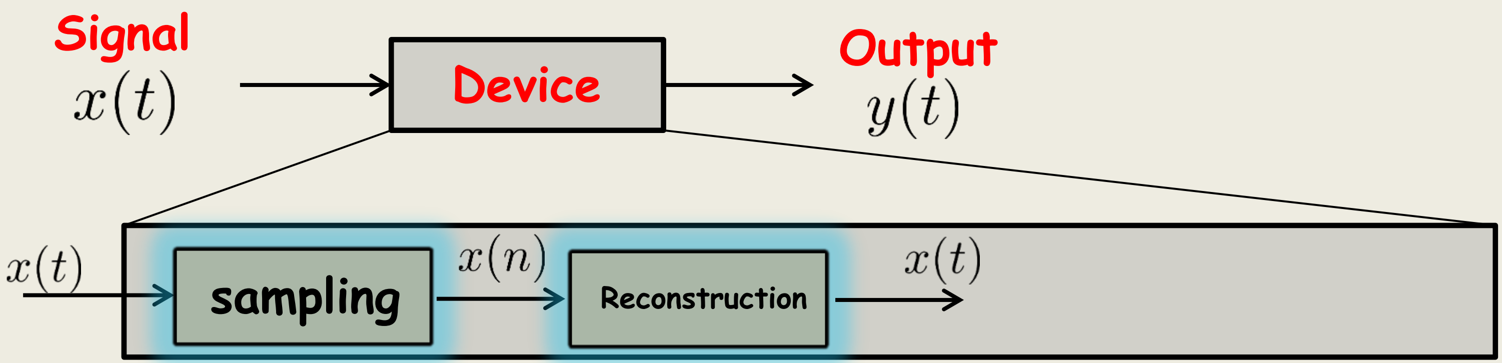

When is the sampling unit optimal?
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Answer 1 (book approach): When it is possible to recover x(t) from x(n)

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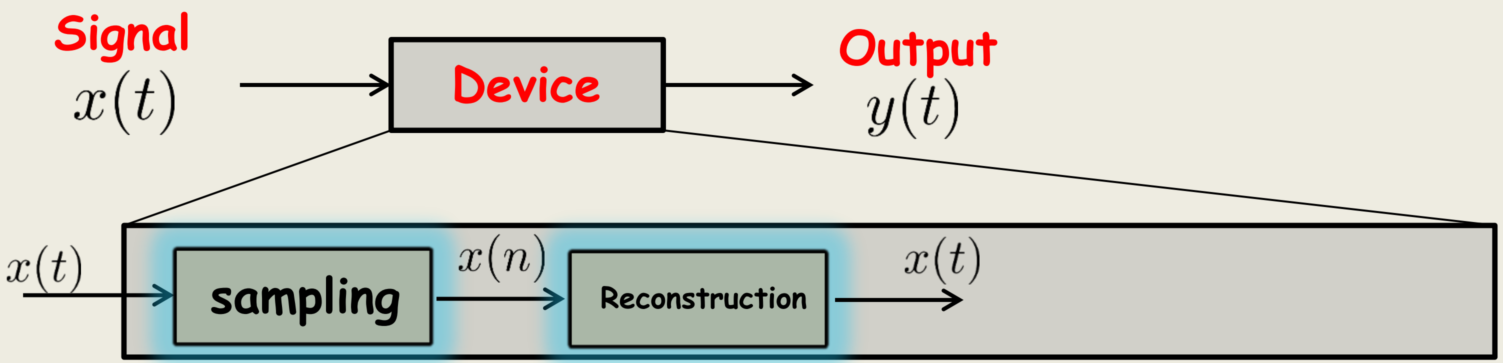

When is the sampling unit optimal?
(optimal means lossless)

Answer 2 (Reality): $x(t)$ contains a **data signal and noise:** $x(t) = s(t) + n(t)$
(No noise in book)

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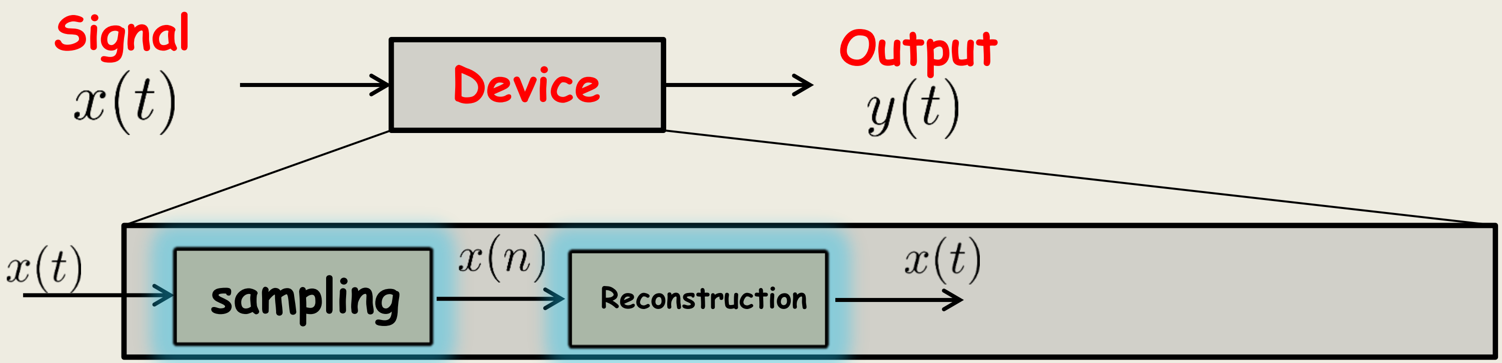

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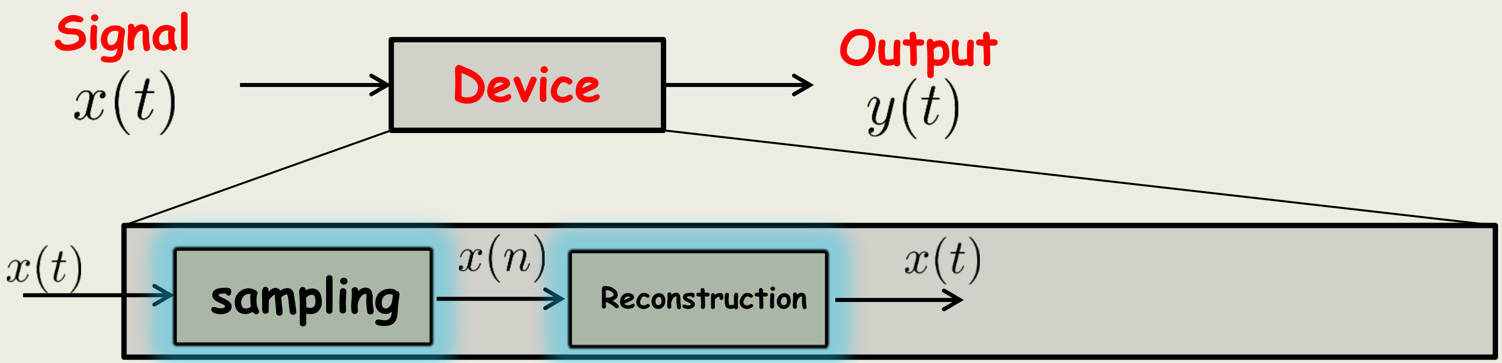

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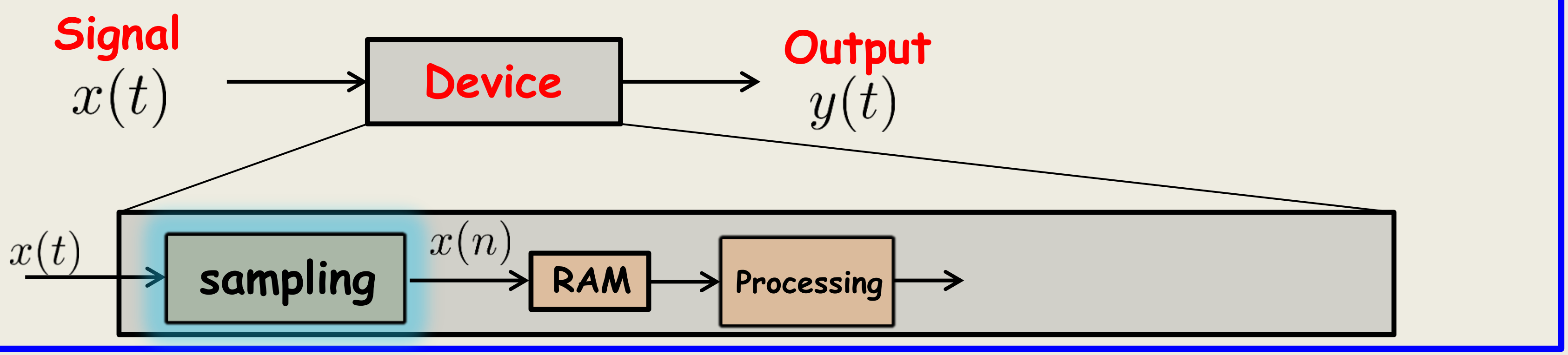

When is the sampling unit optimal?
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Answer 2 (Reality): $x(t)$ contains a **data signal and noise:** $x(t) = s(t) + n(t)$

We want to process the signal $s(t)$, not the noise

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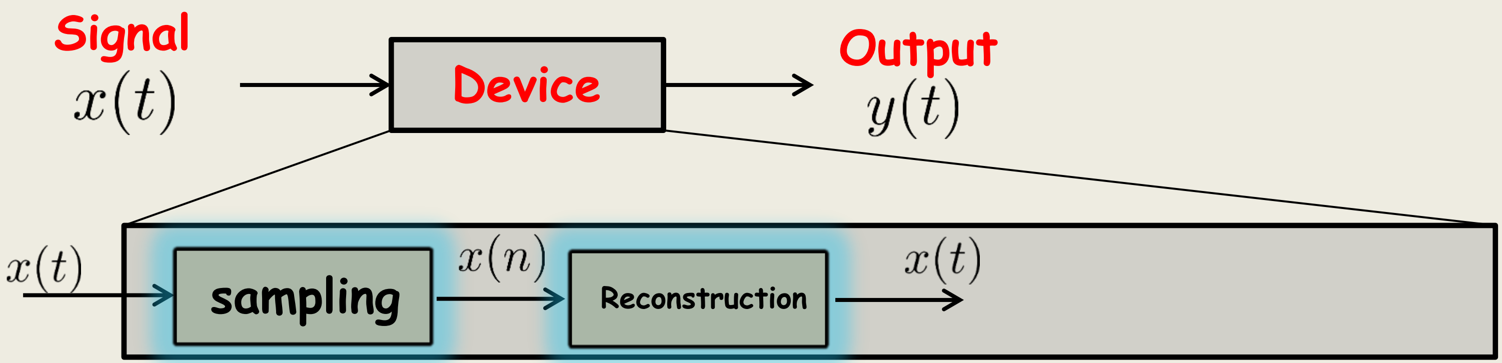
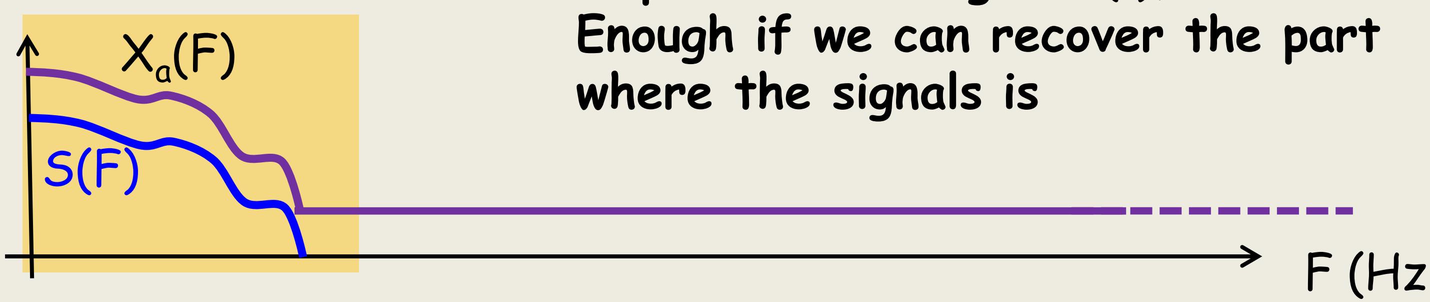


When is the sampling unit optimal?
(optimal means lossless)

Answer 2 (Reality): x(t) contains a **data signal and noise:** x(t) = s(t) + n(t)

We want to process the signal s(t), not the noise
Enough if we can recover the part where the signals is



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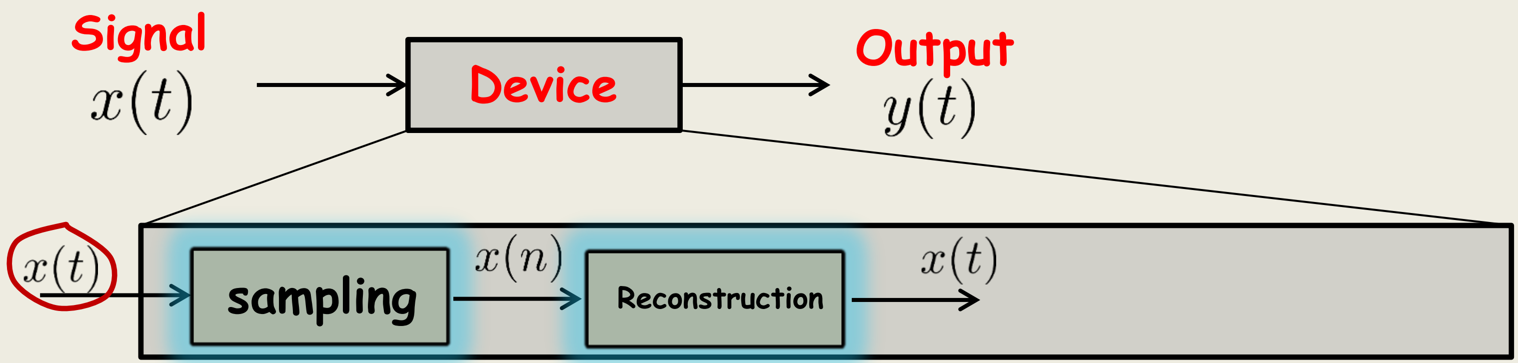

When is the sampling unit optimal?
(optimal means lossless)

Summary:
If input $x(t)$ looks like this

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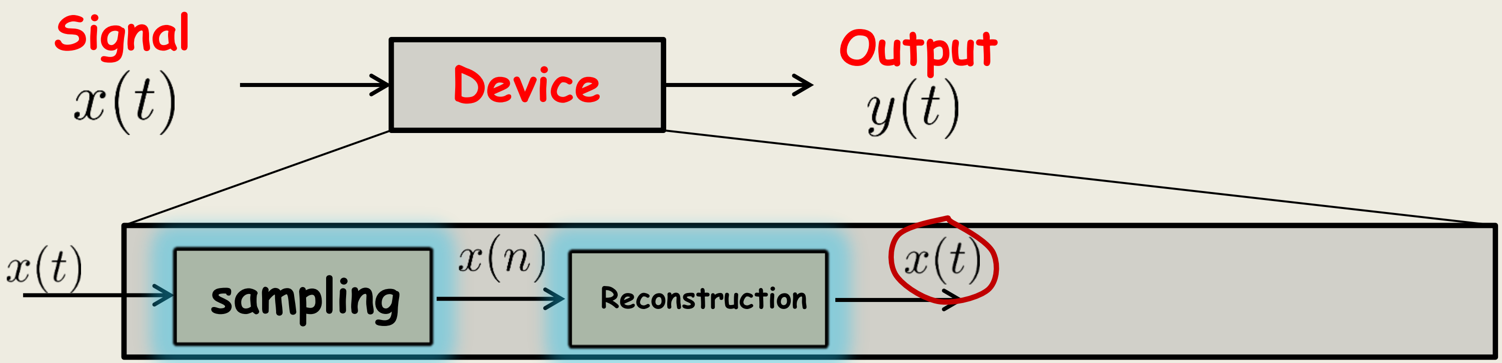
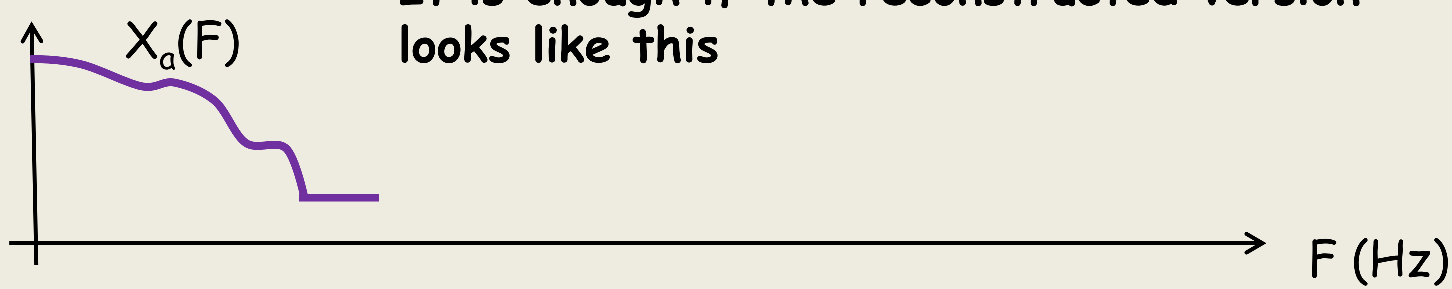


When is the sampling unit optimal?
(optimal means lossless)

Summary:
If input x(t) looks like this
It is enough if the reconstructed version
looks like this



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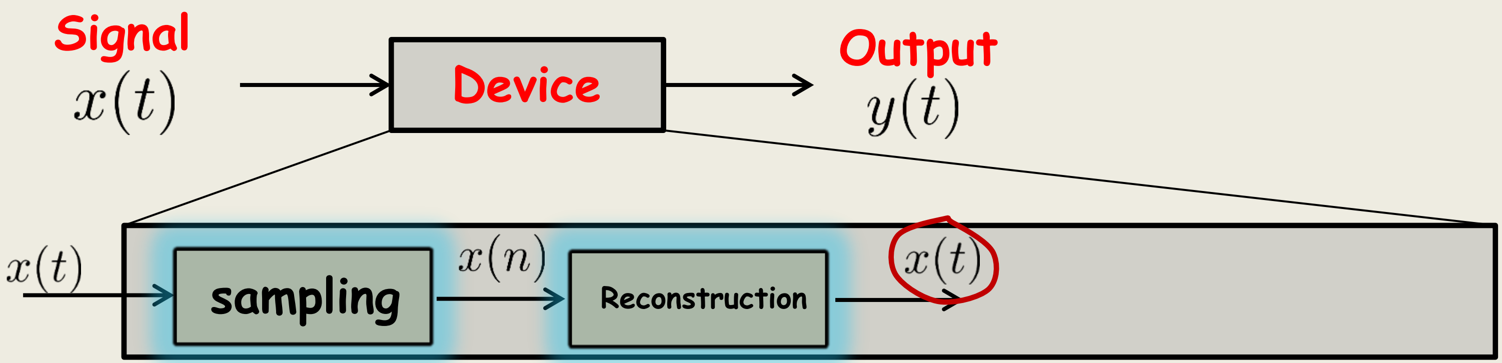


When is the sampling unit optimal?

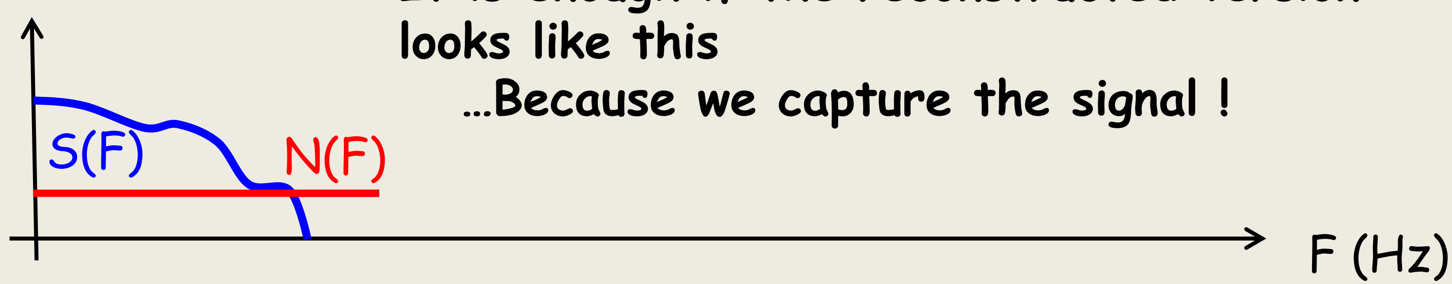
(optimal means lossless)

Summary:

If input x(t) looks like this

It is enough if the reconstructed version looks like this

…Because we capture the signal !



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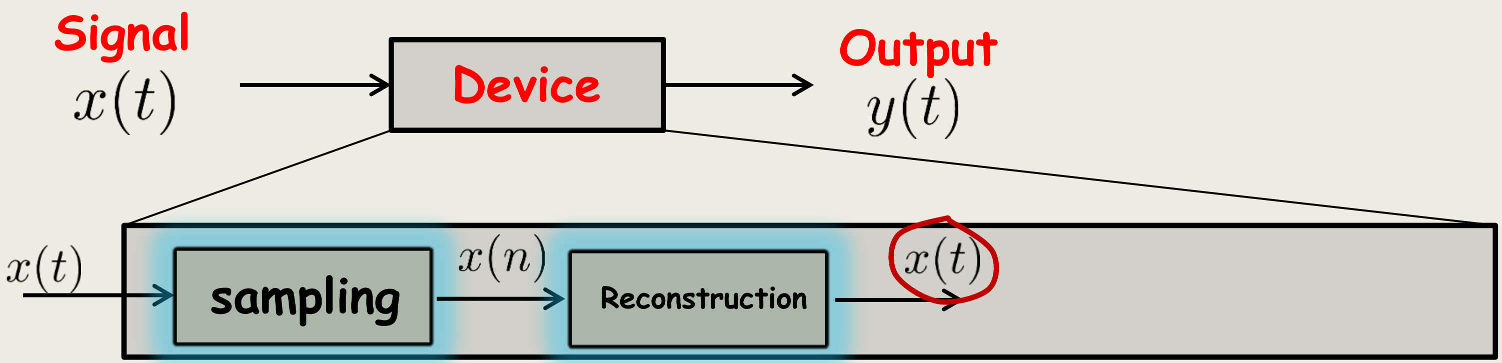


When is the sampling unit optimal?

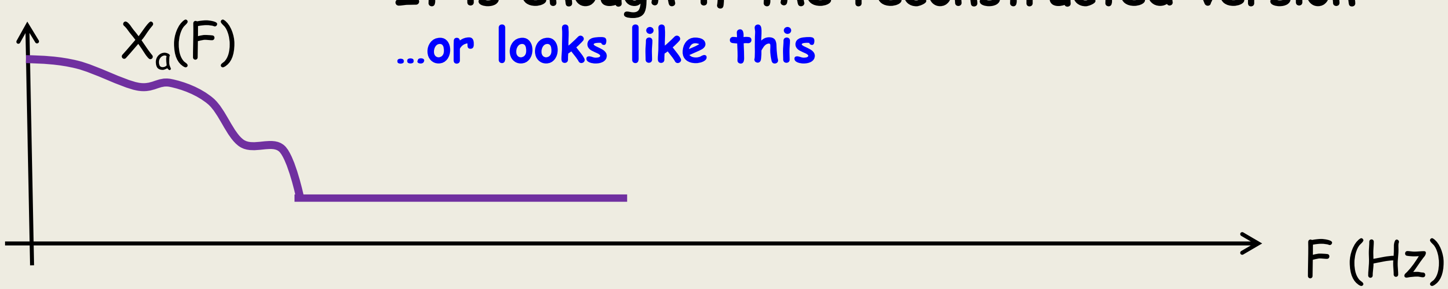
(optimal means lossless)

Summary:

If input x(t) looks like this

It is enough if the reconstructed version

…or looks like this



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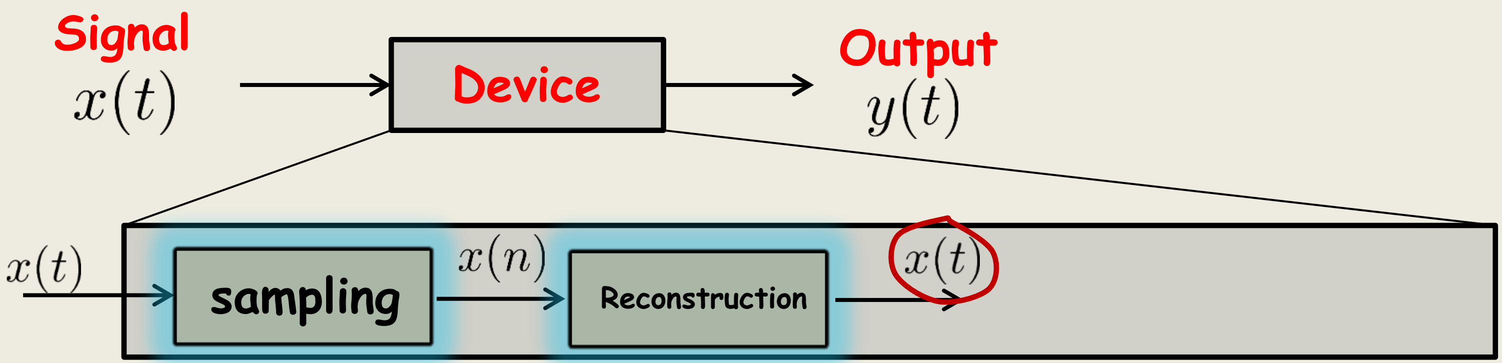

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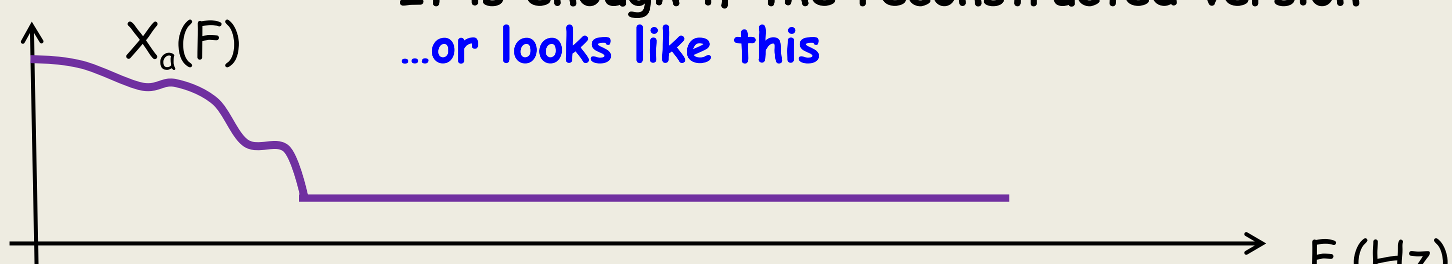

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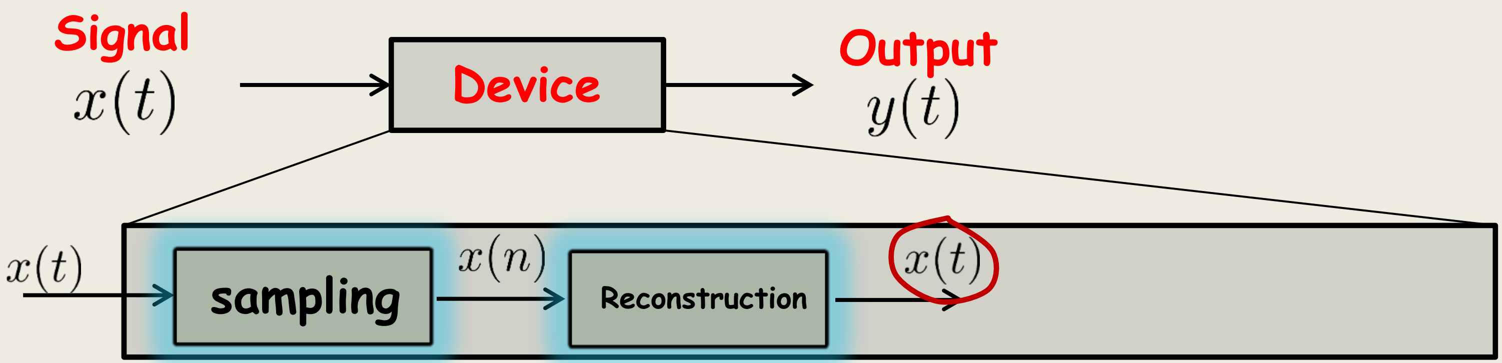
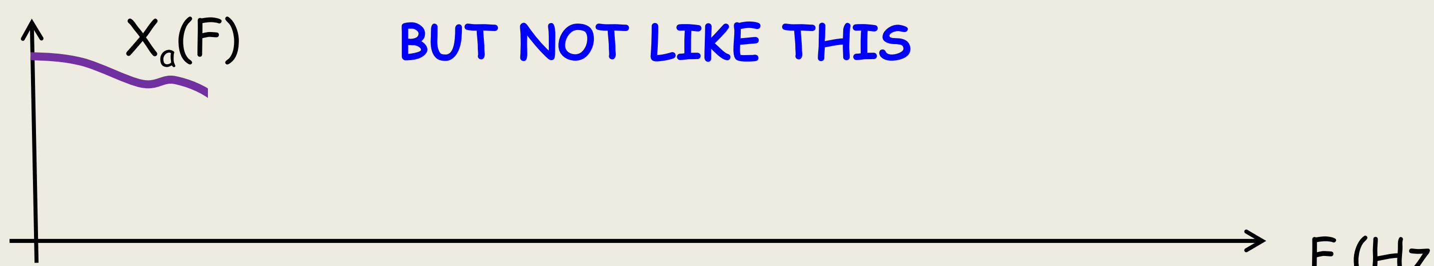


When is the sampling unit optimal?
(optimal means lossless)

Summary:
If input x(t) looks like this

BUT NOT LIKE THIS



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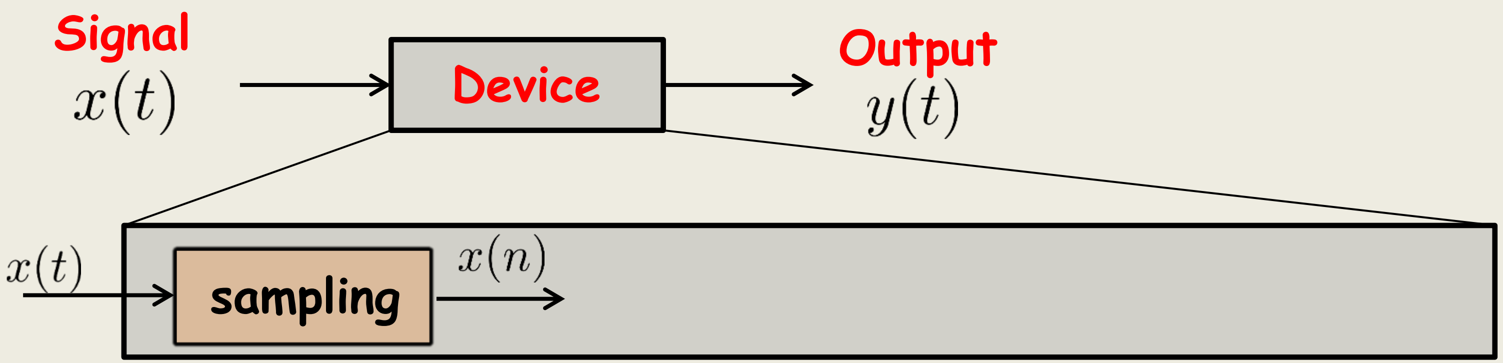

Makes sense to first remove the high frequency noise

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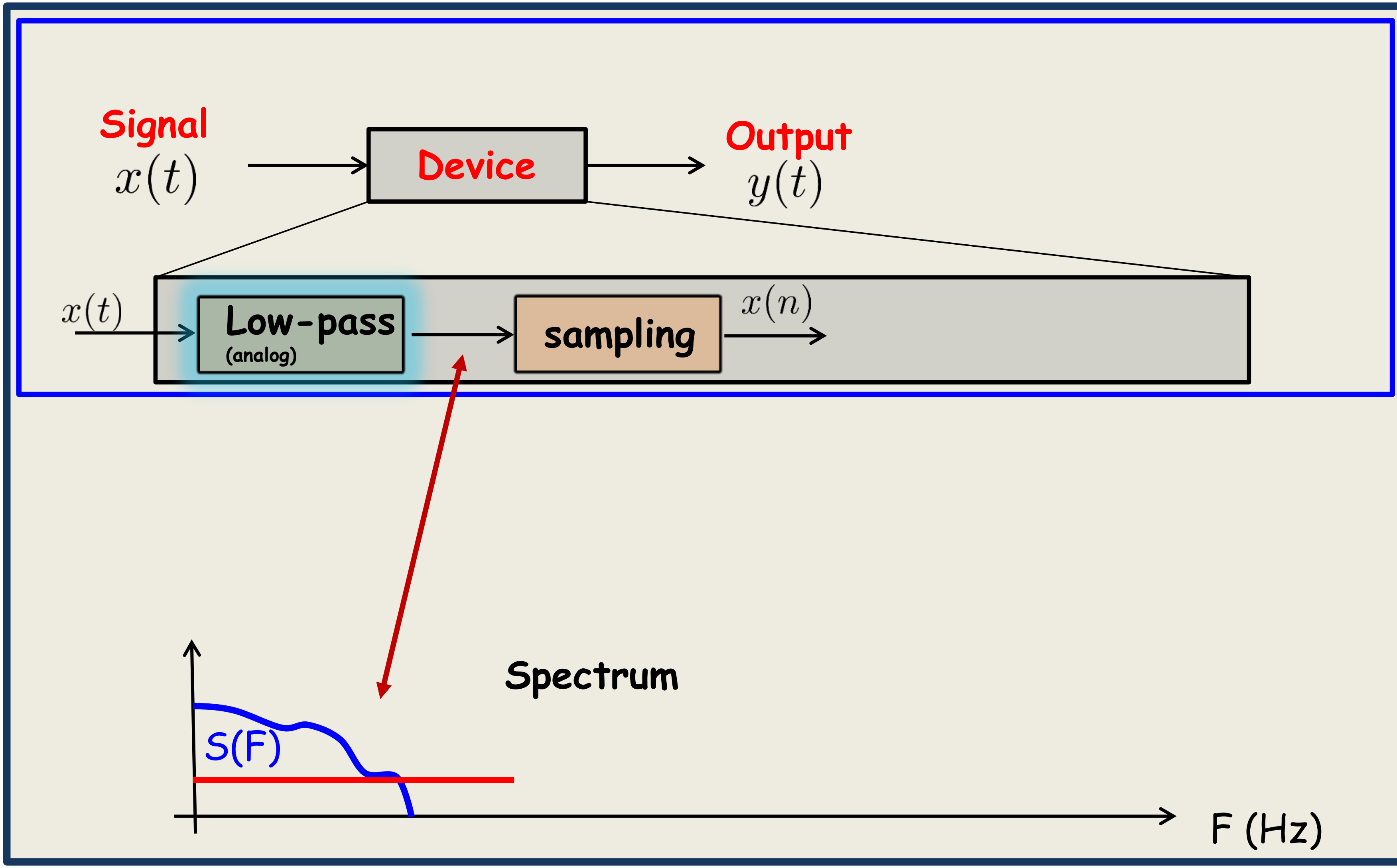

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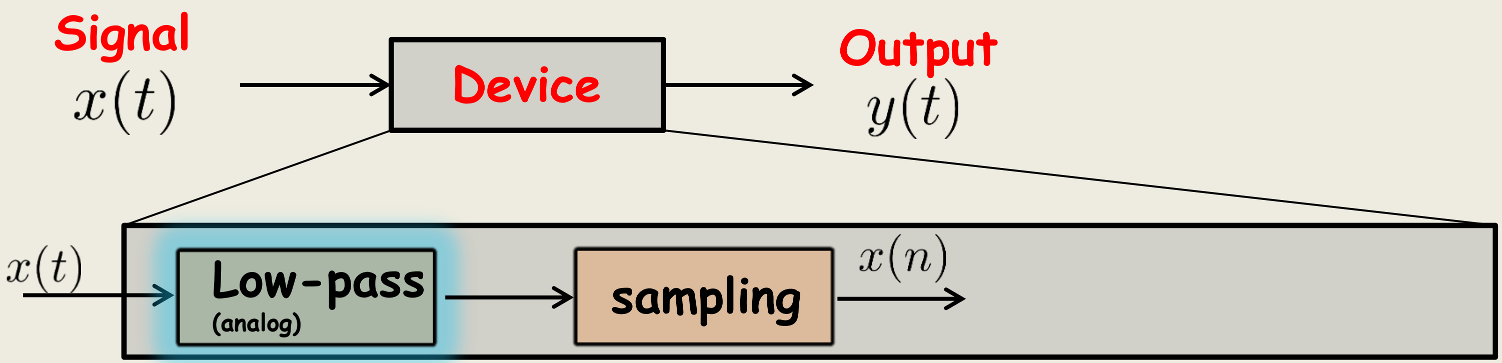

When is the sampling unit optimal?
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Answer 1 (book approach): When it is possible to recover x(t) from x(n)
assuming no noise outside the signal bandwidth

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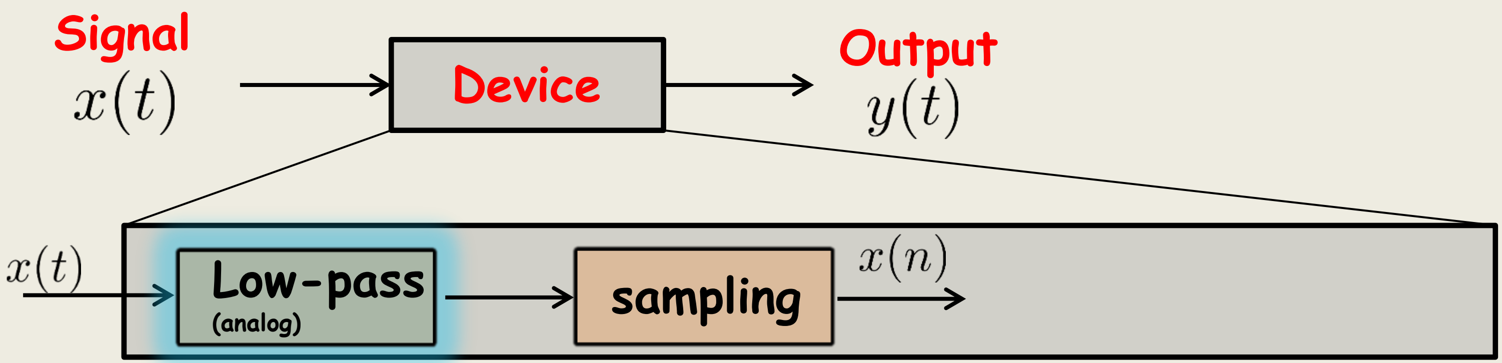

When is it "possible to recover x(t) from x(n) *assuming no noise outside the signal bandwidth*" ?

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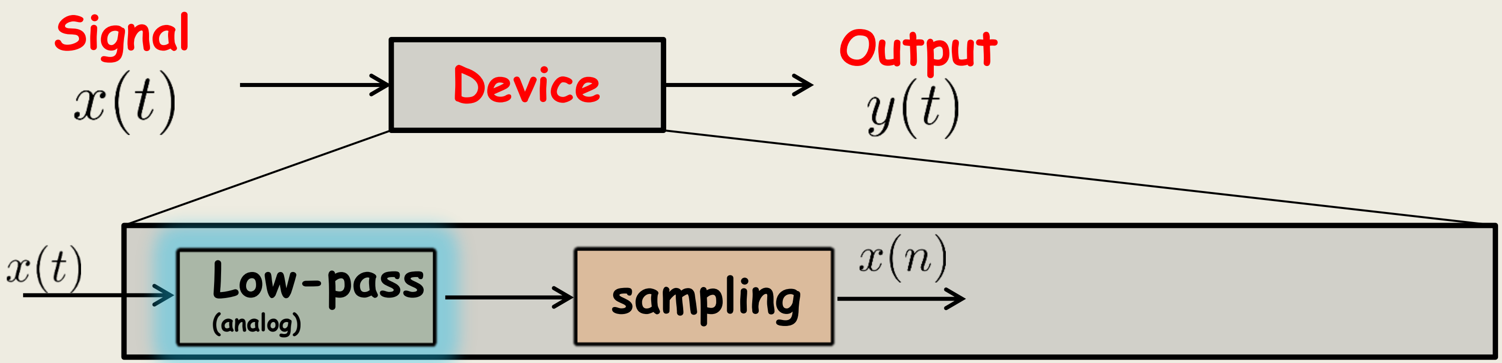

When is it "possible to recover x(t) from x(n) *assuming no noise outside the signal bandwidth*" ?

x(t) aperiodic continuous

x(n) aperiodic discrete

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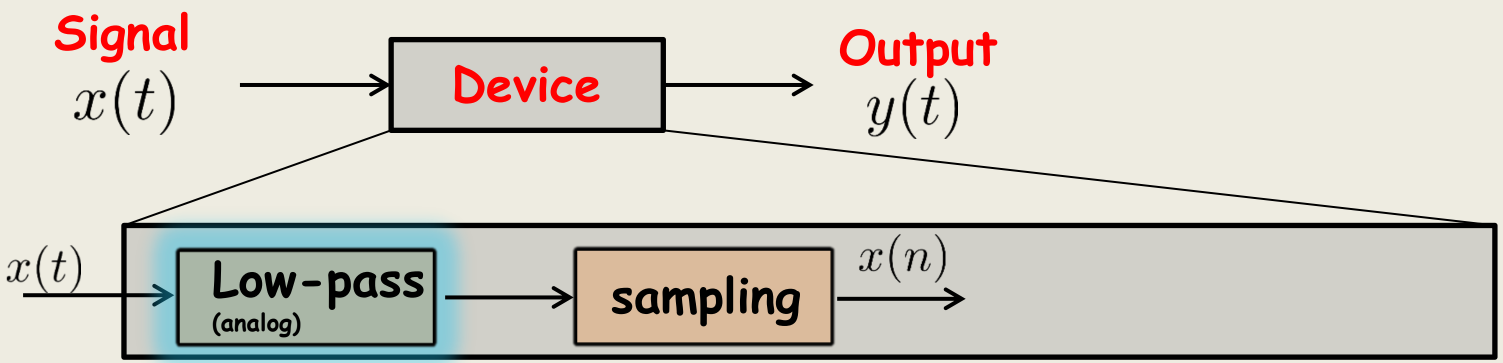

When is it "possible to recover x(t) from x(n) *assuming no noise outside the signal bandwidth*" ?

x(t) aperiodic continuous

$X_a(F)$ aperiodic continuous

x(n) aperiodic discrete

X(f) periodic continuous

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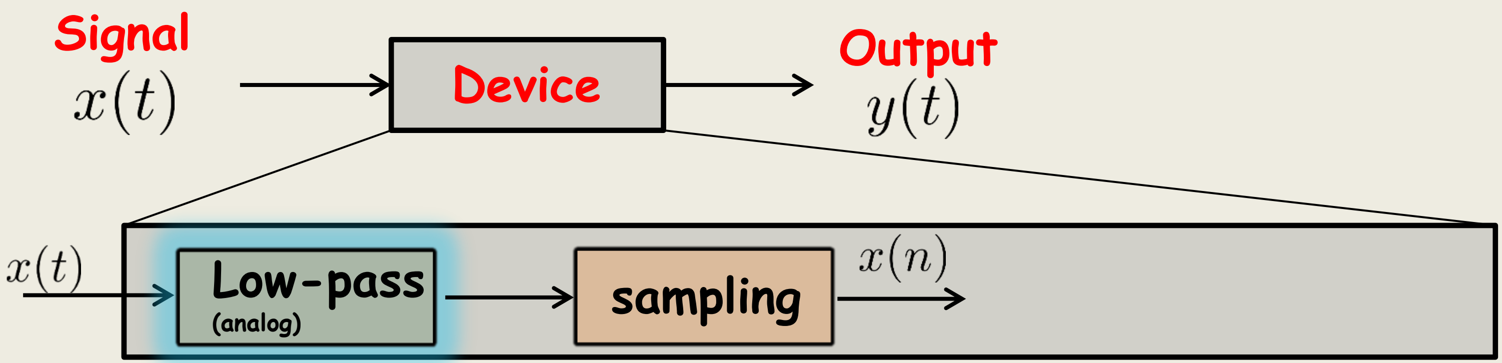
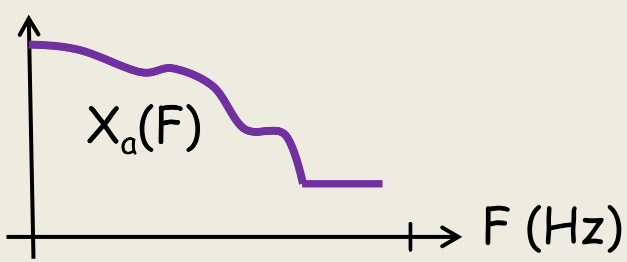


When is it "possible to recover x(t) from x(n) ***assuming no noise outside the signal bandwidth"*** **?**

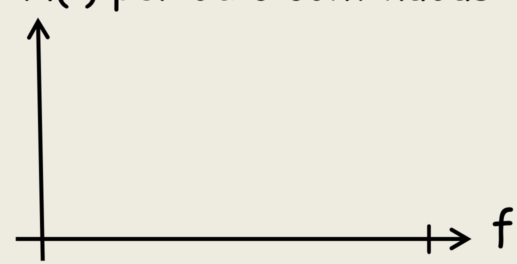
x(t) aperiodic continuous

$X_a(F)$ aperiodic continuous



x(n) aperiodic discrete

X(f) periodic continuous



Ex: F=20.000.000. (20 MHz)

f=1/2

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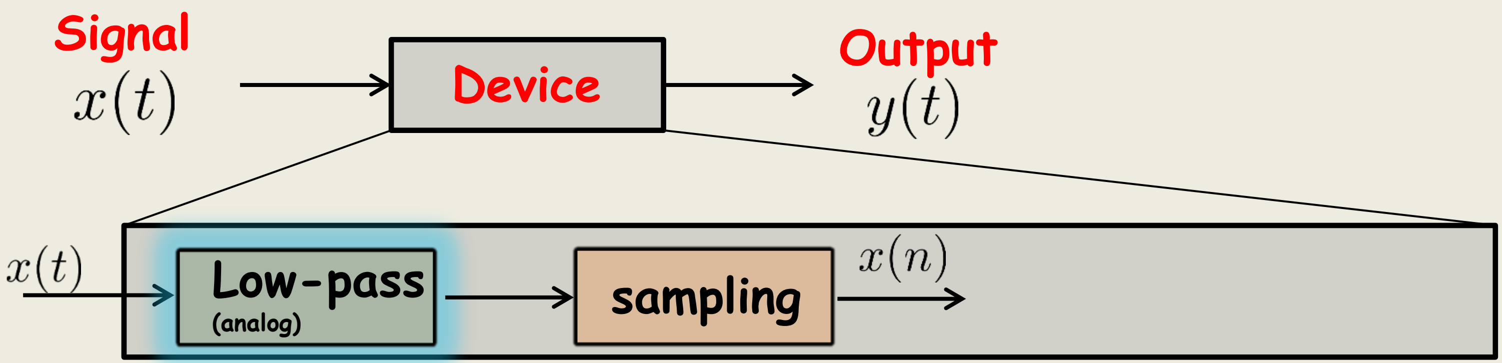
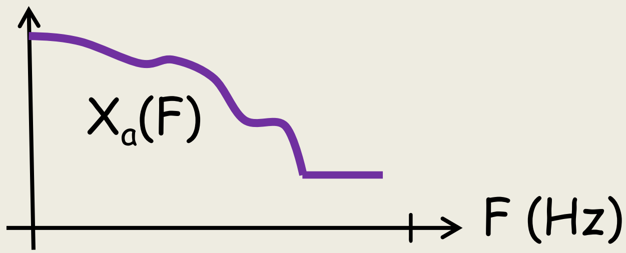


When is it "possible to recover x(t) from x(n) *assuming no noise outside the signal bandwidth*" ? IF X(f) looks the same as $X_a(F)$

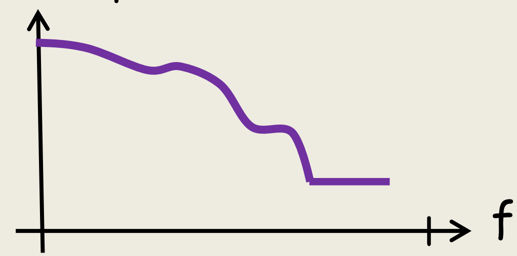
x(t) aperiodic continuous

$X_a(F)$ aperiodic continuous



x(n) aperiodic discrete

X(f) periodic continuous



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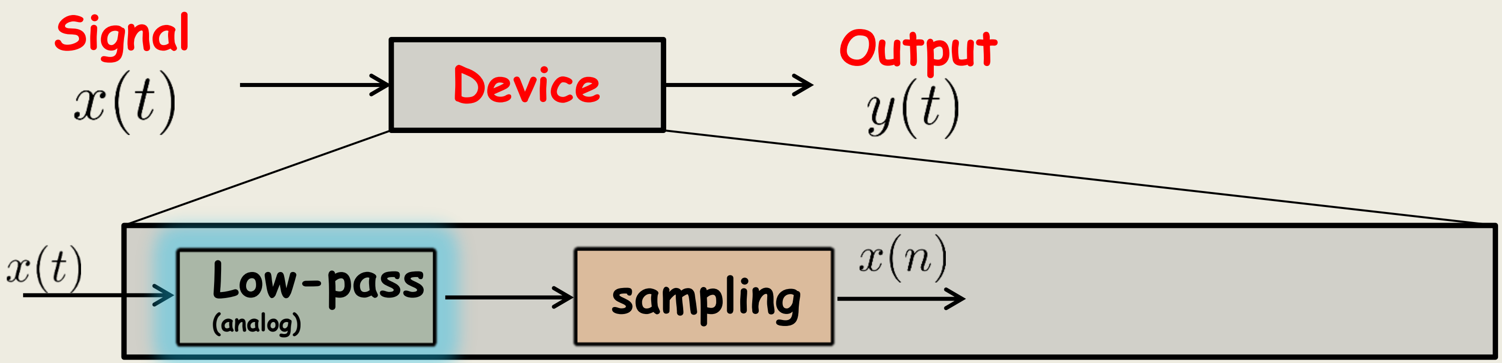

Altogether,

1. $X_a(F)$ contains all information about $x(t)$
2. $X_a(F)$ is apperiodic, but has finite bandwidth
3. $x(n)$ sampled version of $x(t)$
4. $X(f)$ sufficient to recover $X_a(F)$ -> $x(n)$ sufficient to recover $x(t)$
5. -> sampling optimal

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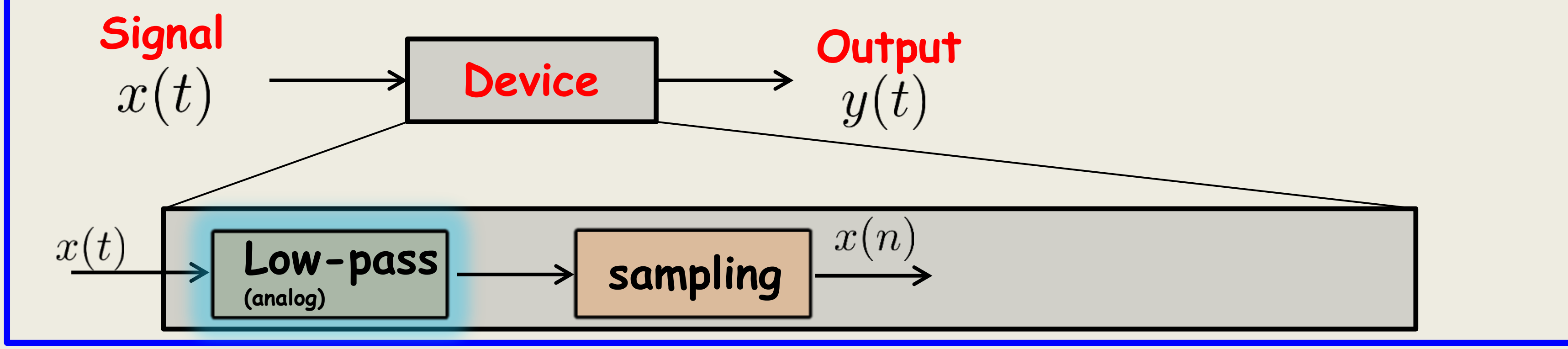

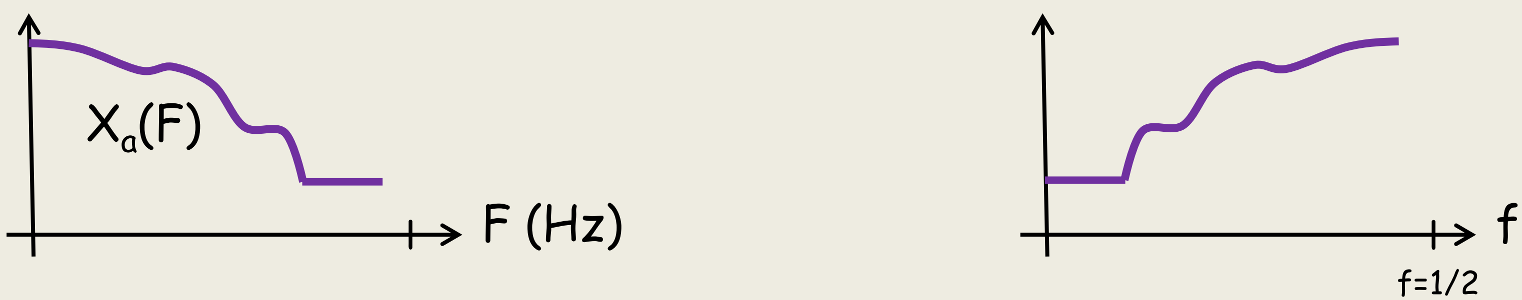

4. $X(f)$ sufficient to recover $X_a(F)$ -> $x(n)$ sufficient to recover $x(t)$
   Still true
5. -> sampling optimal

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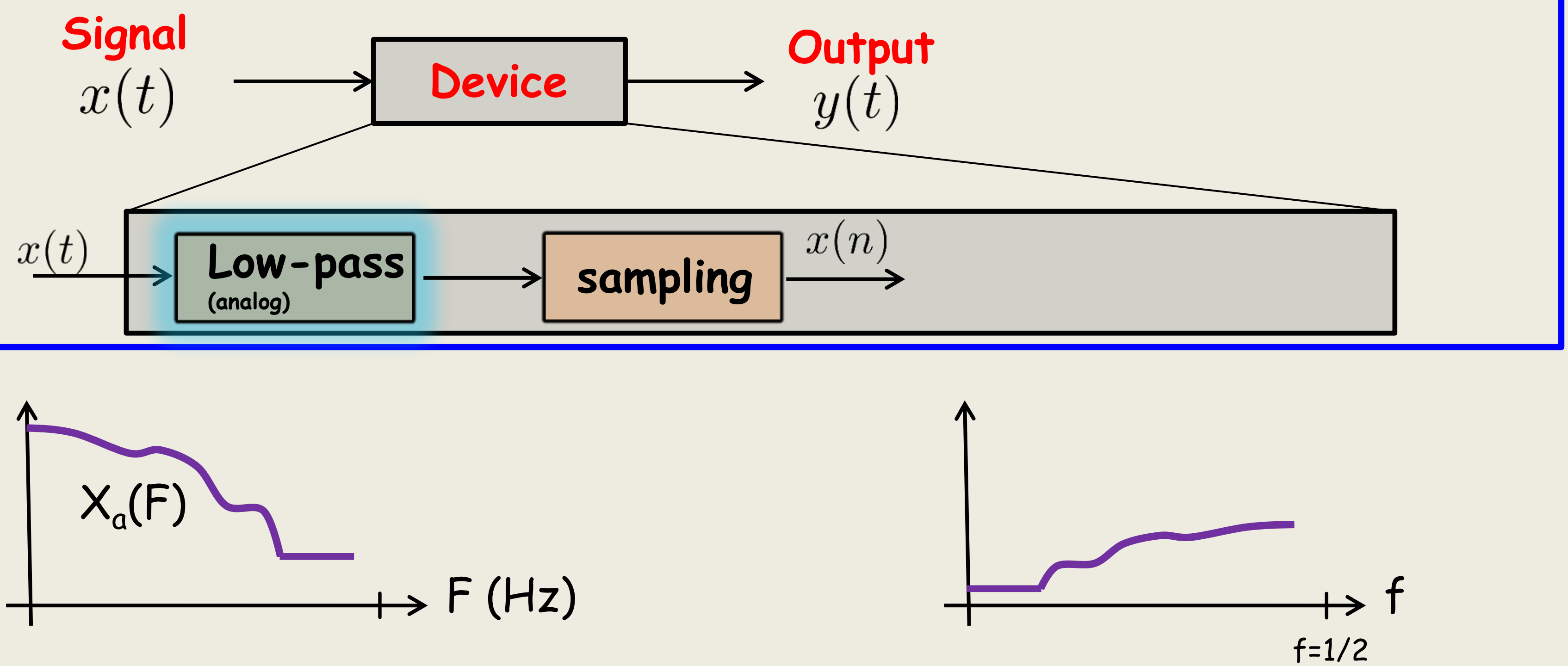

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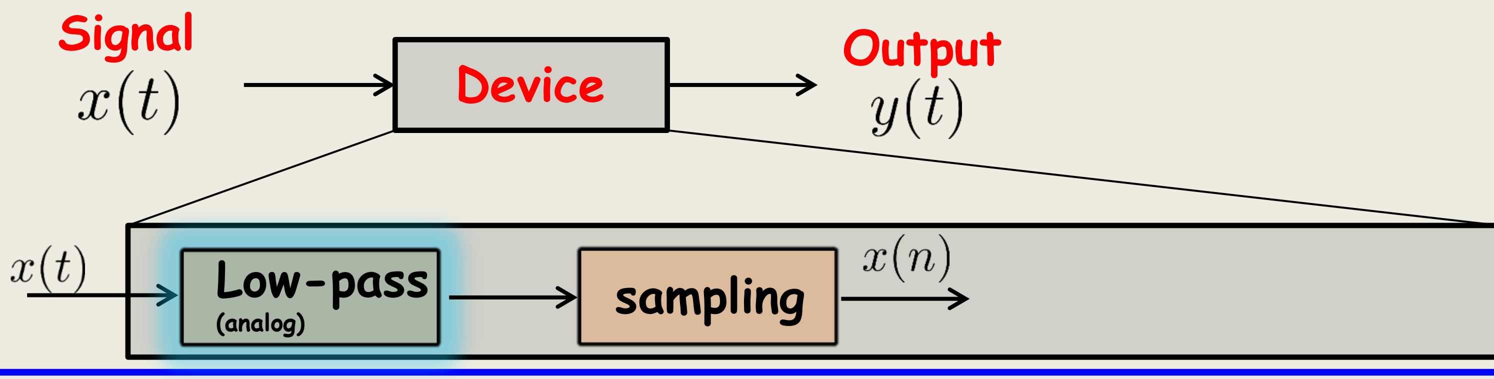

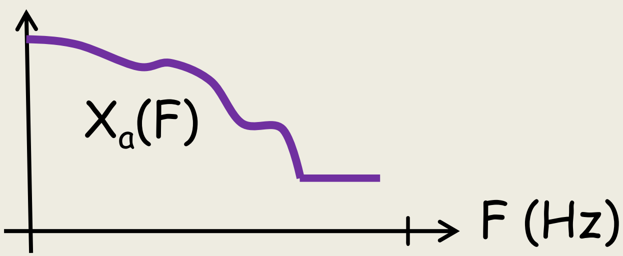

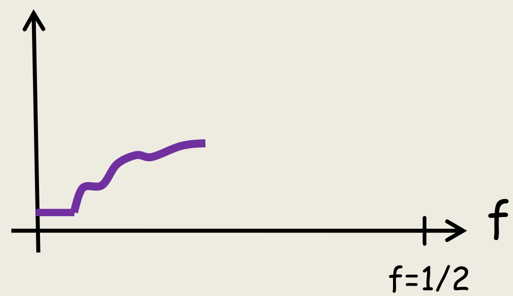

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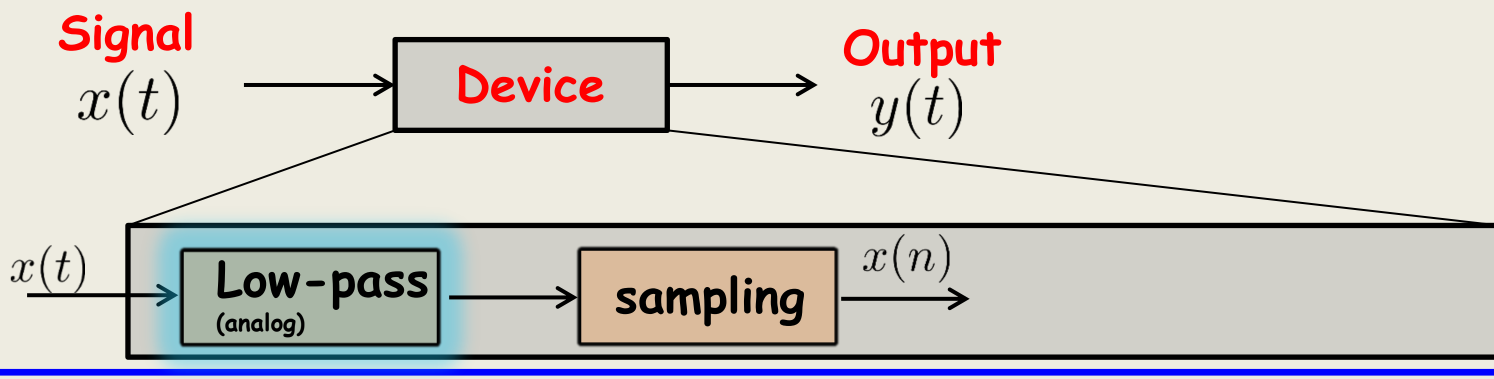

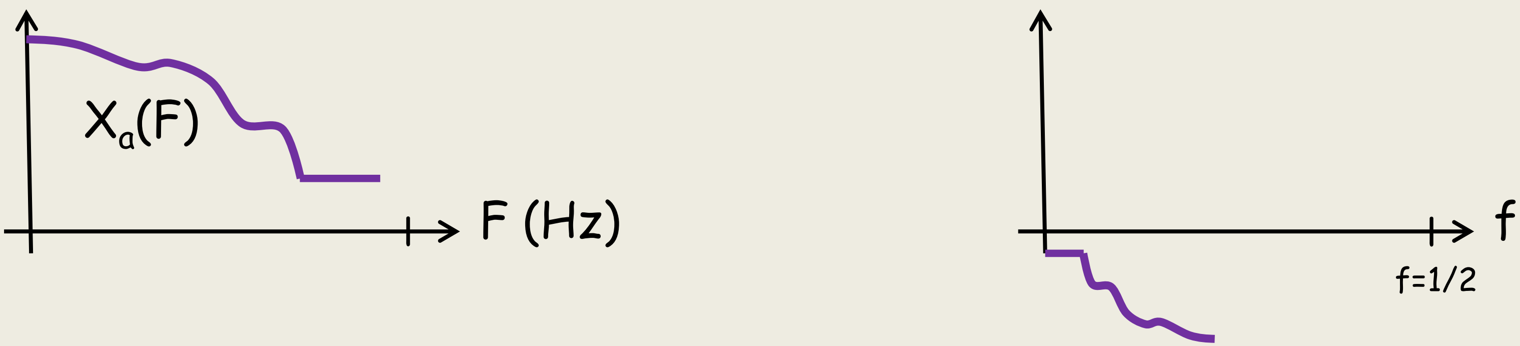

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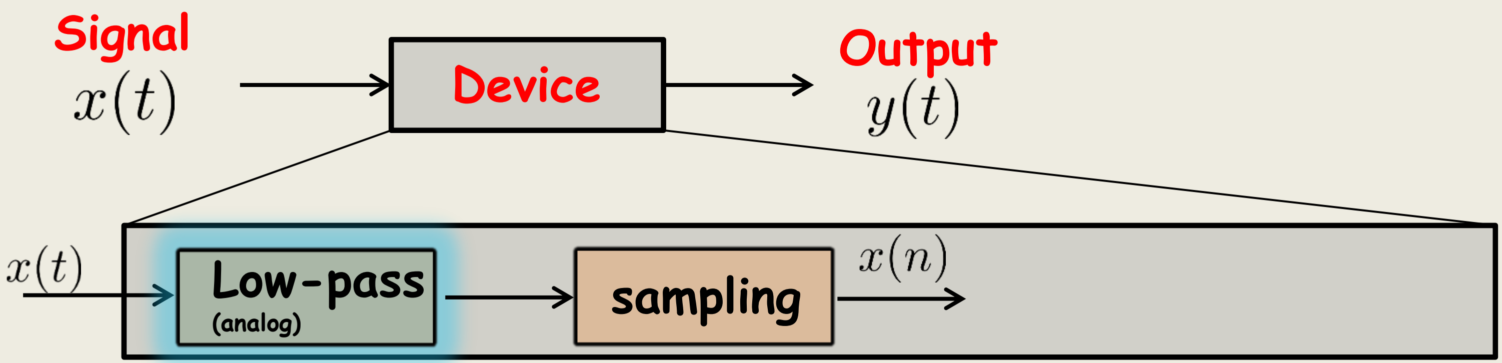

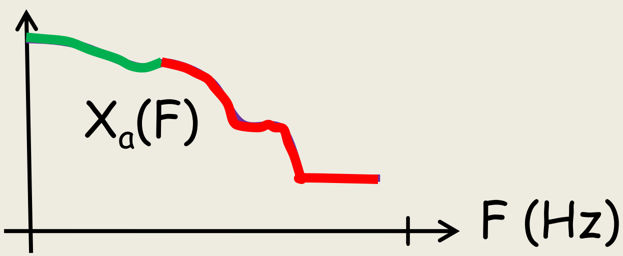

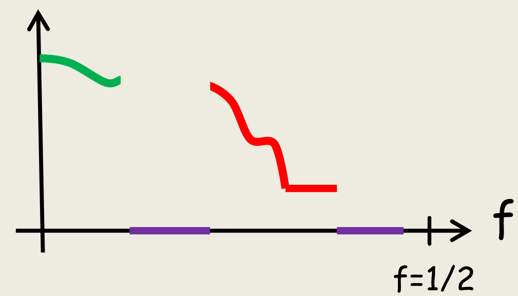

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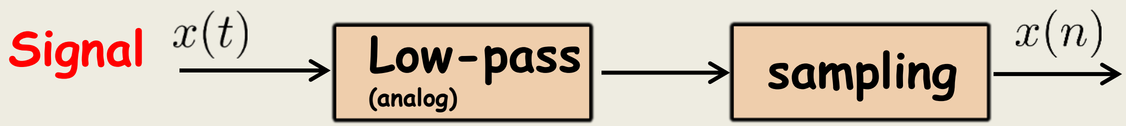

Status

1. We know that we can assume the bandwidth of x(t) to be small

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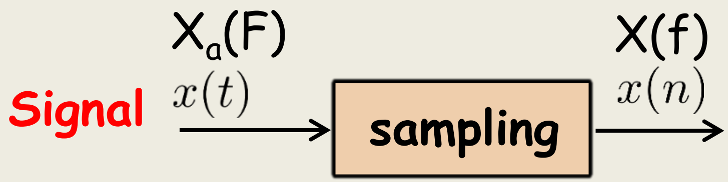

Status

1. We know that we can assume the bandwidth of $x(t)$ to be small

2. We know that if $X_a(F)$ can be recovered from $X(f)$, the sampling is optimal

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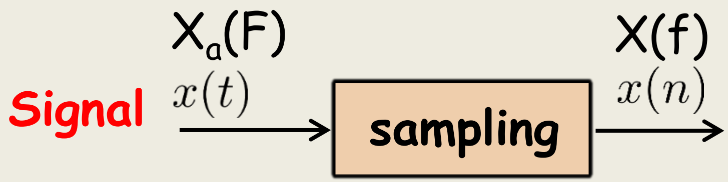

Status

1. We know that we can assume the bandwidth of $x(t)$ to be small
2. We know that if $X_a(F)$ can be recovered from $X(f)$, the sampling is optimal
3. Thus, we must study what $X(f)$ looks like

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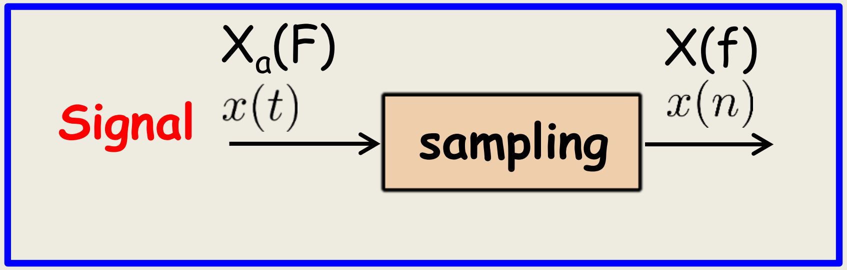

Make a list of everything we know

$$x(t) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi F t} \mathrm{d}F$$

x(t) has Fourier transform $X_a(F)$

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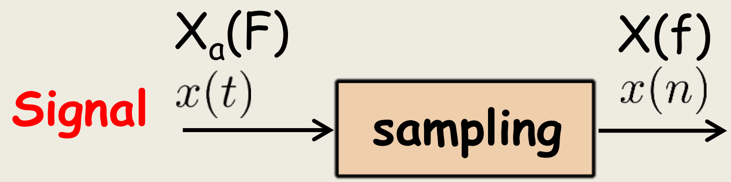

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$$x(n) = \int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f$$

x(n) has DTFT X(f)

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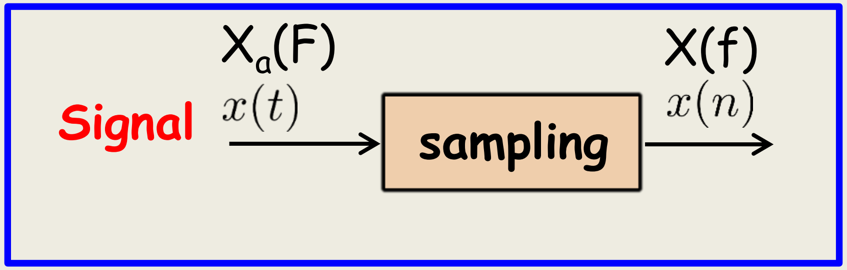

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x(t) has Fourier transform X_a(F)

$$x(n) = \int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f$$

x(n) has DTFT X(f)

$$x(n) = x(t | t = n/F_{\mathrm{s}})$$

x(n) samples of x(t)
F_s is sample rate

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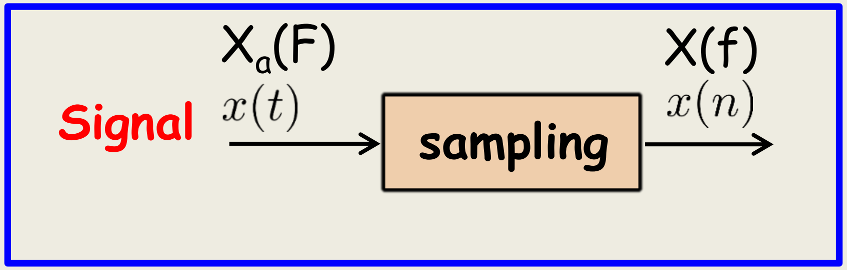

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x(n) has DTFT X(f)

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x(n) samples of x(t)
F_s is sample rate

We don't have any further information

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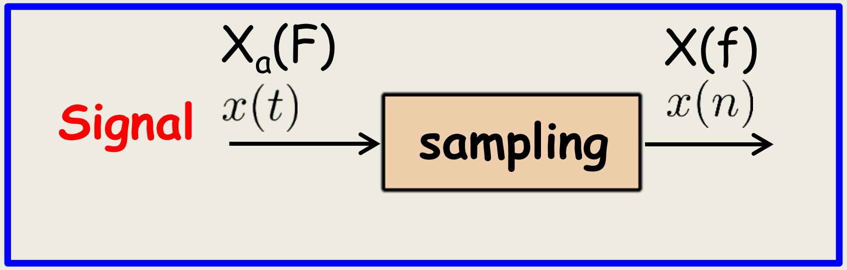

Make a list of everything we know

From this, we should be able to

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x(t) has Fourier transform $X_a(F)$

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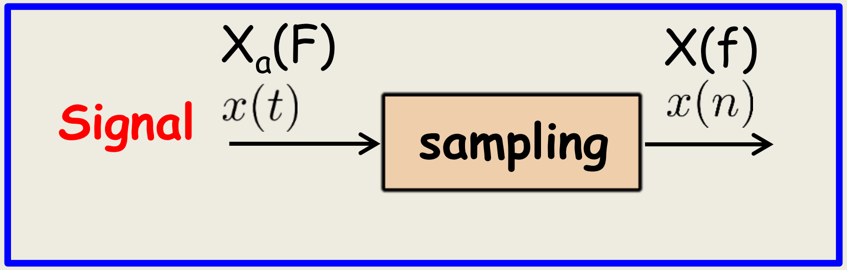


Make a list of everything we know

From this, we should be able to

$$x(t) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi F t} \mathrm{d}F$$

$x(t)$ has Fourier transform $X_a(F)$

compute $X(f)$

$$x(n) = \int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f$$

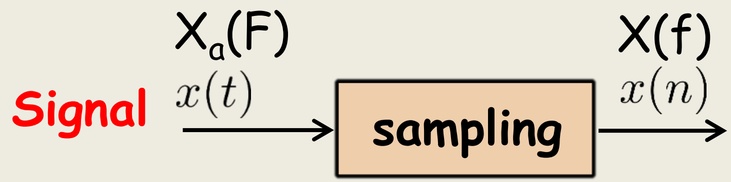
$x(n)$ has DTFT $X(f)$

$$x(n) = x(t|t = n/F_\mathrm{s})$$

$x(n)$ samples of $x(t)$
F_s is sample rate

We don't have any further information

EITF75 Systems and Signals



Goal: $X(f) =$ formula in $X_a(F)$ and F_s

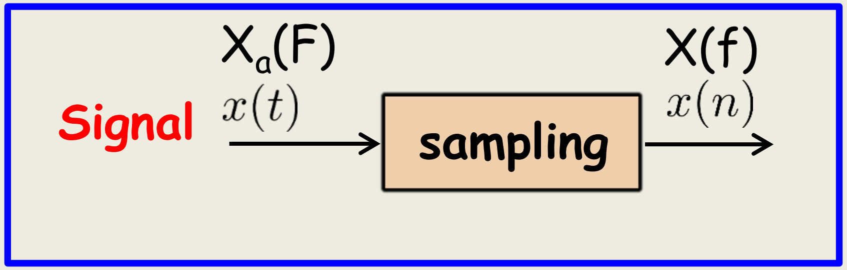
Make a list of everything we know

(1) $$x(t) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi F t} \mathrm{d}F$$

(2) $$x(n) = \int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f$$

(3) $$x(n) = x(t | t = n/F_{\mathrm{s}})$$

EITF75 Systems and Signals



Make a list of everything we know

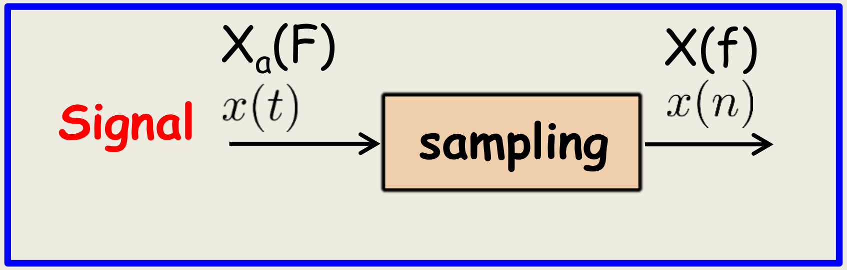
(1) $$x(t) = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi Ft}\mathrm{d}F$$

(2) $$x(n) = \int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi fn}\mathrm{d}f$$

(3) $$x(n) = x(t|t = n/F_\mathrm{s})$$

Use (1) in (3)

EITF75 Systems and Signals



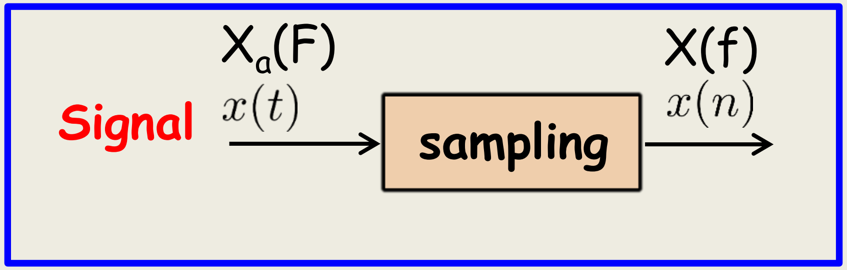
Make a list of everything we know

(1) $$x(t) = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi F t}\mathrm{d}F$$

(2) $$x(n) = \int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi f n}\mathrm{d}f$$

(3) $$x(n) = x(t|t = n/F_\mathrm{s}) = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi n F/F_\mathrm{s}}\mathrm{d}F$$

EITF75 Systems and Signals



Make a list of everything we know

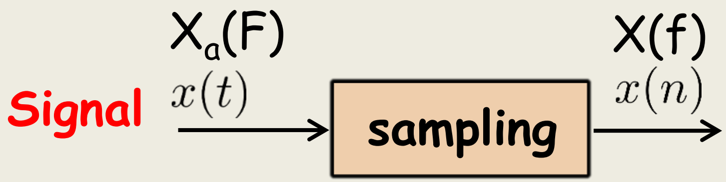
(1) $$x(t) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi F t} \mathrm{d}F$$

(2) $$x(n) = \int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f$$

Rhs of (2) must equal Rhs of (3)

(3) $$x(n) = x(t | t = n/F_\mathrm{s}) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi n F / F_\mathrm{s}} \mathrm{d}F$$

EITF75 Systems and Signals



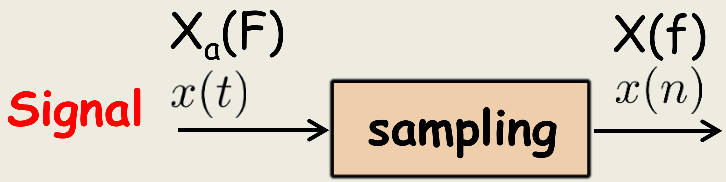
Make a list of everything we know

(1) $$x(t) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi F t} \mathrm{d}F$$

(2) $$x(n) = \int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f \ = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi n F/F_\mathrm{s}} \mathrm{d}F$$

(3) $$x(n) = x(t|t = n/F_\mathrm{s}) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi n F/F_\mathrm{s}} \mathrm{d}F$$

EITF75 Systems and Signals



Make a list of everything we know

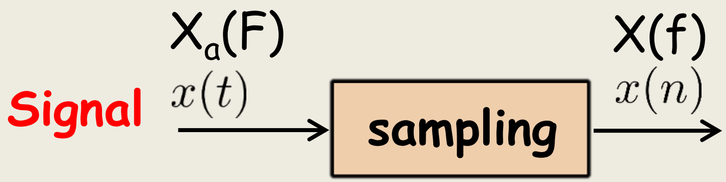
(1) $$x(t) = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi F t}\mathrm{d}F$$

Contains all information we have

(2) $$x(n) = \boxed{\int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi f n}\mathrm{d}f = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi n F/F_s}\mathrm{d}F}$$

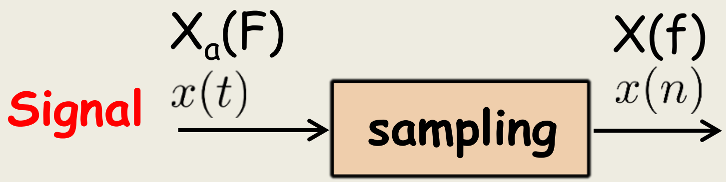
(3) $$x(n) = x(t|t = n/F_s) = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi n F/F_s}\mathrm{d}F$$

EITF75 Systems and Signals



$$\int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi f n}\mathrm{d}f = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

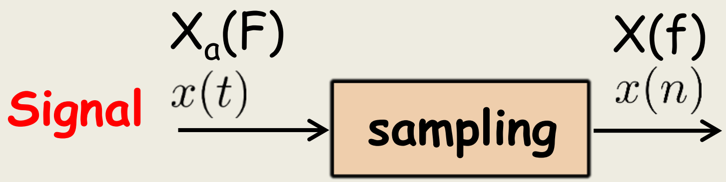
EITF75 Systems and Signals



$$\int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi fn}\mathrm{d}f \;=\int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

Make variable change in lhs (pure calculus) $f = \dfrac{F}{F_\mathrm{s}}$

EITF75 Systems and Signals



$$\int_{-0.5}^{0.5} X(f) \mathrm{e}^{i2\pi f n} \mathrm{d}f \;=\; \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi n F/F_\mathrm{s}} \mathrm{d}F$$

Make variable change in lhs (pure calculus) $f = \dfrac{F}{F_\mathrm{s}}$ $\mathrm{d}f = \mathrm{d}F \dfrac{1}{F_\mathrm{s}}$

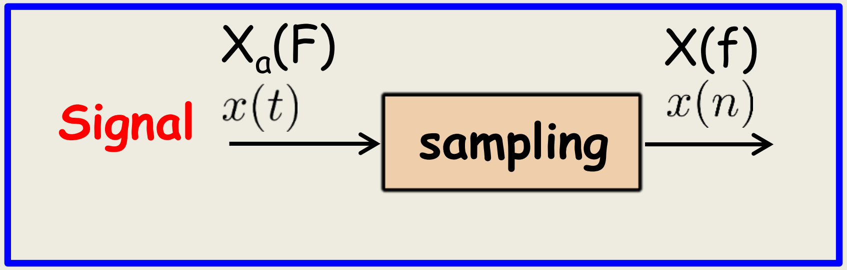
$$f = 0.5 \rightarrow F = \frac{F_\mathrm{s}}{2}$$

$$f = -0.5 \rightarrow F = -\frac{F_\mathrm{s}}{2}$$

Lhs becomes

$$\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2}$$

EITF75 Systems and Signals



$$\int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi f n}\mathrm{d}f \;=\; \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

Make variable change in lhs (pure calculus) $f = \dfrac{F}{F_\mathrm{s}}$ $\qquad \mathrm{d}f = \mathrm{d}F\dfrac{1}{F_\mathrm{s}}$

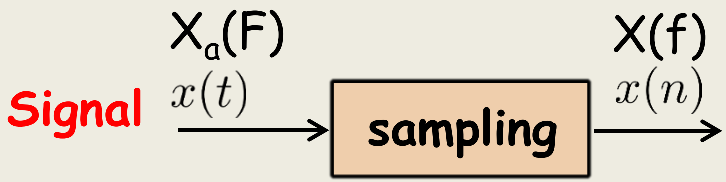
$$f = 0.5 \rightarrow F = \frac{F_\mathrm{s}}{2}$$

$$f = -0.5 \rightarrow F = -\frac{F_\mathrm{s}}{2}$$

Lhs becomes

$$\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)$$

EITF75 Systems and Signals



$$\int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi f n}\mathrm{d}f \;=\; \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

Make variable change in lhs (pure calculus) $f = \dfrac{F}{F_\mathrm{s}}$ $\qquad \mathrm{d}f = \mathrm{d}F\dfrac{1}{F_\mathrm{s}}$

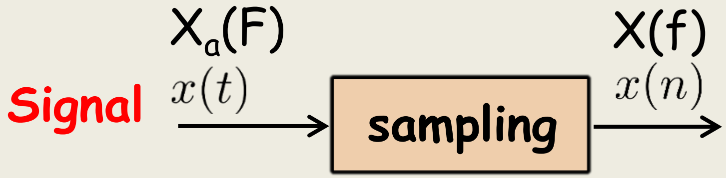
$$f = 0.5 \rightarrow F = \frac{F_\mathrm{s}}{2}$$

Lhs becomes

$$f = -0.5 \rightarrow F = -\frac{F_\mathrm{s}}{2}$$

$$\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}$$

EITF75 Systems and Signals



$$\int_{-0.5}^{0.5} X(f)\mathrm{e}^{i2\pi f n}\mathrm{d}f = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

Make variable change in lhs (pure calculus) $f = \dfrac{F}{F_\mathrm{s}}$ $\quad \mathrm{d}f = \mathrm{d}F\dfrac{1}{F_\mathrm{s}}$

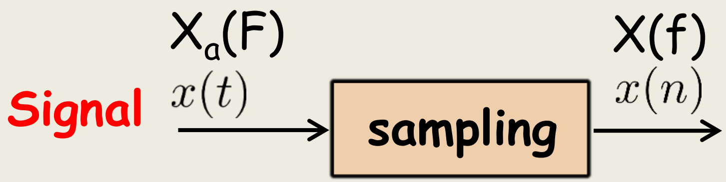
$$f = 0.5 \rightarrow F = \frac{F_\mathrm{s}}{2}$$

Lhs becomes

$$f = -0.5 \rightarrow F = -\frac{F_\mathrm{s}}{2}$$

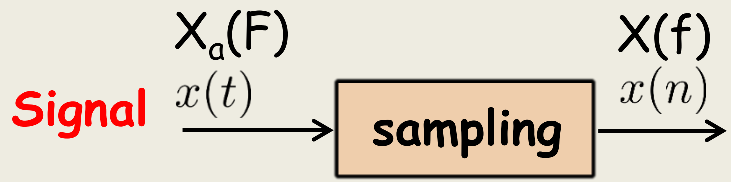
$$\frac{1}{F_\mathrm{s}}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

EITF75 Systems and Signals



$$\frac{1}{F_{\mathrm{s}}}\int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2} X\left(\frac{F}{F_{\mathrm{s}}}\right) \mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F$$

EITF75 Systems and Signals

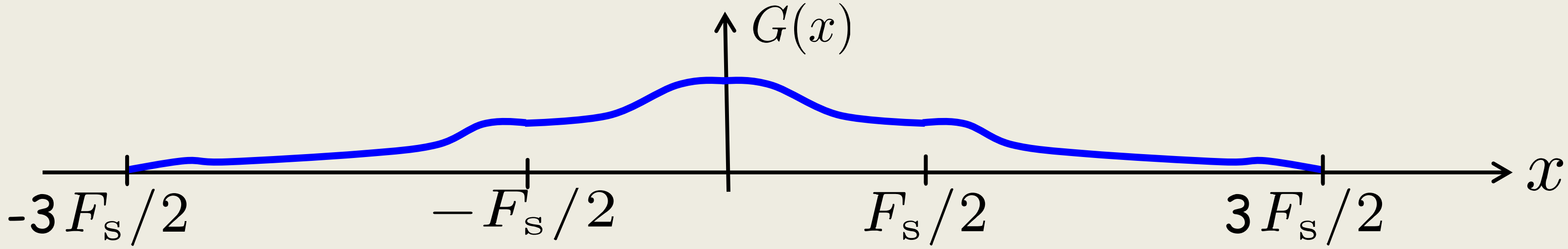


$$\frac{1}{F_\mathrm{s}}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

We next concentrate on this part

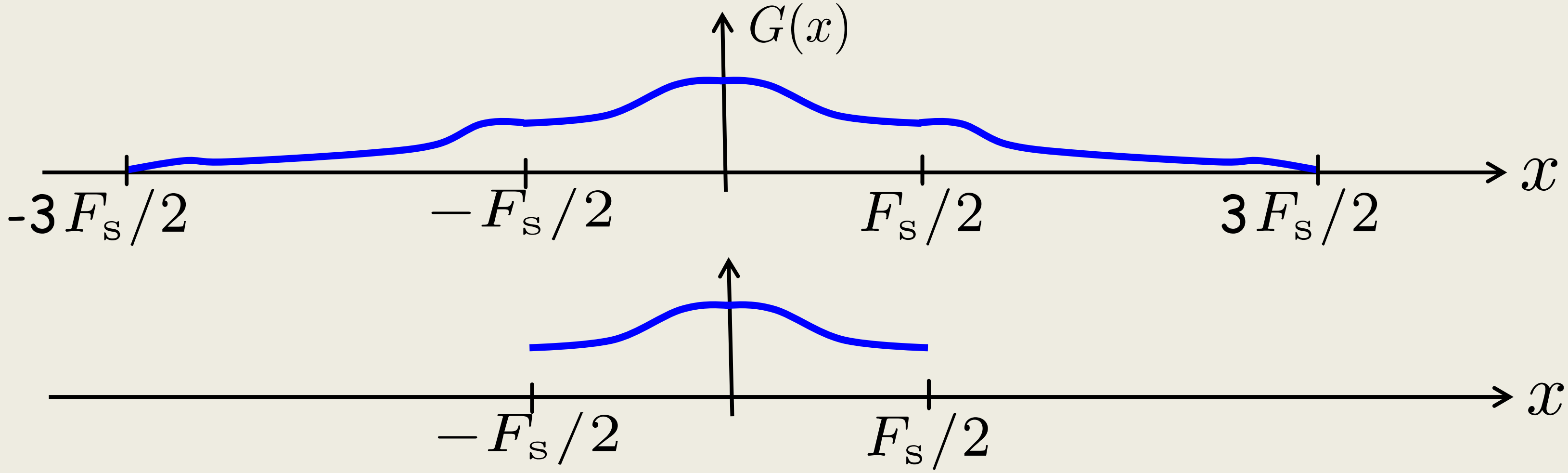
EITF75 Systems and Signals

Interlude Assume an integral $\int_{-\infty}^{\infty} G(x)\mathrm{d}x$



EITF75 Systems and Signals

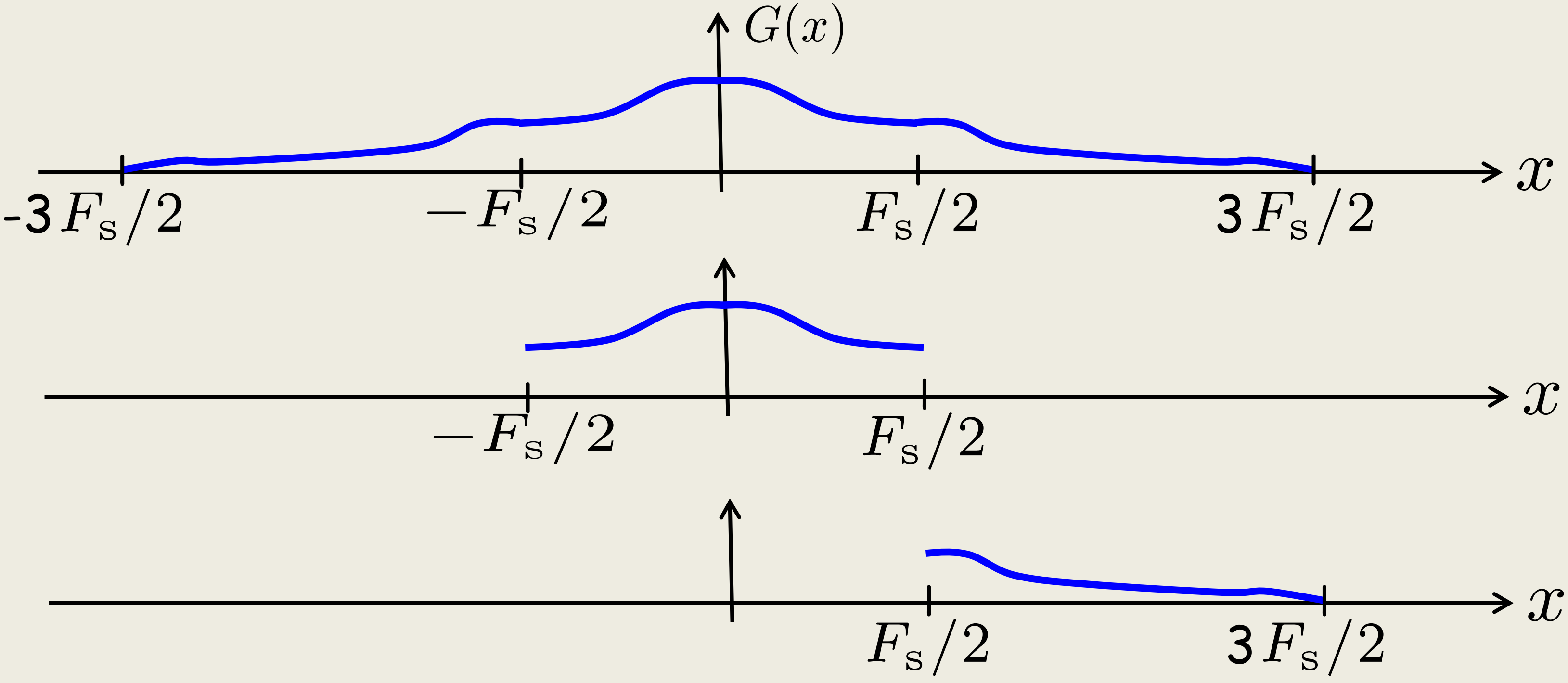
Interlude $\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} G(x)\mathrm{d}x +$



EITF75 Systems and Signals

Interlude

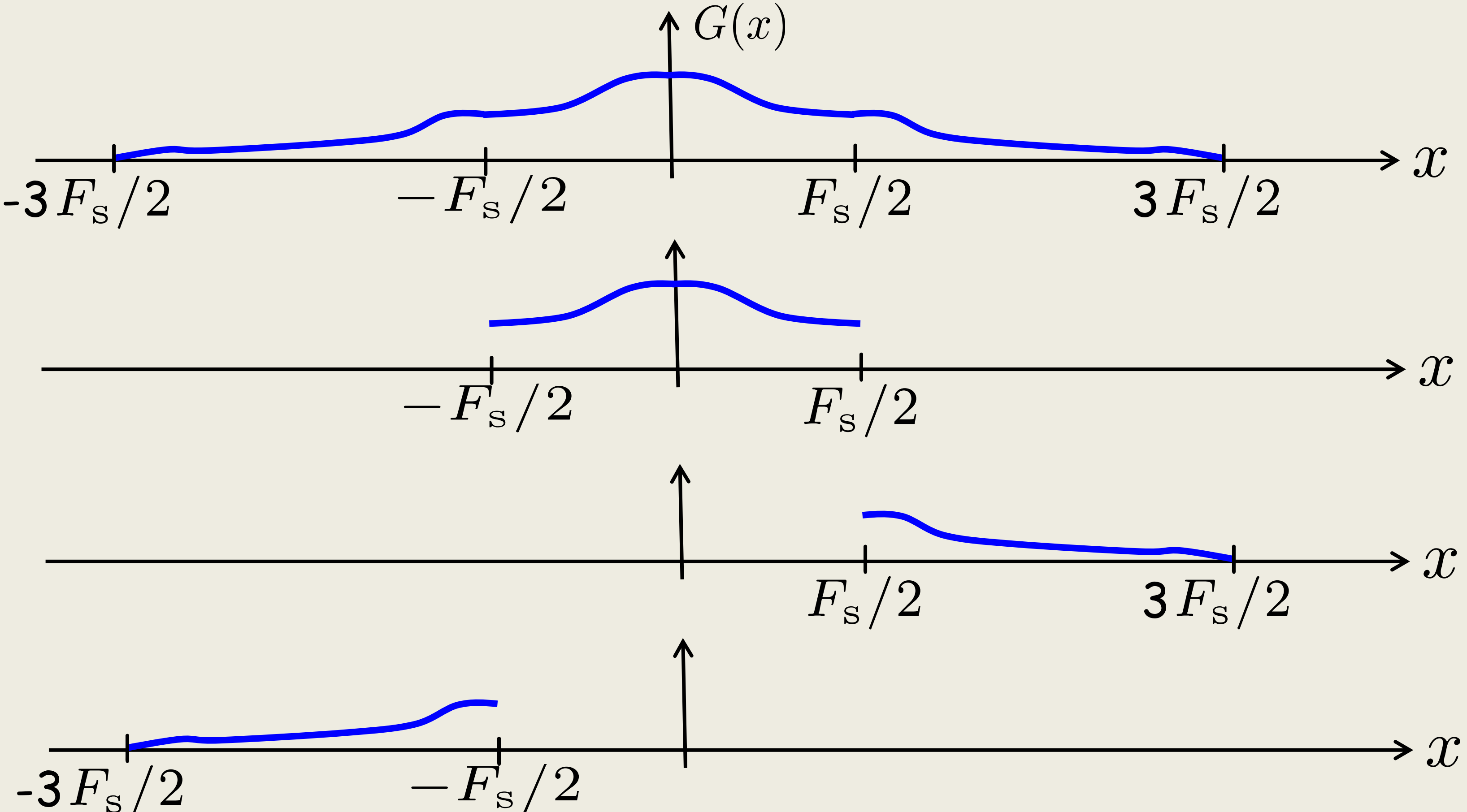
$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} G(x)\mathrm{d}x + \int_{F_\mathrm{s}/2}^{3F_\mathrm{s}/2} G(x)\mathrm{d}x +$$



EITF75 Systems and Signals

Interlude

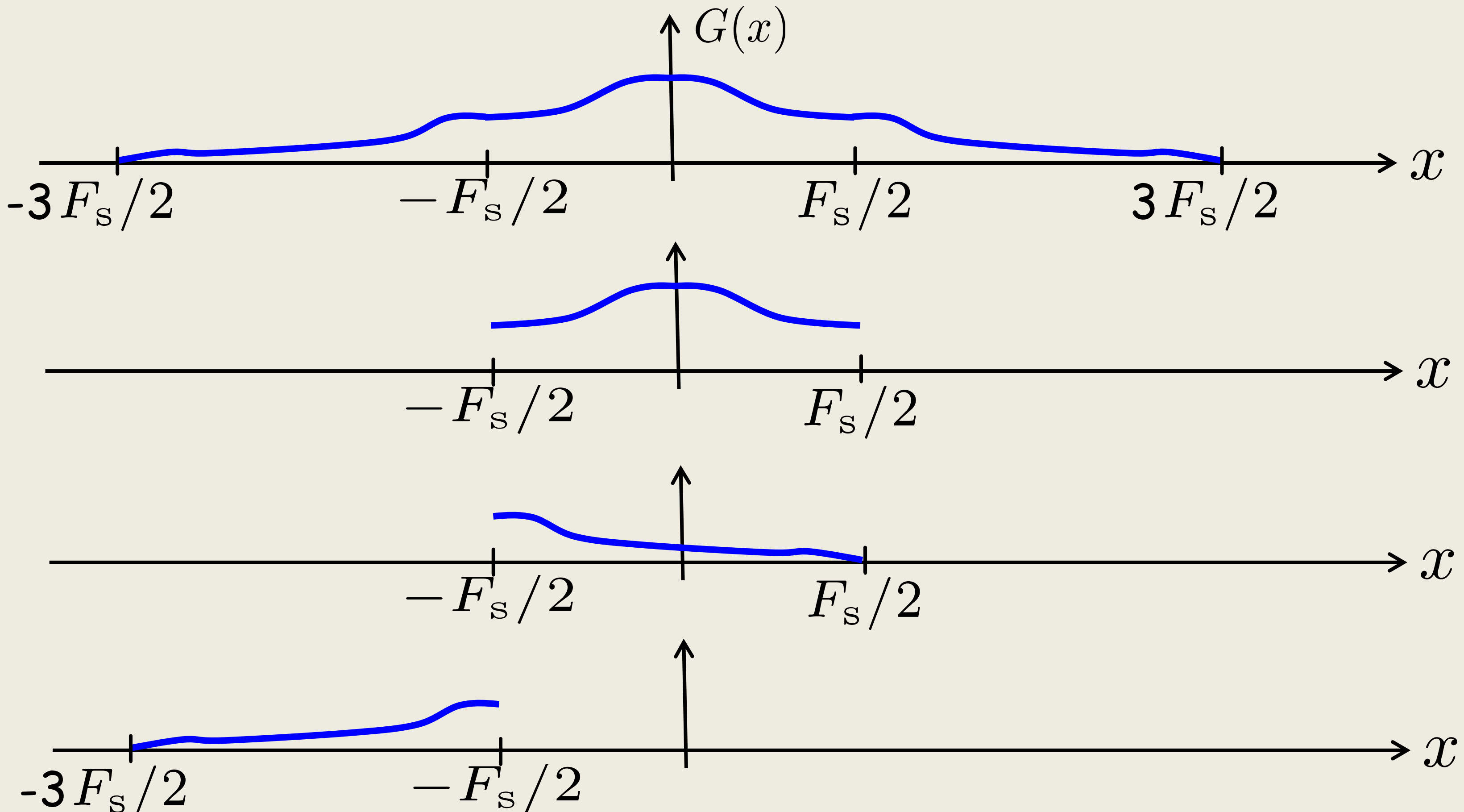
$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \int_{-F_s/2}^{F_s/2} G(x)\mathrm{d}x + \int_{F_s/2}^{3F_s/2} G(x)\mathrm{d}x + \int_{-3F_s/2}^{-F_s/2} G(x)\mathrm{d}x$$



EITF75 Systems and Signals

Interlude

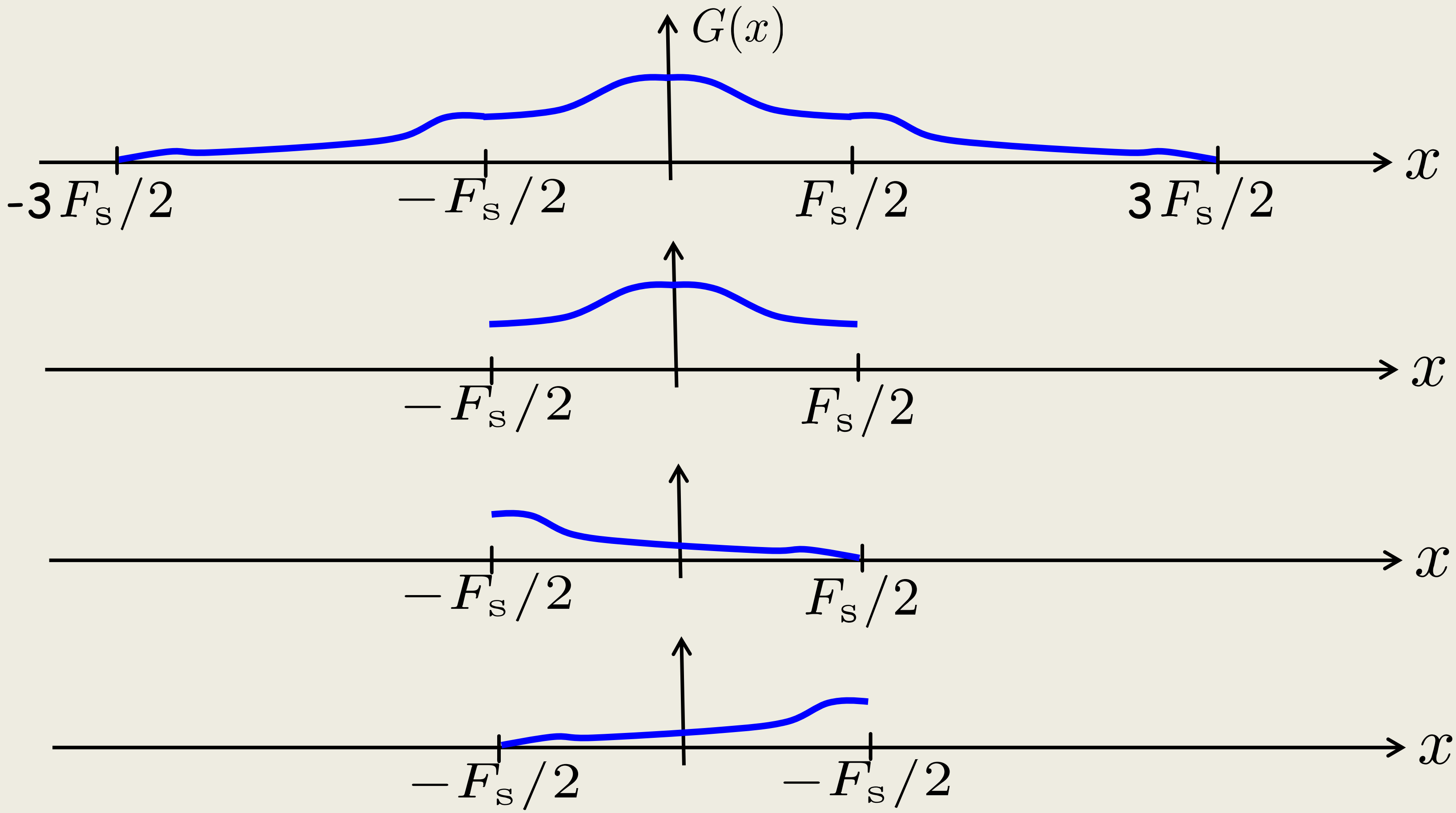
$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \int_{-F_s/2}^{F_s/2} G(x)\mathrm{d}x + \int_{-F_s/2}^{F_s/2} G(x+F_s)\mathrm{d}x + \int_{-3F_s/2}^{-F_s/2} G(x)\mathrm{d}x$$



EITF75 Systems and Signals

Interlude

$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \int_{-F_s/2}^{F_s/2} G(x)\mathrm{d}x + \int_{-F_s/2}^{F_s/2} G(x+F_s)\mathrm{d}x + \int_{-F_s/2}^{F_s/2} G(x-F_s)\mathrm{d}x$$



EITF75 Systems and Signals

Interlude

$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \int_{-F_s/2}^{F_s/2} G(x)\mathrm{d}x + \int_{-F_s/2}^{F_s/2} G(x+F_s)\mathrm{d}x + \int_{-F_s/2}^{F_s/2} G(x-F_s)\mathrm{d}x$$

Lesson learned

$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty} \int_{-F_s/2}^{F_s/2} G(x - kF_s)\mathrm{d}x$$

EITF75 Systems and Signals

Signal $\xrightarrow[x(t)]{X_a(F)}$ **sampling** $\xrightarrow[x(n)]{X(f)}$

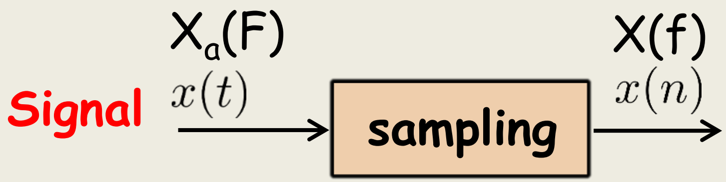
$$\frac{1}{F_\mathrm{s}}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

We next concentrate on this part

Lesson learned

$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} G(x - kF_\mathrm{s})\mathrm{d}x$$

EITF75 Systems and Signals



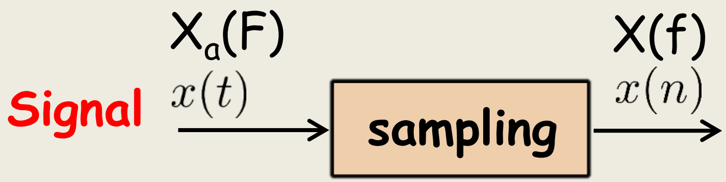
$$\frac{1}{F_s}\int_{-F_s/2}^{F_s/2} X\left(\frac{F}{F_s}\right)\mathrm{e}^{i2\pi nF/F_s}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_s}\mathrm{d}F$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_s/2}^{F_s/2} X_a(F-kF_s)\mathrm{e}^{i2\pi n(F-kF_s)/F_s}\mathrm{d}F$$

Lesson learned

$$\int_{-\infty}^{\infty} G(x)\mathrm{d}x = \sum_{k=-\infty}^{\infty}\int_{-F_s/2}^{F_s/2} G(x - kF_s)\mathrm{d}x$$

EITF75 Systems and Signals

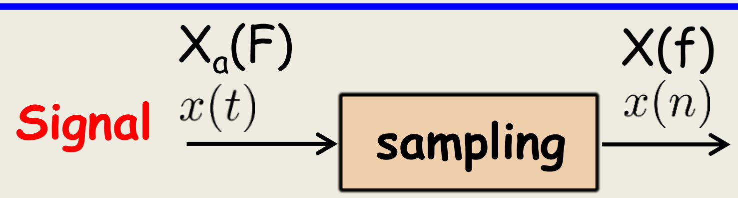


$$\frac{1}{F_{\mathrm{s}}}\int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2} X\left(\frac{F}{F_{\mathrm{s}}}\right)\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F \;=\int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F$$

$$=\sum_{k=-\infty}^{\infty}\int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2} X_a(F-kF_{\mathrm{s}})\mathrm{e}^{i2\pi n(F-kF_{\mathrm{s}})/F_{\mathrm{s}}}\mathrm{d}F$$

$$=\sum_{k=-\infty}^{\infty}\int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2} X_a(F-kF_{\mathrm{s}})\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F$$

EITF75 Systems and Signals



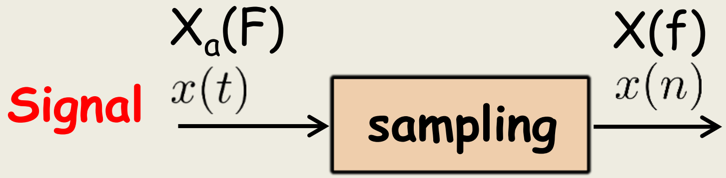
$$\frac{1}{F_\mathrm{s}}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X_a(F-kF_\mathrm{s})\mathrm{e}^{i2\pi n(F-kF_\mathrm{s})/F_\mathrm{s}}\mathrm{d}F$$

Not dependent on k – pull out

$$= \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X_a(F-kF_\mathrm{s})\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

EITF75 Systems and Signals



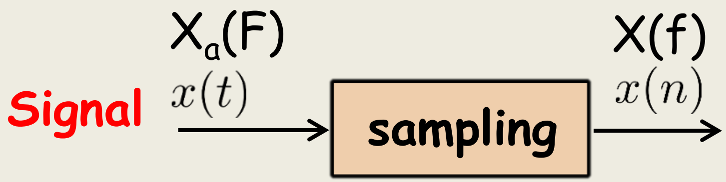
$$\frac{1}{F_\mathrm{s}}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X_a(F-kF_\mathrm{s})\mathrm{e}^{i2\pi n(F-kF_\mathrm{s})/F_\mathrm{s}}\mathrm{d}F$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X_a(F-kF_\mathrm{s})\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

$$= \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_\mathrm{s})\right]\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

EITF75 Systems and Signals



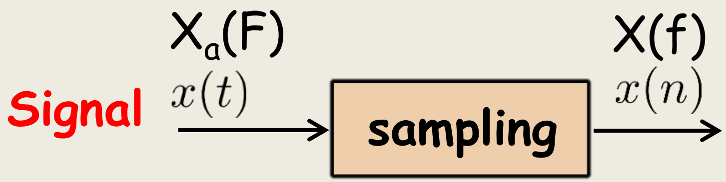
$$\frac{1}{F_\mathrm{s}}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F = \int_{-\infty}^{\infty} X_a(F)\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X_a(F-kF_\mathrm{s})\mathrm{e}^{i2\pi n(F-kF_\mathrm{s})/F_\mathrm{s}}\mathrm{d}F$$

$$= \sum_{k=-\infty}^{\infty}\int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X_a(F-kF_\mathrm{s})\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

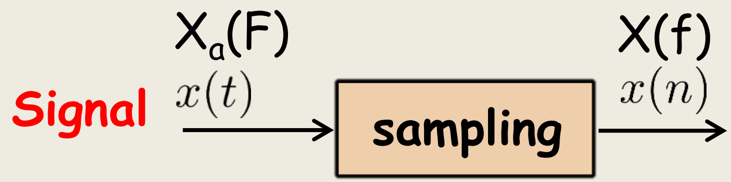
$$= \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_\mathrm{s})\right]\mathrm{e}^{i2\pi nF/F_\mathrm{s}}\mathrm{d}F$$

EITF75 Systems and Signals



$$
\begin{aligned}
&\frac{1}{F_{\mathrm{s}}}\int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2} X\left(\frac{F}{F_{\mathrm{s}}}\right)\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F \\
&= \int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_{\mathrm{s}})\right]\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F
\end{aligned}
$$

EITF75 Systems and Signals



x(n) = $$\frac{1}{F_s}\int_{-F_s/2}^{F_s/2} X\left(\frac{F}{F_s}\right) e^{i2\pi nF/F_s}\,\mathrm{d}F$$

$$= \int_{-F_s/2}^{F_s/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_s)\right] e^{i2\pi nF/F_s}\,\mathrm{d}F$$

Remember what this is:
Top line: DTFT representation of x(n)

EITF75 Systems and Signals

Signal

$$\underset{x(t)}{\overset{X_a(F)}{\longrightarrow}} \boxed{\textbf{sampling}} \underset{x(n)}{\overset{X(f)}{\longrightarrow}}$$

$$\frac{1}{F_\mathrm{s}} \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right) \mathrm{e}^{i2\pi nF/F_\mathrm{s}} \mathrm{d}F$$

$$x(n) = \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{s})\right] \mathrm{e}^{i2\pi nF/F_\mathrm{s}} \mathrm{d}F$$

Remember what this is:
Top line: DTFT representation of x(n)
Bottom: Fourier representation of x(t | t=n/Fs) = x(n)

EITF75 Systems and Signals

Signal

$X_a(F)$ $X(f)$

$x(t) \longrightarrow$ **sampling** $\longrightarrow x(n)$

$$\frac{1}{F_\mathrm{s}} \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} X\left(\frac{F}{F_\mathrm{s}}\right) \mathrm{e}^{i2\pi nF/F_\mathrm{s}} \mathrm{d}F$$

$$= \int_{-F_\mathrm{s}/2}^{F_\mathrm{s}/2} \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{s})\right] \mathrm{e}^{i2\pi nF/F_\mathrm{s}} \mathrm{d}F$$

$$\frac{1}{F_\mathrm{s}} X\left(\frac{F}{F_\mathrm{s}}\right) = \left[\sum_{k=-\infty}^{\infty} X_a(F - kF_\mathrm{s})\right]$$

By inspection

DTFT of x(n)   Fourier transform of x(t)

EITF75 Systems and Signals

Signal

$$
\begin{aligned}
&\frac{1}{F_{\mathrm{s}}}\int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2} X\left(\frac{F}{F_{\mathrm{s}}}\right) \mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F \\
&= \int_{-F_{\mathrm{s}}/2}^{F_{\mathrm{s}}/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_{\mathrm{s}})\right]\mathrm{e}^{i2\pi nF/F_{\mathrm{s}}}\mathrm{d}F
\end{aligned}
$$

$$
X\left(\frac{F}{F_{\mathrm{s}}}\right) = F_{\mathrm{s}}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_{\mathrm{s}})\right]
$$

manipulation

EITF75 Systems and Signals

Signal

$X_a(F)$ $\qquad$ $X(f)$

$x(t) \longrightarrow$ **sampling** $\longrightarrow x(n)$

$$\frac{1}{F_s}\int_{-F_s/2}^{F_s/2} X\left(\frac{F}{F_s}\right) \mathrm{e}^{i2\pi nF/F_s}\mathrm{d}F$$

$$= \int_{-F_s/2}^{F_s/2}\left[\sum_{k=-\infty}^{\infty} X_a(F-kF_s)\right]\mathrm{e}^{i2\pi nF/F_s}\mathrm{d}F$$

$$X(f) = F_s\sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

Variable change

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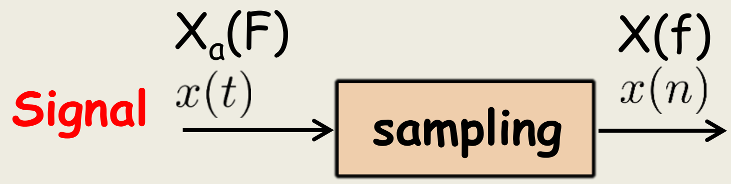


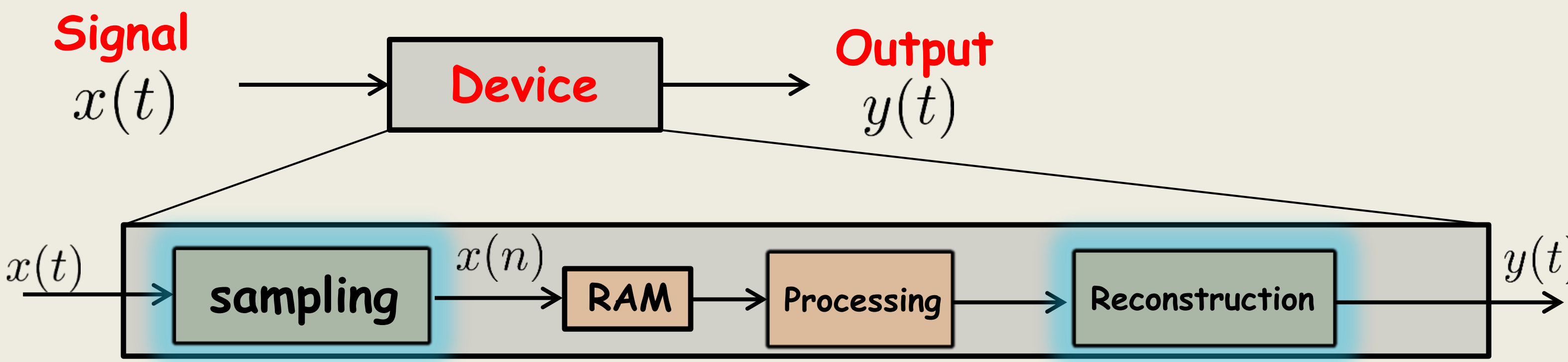
Goal: $X(f)$ = formula in $X_a(F)$ and F_s

Completed

$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

Very important formula

SLIDE 3



An important part of digital signal processing:
Process time-continuous signals digitally

To do so, we need

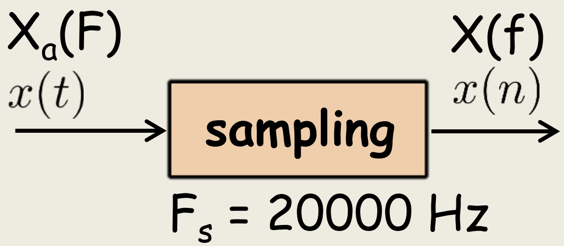
1. Study effects of sampling (A/D). DONE $X(f) = F_{\mathrm{s}} \sum_{k=-\infty}^{\infty} X_a((f-k)F_{\mathrm{s}})$

2. Study ways to implement (D/A)

3. Understand when 1&2 are optimally done

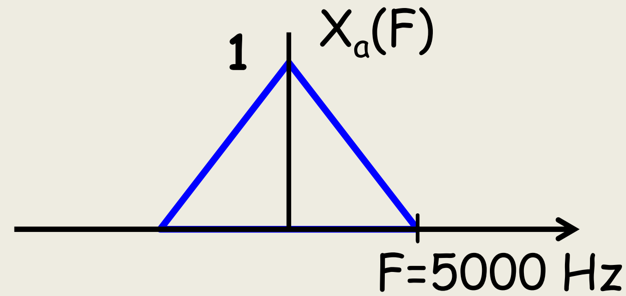
EITF75 Systems and Signals

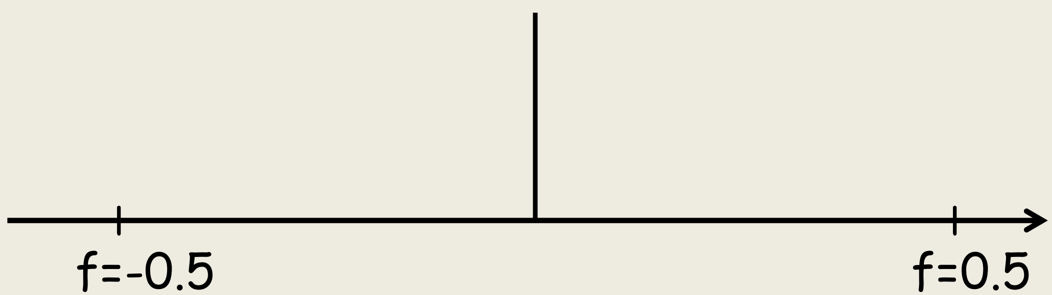
Signal



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

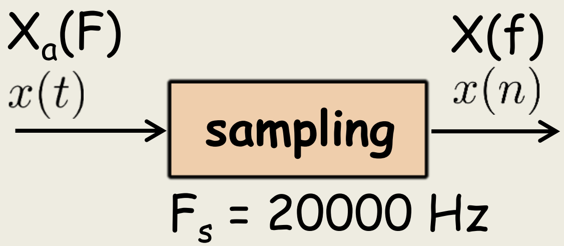
Example





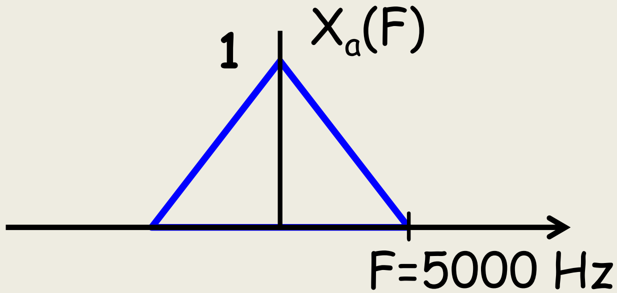
EITF75 Systems and Signals

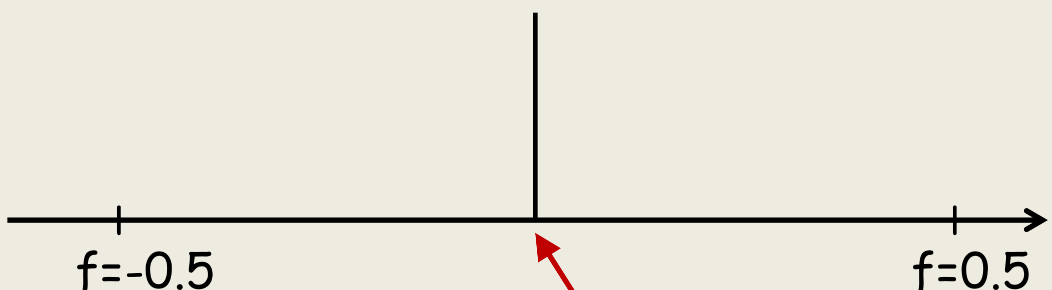
Signal



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

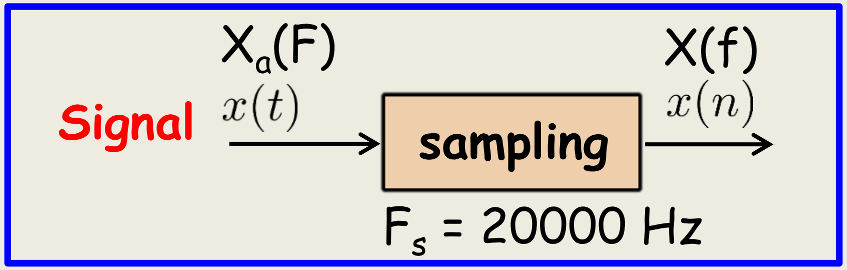
Example





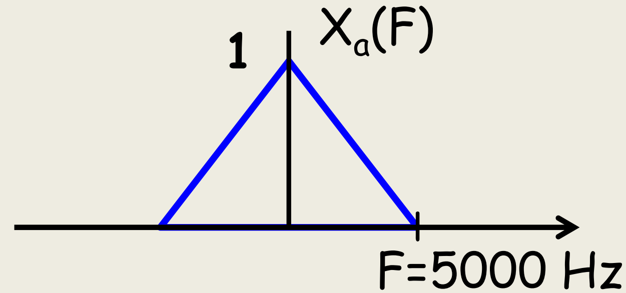
Compute X(0)

EITF75 Systems and Signals

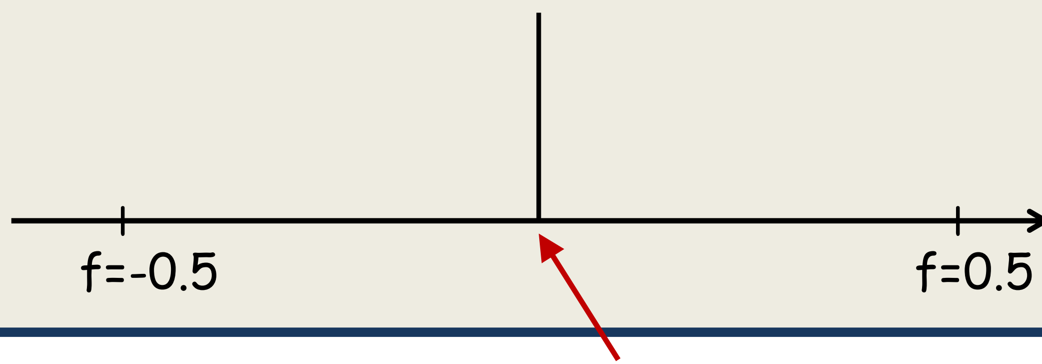


$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

Example



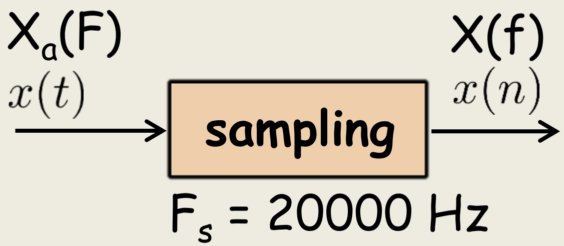
$$X(0) = F_s\left[\cdots + X_a(-2F_s) + X_a(-F_s) + X_a(0) + X_a(F_s) + X_a(2F_s) + \cdots\right]$$



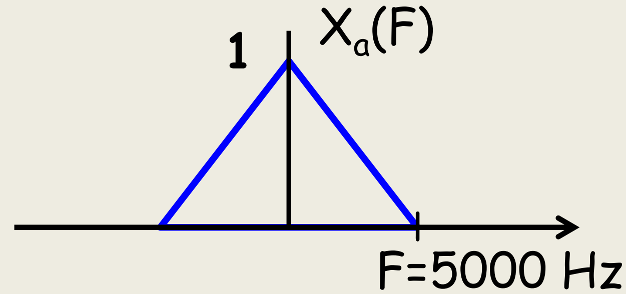
Compute X(0)

EITF75 Systems and Signals

Signal



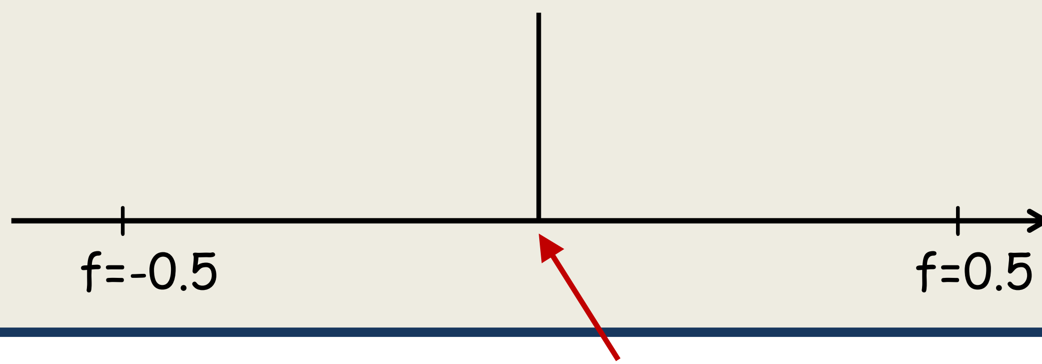
Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

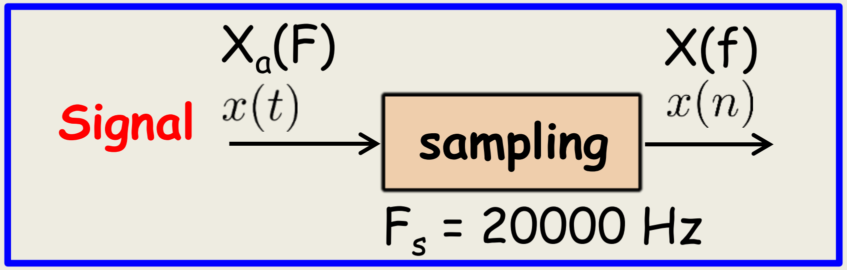
$$X(0) = F_s\left[\cdots + X_a(-2F_s) + X_a(-F_s) + X_a(0) + X_a(F_s) + X_a(2F_s) + \cdots\right]$$

$X_a(-40000) = 0$

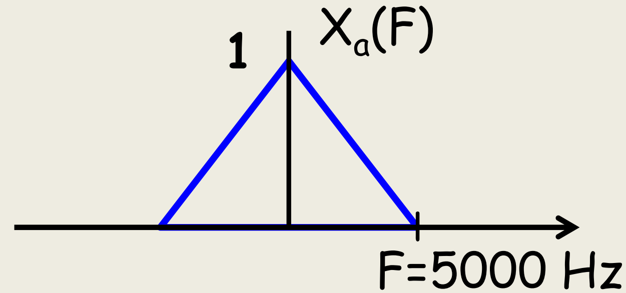


Compute X(0)

EITF75 Systems and Signals



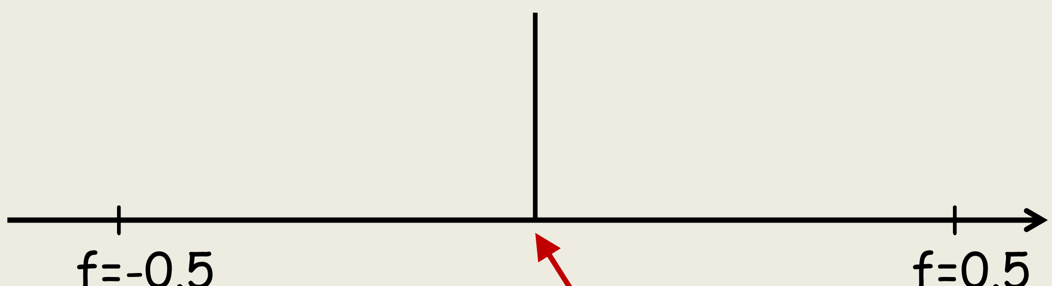
Example



$$\boxed{X(f) = F_\mathrm{s} \sum_{k=-\infty}^{\infty} X_a((f-k)F_\mathrm{s})}$$

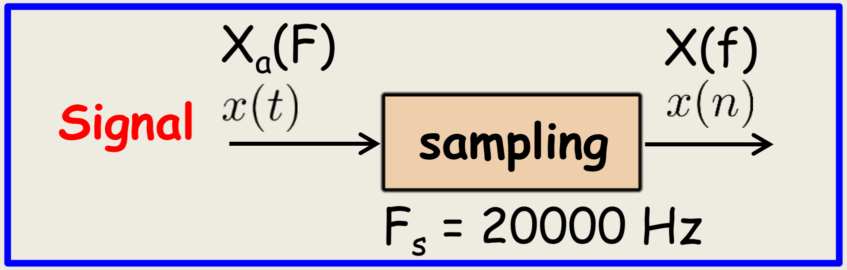
$$X(0) = F_\mathrm{s}\left[\cdots + \cancel{X_a(-2F_\mathrm{s})} + X_a(-F_\mathrm{s}) + X_a(0) + X_a(F_\mathrm{s}) + X_a(2F_\mathrm{s}) + \cdots\right]$$

$X_a(-20000) = 0$

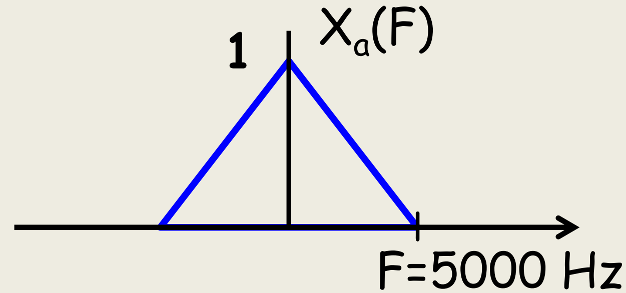


Compute X(0)

EITF75 Systems and Signals



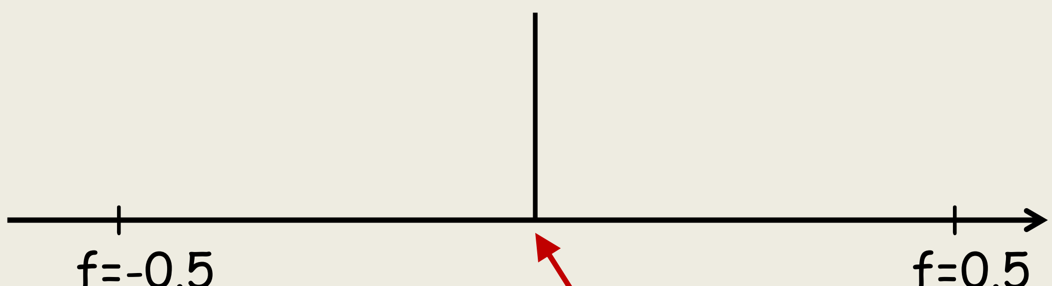
Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

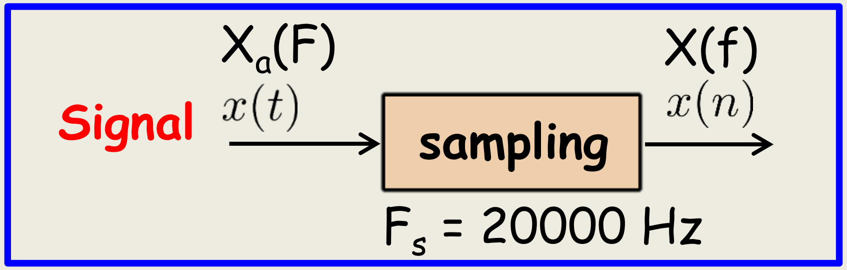
$$X(0) = F_s\left[\cdots + \cancel{X_a(-2F_s)} + \cancel{X_a(-F_s)} + X_a(0) + X_a(F_s) + X_a(2F_s) + \cdots\right]$$

$X_a(0) = 1$

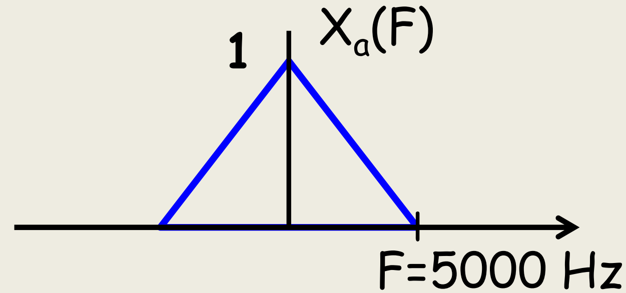


Compute X(0)

EITF75 Systems and Signals



Example

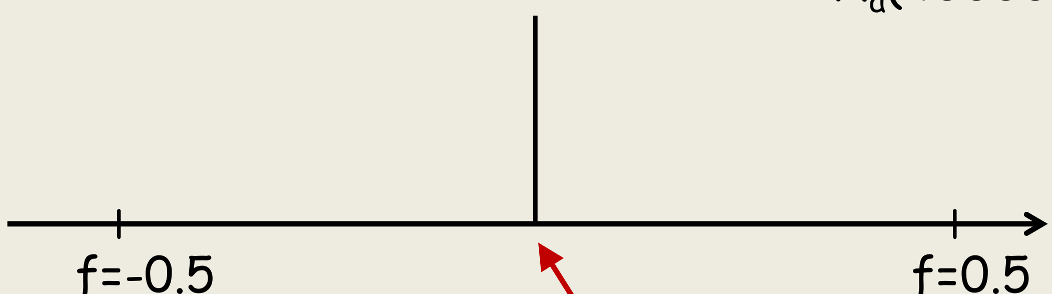


$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0) = F_s\left[\cdots + \cancel{X_a(-2F_s)} + \cancel{X_a(-F_s)} + 1 + \cancel{X_a(F_s)} + \cancel{X_a(2F_s)} + \cdots\right]$$

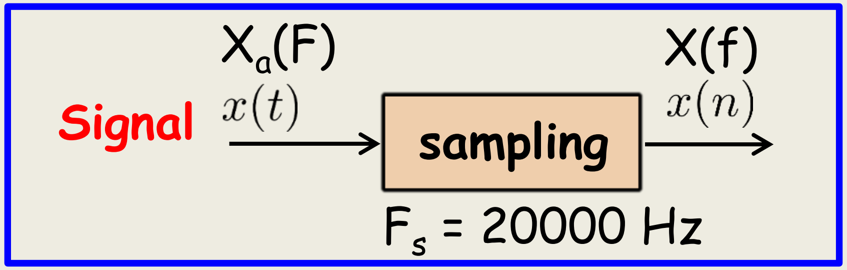
$X_a(20000) = 0$

$X_a(40000) = 0$

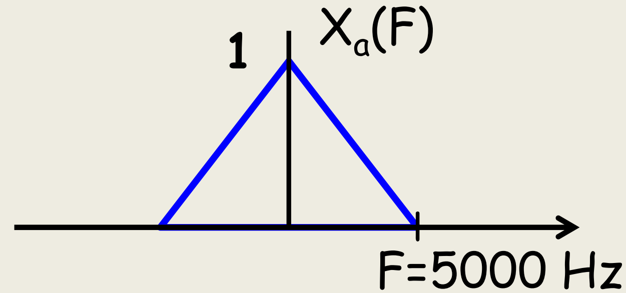


Compute X(0)

EITF75 Systems and Signals

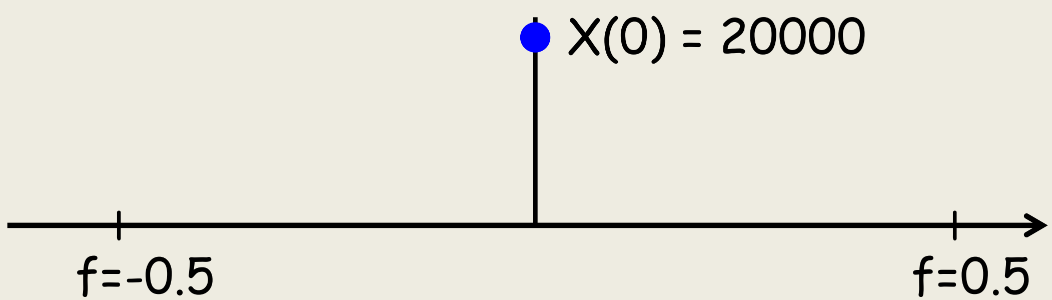


Example



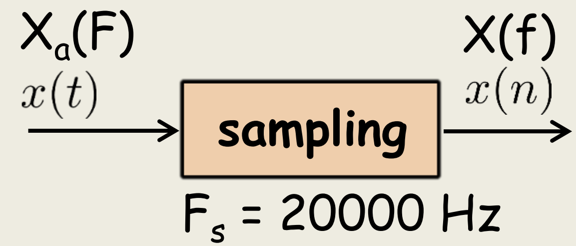
$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0) = F_s \left[\cdots + \cancel{X_a(-2F_s)} + \cancel{X_a(-F_s)} + \quad 1 \quad + \cancel{X_a(F_s)} + \cancel{X_a(2F_s)} + \cdots\right]$$



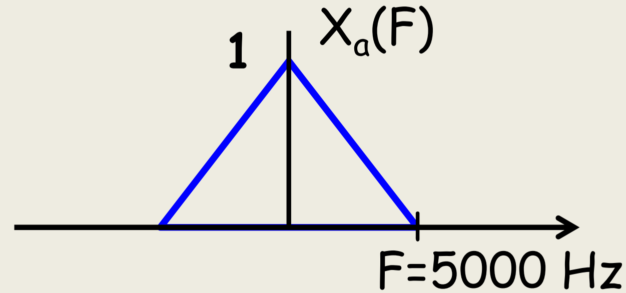
EITF75 Systems and Signals

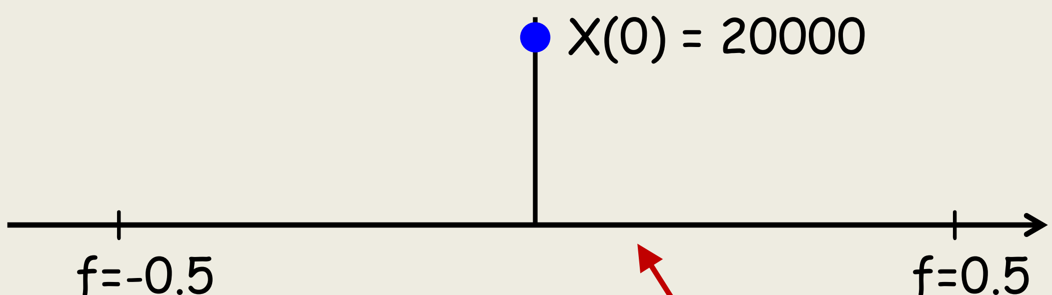
Signal



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

Example

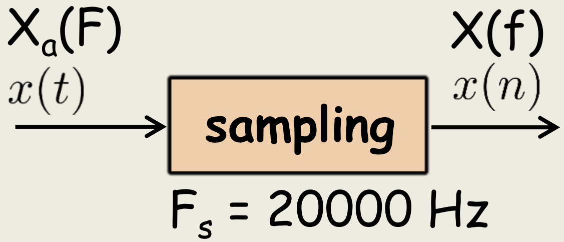




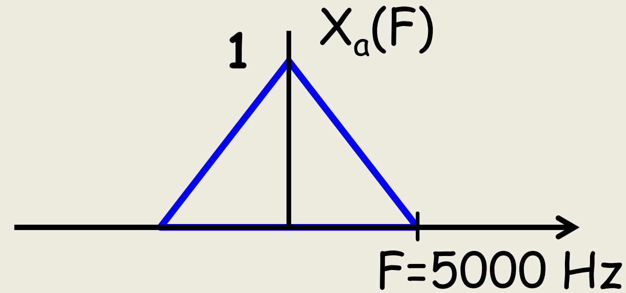
Compute X(1/8)

EITF75 Systems and Signals

Signal

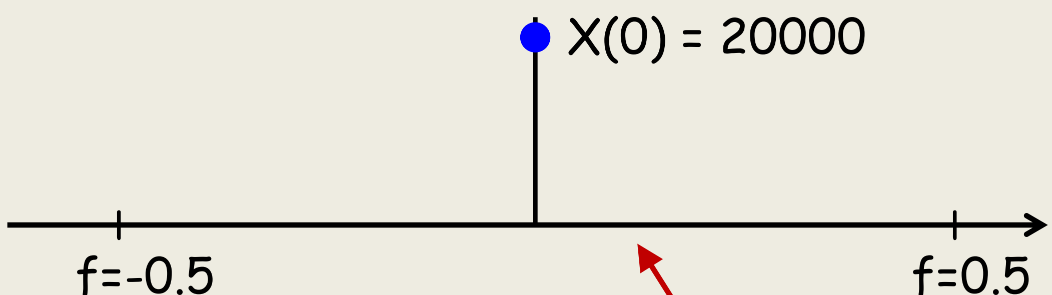


Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

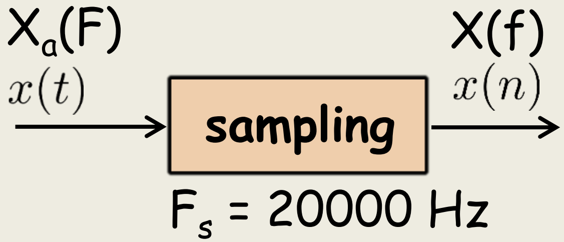
$$X(1/8) = F_s\left[\cdots + X_a((1/8-1)F_s) + X_a(F_s/8) + X_a((1/8+1)F_s) + \cdots\right]$$



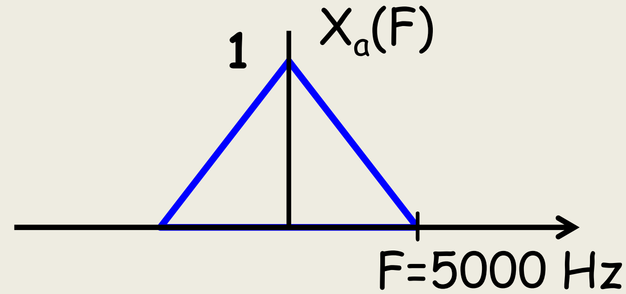
Compute X(1/8)

EITF75 Systems and Signals

Signal



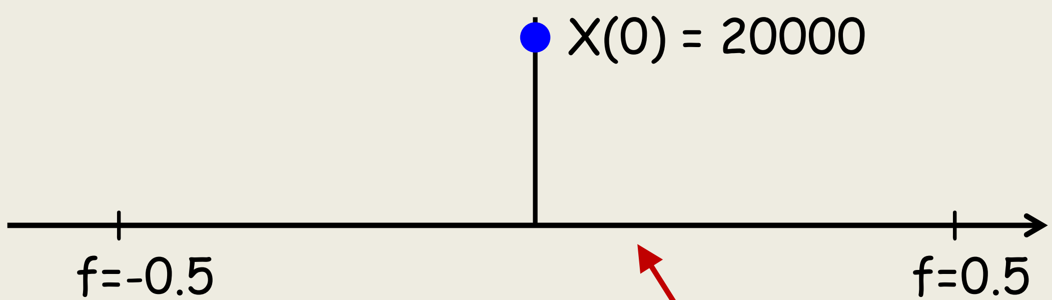
Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(1/8) = F_s\left[\cdots + X_a((1/8-1)F_s) + X_a(F_s/8) + X_a((1/8+1)F_s) + \cdots\right]$$

$$X(1/8) = F_s\left[\cdots + X_a(-7F_s/8) + X_a(F_s/8) + X_a(9F_s/8) + \cdots\right]$$



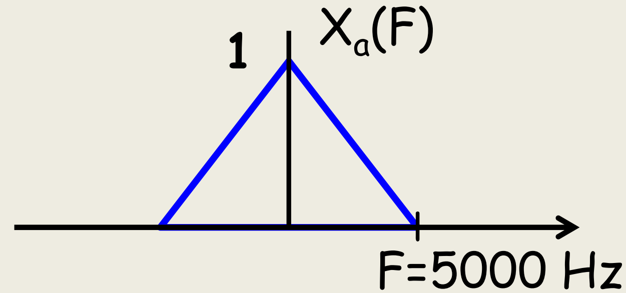
Compute X(1/8)

EITF75 Systems and Signals

Signal

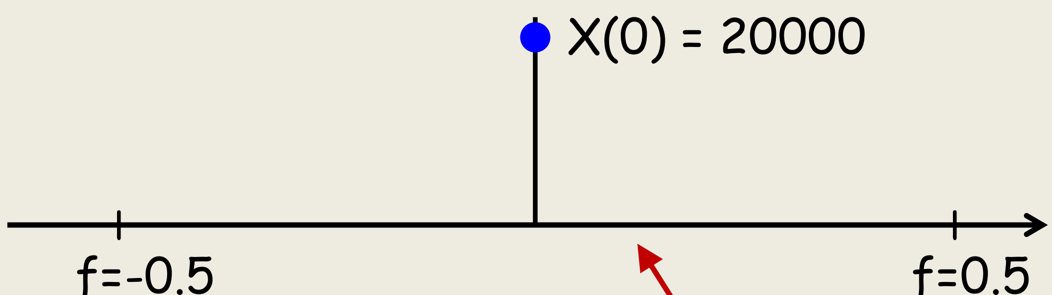
F_s = 20000 Hz

Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

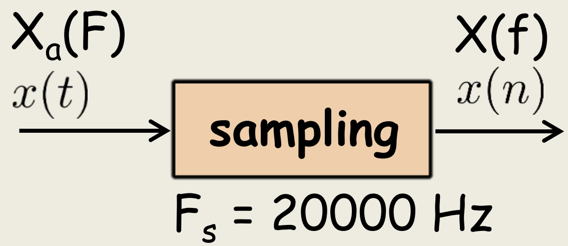
$$X(1/8) = F_s\left[\cdots + X_a(-17500) + X_a(2500) + X_a(22500) + \cdots\right]$$



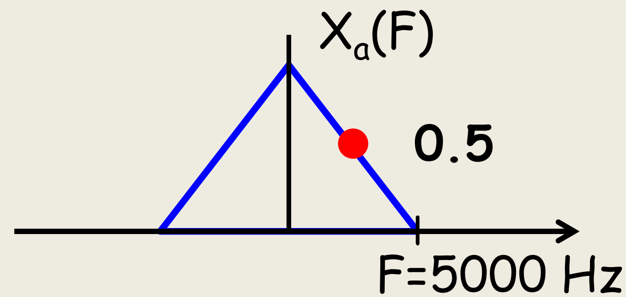
Compute X(1/8)

EITF75 Systems and Signals

Signal

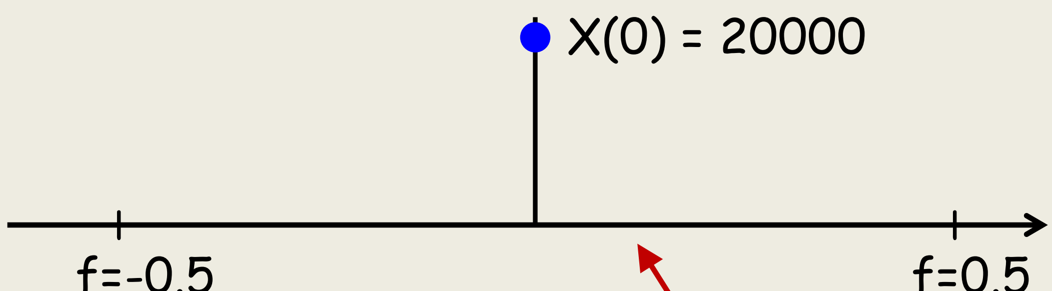


Example



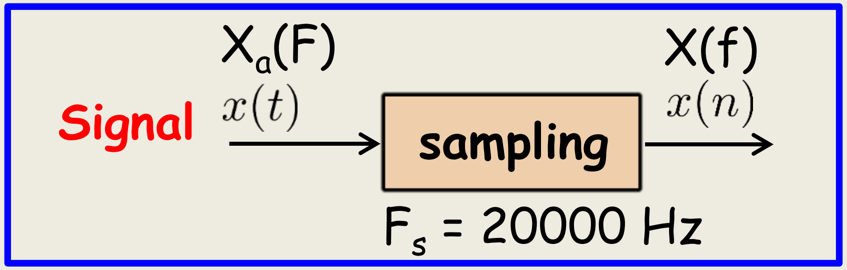
$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(1/8) = F_s\left[\cdots + \cancel{X_a(-17500)} + X_a(2500) + \cancel{X_a(22500)} + \cdots\right]$$

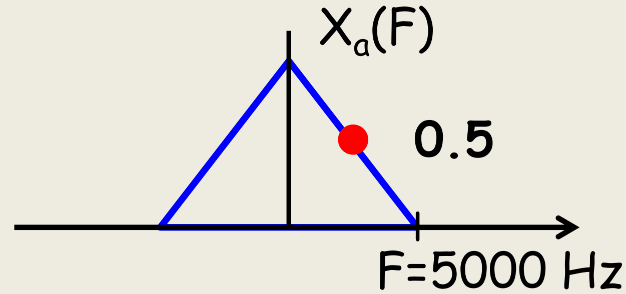


Compute X(1/8)

EITF75 Systems and Signals

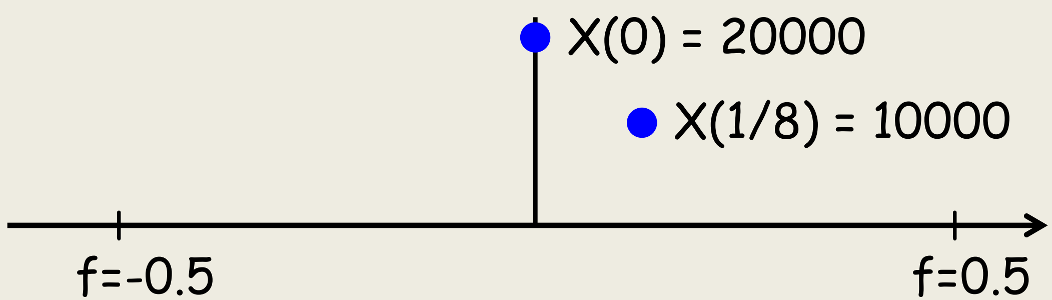


Example

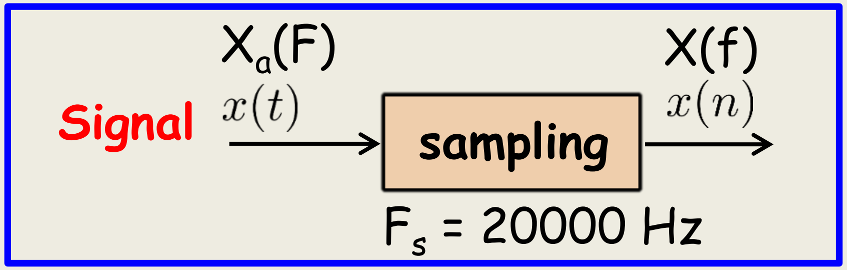


$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

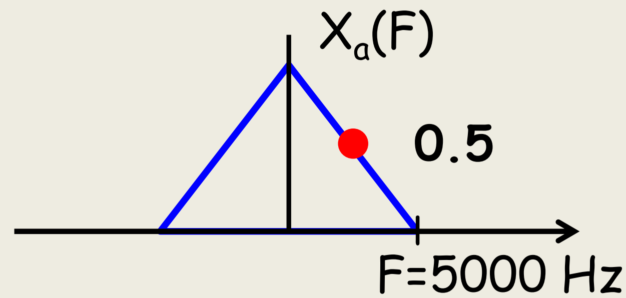
$$X(1/8) = F_s\left[\cdots + \cancel{X_a(-17500)} + X_a(2500) + \cancel{X_a(22500) +} \cdots\right]$$



EITF75 Systems and Signals

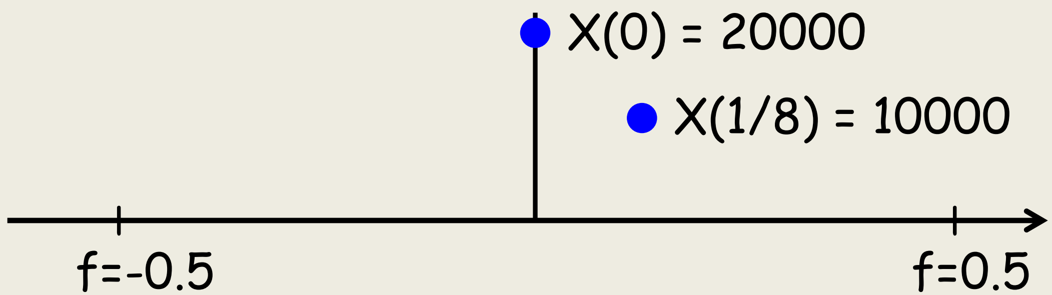


Example

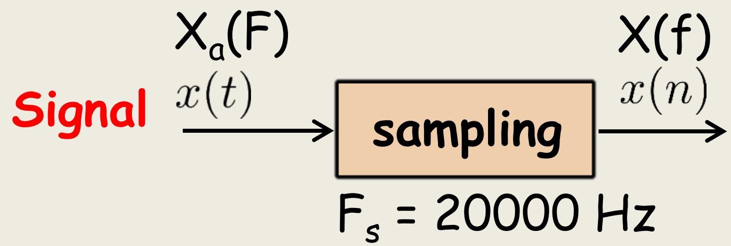


$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

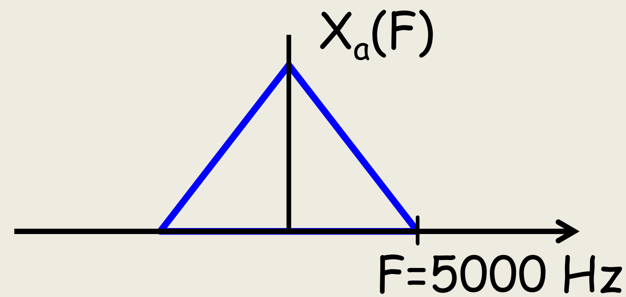
$$X(1/8) = F_s\left[\cdots + \underset{-F_s\,+\,f\,\times\,F_s}{\cancel{X_a(-17500)}} + \overset{f\,\times\,F_s}{X_a(2500)} + \overset{F_s\,+\,f\,\times\,F_s}{\cancel{X_a(22500) +}} \cdots\right]$$



EITF75 Systems and Signals

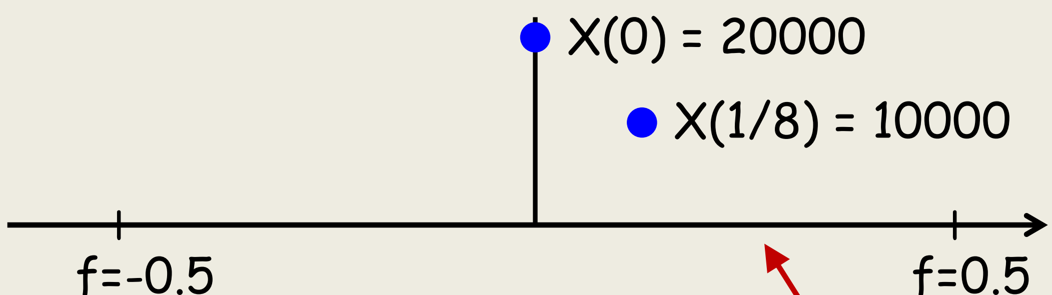


Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(1/4) = F_s\left[\cdots + X_a(-F_s + f \times F_s) + X_a(f \times F_s) + X_a(F_s + f \times F_s) + \cdots\right]$$



Compute X(1/4)

EITF75 Systems and Signals

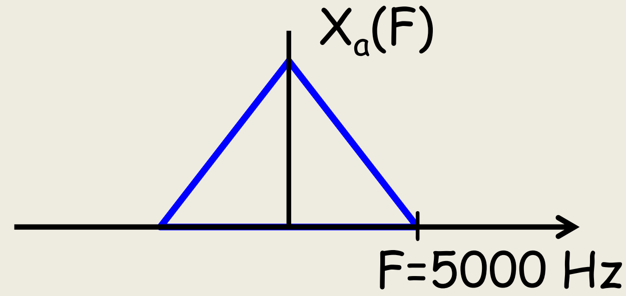
Signal

$X_a(F)$ $X(f)$

$x(t) \rightarrow$ **sampling** $\rightarrow x(n)$

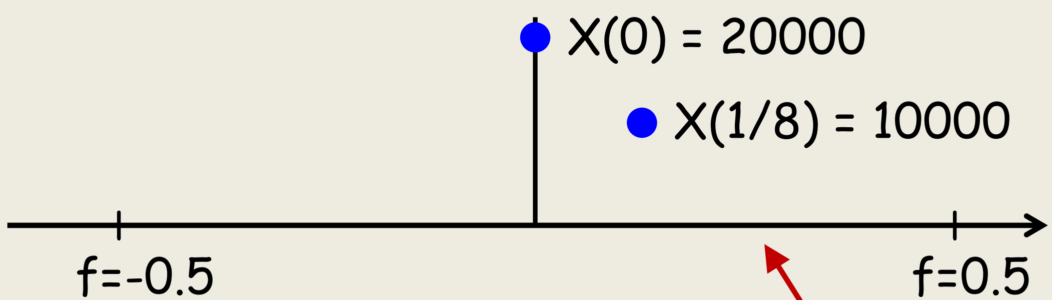
F_s = 20000 Hz

Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

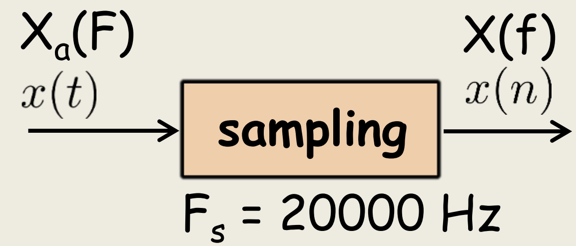
$$X(1/4) = F_s\left[\cdots + X_a(\underbrace{-15000}_{-F_s + f \times F_s}) + X_a(\overbrace{5000}^{f \times F_s}) + X_a(\overbrace{25000}^{F_s + f \times F_s}) + \cdots\right]$$



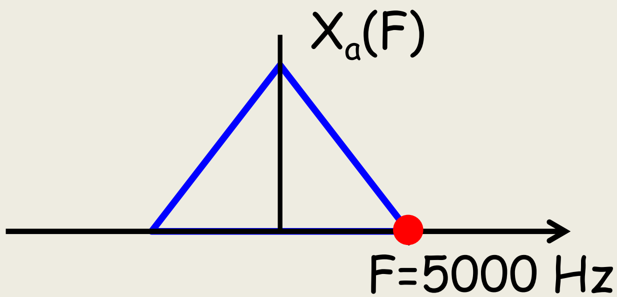
Compute X(1/4)

EITF75 Systems and Signals

Signal

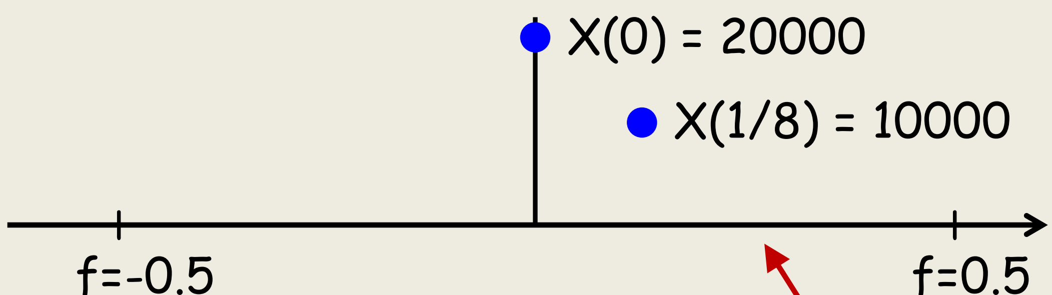


Example



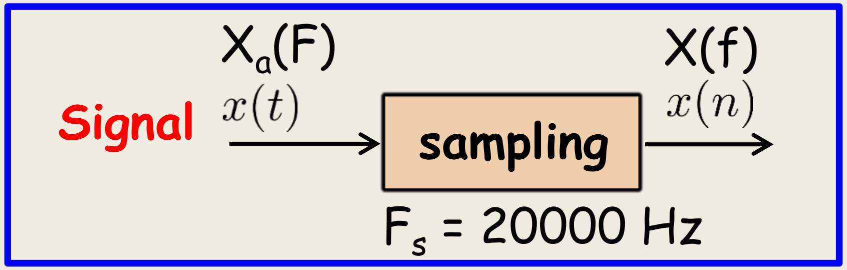
$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(1/4) = F_s\left[\cdots + \underset{-F_s + f \times F_s}{\cancel{X_a(-15000)}} + X_a(\overset{f \times F_s}{5000}) + \cancel{X_a(\overset{F_s + f \times F_s}{25000})} + \cdots\right]$$

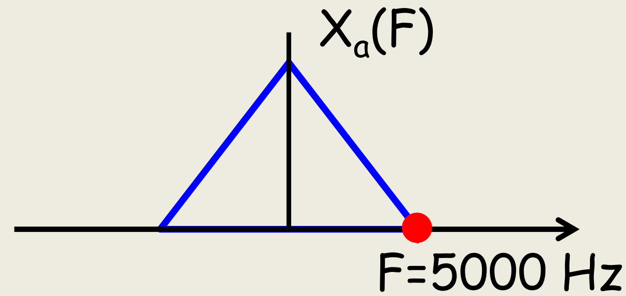


Compute X(1/4)

EITF75 Systems and Signals

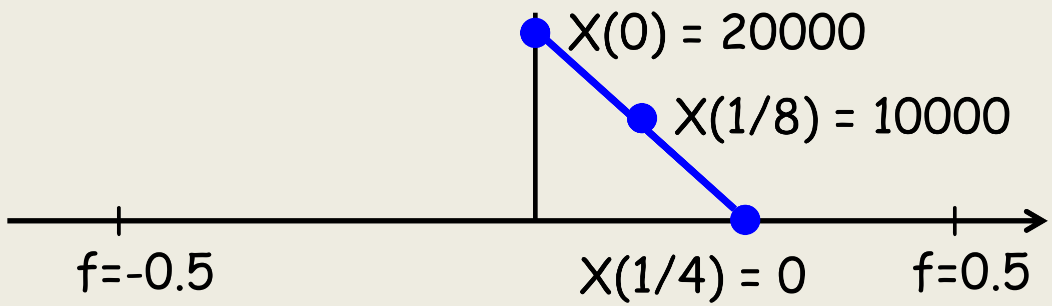


Example



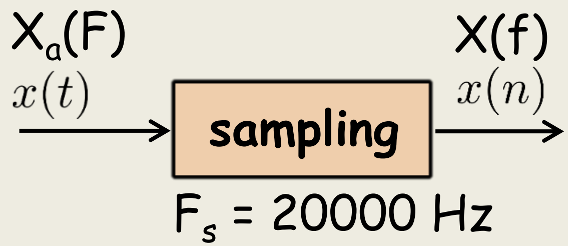
$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(1/4) = F_s\left[\cdots + \cancel{X_a(\underbrace{-15000}_{-F_s + f \times F_s})} + X_a(\overbrace{5000}^{f \times F_s}) + \cancel{X_a(\overbrace{25000}^{F_s + f \times F_s})} + \cdots\right]$$

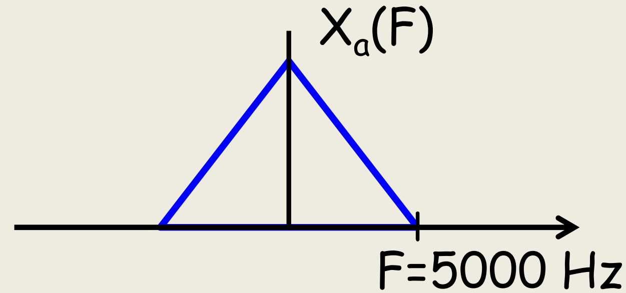


EITF75 Systems and Signals

Signal

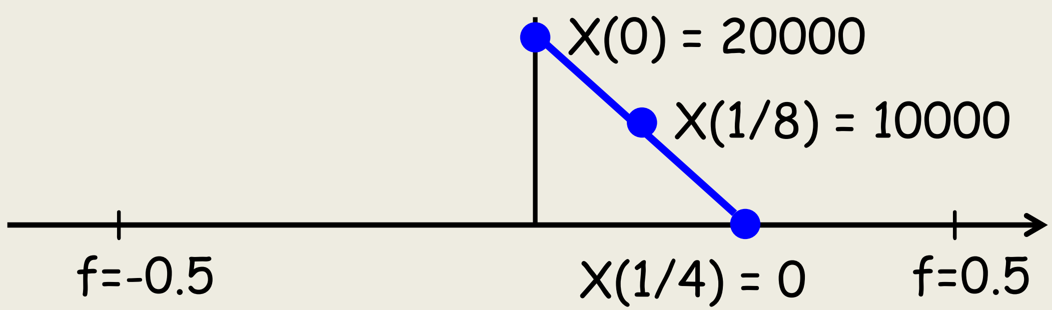


Example

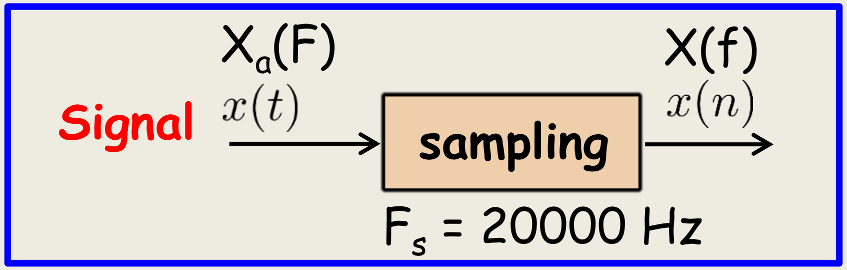


$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

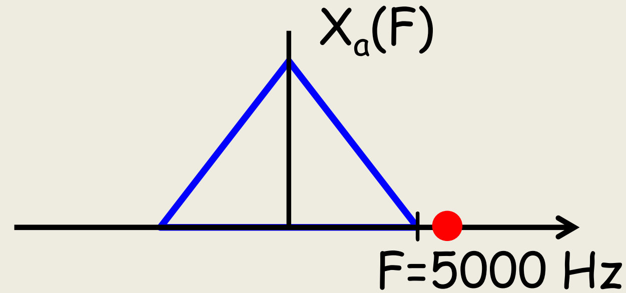
$$X(0.3) = F_s\left[\cdots + X_a(\underbrace{-14000}_{-F_s + f \times F_s}) + X_a(\overbrace{6000}^{f \times F_s}) + X_a(\overbrace{26000}^{F_s + f \times F_s}) + \cdots\right]$$



EITF75 Systems and Signals

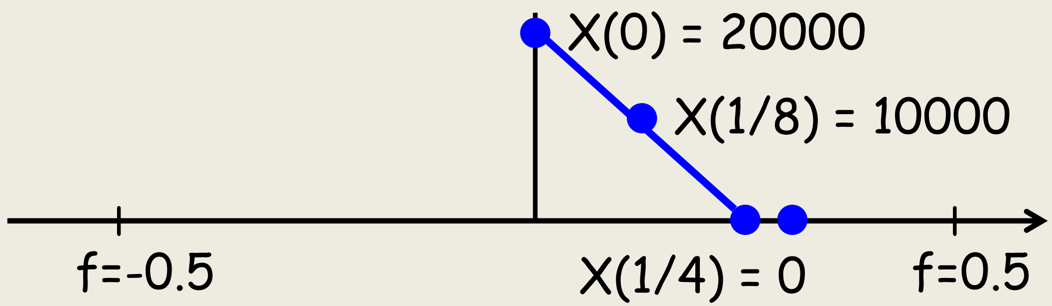


Example



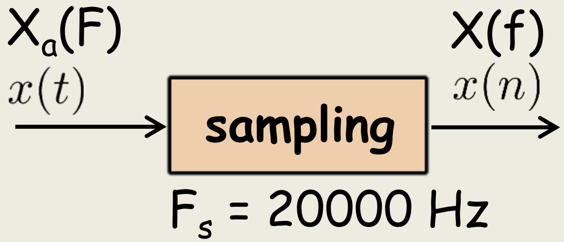
$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0.3) = F_s\left[\cdots + \underbrace{\cancel{X_a(-14000)}}_{-F_s\,+\,f\,\times\,F_s} + \overbrace{\cancel{X_a(6000)}}^{f\,\times\,F_s} + \overbrace{\cancel{X_a(26000)}}^{F_s\,+\,f\,\times\,F_s} + \cdots\right]$$

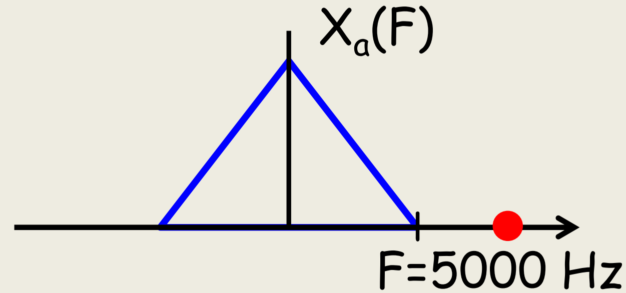


EITF75 Systems and Signals

Signal

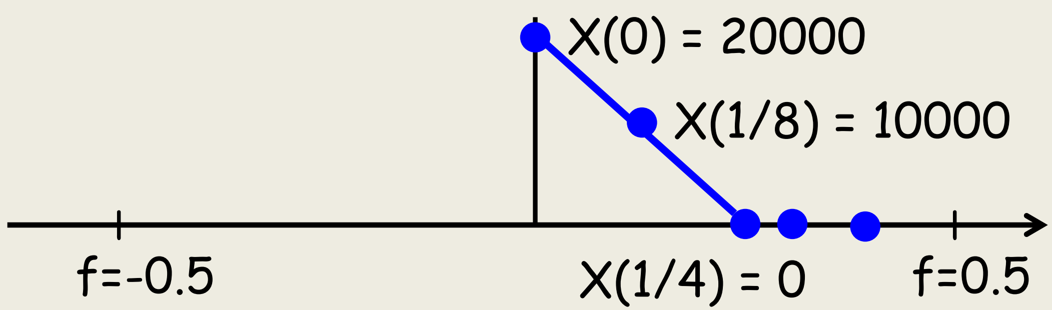


Example

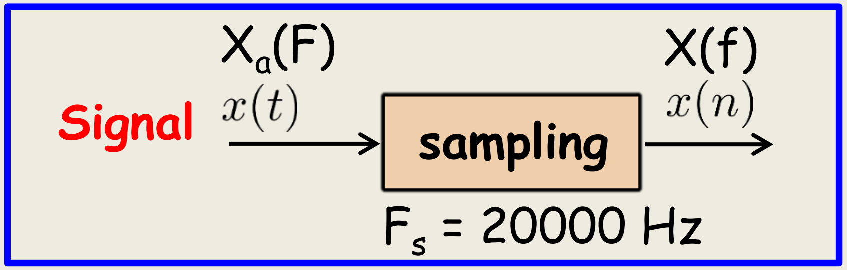


$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

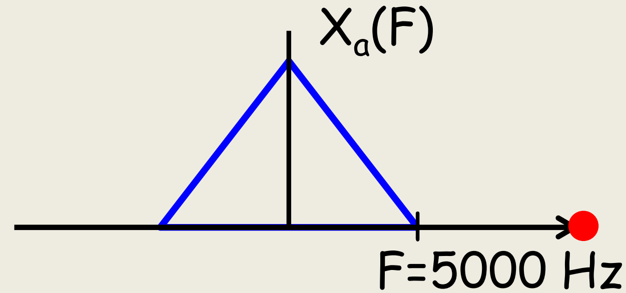
$$X(0.4) = F_s\left[\cdots + X_a(\underbrace{-12000}_{-F_s + f \times F_s}) + X_a(\overbrace{8000}^{f \times F_s}) + X_a(\overbrace{28000}^{F_s + f \times F_s}) + \cdots\right]$$



EITF75 Systems and Signals

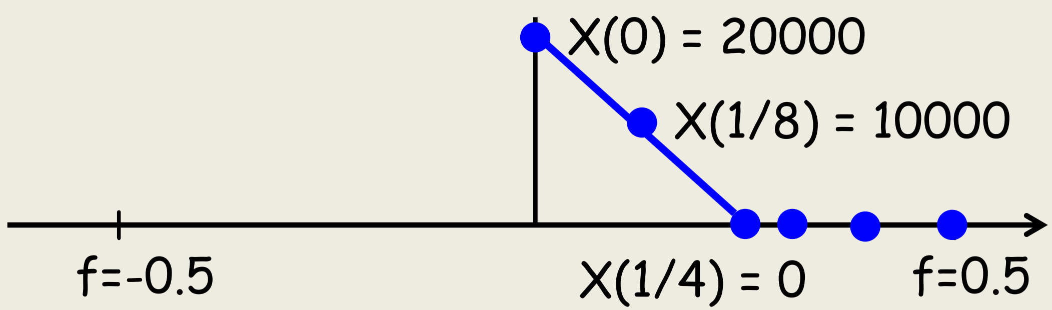


Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(0.5) = F_s\left[\cdots + X_a(\underbrace{-10000}_{-F_s + f \times F_s}) + X_a(\overbrace{10000}^{f \times F_s}) + X_a(\overbrace{30000}^{F_s + f \times F_s}) + \cdots\right]$$



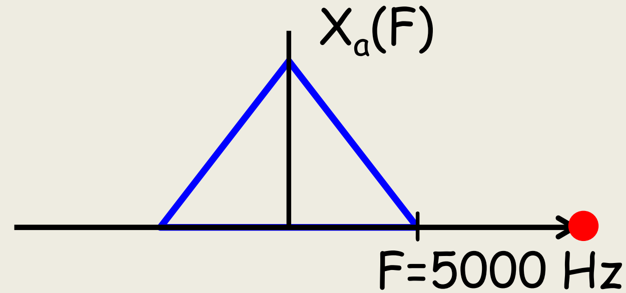
EITF75 Systems and Signals

Signal

$X_a(F)$, $x(t)$ → **sampling** → $X(f)$, $x(n)$

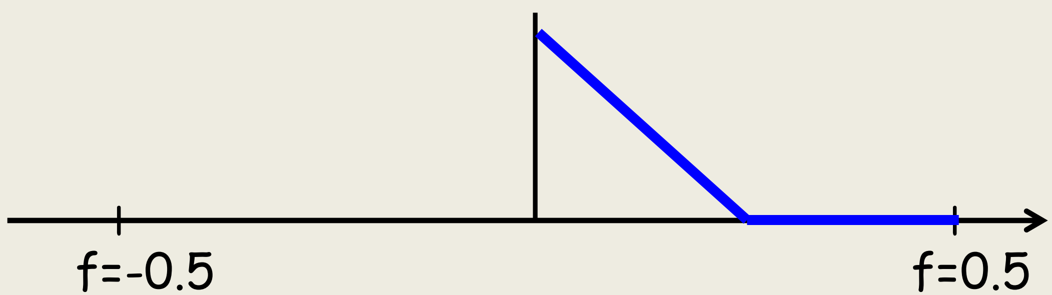
$F_s = 20000$ Hz

Example

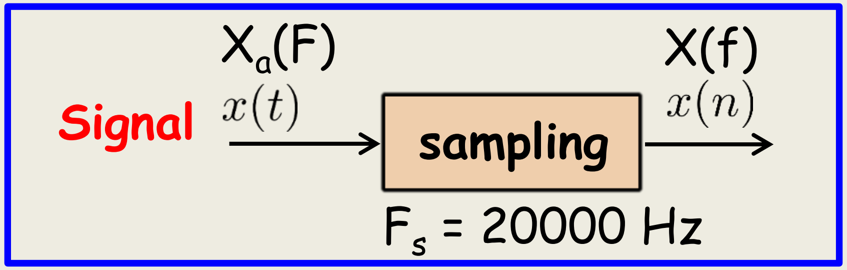


$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

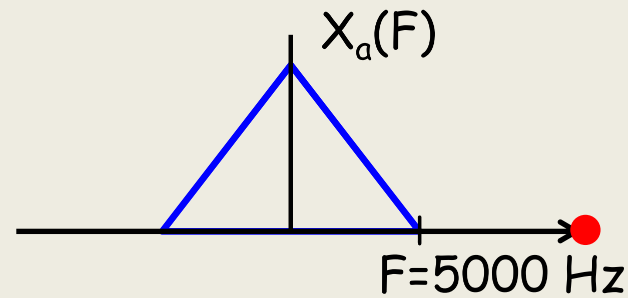
$$X(0.5) = F_s\,[\cdots + X_a(\underbrace{-10000}_{-F_s + f \times F_s}) + X_a(\overbrace{10000}^{f \times F_s}) + X_a(\overbrace{30000}^{F_s + f \times F_s}) + \cdots]$$



EITF75 Systems and Signals

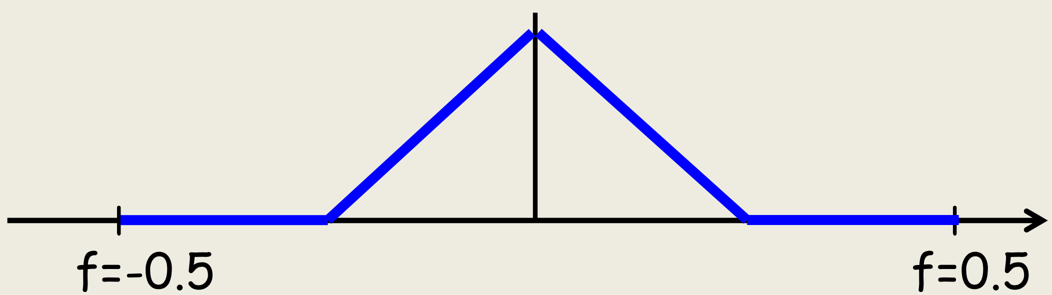


Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

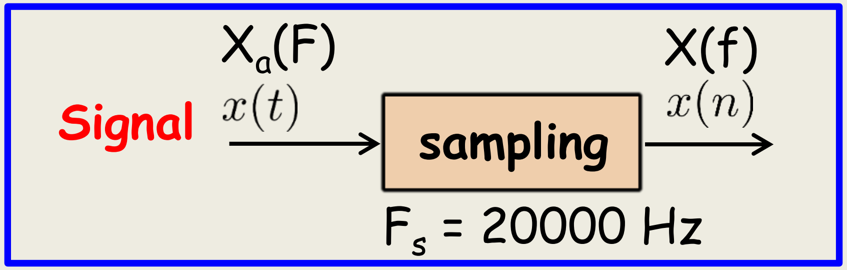
$$X(0.5) = F_s\left[\cdots + X_a(\underbrace{-10000}_{-F_s + f \times F_s}) + X_a(\overbrace{10000}^{f \times F_s}) + X_a(\overbrace{30000}^{F_s + f \times F_s}) + \cdots\right]$$



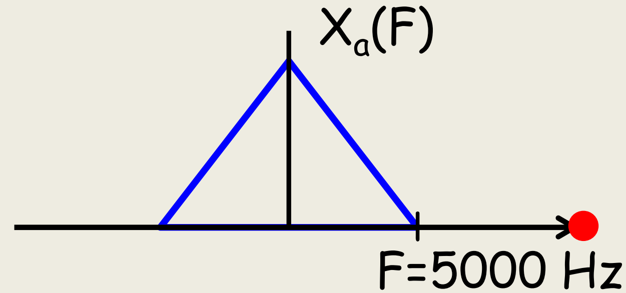
From symmetry

EITF75 Systems and Signals

Signal

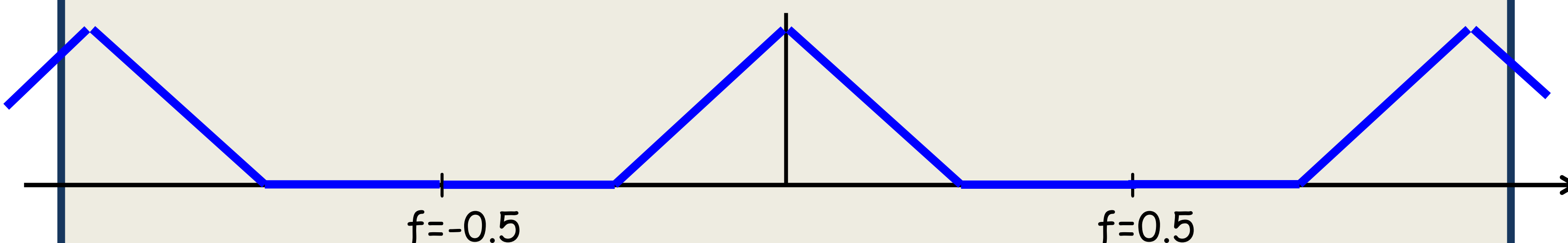


Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

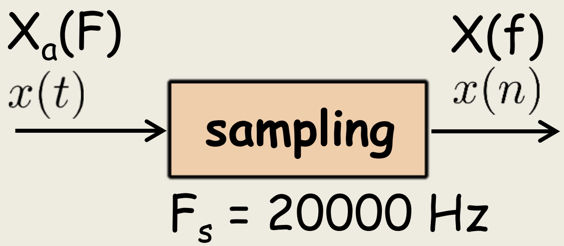
$$X(0.5) = F_s\left[\cdots + X_a(\underset{-F_s + f \times F_s}{-10000}) + X_a(\overset{f \times F_s}{10000}) + X_a(\overset{F_s + f \times F_s}{30000}) + \cdots\right]$$



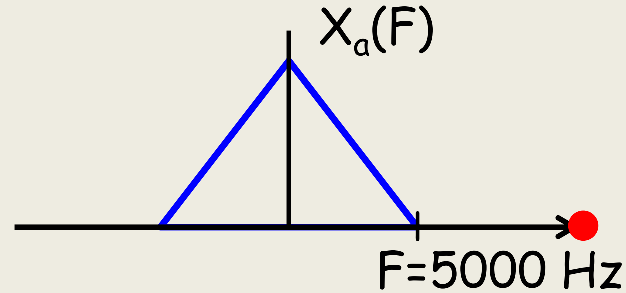
From periodicity

EITF75 Systems and Signals

Signal

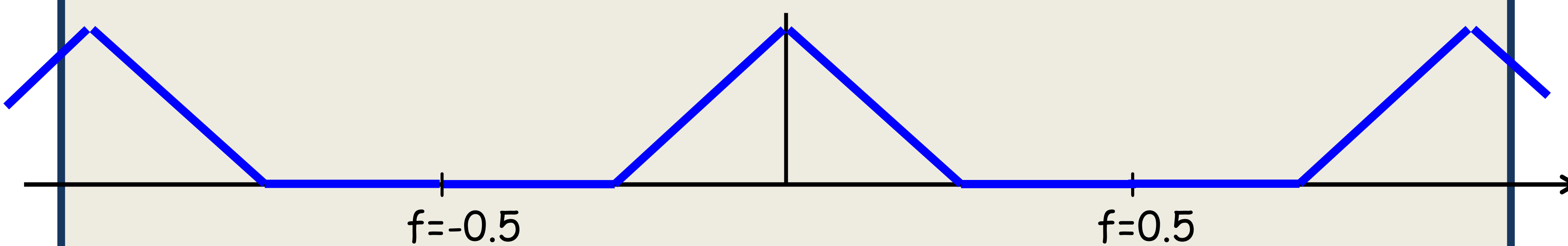


Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

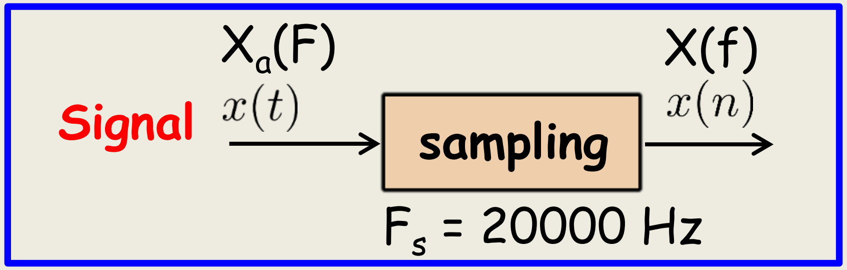
$$X(0.5) = F_s\left[\cdots + X_a(\underset{-F_s + f \times F_s}{-10000}) + X_a(\overset{f \times F_s}{10000}) + X_a(\overset{F_s + f \times F_s}{30000}) + \cdots\right]$$



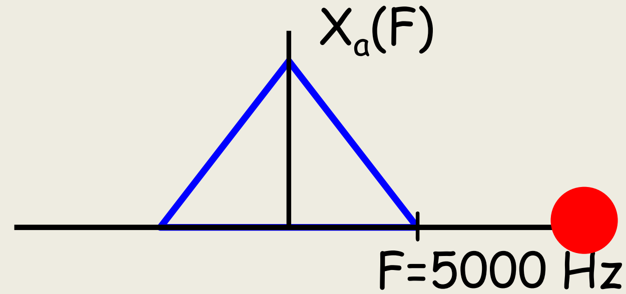
The same shape appears in the DTFT as in the analog Fourier transform

EITF75 Systems and Signals

Signal



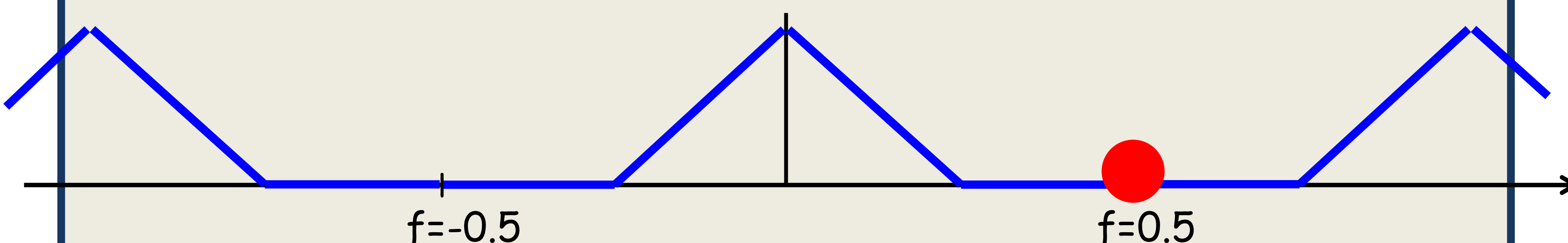
Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

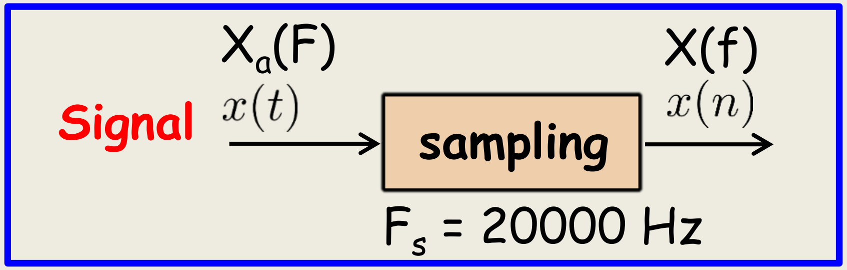
$$X(0.5) = F_s\left[\cdots + X_a(-10000) + X_a(\overset{f \times F_s}{10000}) + X_a(30000) + \cdots\right]$$

10000 Hz (analog) ends up at 0.5

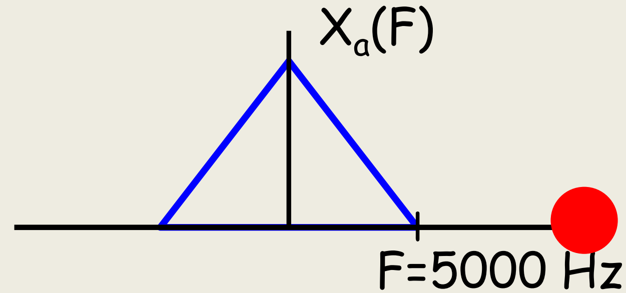


EIT F75 Systems and Signals

Signal



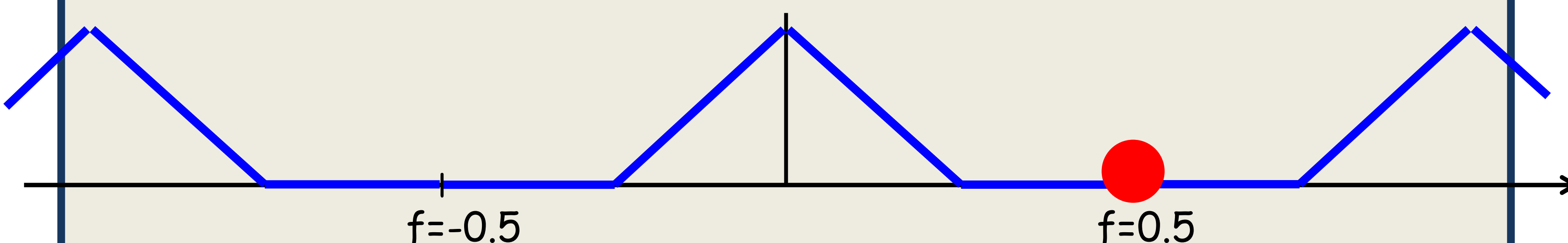
Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0.5) = F_s\left[\cdots + X_a(-10000) + X_a(\overbrace{10000}^{f \times F_s}) + X_a(30000) + \cdots\right]$$

$F_s/2$ Hz (analog) ends up at 0.5

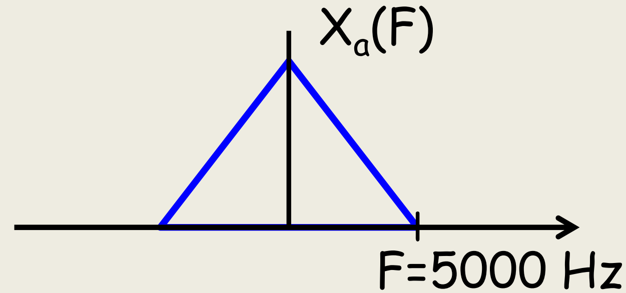


EITF75 Systems and Signals

Signal $\xrightarrow[x(t)]{X_a(F)}$ **sampling** $\xrightarrow[x(n)]{X(f)}$

$F_s = 10000$ Hz

Example

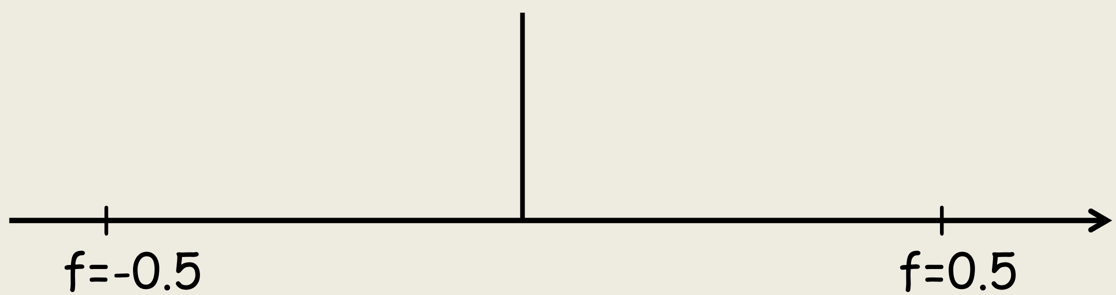


$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$f \times F_s$ $\quad$ $F_s + f \times F_s$

$$X(\quad) = F_s\left[\cdots + X_a(\qquad) + X_a(\qquad) + X_a(\qquad) + \cdots\right]$$

$-F_s + f \times F_s$



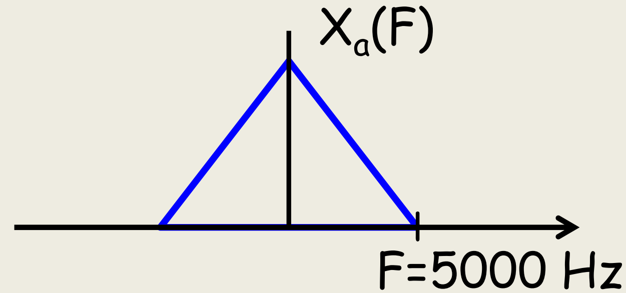
EITF75 Systems and Signals

Signal

F_s = 10000 Hz

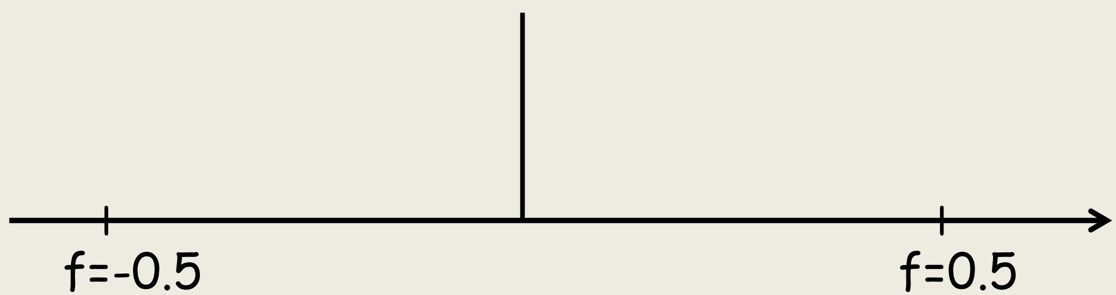
$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

Example



$$X(\quad) = F_s\left[\cdots + X_a(-F_s + f \times F_s) + X_a(f \times F_s) + X_a(F_s + f \times F_s) + \cdots\right]$$

F_s/2 Hz (analog) ends up at 0.5



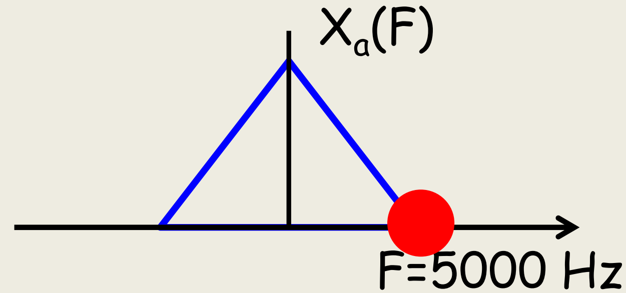
EITF75 Systems and Signals

Signal

$X_a(F)$, $x(t)$ → **sampling** → $X(f)$, $x(n)$

$F_s = 10000$ Hz

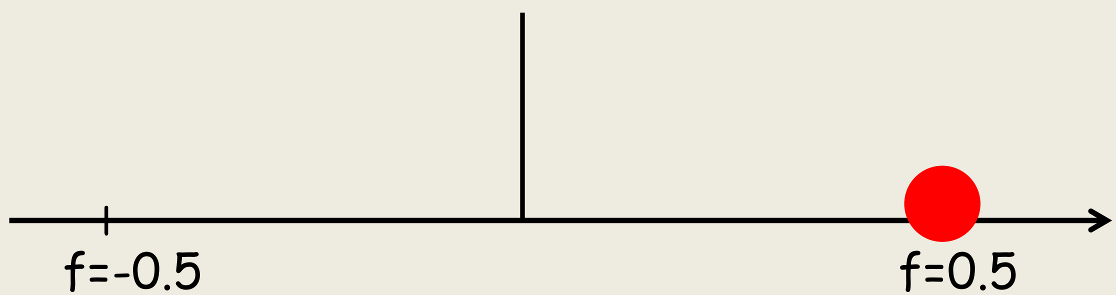
Example



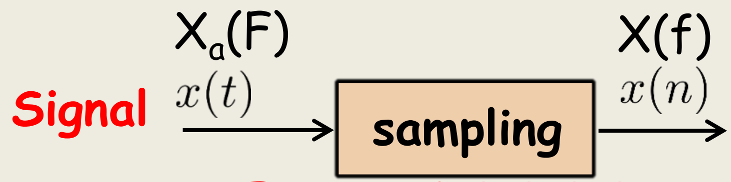
$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0.5) = F_s\left[\cdots + X_a(\underbrace{-5000}_{-F_s + f \times F_s}) + X_a(\overbrace{5000}^{f \times F_s}) + X_a(\overbrace{15000}^{F_s + f \times F_s}) + \cdots\right]$$

$F_s/2$ Hz (analog) ends up at 0.5

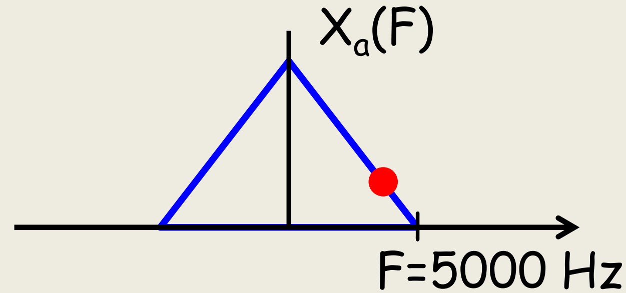


EITF75 Systems and Signals



$F_s = 10000$ Hz

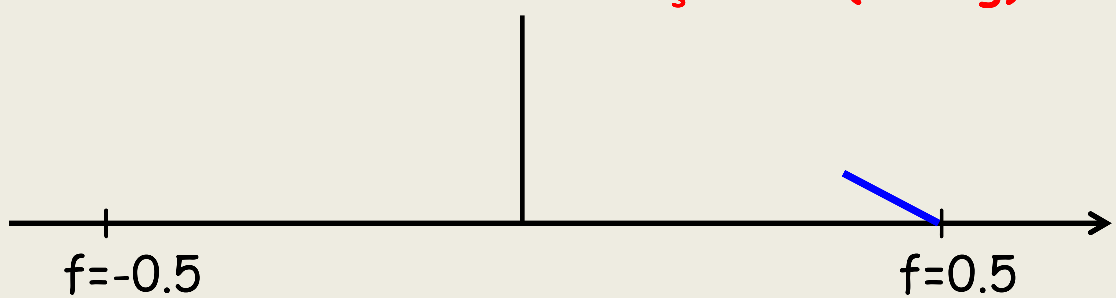
Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(0.4) = F_s\left[\cdots + X_a(\underbrace{-6000}_{-F_s + f \times F_s}) + X_a(\overbrace{4000}^{f \times F_s}) + X_a(\overbrace{14000}^{F_s + f \times F_s}) + \cdots\right]$$

$0.8F_s/2$ Hz (analog) ends up at 0.4

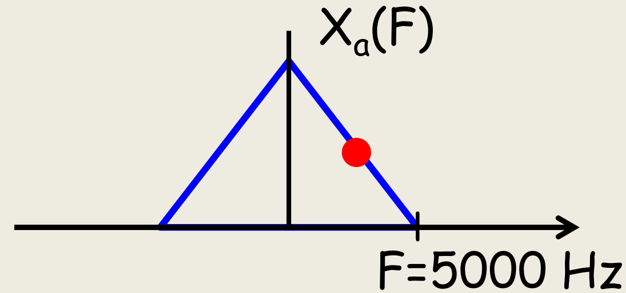


EITF75 Systems and Signals

Signal

$F_s = 10000$ Hz

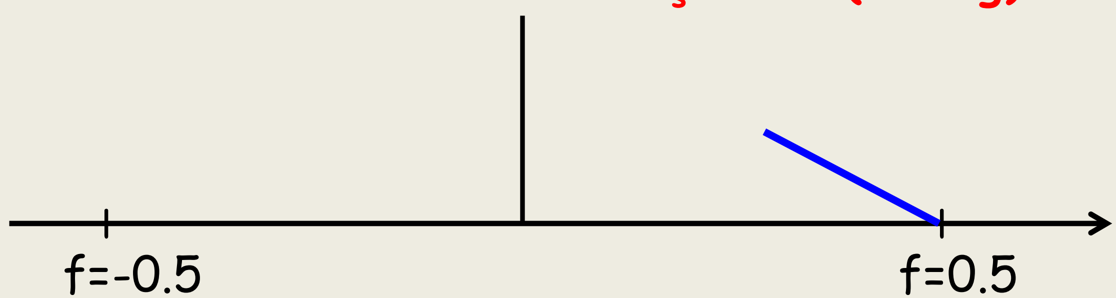
Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(0.3) = F_s\left[\cdots + X_a(\underbrace{-7000}_{-F_s + f \times F_s}) + X_a(\overbrace{3000}^{f \times F_s}) + X_a(\overbrace{13000}^{F_s + f \times F_s}) + \cdots\right]$$

$0.6F_s/2$ Hz (analog) ends up at 0.3

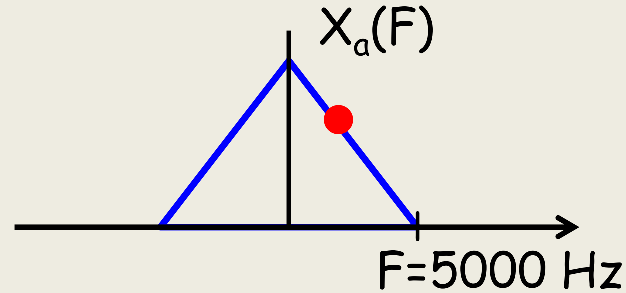


EITF75 Systems and Signals

Signal

F_s = 10000 Hz

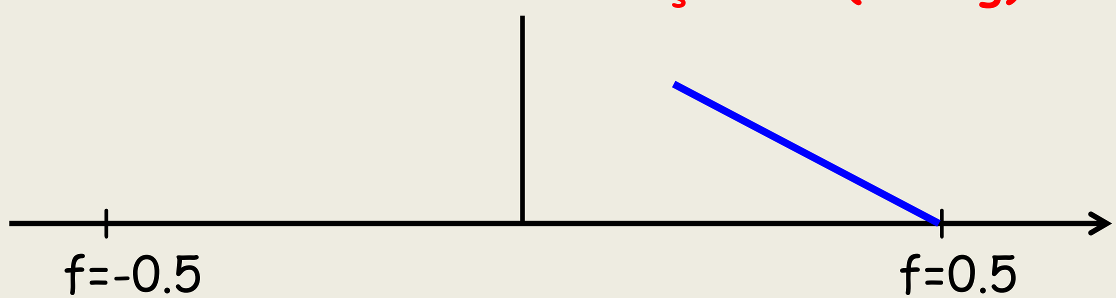
Example



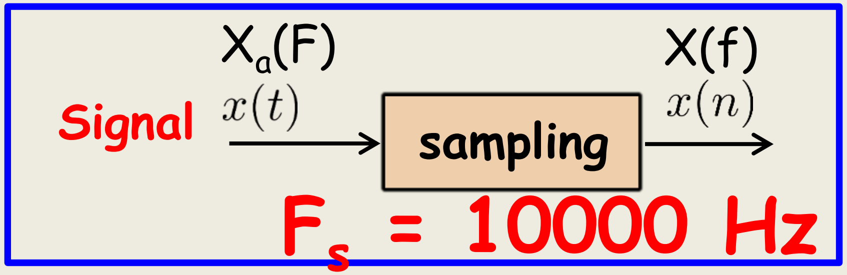
$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(0.2) = F_s\left[\cdots + X_a(\underbrace{-8000}_{-F_s + f \times F_s}) + X_a(\overbrace{2000}^{f \times F_s}) + X_a(\overbrace{12000}^{F_s + f \times F_s}) + \cdots\right]$$

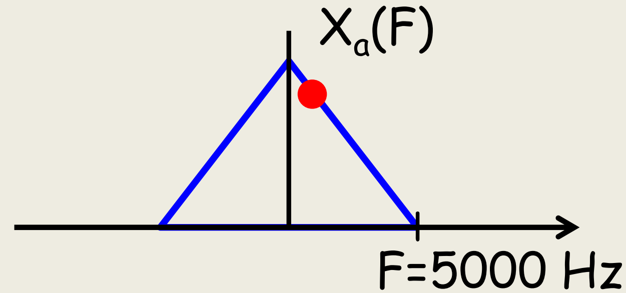
$0.4F_s/2$ Hz (analog) ends up at 0.2



EITF75 Systems and Signals



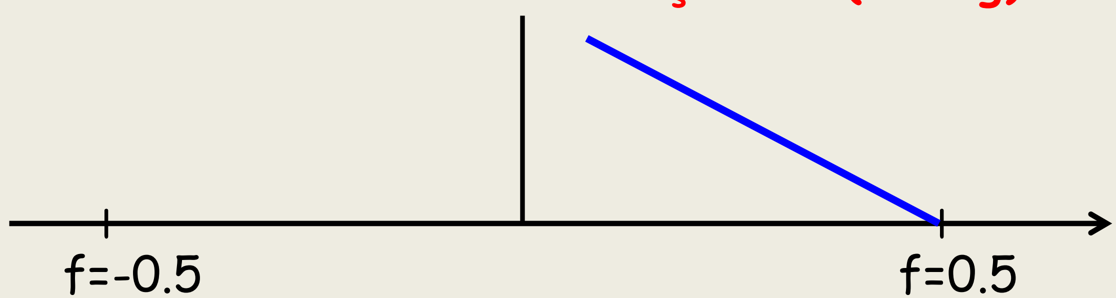
Example



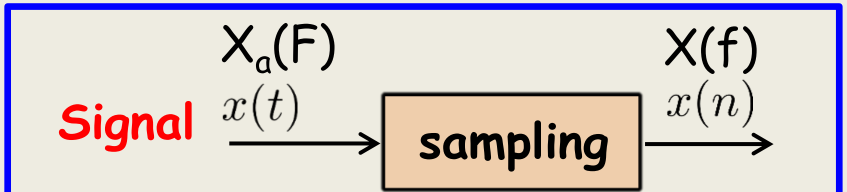
$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0.1) = F_s\left[\cdots + X_a(\underbrace{-9000}_{-F_s + f \times F_s}) + X_a(\overbrace{1000}^{f \times F_s}) + X_a(\overbrace{11000}^{F_s + f \times F_s}) + \cdots\right]$$

0.2F_s/2 Hz (analog) ends up at 0.1

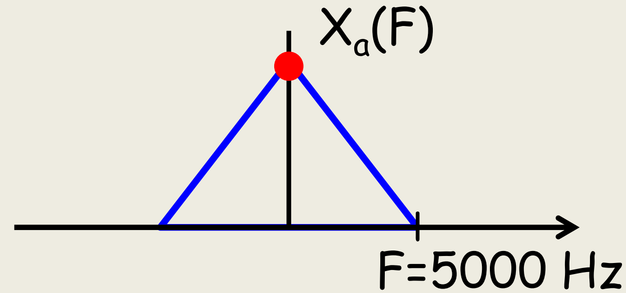


EITF75 Systems and Signals



F_s = 10000 Hz

Example



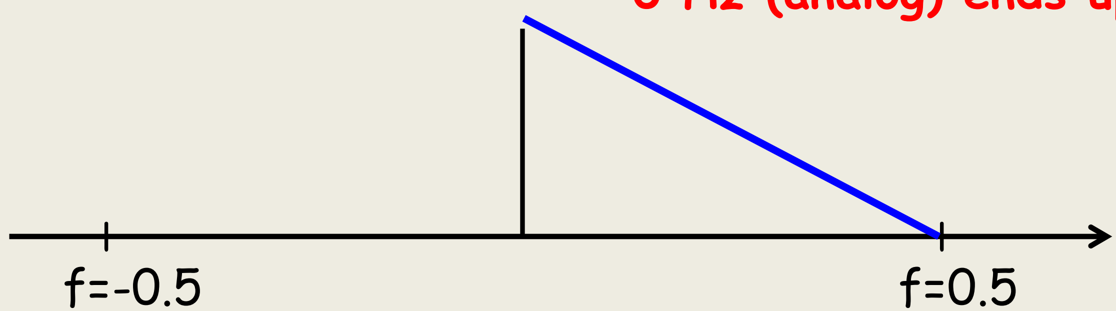
$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$f \times F_s$ $F_s + f \times F_s$

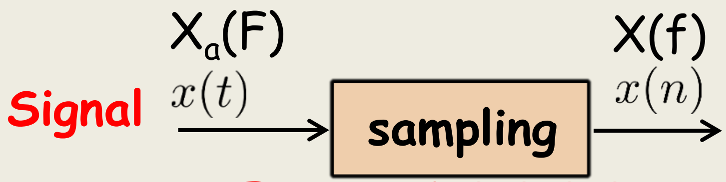
$$X(0) = F_s\left[\cdots + X_a(-10000) + X_a(0) + X_a(10000) + \cdots\right]$$

$-F_s + f \times F_s$

0 Hz (analog) ends up at 0

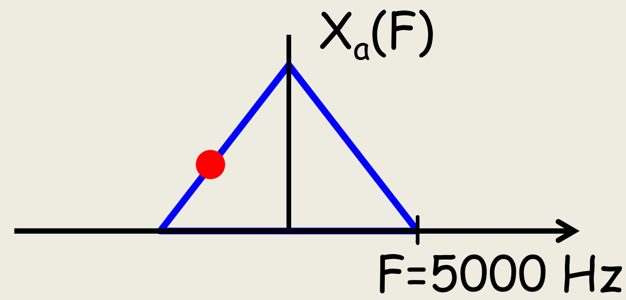


EITF75 Systems and Signals



F_s = 10000 Hz

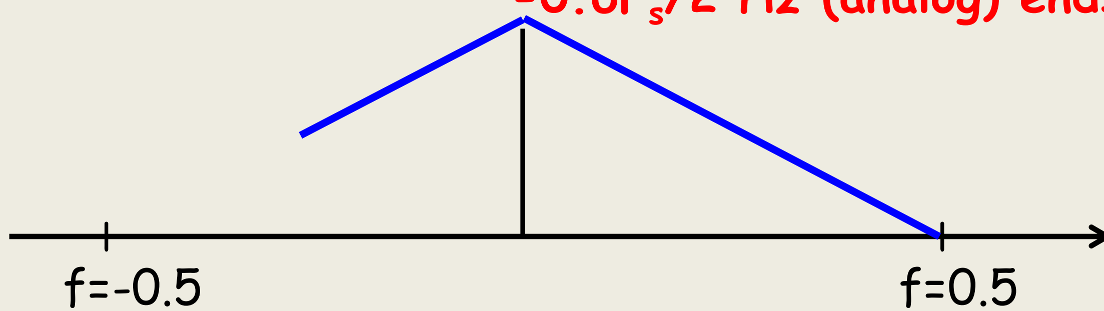
Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(-0.3) = F_s\left[\cdots + X_a(\underbrace{-13000}_{-F_s + f \times F_s}) + X_a(\overbrace{-3000}^{f \times F_s}) + X_a(\overbrace{7000}^{F_s + f \times F_s}) + \cdots\right]$$

$-0.6F_s/2$ Hz (analog) ends up at -0.3



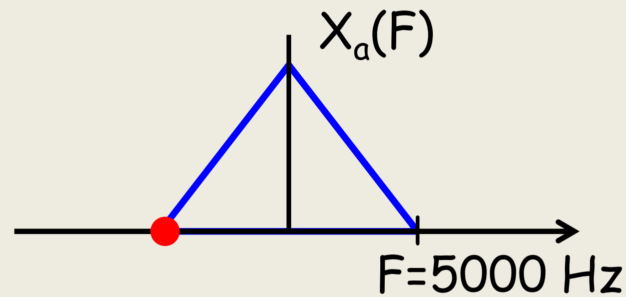
EITF75 Systems and Signals

Signal

F_s = 10000 Hz

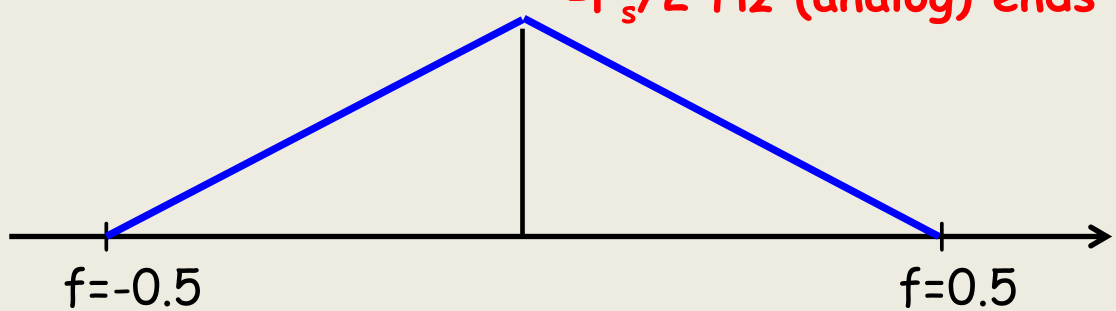
$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

Example

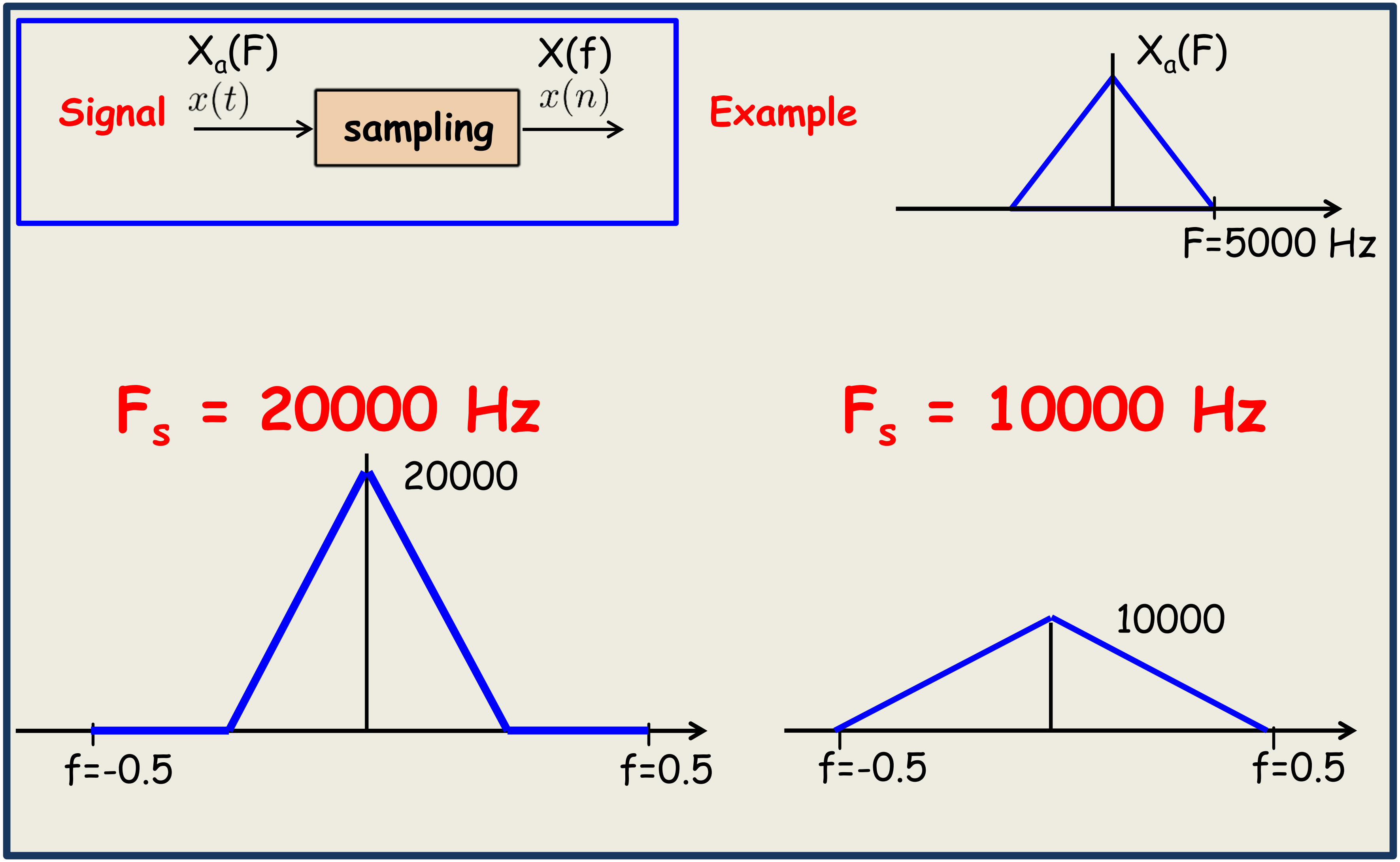


$$X(-0.5) = F_s\left[\cdots + X_a(\underbrace{-15000}_{-F_s + f \times F_s}) + X_a(\overbrace{-5000}^{f \times F_s}) + X_a(\overbrace{5000}^{F_s + f \times F_s}) + \cdots\right]$$

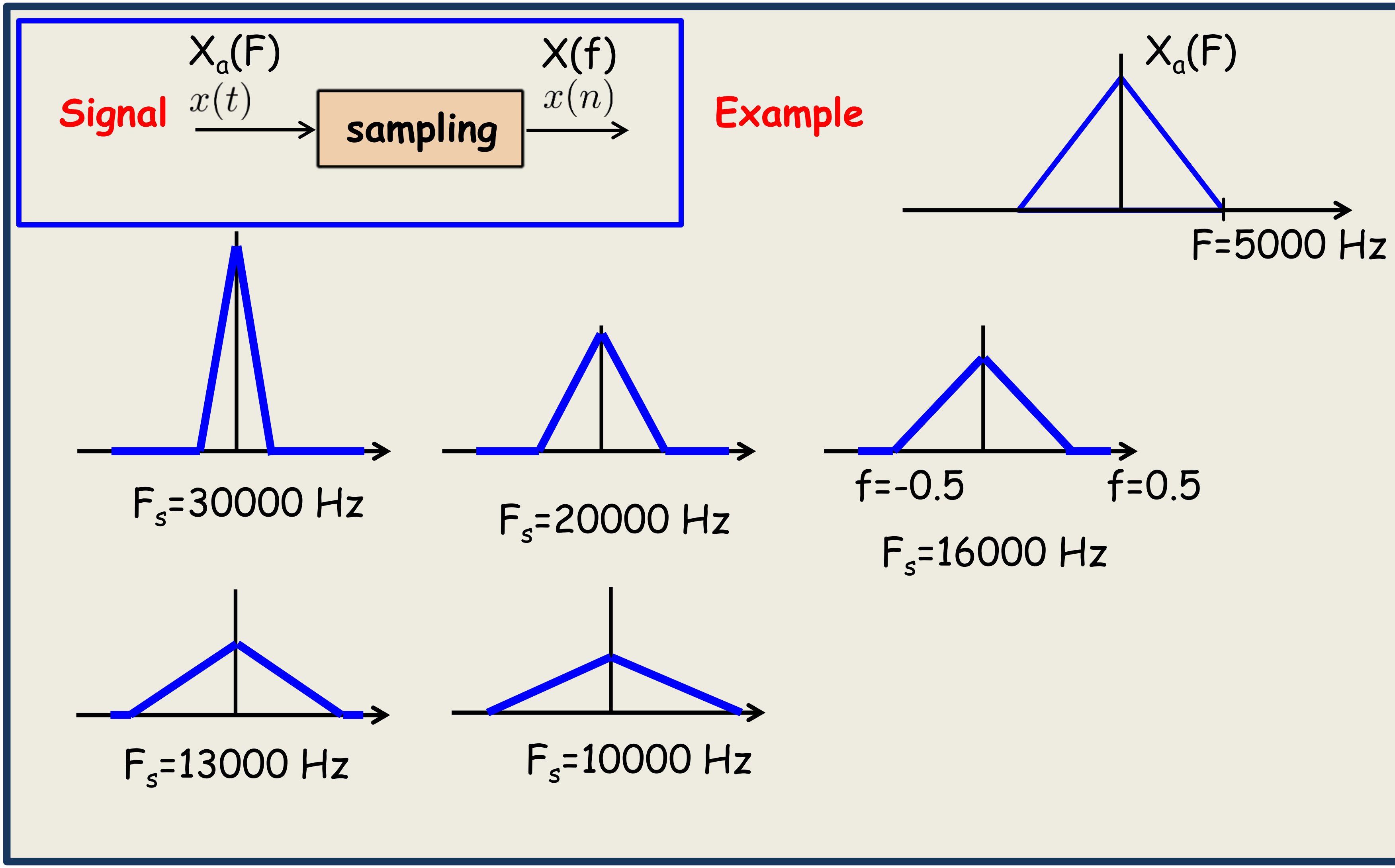
$-F_s/2$ Hz (analog) ends up at -0.5



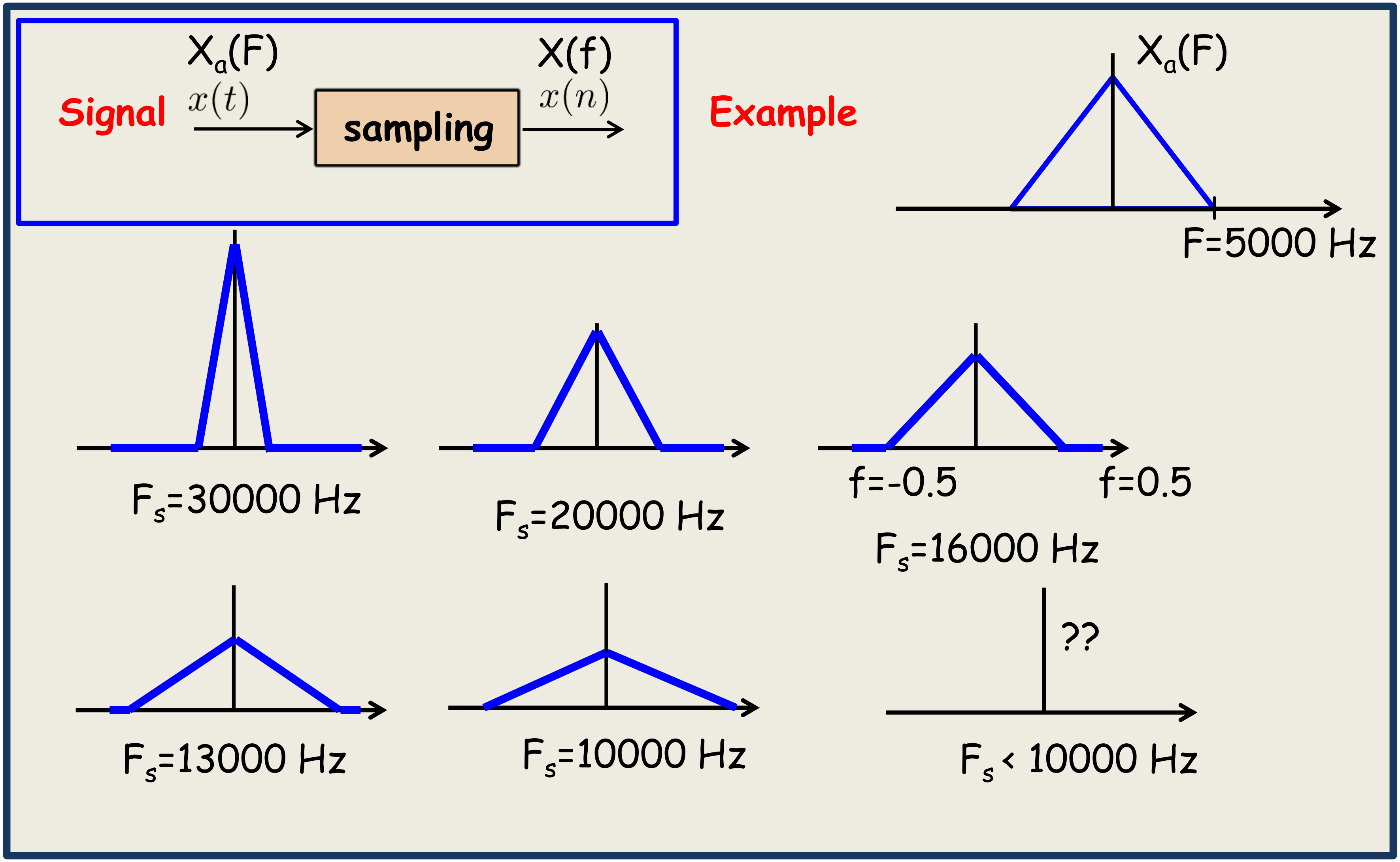
EITF75 Systems and Signals



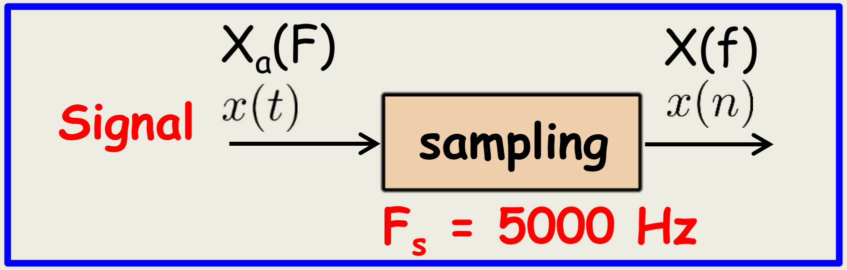
EITF75 Systems and Signals



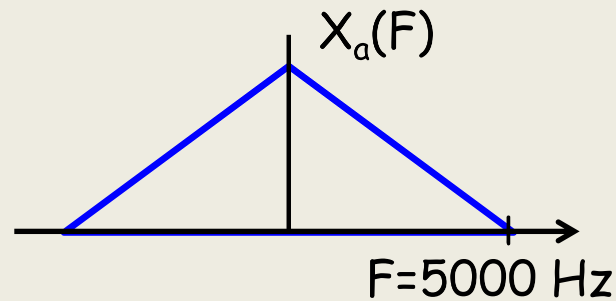
EITF75 Systems and Signals



EITF75 Systems and Signals

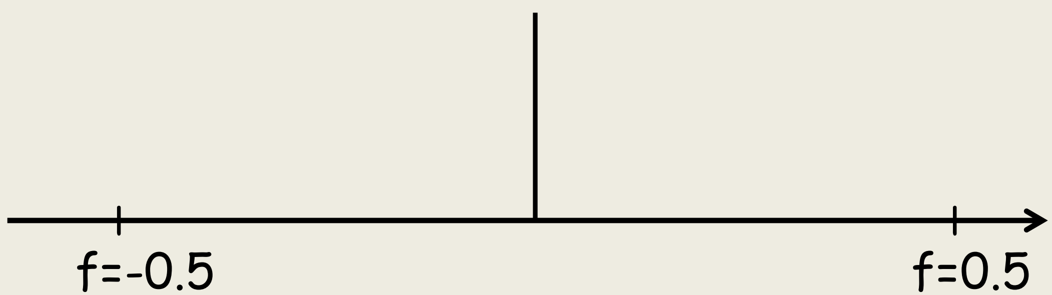


Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(\quad) = F_s\left[\cdots + X_a(\underset{-F_s + f \times F_s}{\qquad}) + X_a(\overset{f \times F_s}{\qquad}) + X_a(\overset{F_s + f \times F_s}{\qquad}) + \cdots\right]$$

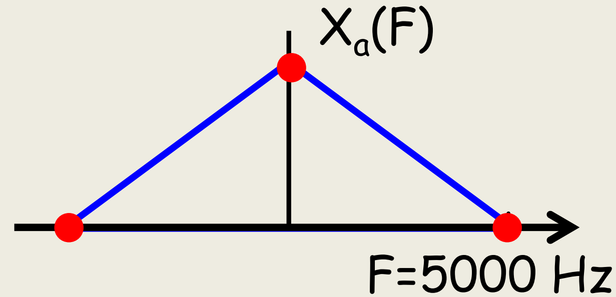


EITF75 Systems and Signals

Signal

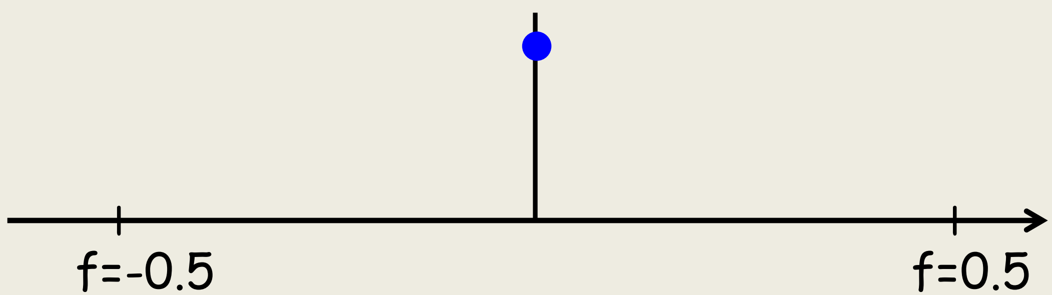
F_s = 5000 Hz

Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0) = F_s\left[\cdots + X_a(\underbrace{-5000}_{-F_s + f \times F_s}) + X_a(\overbrace{0}^{f \times F_s}) + X_a(\overbrace{5000}^{F_s + f \times F_s}) + \cdots\right]$$

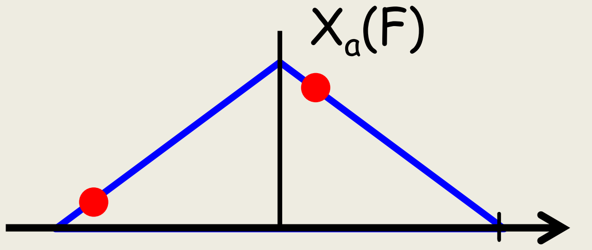


EITF75 Systems and Signals

Signal

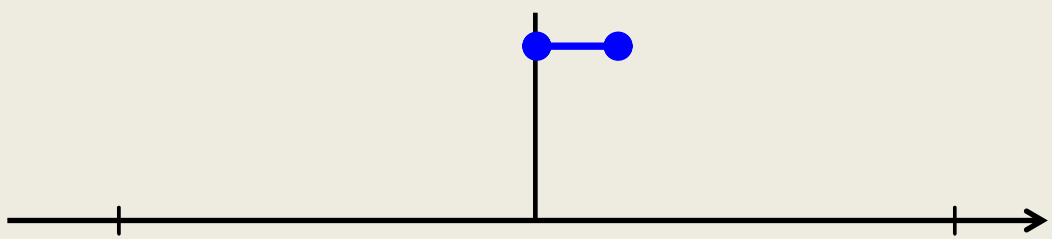
$F_s = 5000$ Hz

Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0.1) = F_s\left[\cdots + X_a(\underset{-F_s + f \times F_s}{-4500}) + X_a(\overset{f \times F_s}{500}) + X_a(\overset{F_s + f \times F_s}{5500}) + \cdots\right]$$

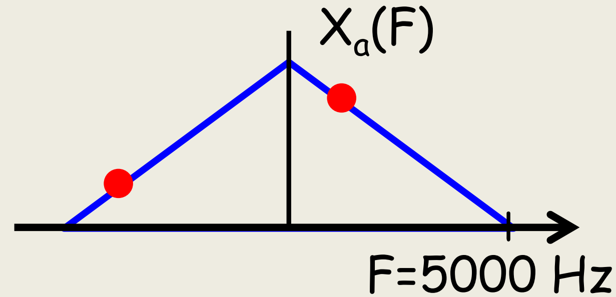


EITF75 Systems and Signals

Signal $x(t)$, $X_a(F)$ $\rightarrow$ **sampling** $\rightarrow$ $x(n)$, $X(f)$

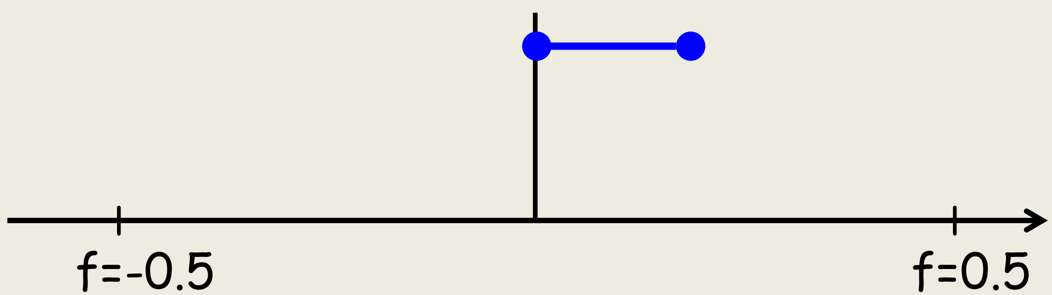
$F_s = 5000$ Hz

Example



$$\boxed{X(f) = F_{\mathrm{s}} \sum_{k=-\infty}^{\infty} X_a((f-k)F_{\mathrm{s}})}$$

$$X(0.2) = F_{\mathrm{s}}\left[\cdots + X_a(\underbrace{-4000}_{-F_s + f \times F_s}) + X_a(\overbrace{1000}^{f \times F_s}) + X_a(\overbrace{6000}^{F_s + f \times F_s}) + \cdots\right]$$



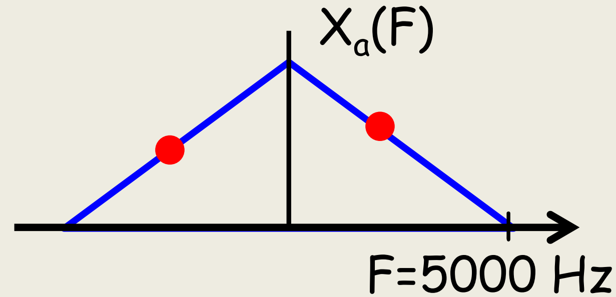
EITF75 Systems and Signals

Signal

$$\underset{X_a(F)}{x(t)} \longrightarrow \boxed{\text{sampling}} \longrightarrow \underset{X(f)}{x(n)}$$

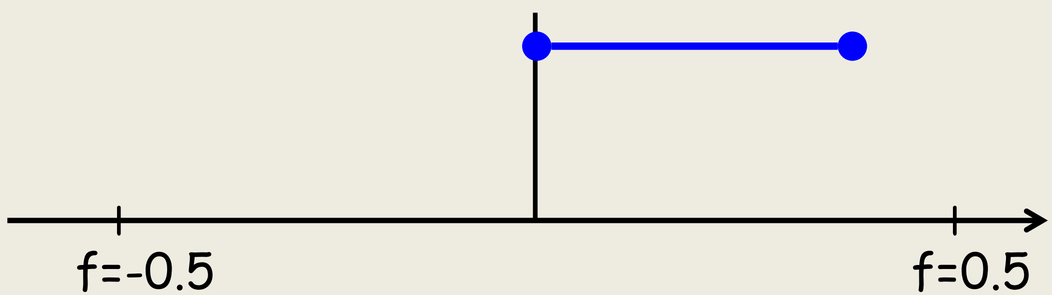
F_s = 5000 Hz

Example



$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(0.4) = F_s\left[\cdots + X_a(\underbrace{-3000}_{-F_s + f \times F_s}) + X_a(\overbrace{2000}^{f \times F_s}) + X_a(\overbrace{7000}^{F_s + f \times F_s}) + \cdots\right]$$

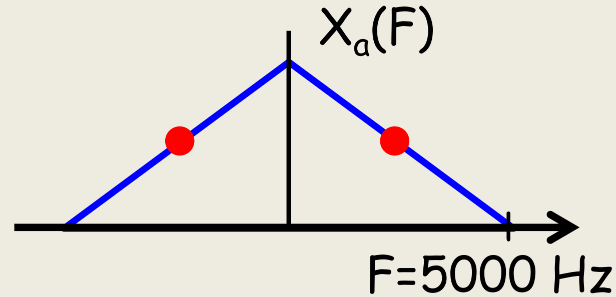


EITF75 Systems and Signals

Signal

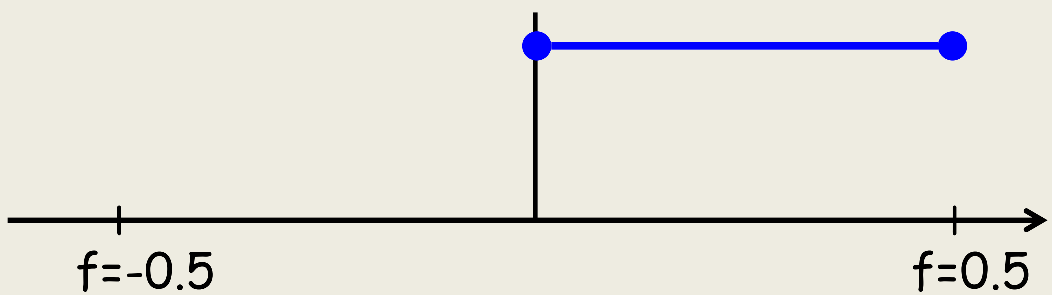
F_s = 5000 Hz

Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(0.5) = F_s\left[\cdots + X_a(\underbrace{-2500}_{-F_s + f \times F_s}) + X_a(\overbrace{2500}^{f \times F_s}) + X_a(\overbrace{7500}^{F_s + f \times F_s}) + \cdots\right]$$



EITF75 Systems and Signals

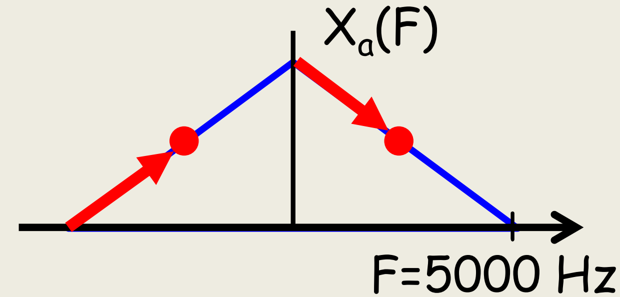
Signal

$X_a(F)$ $\quad$ $X(f)$

$x(t) \longrightarrow$ **sampling** $\longrightarrow x(n)$

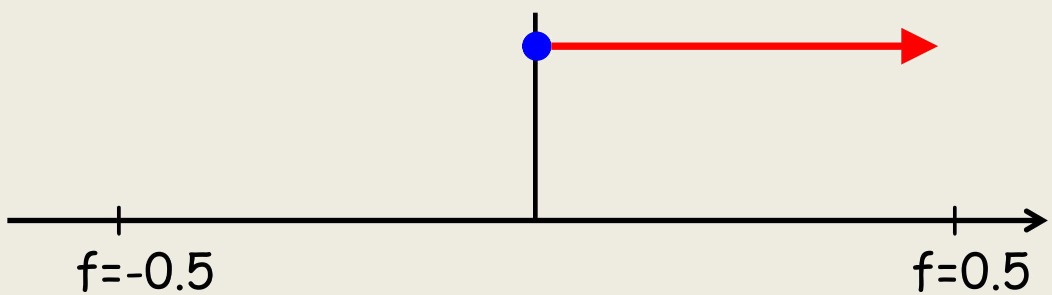
F_s = 5000 Hz

Example

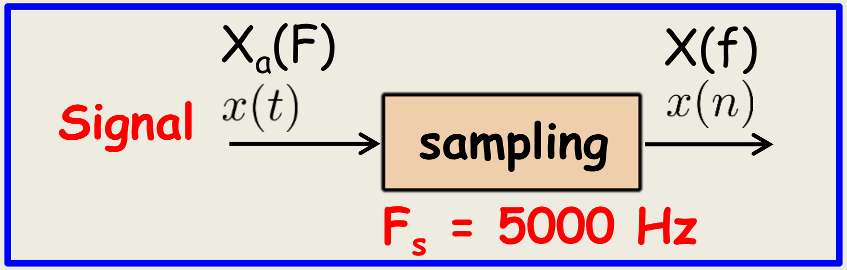


$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

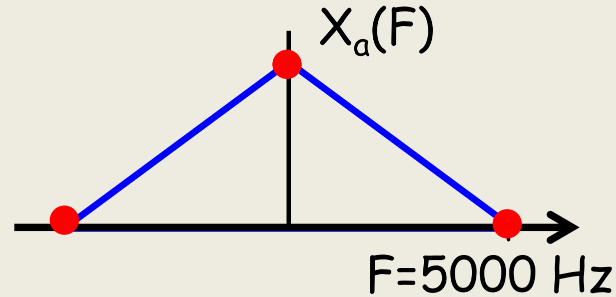
$$X(0.5) = F_s\left[\cdots + X_a(\underbrace{-2500}_{-F_s + f \times F_s}) + X_a(\overbrace{2500}^{f \times F_s}) + X_a(\overbrace{7500}^{F_s + f \times F_s}) + \cdots\right]$$



EITF75 Systems and Signals

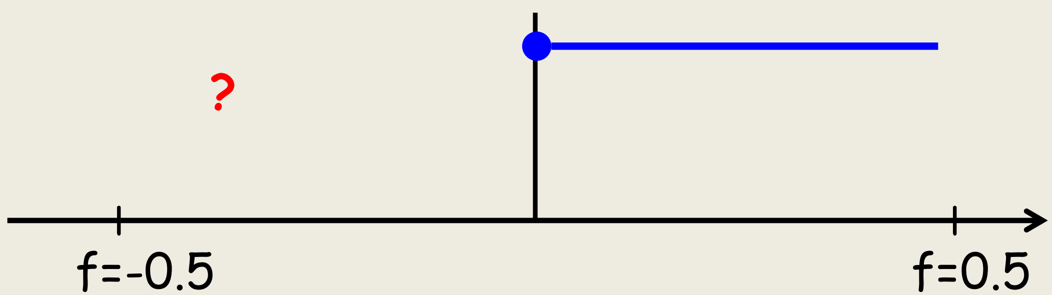


Example

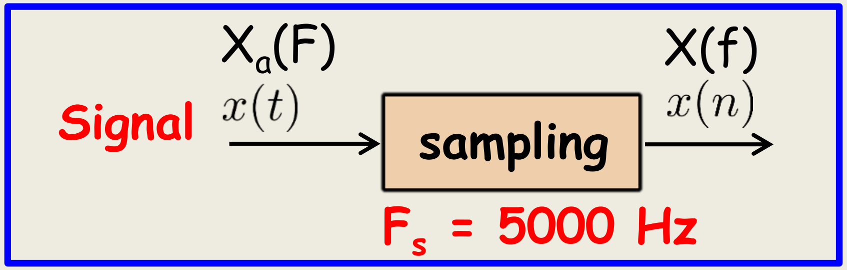


$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

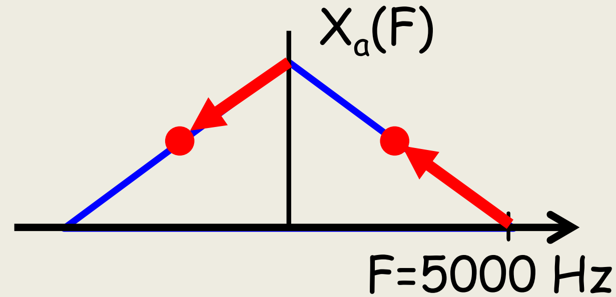
$$X(0) = F_s\left[\cdots + X_a(\underset{-F_s + f \times F_s}{-5000}) + X_a(\overset{f \times F_s}{0}) + X_a(\overset{F_s + f \times F_s}{5000}) + \cdots\right]$$



EITF75 Systems and Signals

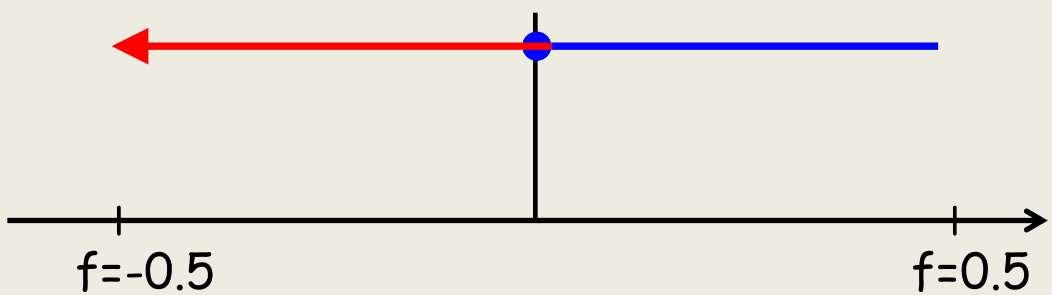


Example



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

$$X(<0) = F_s\left[\cdots + X_a(-F_s + f \times F_s) + X_a(f \times F_s) + X_a(F_s + f \times F_s) + \cdots\right]$$



EITF75 Systems and Signals

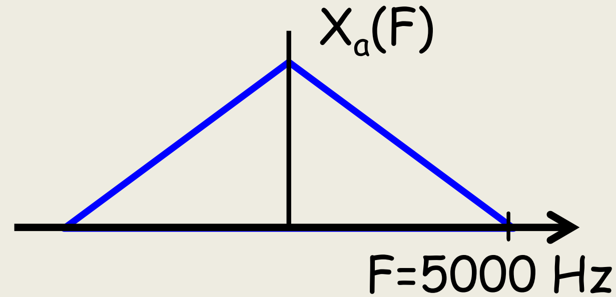
Signal

$X_a(F)$ $\quad$ $X(f)$

$x(t) \longrightarrow$ **sampling** $\longrightarrow x(n)$

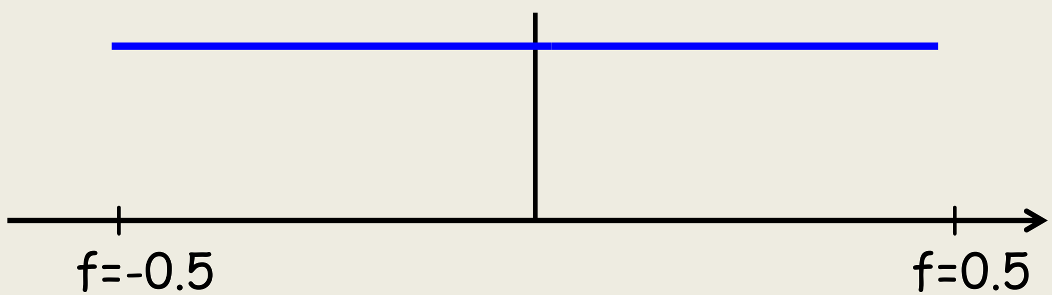
F_s = 5000 Hz

Example

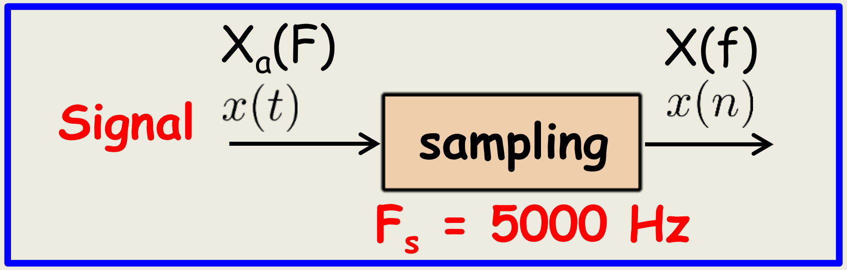


$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(f) = F_s\left[\cdots + X_a(-F_s + f \times F_s) + X_a(f \times F_s) + X_a(F_s + f \times F_s) + \cdots\right]$$

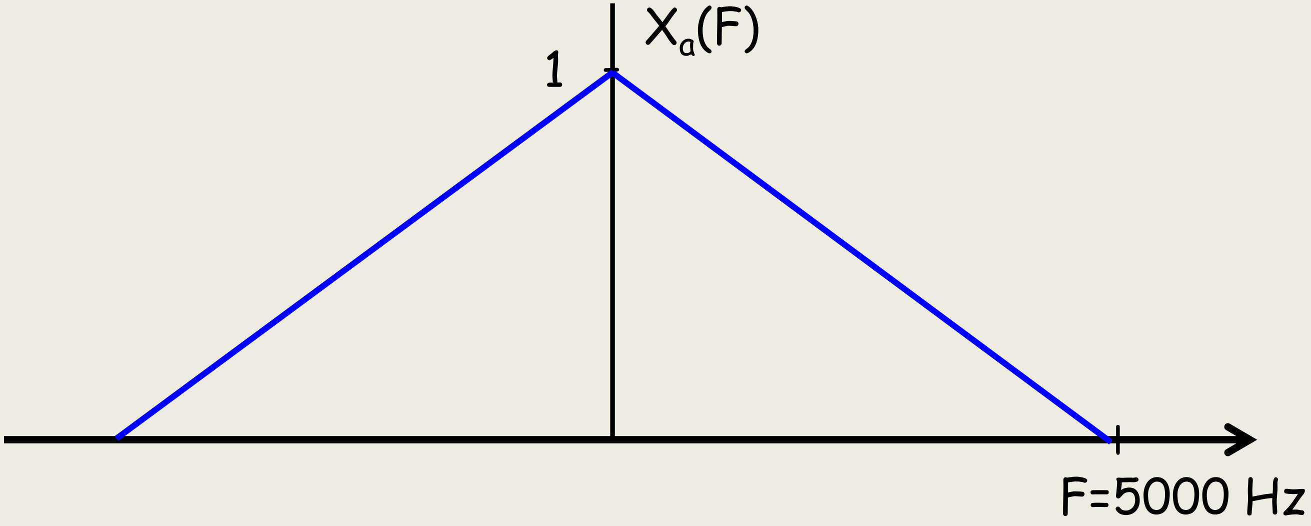


EITF75 Systems and Signals

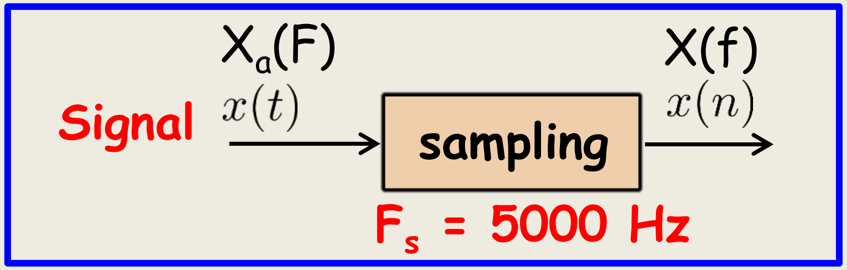


Example

Folding



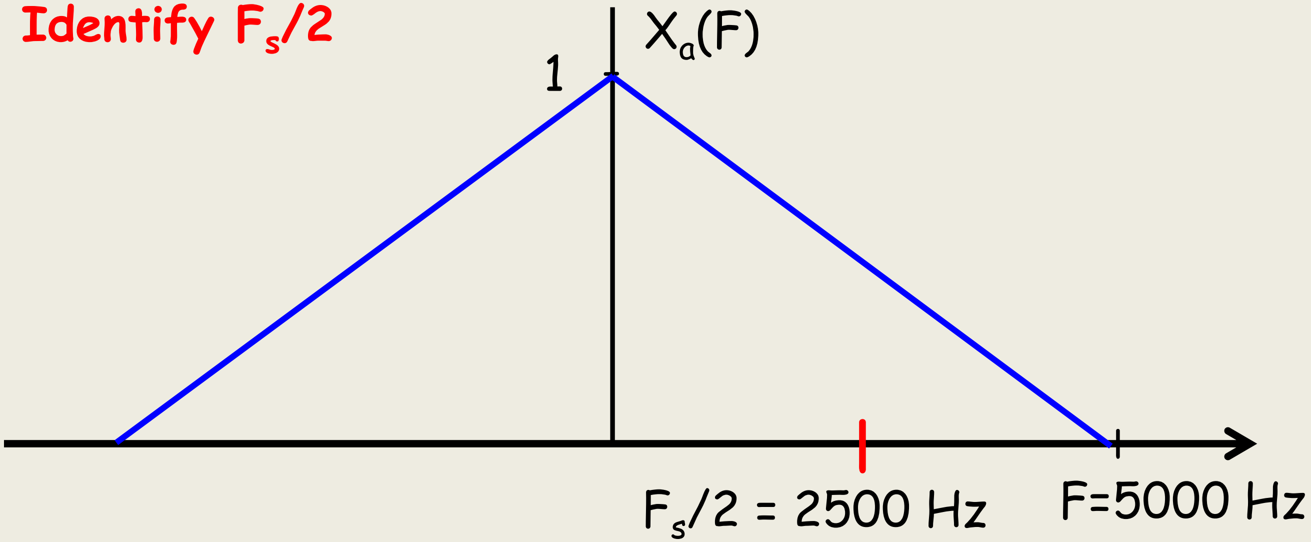
EITF75 Systems and Signals



Example

Folding

Step 1: Identify $F_s/2$



EITF75 Systems and Signals

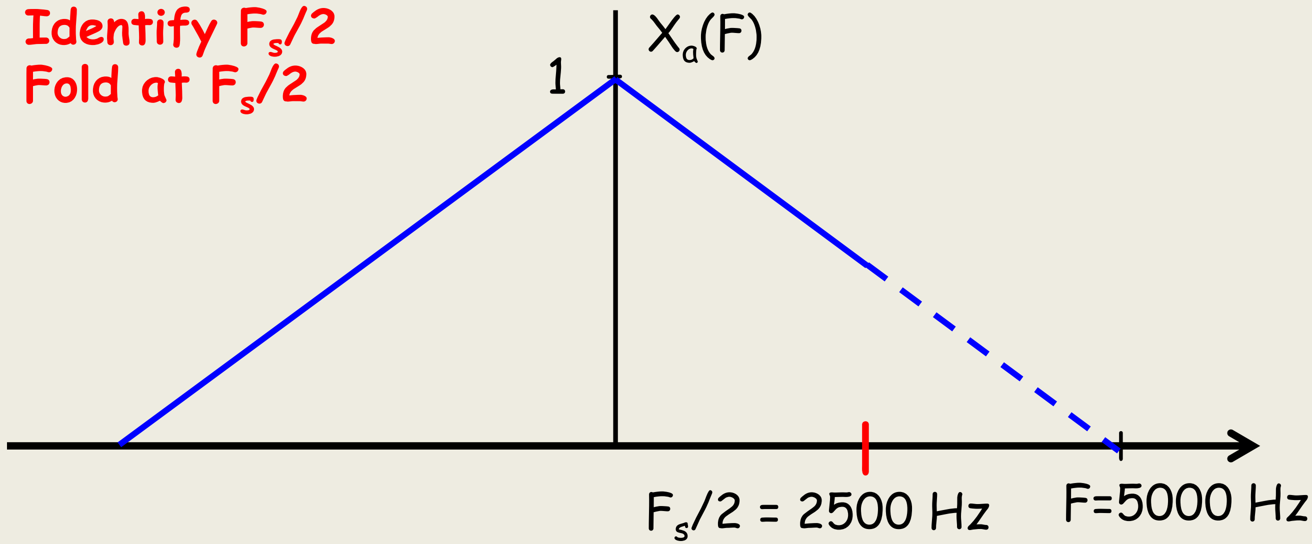
Signal

F_s = 5000 Hz

Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$



EITF75 Systems and Signals

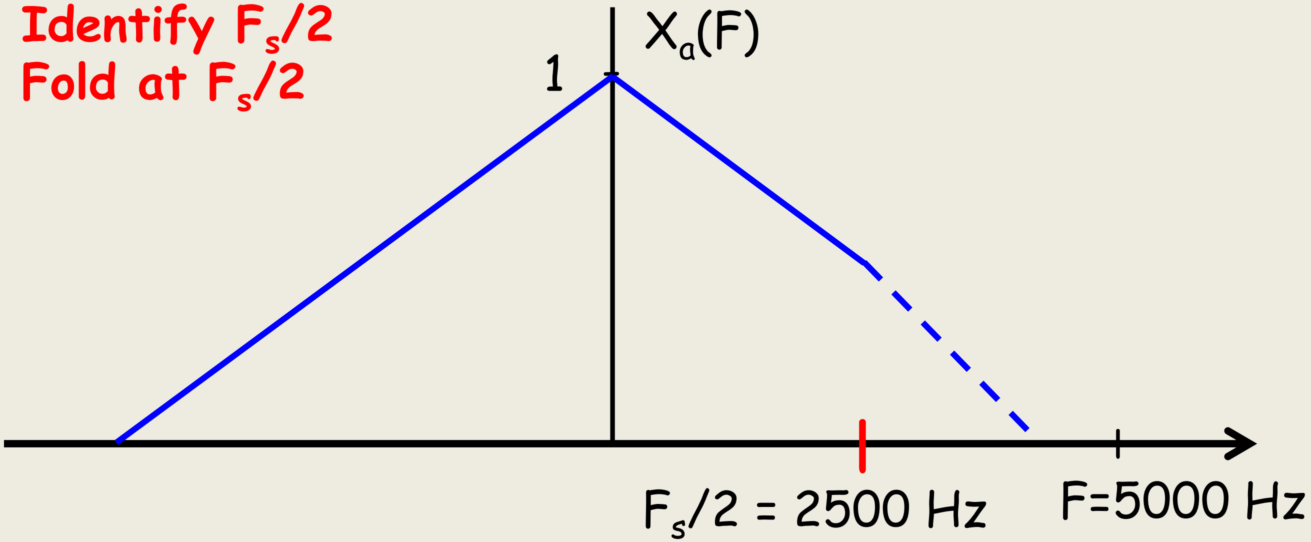
Signal

F_s = 5000 Hz

Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$



EITF75 Systems and Signals

Signal

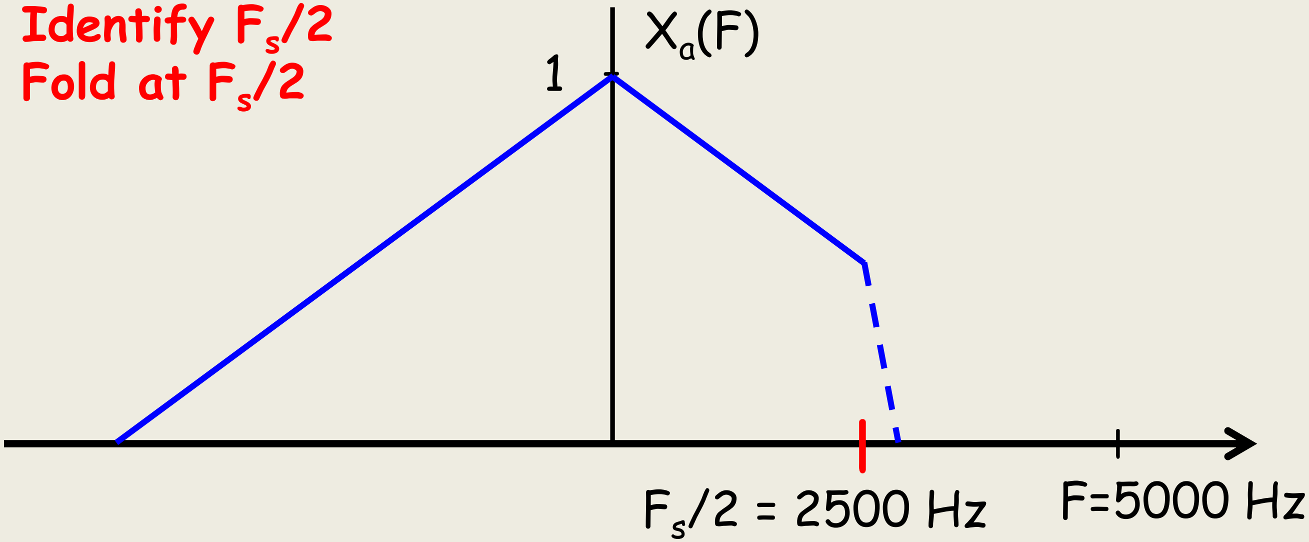
F_s = 5000 Hz

Example

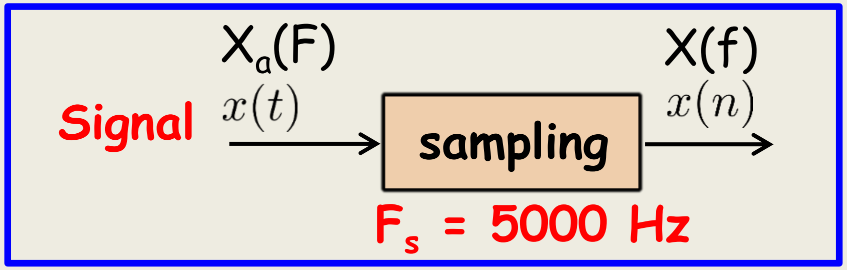
Folding

Step 1: Identify $F_s/2$

Step 2: Fold at $F_s/2$



EITF75 Systems and Signals

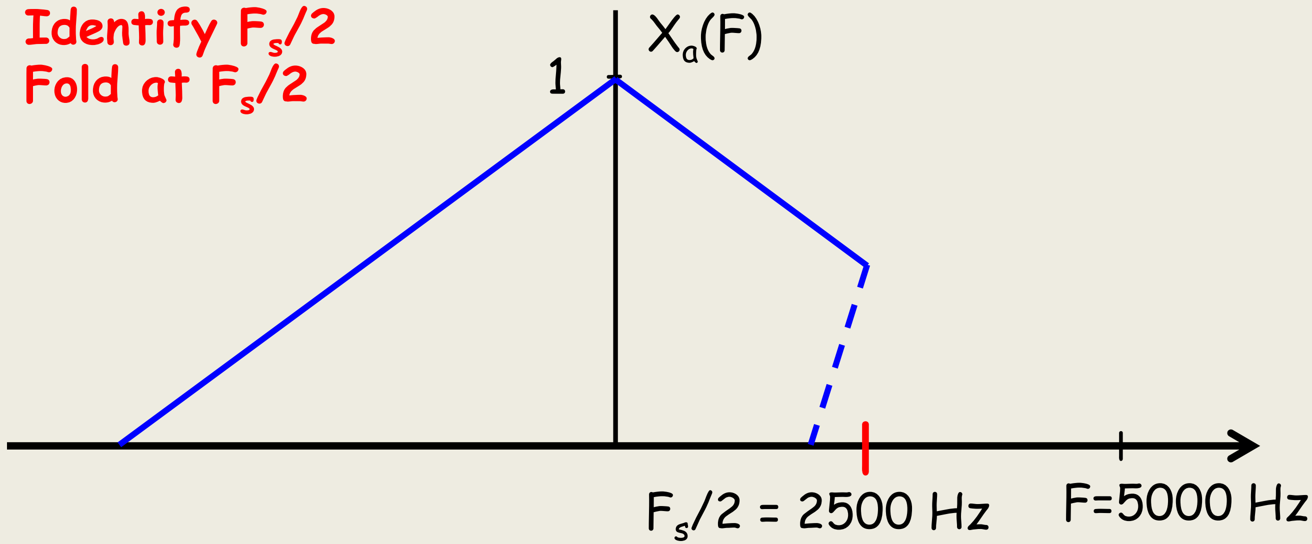


Example

Folding

Step 1: Identify $F_s/2$

Step 2: Fold at $F_s/2$



EITF75 Systems and Signals

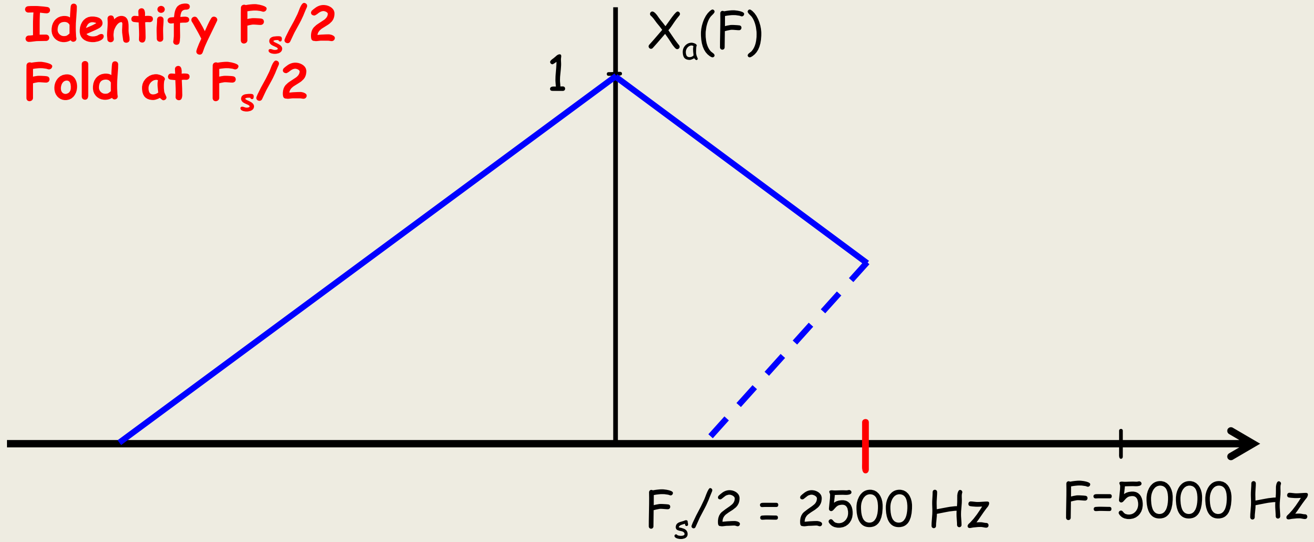
Signal

F_s = 5000 Hz

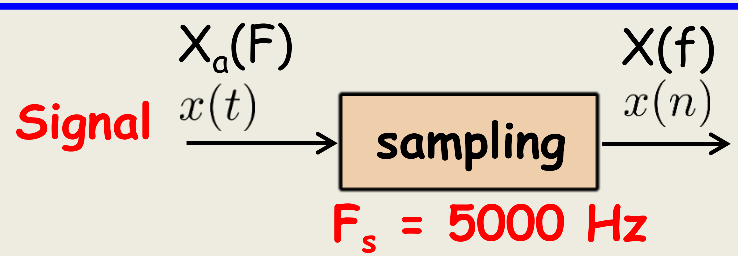
Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$



EITF75 Systems and Signals

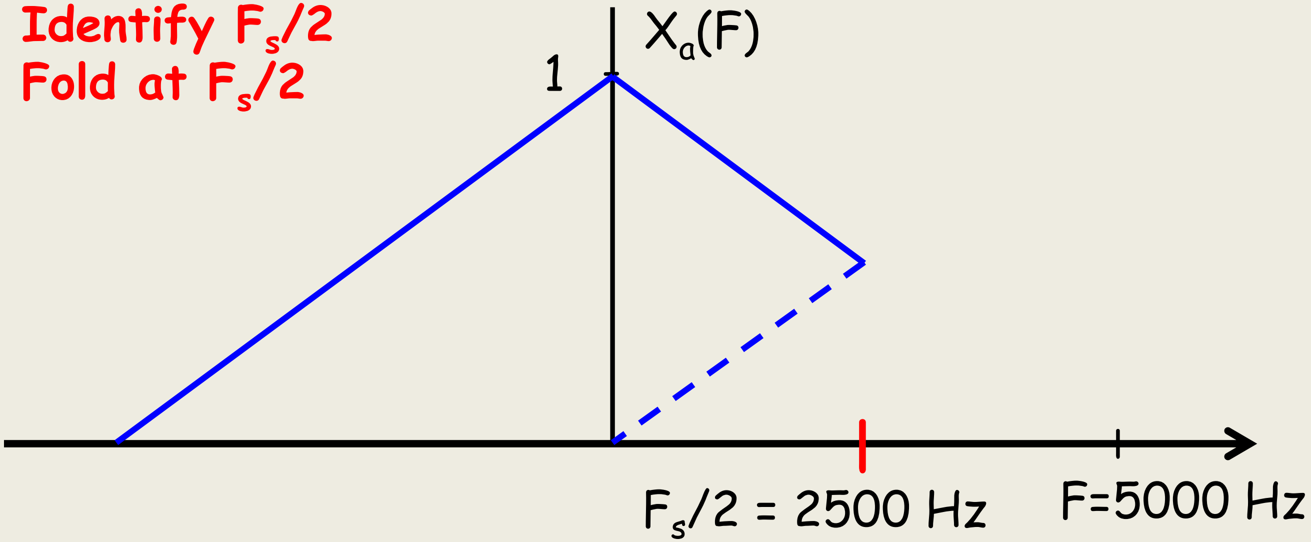


Example

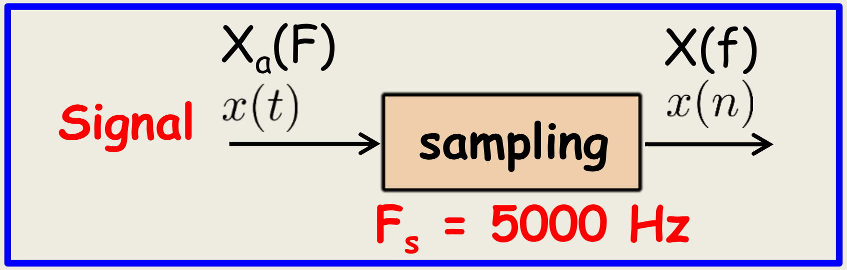
Folding

Step 1: Identify $F_s/2$

Step 2: Fold at $F_s/2$



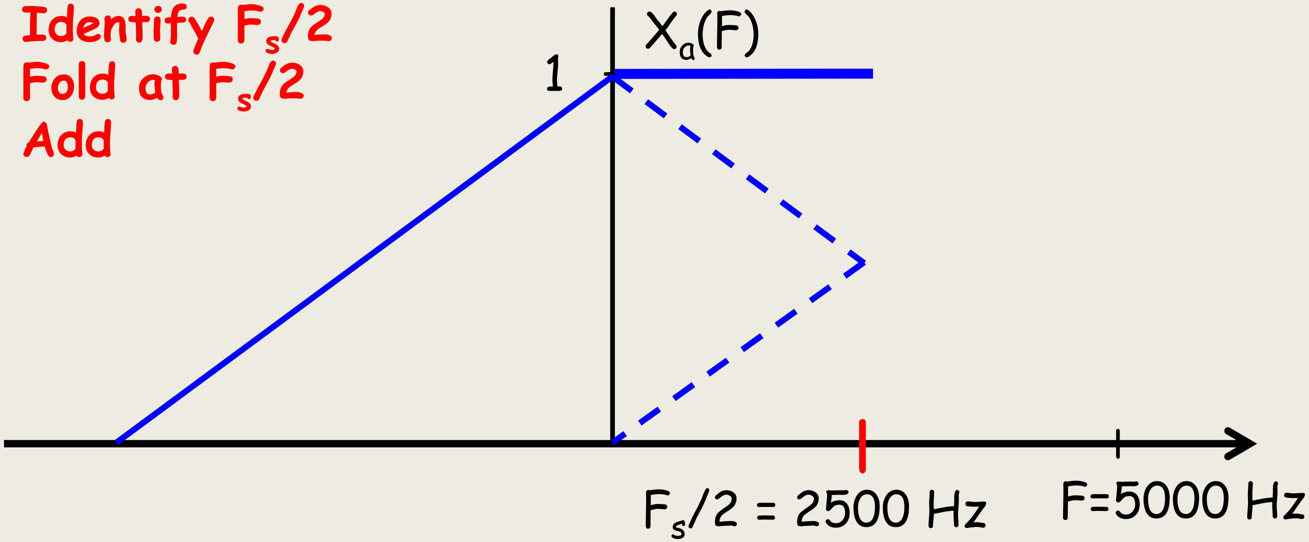
EITF75 Systems and Signals



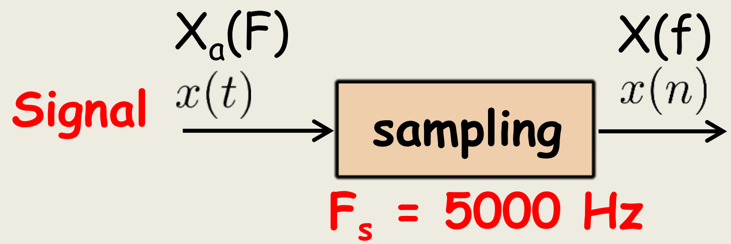
Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$
Step 3: Add



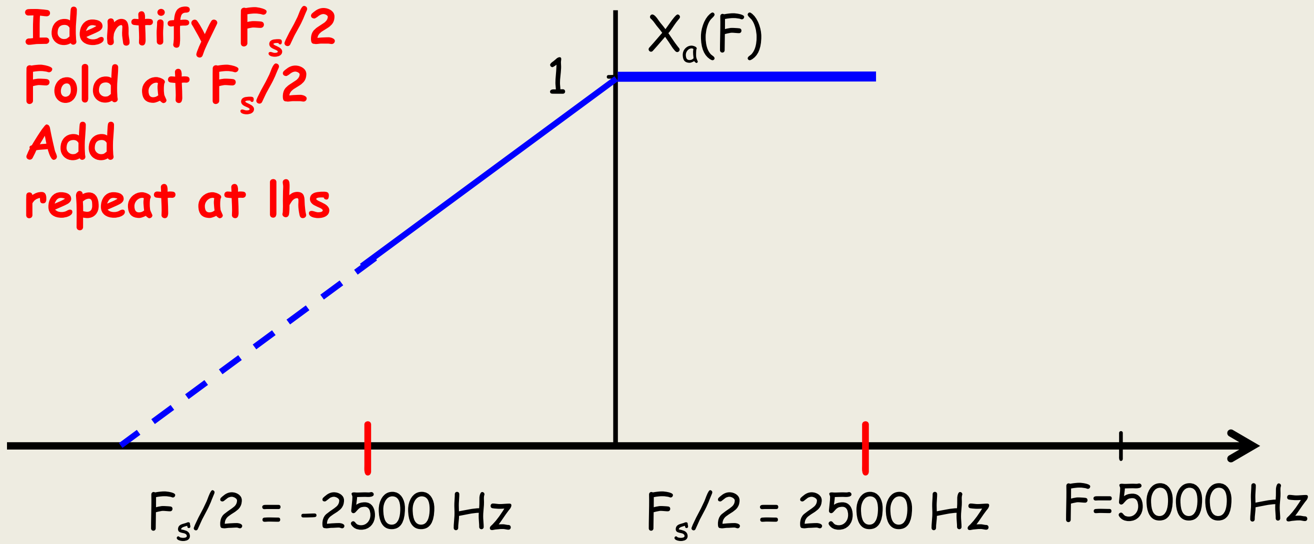
EITF75 Systems and Signals



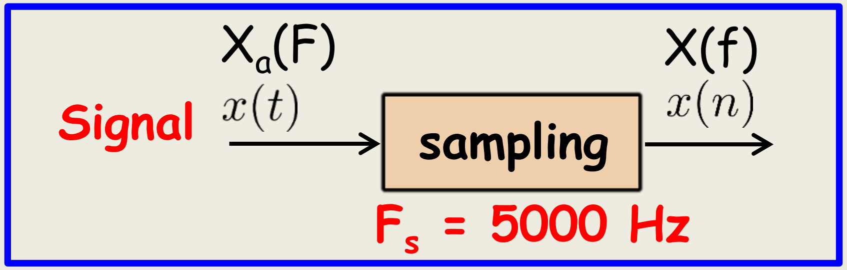
Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$
Step 3: Add
Step 4: repeat at lhs



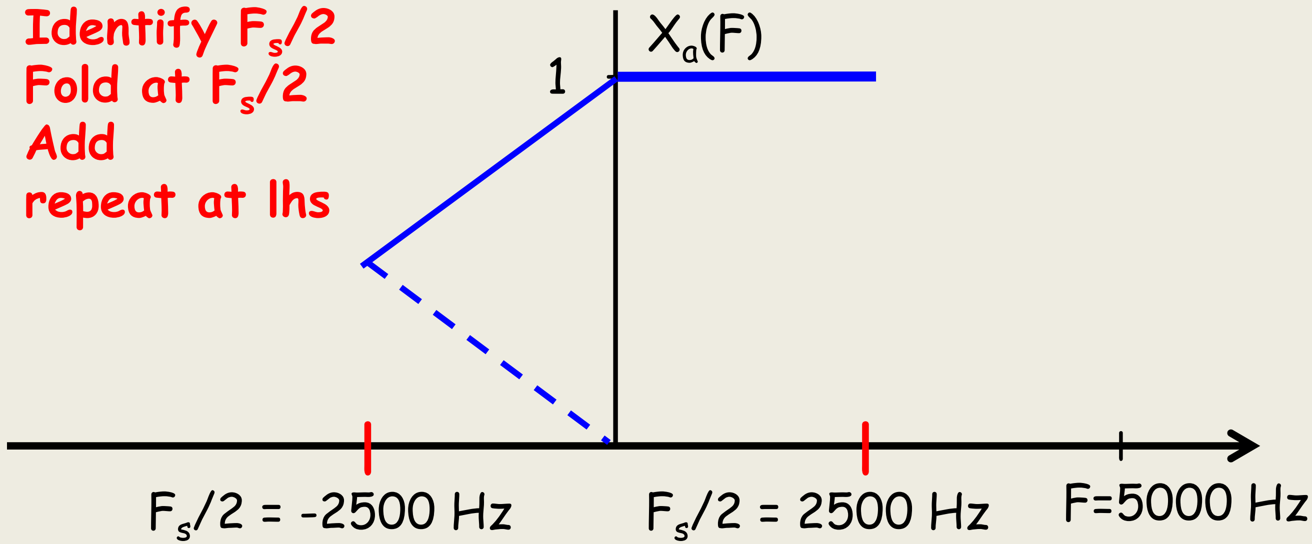
EITF75 Systems and Signals



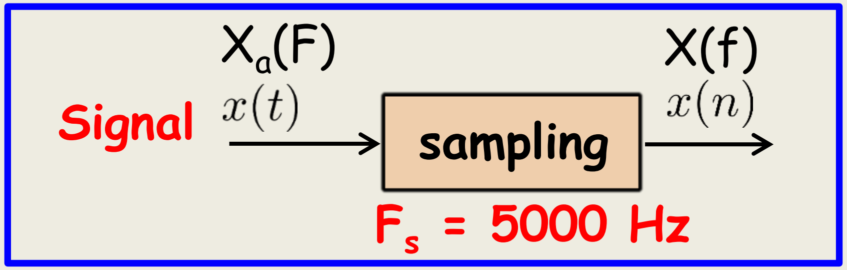
Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$
Step 3: Add
Step 4: repeat at lhs



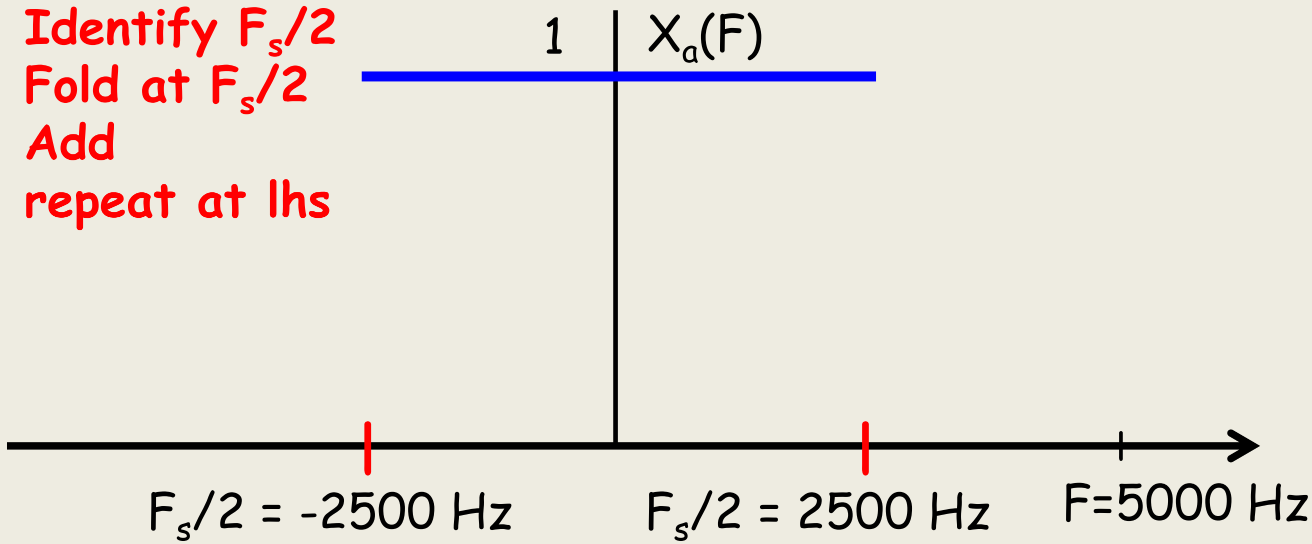
EITF75 Systems and Signals



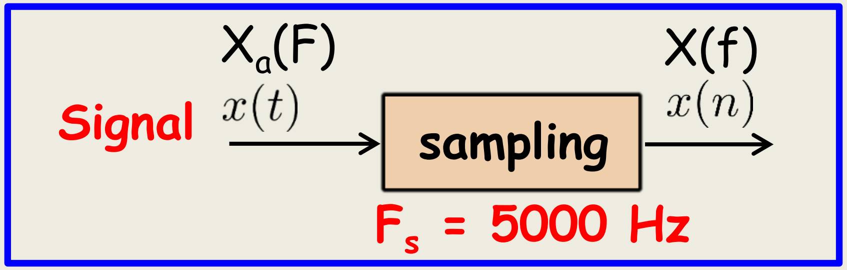
Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$
Step 3: Add
Step 4: repeat at lhs



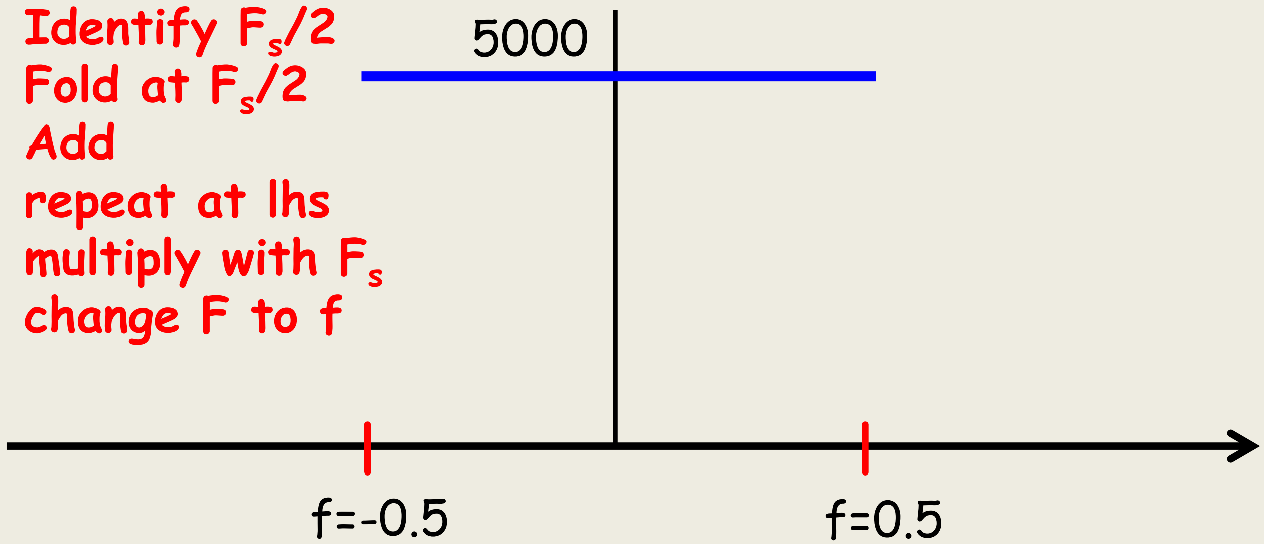
EITF75 Systems and Signals



Example

Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$
Step 3: Add
Step 4: repeat at lhs
Step 5: multiply with F_s
Step 6: change F to f



EITF75 Systems and Signals

Signal

$F_s = 4000$ Hz

Example

Folding

Step 1: Identify $F_s/2$

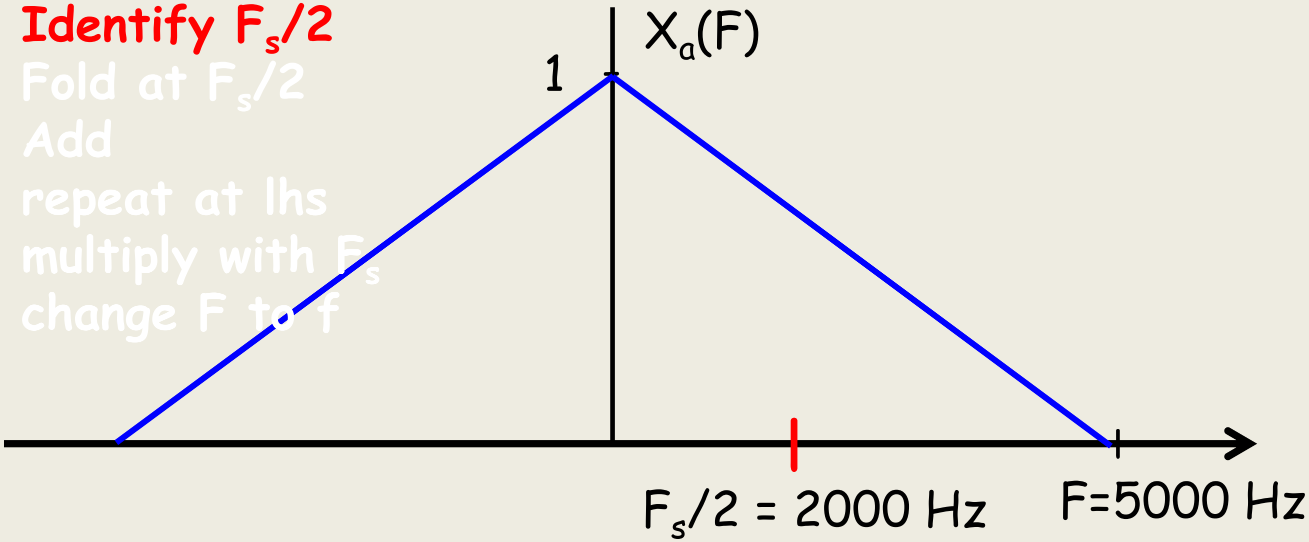
Step 2: Fold at $F_s/2$

Step 3: Add

Step 4: repeat at lhs

Step 5: multiply with F_s

Step 6: change F to f



EITF75 Systems and Signals

Signal

F_s = 4000 Hz

Example

Folding

Step 1: Identify $F_s/2$

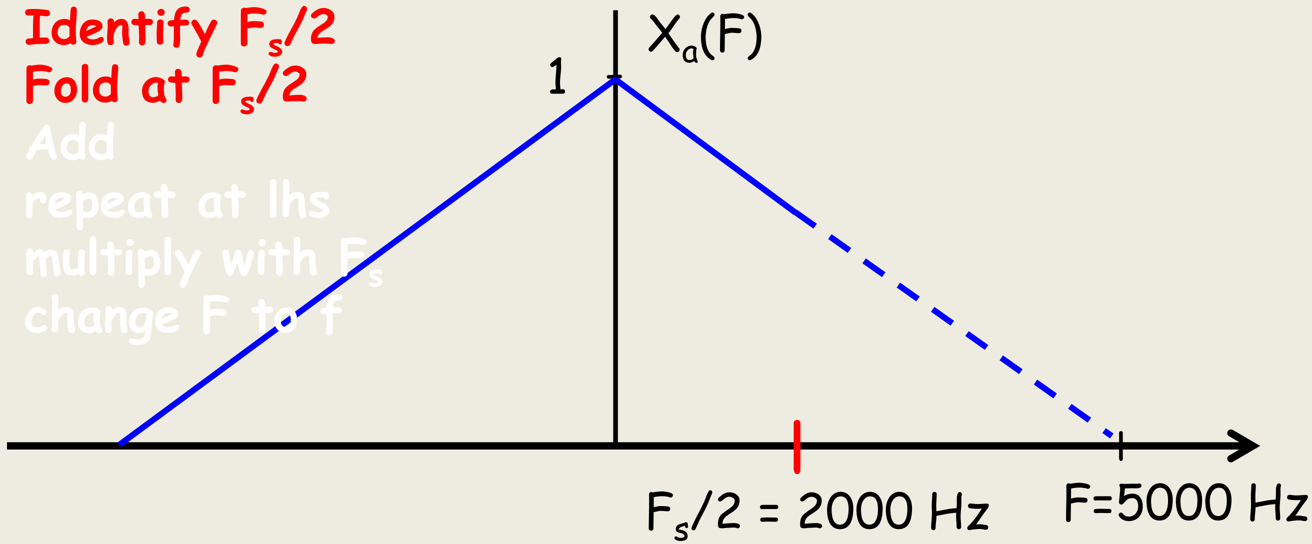
Step 2: Fold at $F_s/2$

Step 3: Add

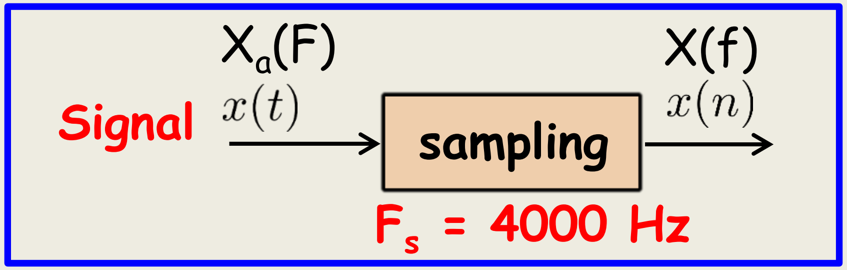
Step 4: repeat at lhs

Step 5: multiply with F_s

Step 6: change F to f



EITF75 Systems and Signals



Example

Folding

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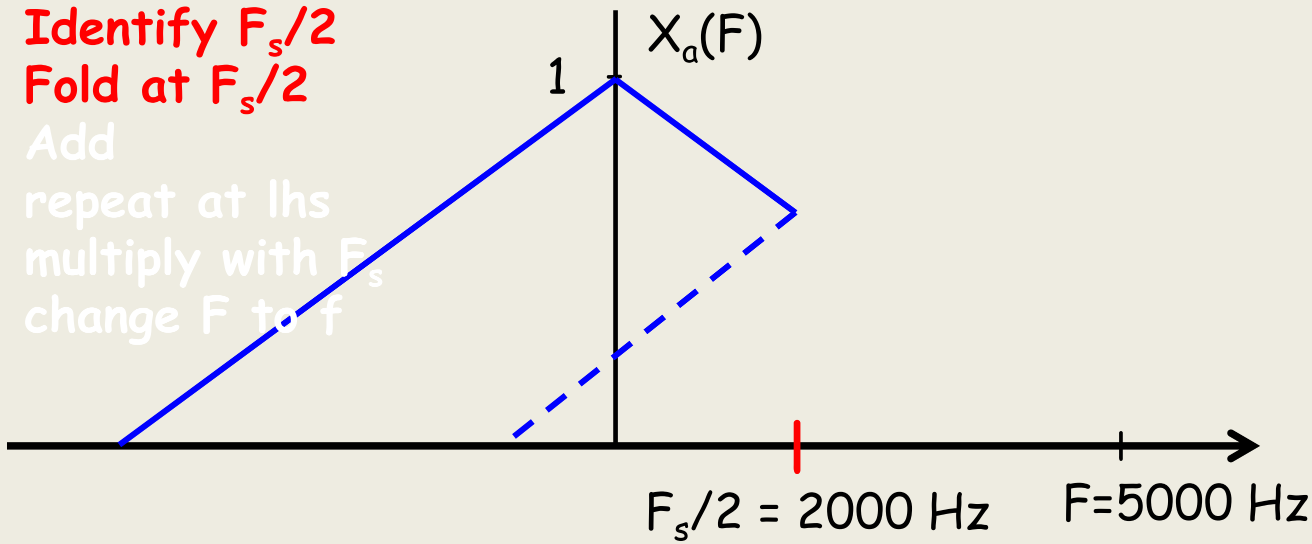
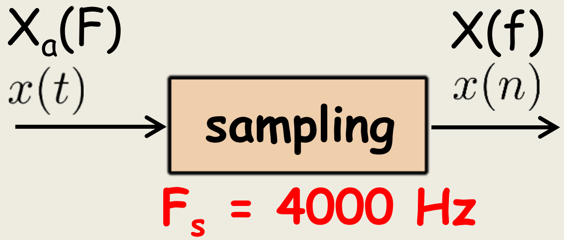


EITF75 Systems and Signals

Signal



Example

Folding

Step 1: Identify $F_s/2$

Step 2: Fold at $F_s/2$

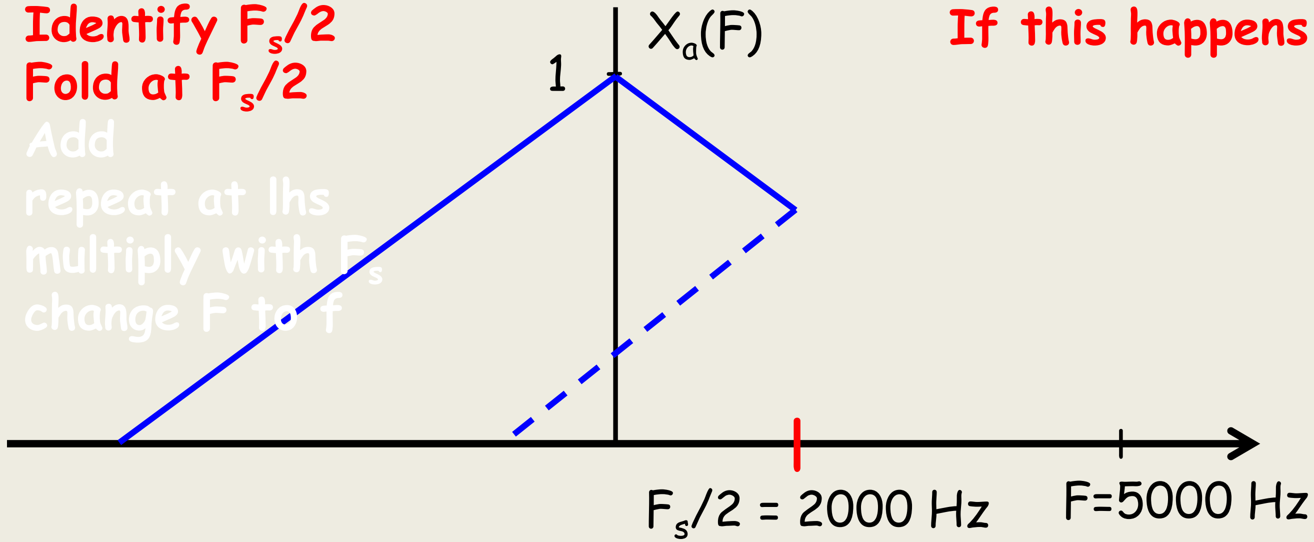
Step 3: Add

Step 4: repeat at lhs

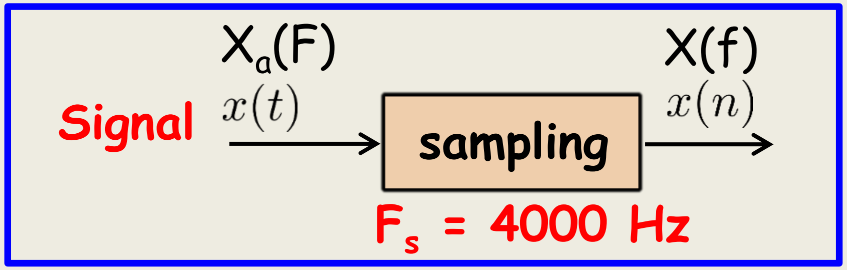
Step 5: multiply with F_s

Step 6: change F to f

If this happens:



EITF75 Systems and Signals



Example

Folding

Step 1: Identify $F_s/2$
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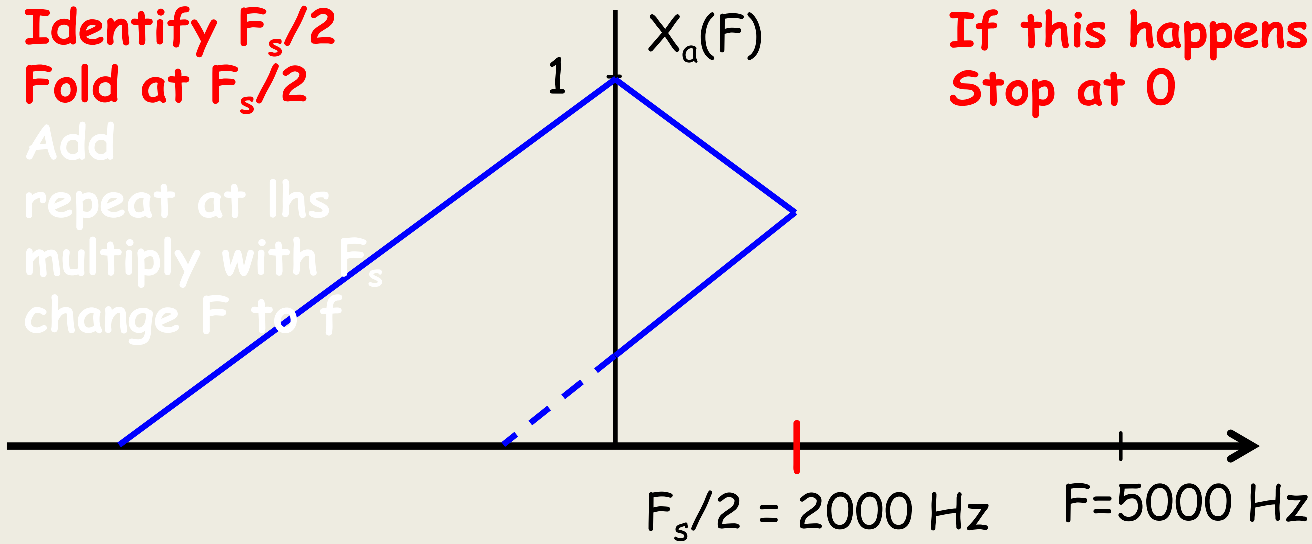


If this happens:
Stop at 0

EITF75 Systems and Signals

Signal

F_s = 4000 Hz

Example

Folding

Step 1: Identify $F_s/2$

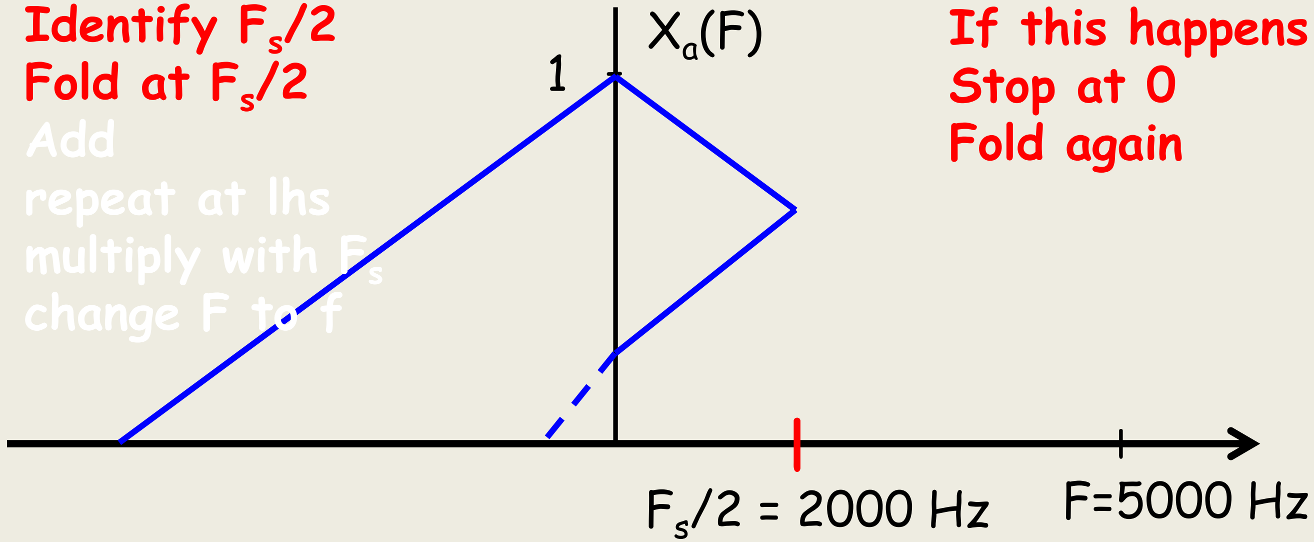
Step 2: Fold at $F_s/2$

Step 3: Add

Step 4: repeat at lhs

Step 5: multiply with F_s

Step 6: change F to f



If this happens:

Stop at 0

Fold again

EITF75 Systems and Signals

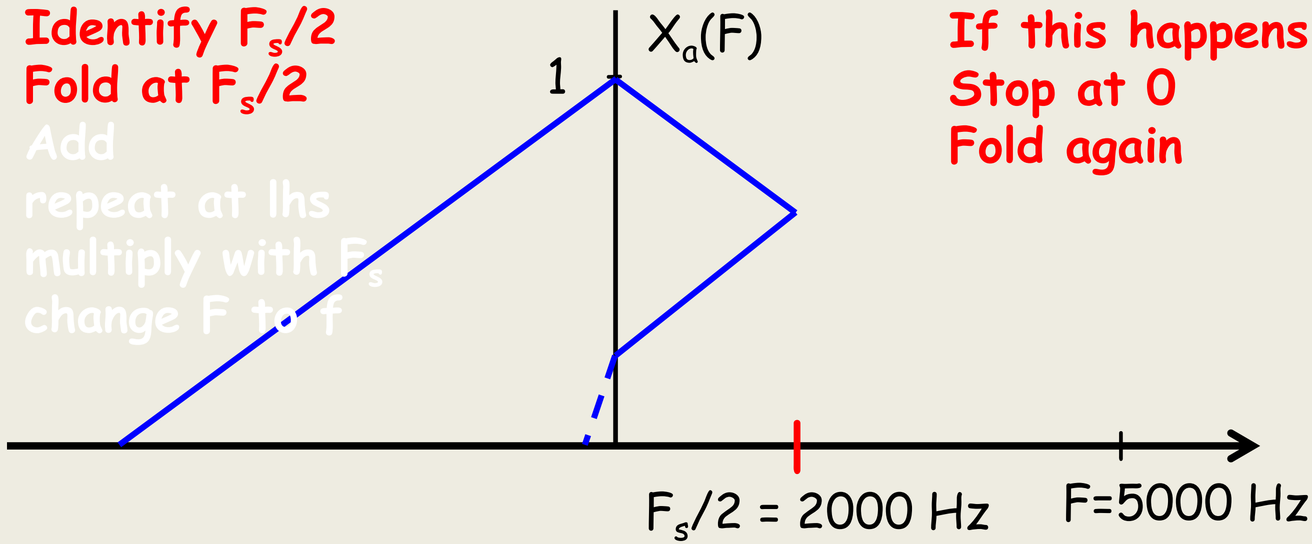
Signal

F_s = 4000 Hz

Example

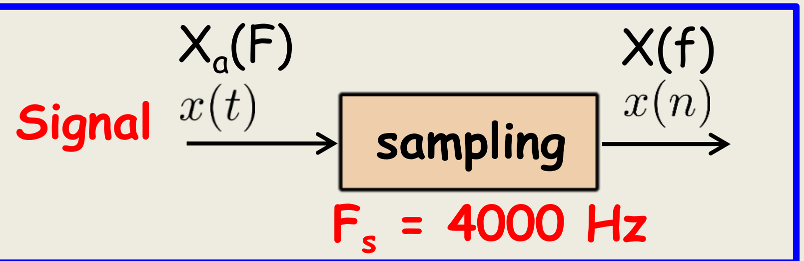
Folding

Step 1: Identify $F_s/2$
Step 2: Fold at $F_s/2$
Step 3: Add
Step 4: repeat at lhs
Step 5: multiply with F_s
Step 6: change F to f



If this happens:
Stop at 0
Fold again

EITF75 Systems and Signals



Example

Folding

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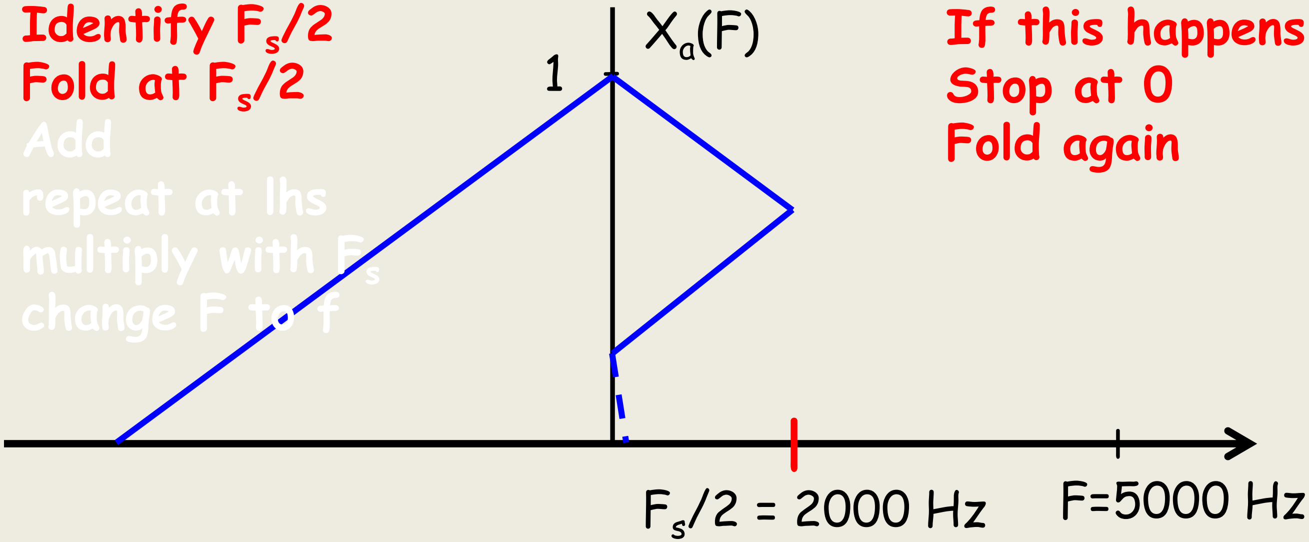
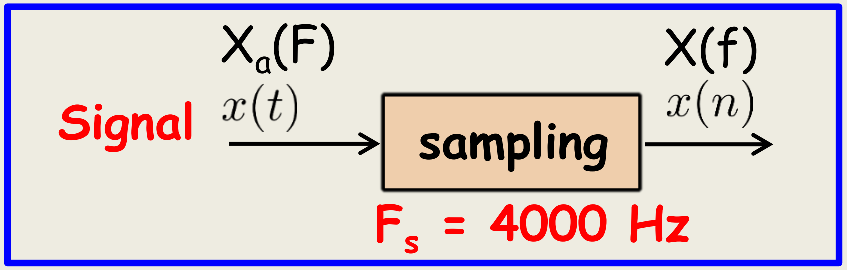


If this happens:
Stop at 0
Fold again

EITF75 Systems and Signals



Example

Folding

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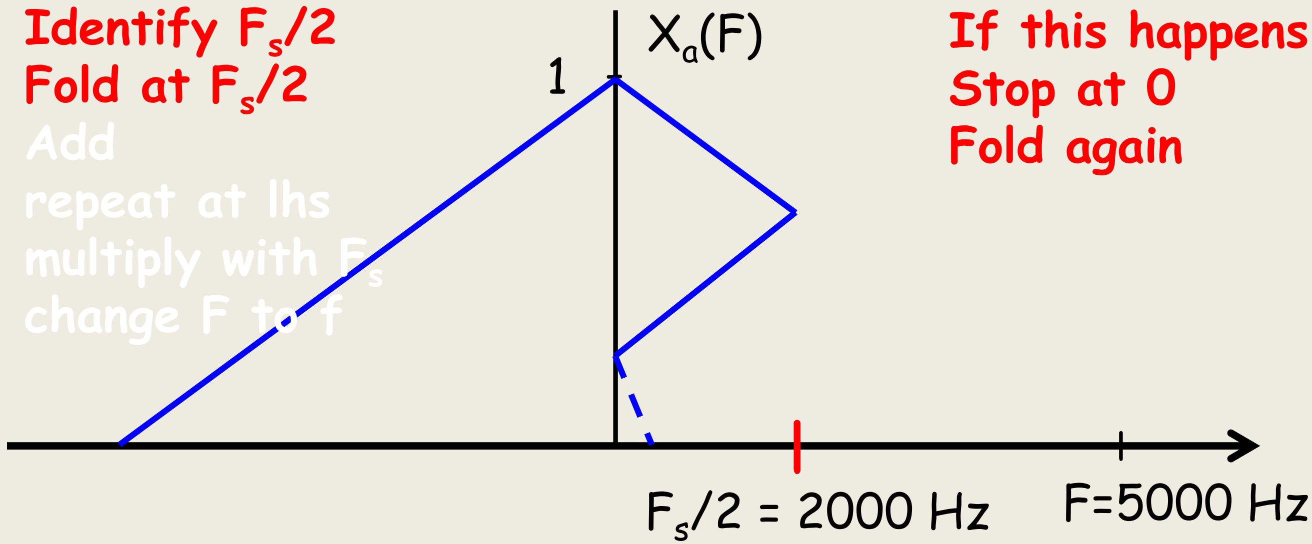


If this happens:
Stop at 0
Fold again

EITF75 Systems and Signals

Signal

F_s = 4000 Hz

Example

Folding

Step 1: Identify $F_s/2$

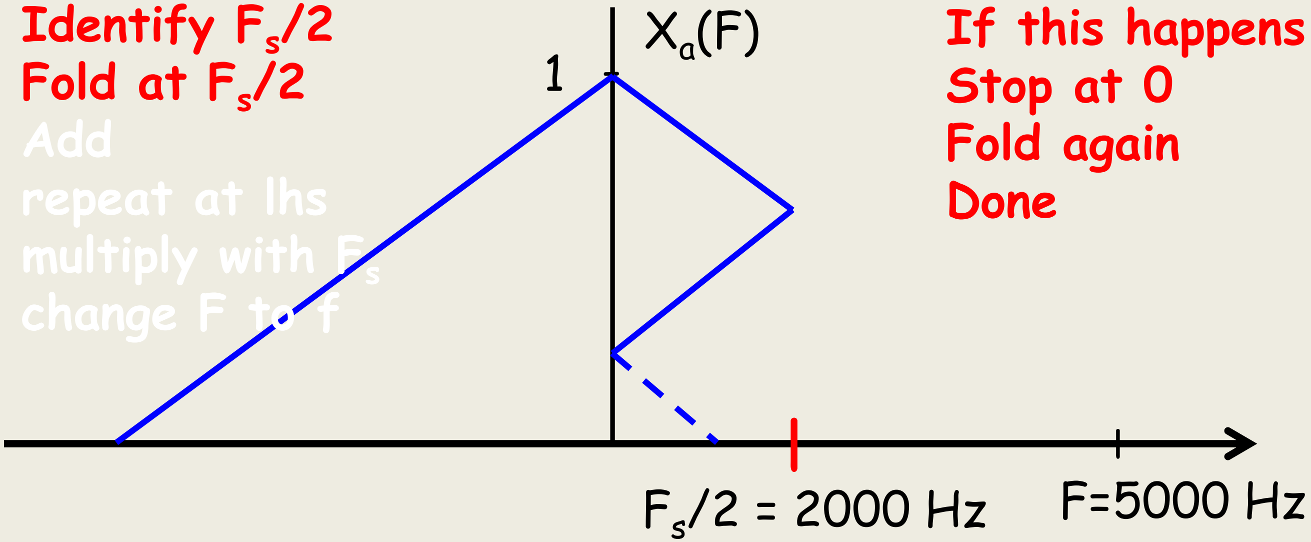
Step 2: Fold at $F_s/2$

Step 3: Add

Step 4: repeat at lhs

Step 5: multiply with F_s

Step 6: change F to f



If this happens:

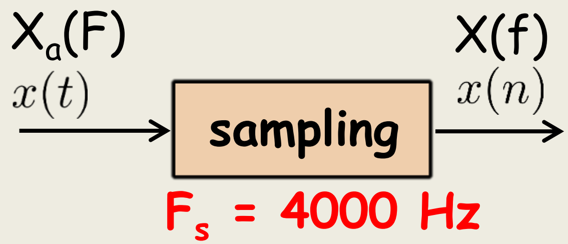
Stop at 0

Fold again

Done

EITF75 Systems and Signals

Signal



Example

Folding

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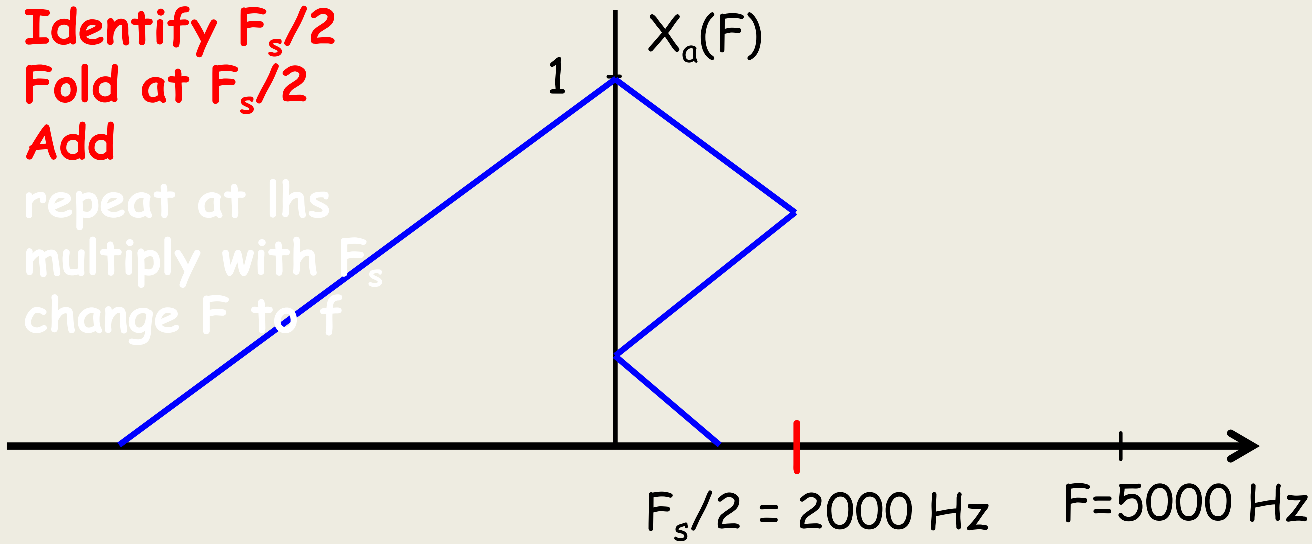

EITF75 Systems and Signals

Signal $x(t)$ → **sampling** → $x(n)$

$X_a(F)$ $X(f)$

F_s = 4000 Hz

Example

Folding

Step 1: Identify $F_s/2$
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Step 3: Add
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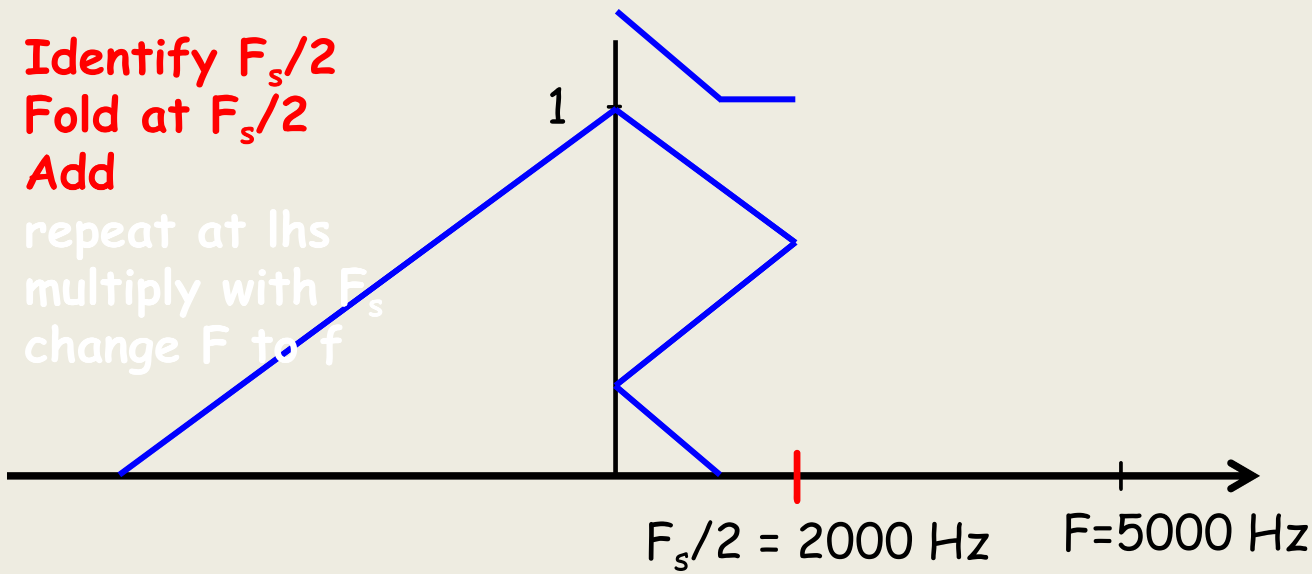
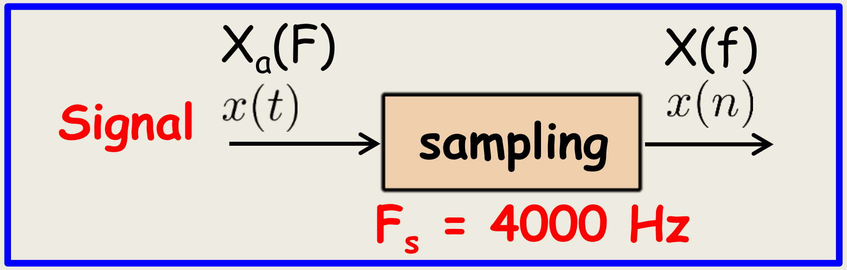


EITF75 Systems and Signals



Example

Folding

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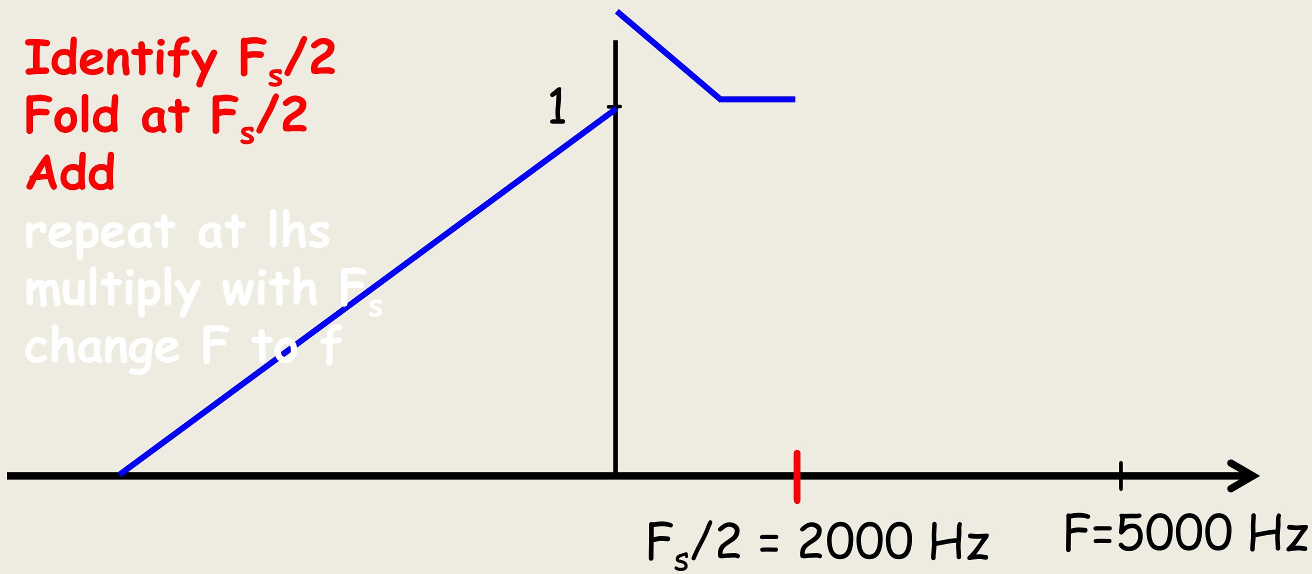

EITF75 Systems and Signals

Signal

F_s = 4000 Hz

Example

Folding

Step 1: Identify $F_s/2$
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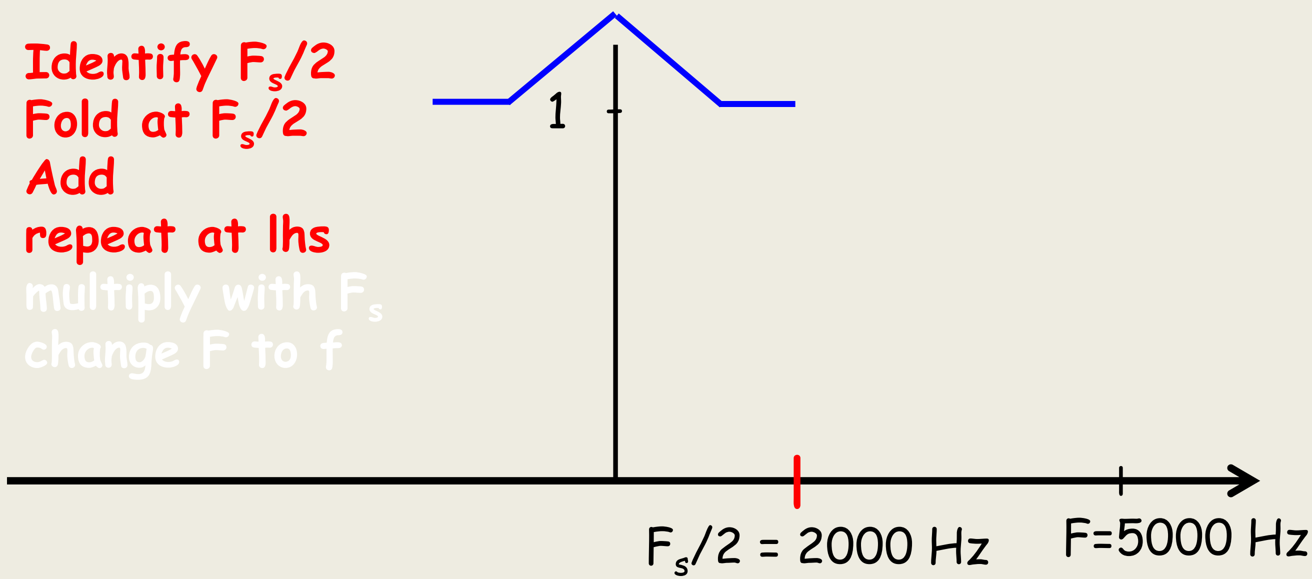
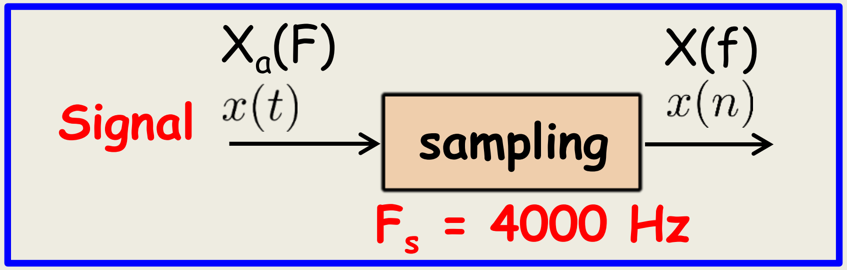


EITF75 Systems and Signals



Example

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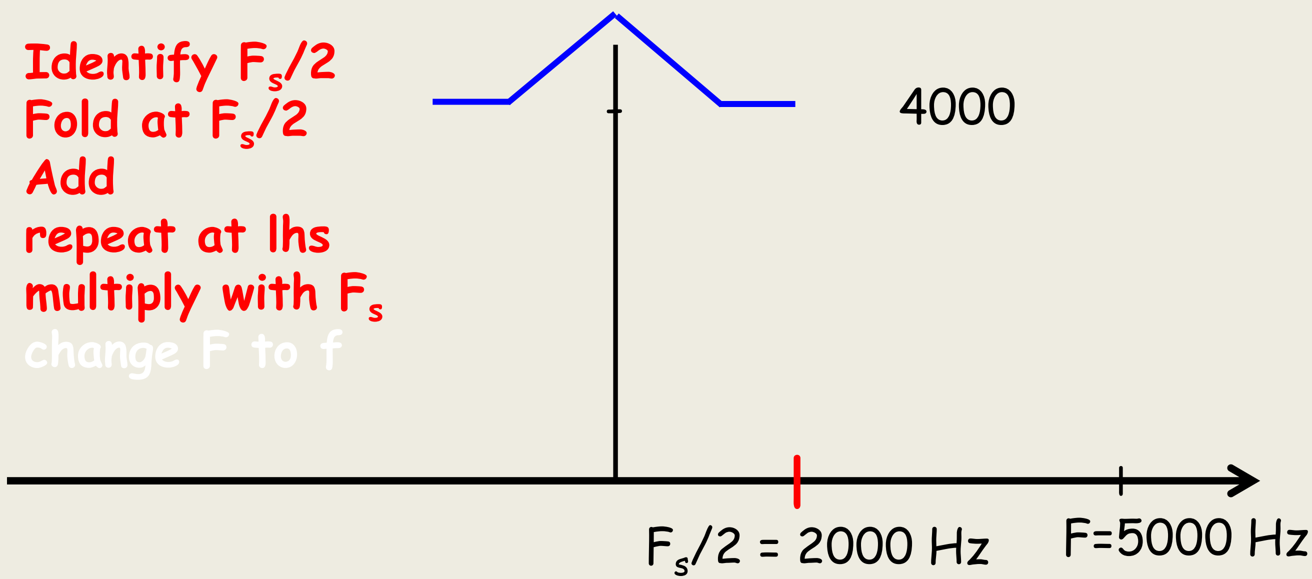

EITF75 Systems and Signals

Signal

F_s = 4000 Hz

Example

Folding

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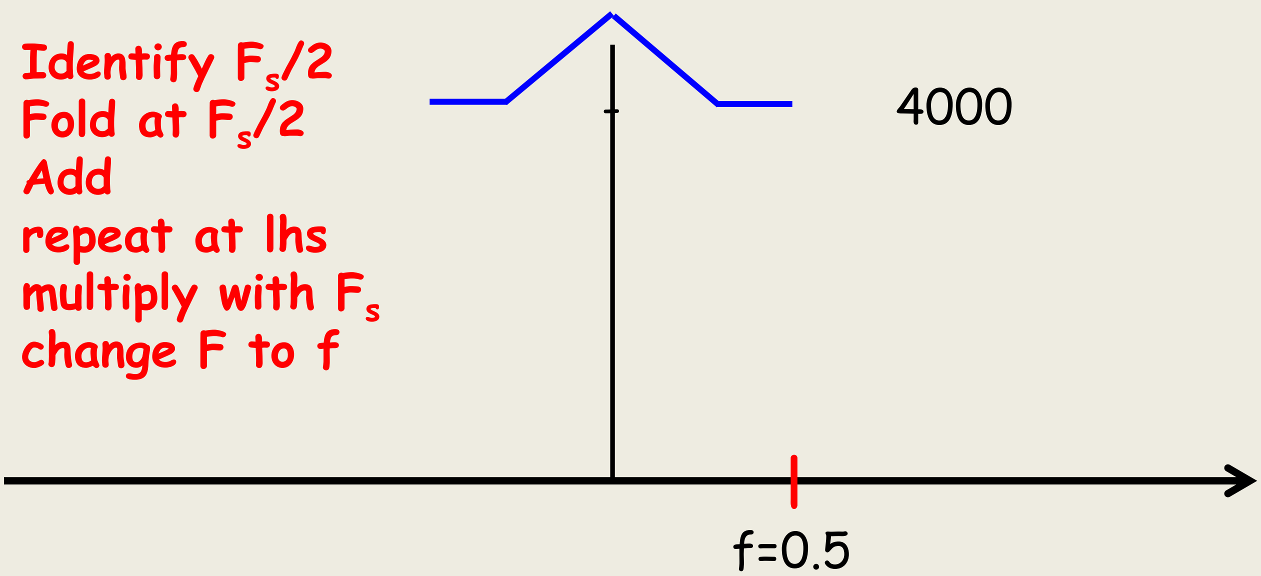
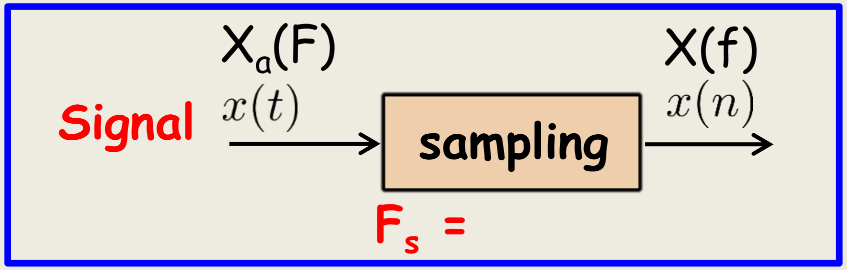
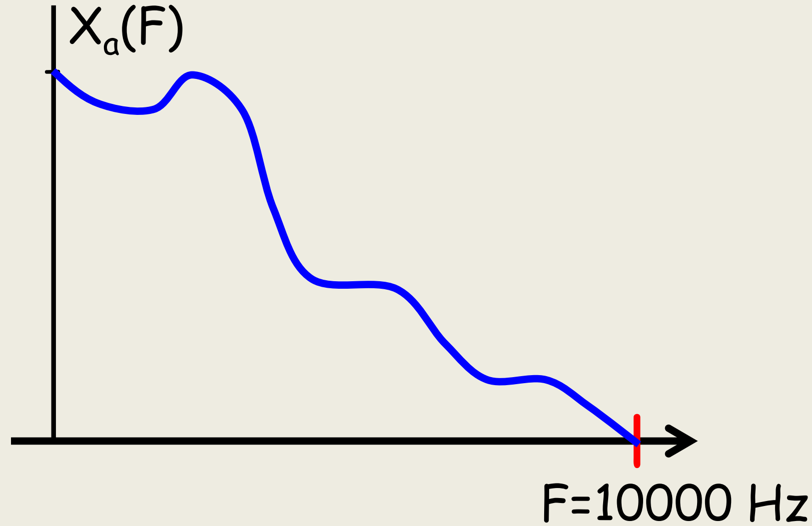


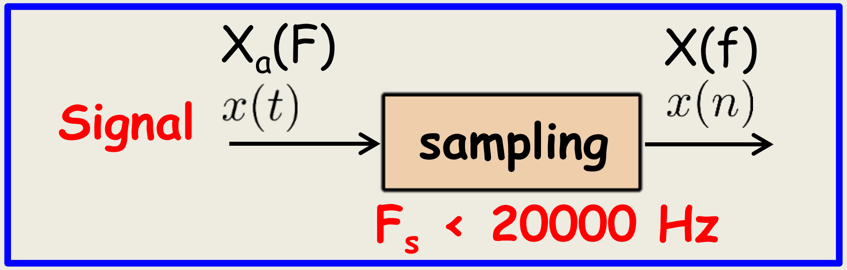
EITF75 Systems and Signals



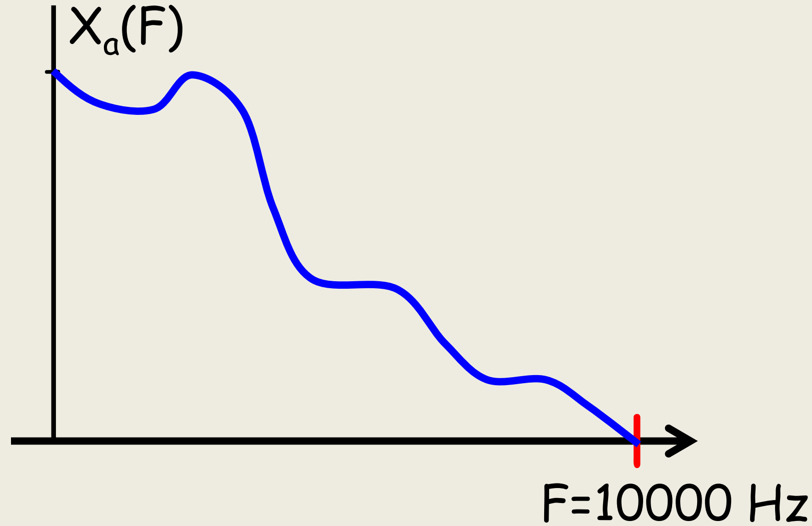
When is sampling lossless?

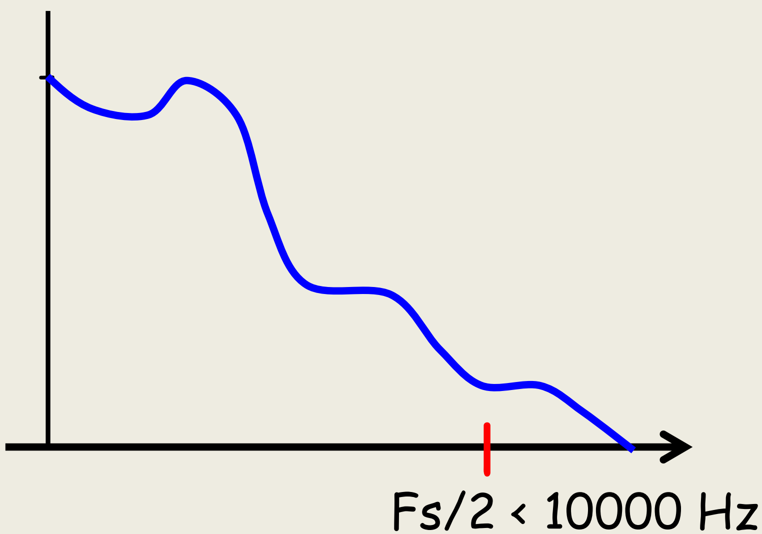


EITF75 Systems and Signals

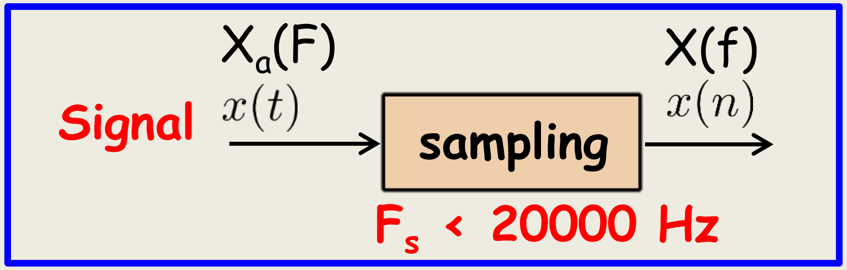


When is sampling lossless?





EITF75 Systems and Signals



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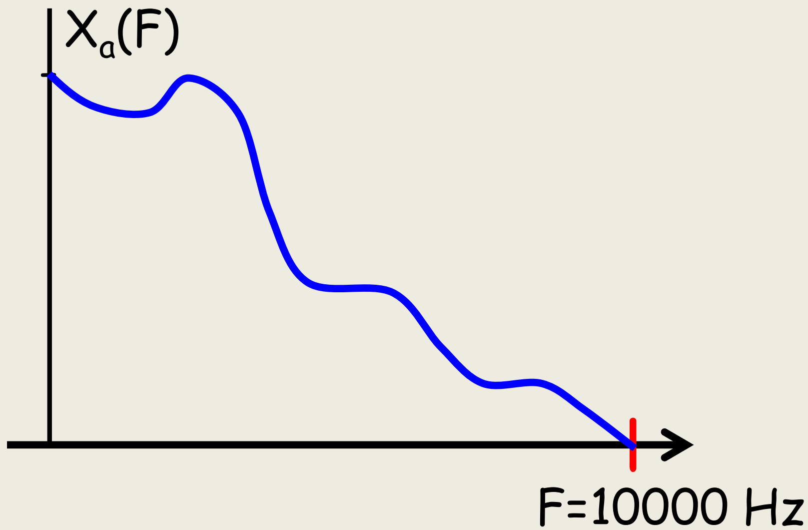
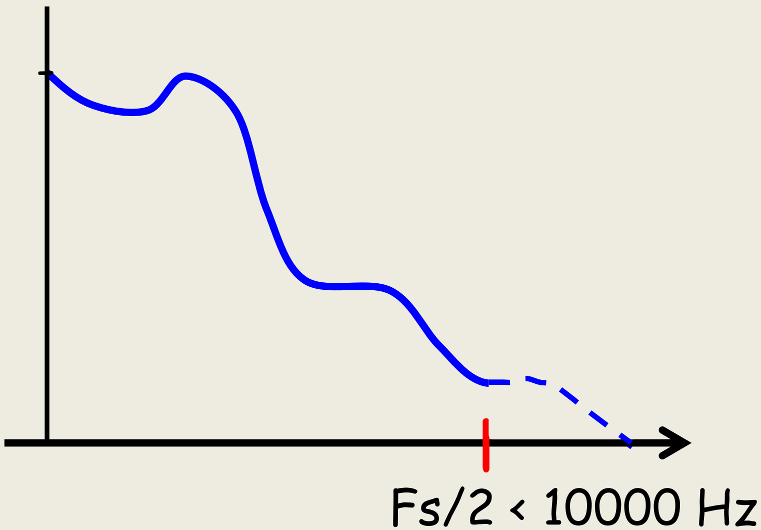
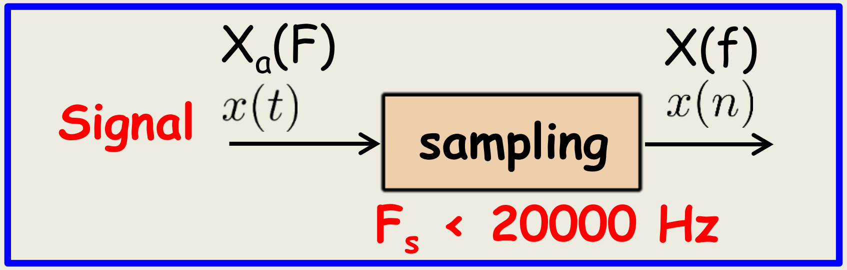




EITF75 Systems and Signals



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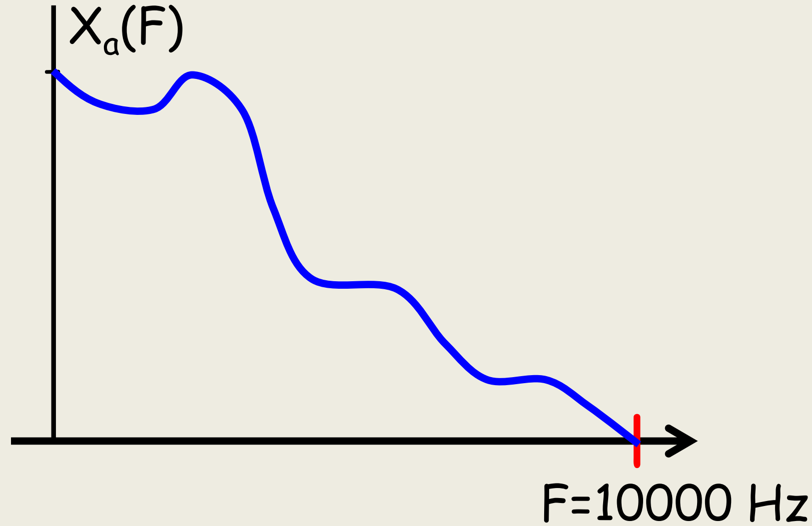
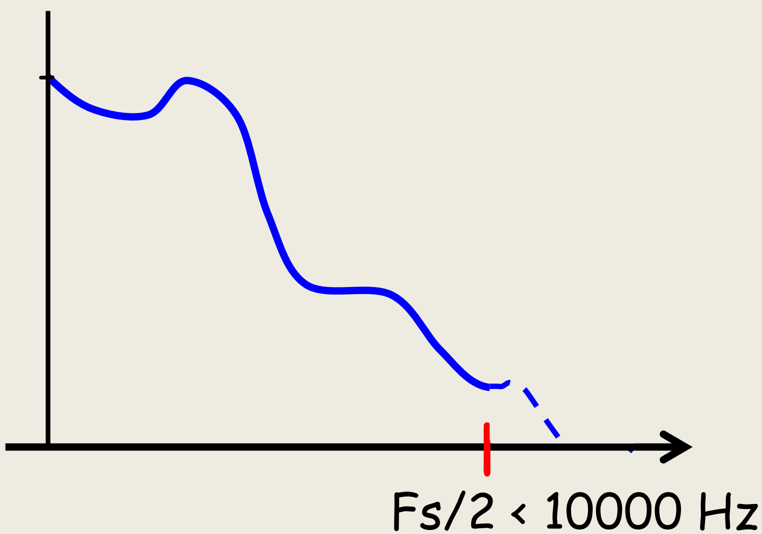
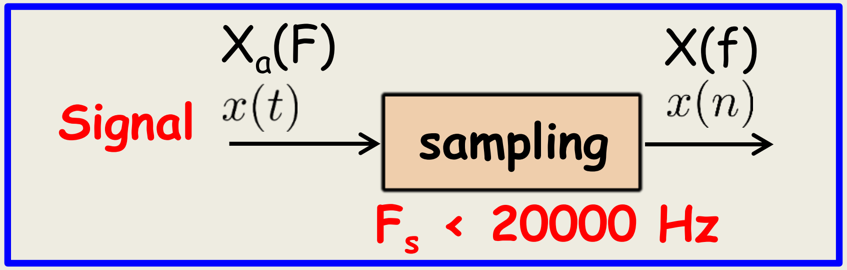




EITF75 Systems and Signals



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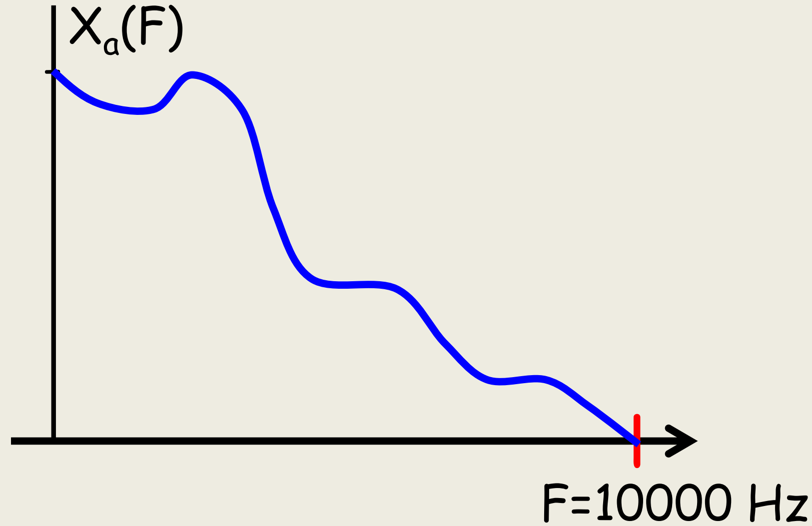
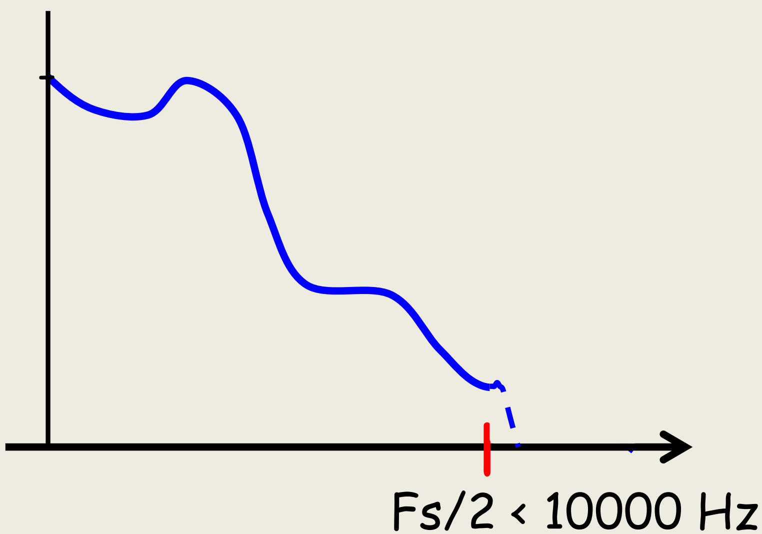
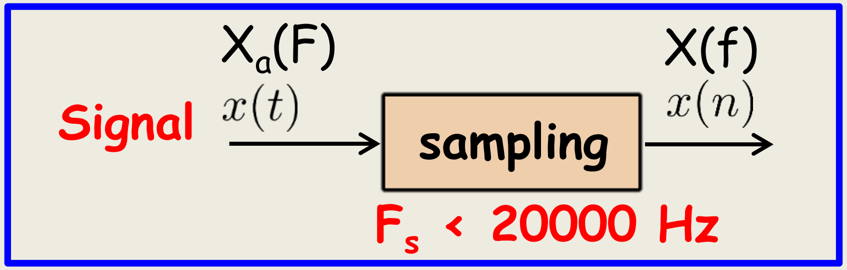




EITF75 Systems and Signals



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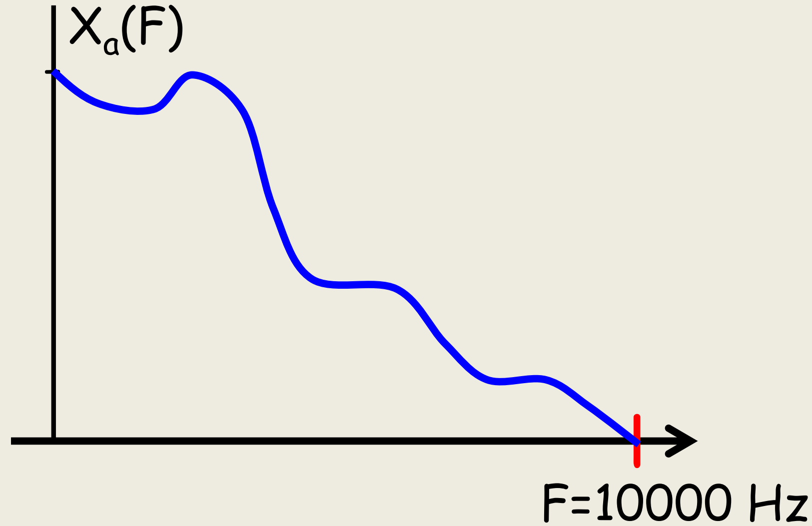
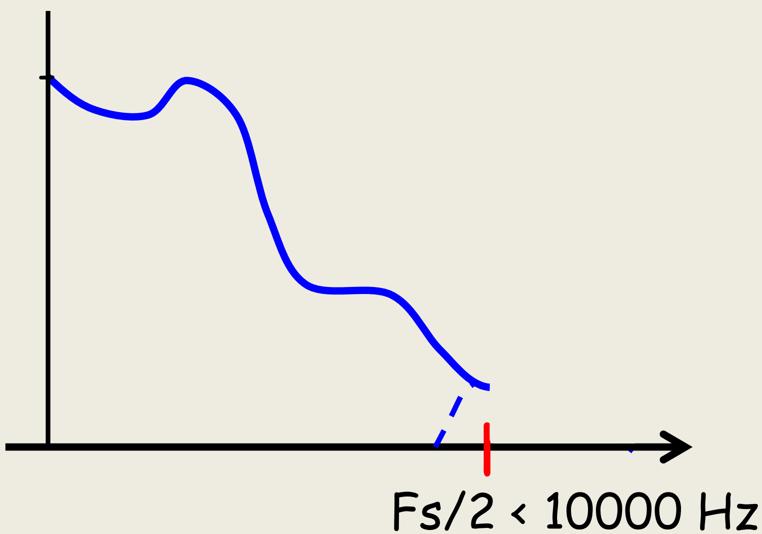
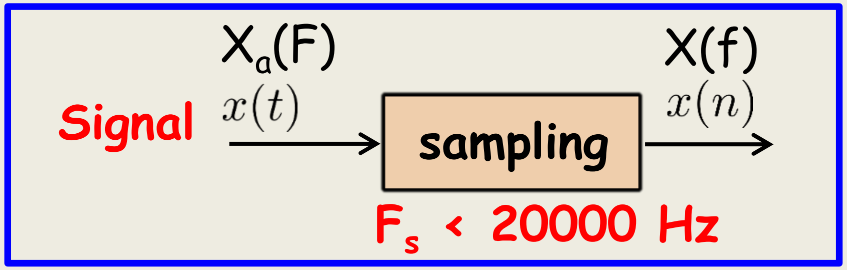




EITF75 Systems and Signals



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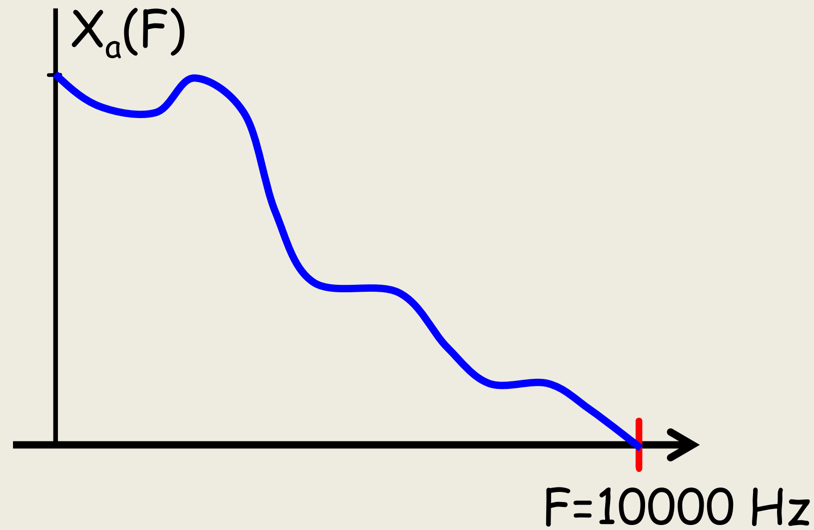
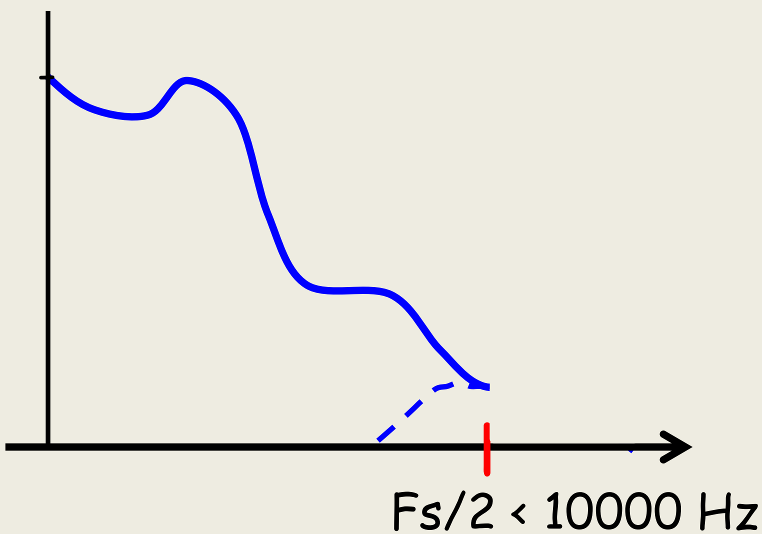
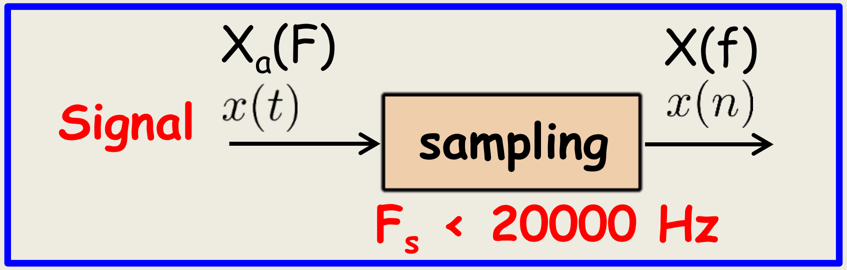




EITF75 Systems and Signals



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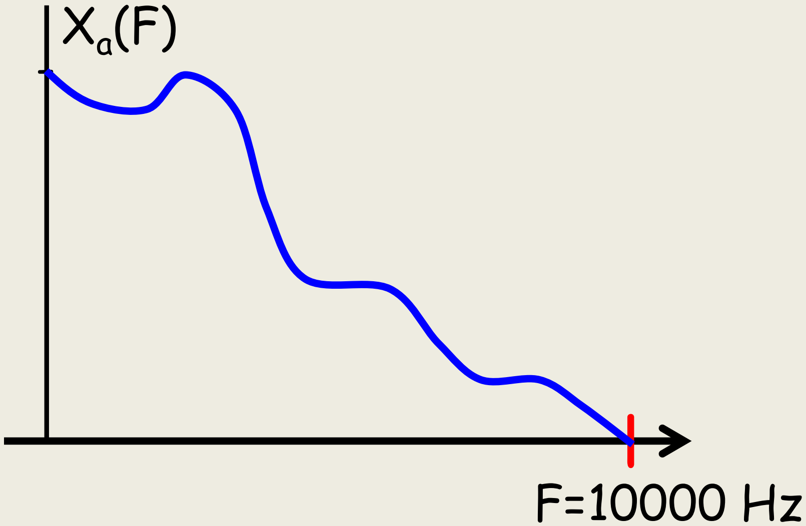
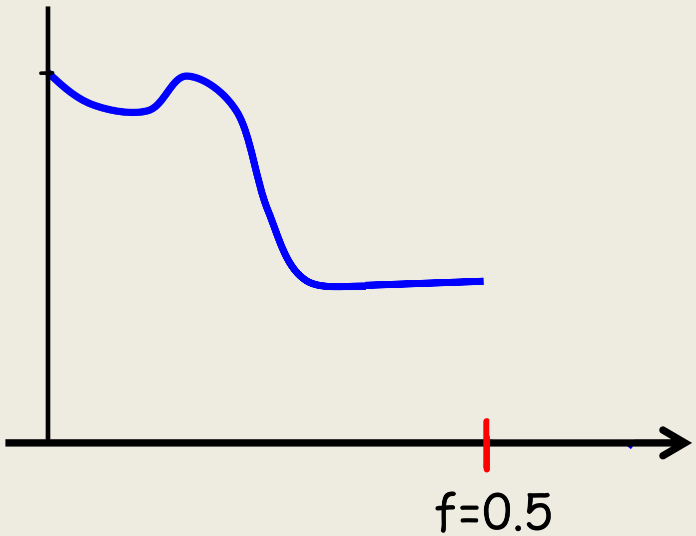
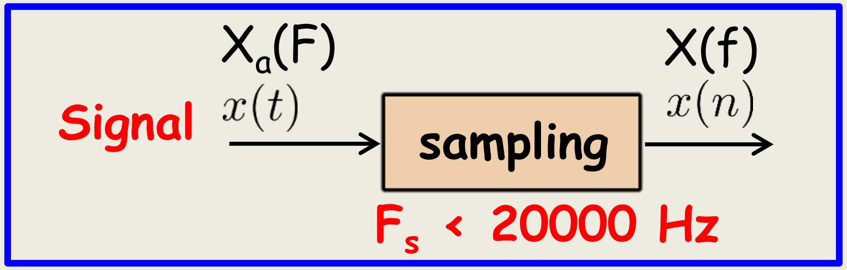




EITF75 Systems and Signals



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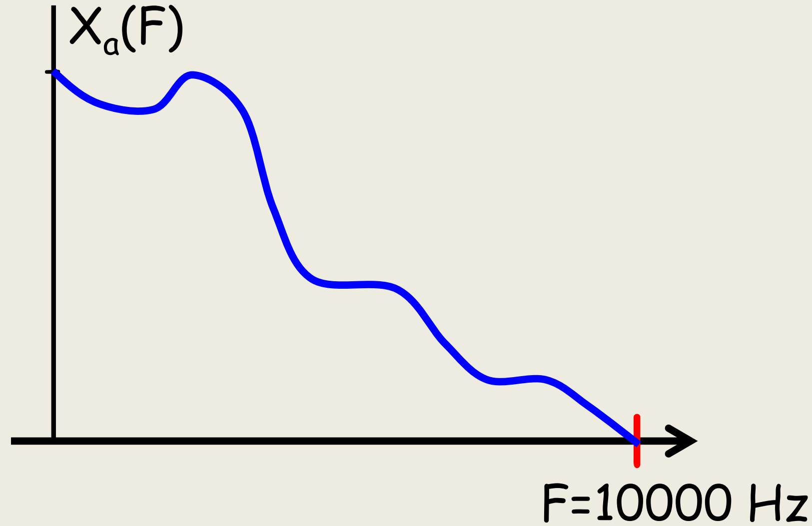
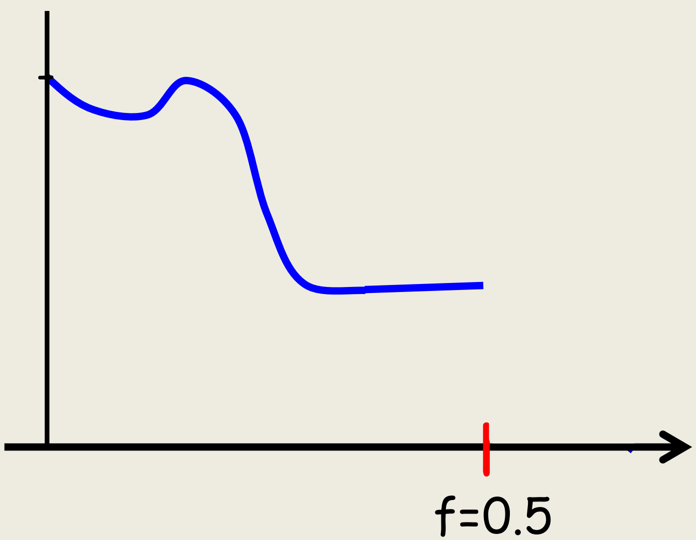
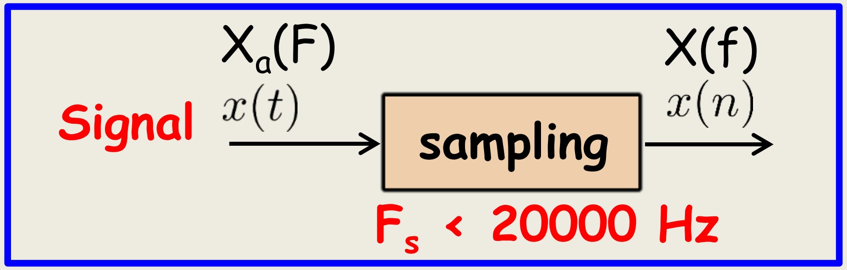




We cannot recover x(t) from x(n)

EITF75 Systems and Signals



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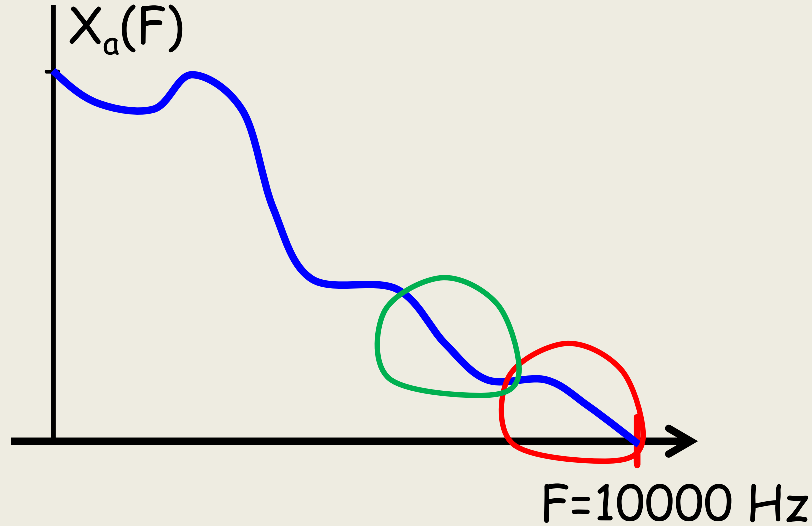
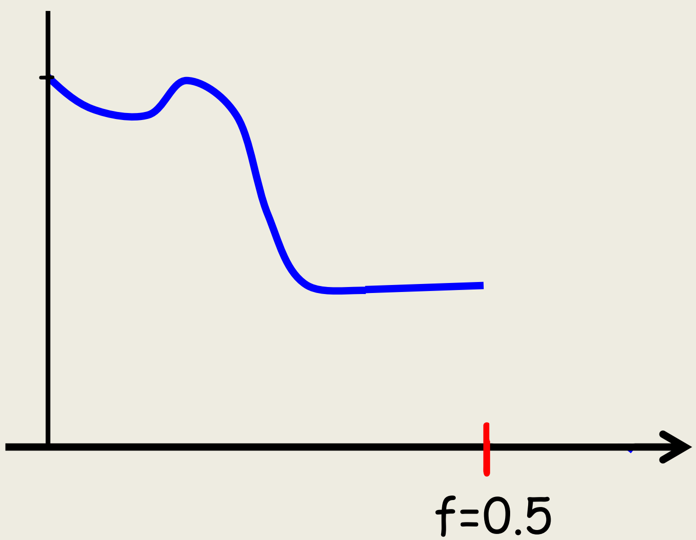
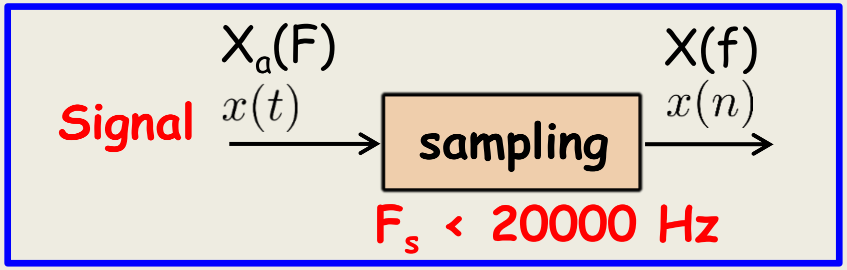




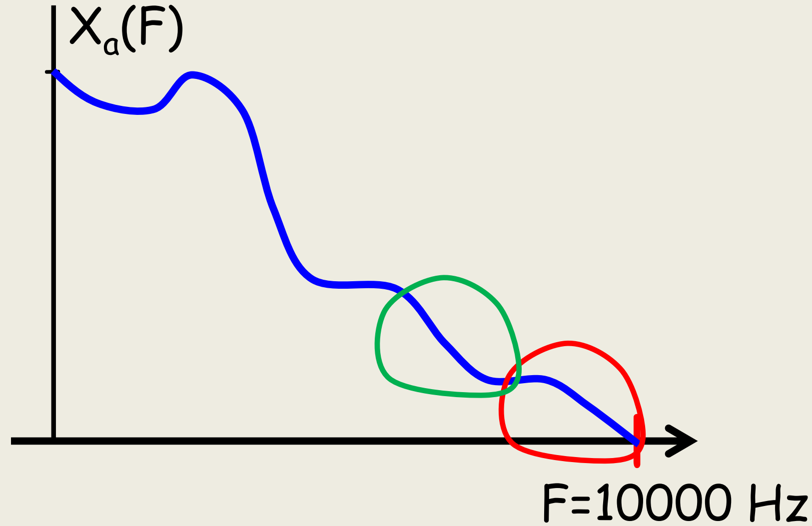
We cannot recover x(t) from x(n)

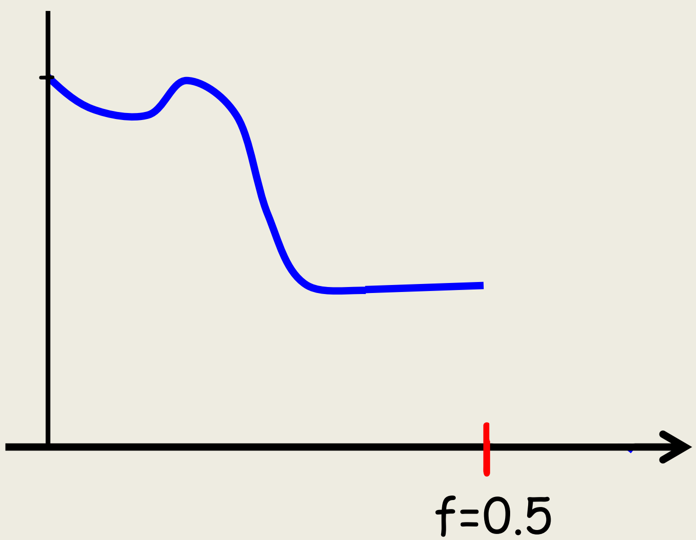
Red frequencies have been mixed with green

EITF75 Systems and Signals



When is sampling lossless?



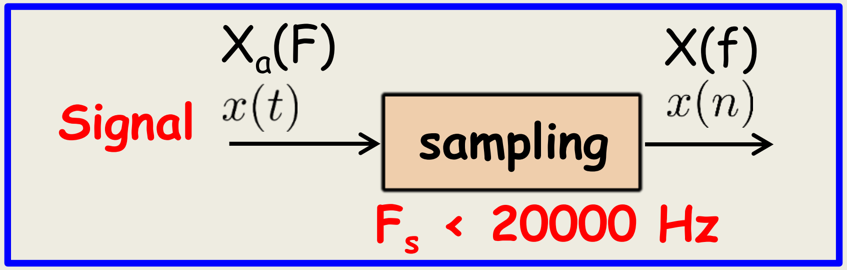


We cannot recover x(t) from x(n)

Red frequencies have been mixed with green

This effect is called Aliasing

EITF75 Systems and Signals



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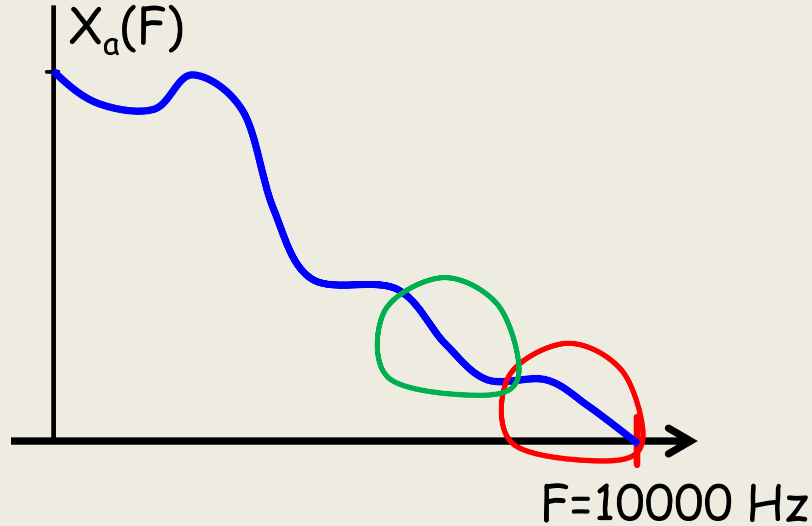

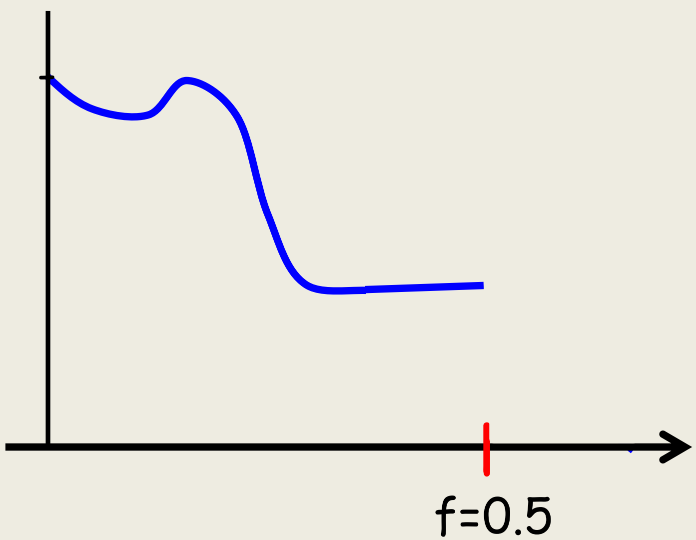

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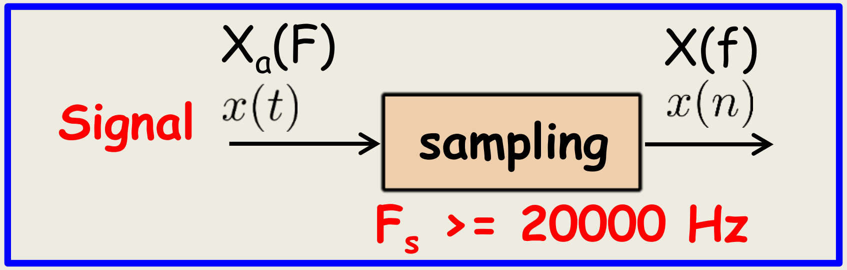
Method to find X(f) is called folding

EITF75 Systems and Signals



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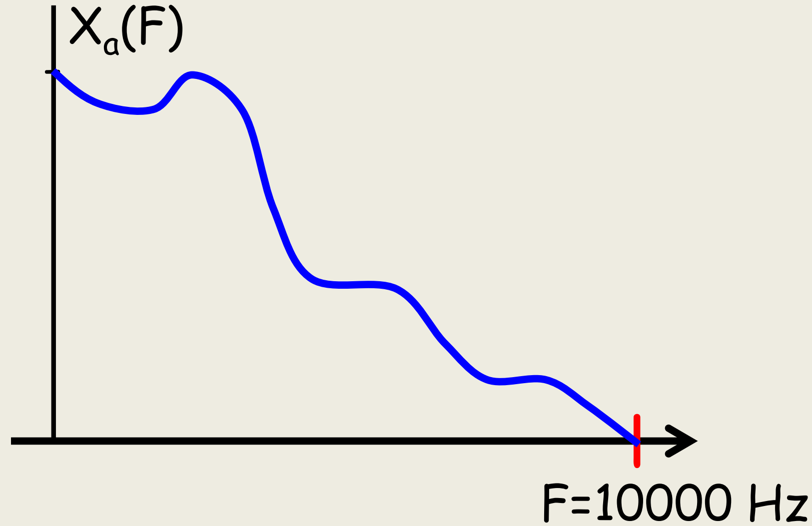
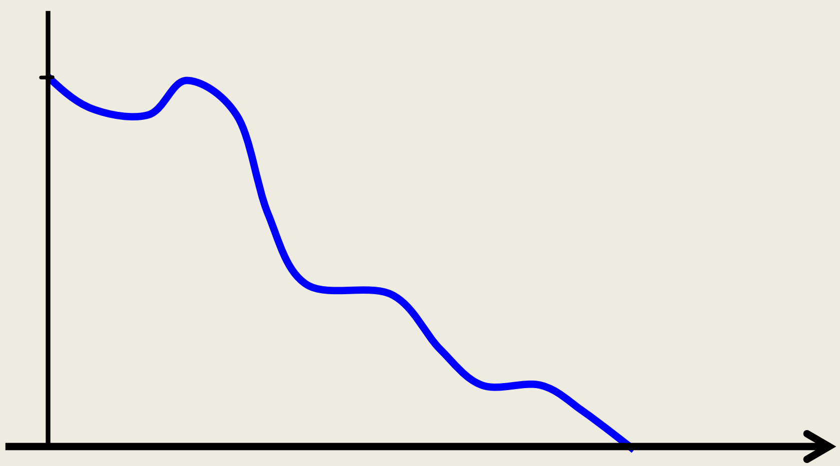
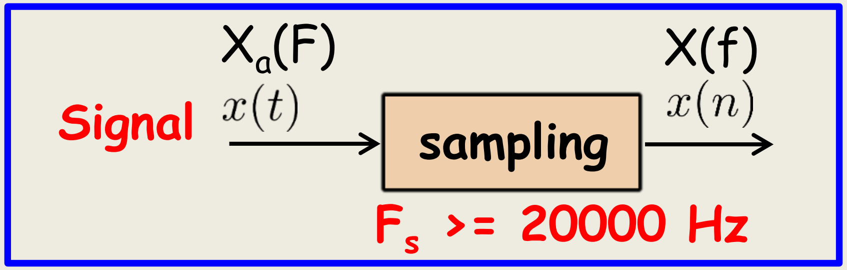




EITF75 Systems and Signals



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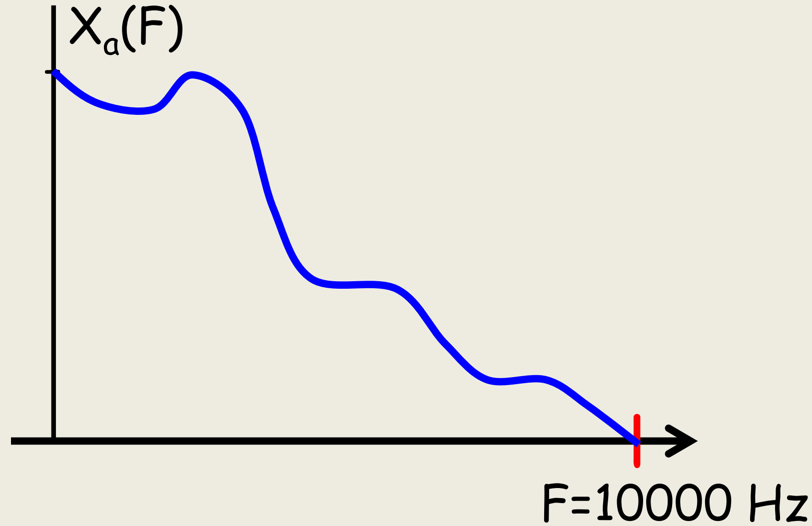
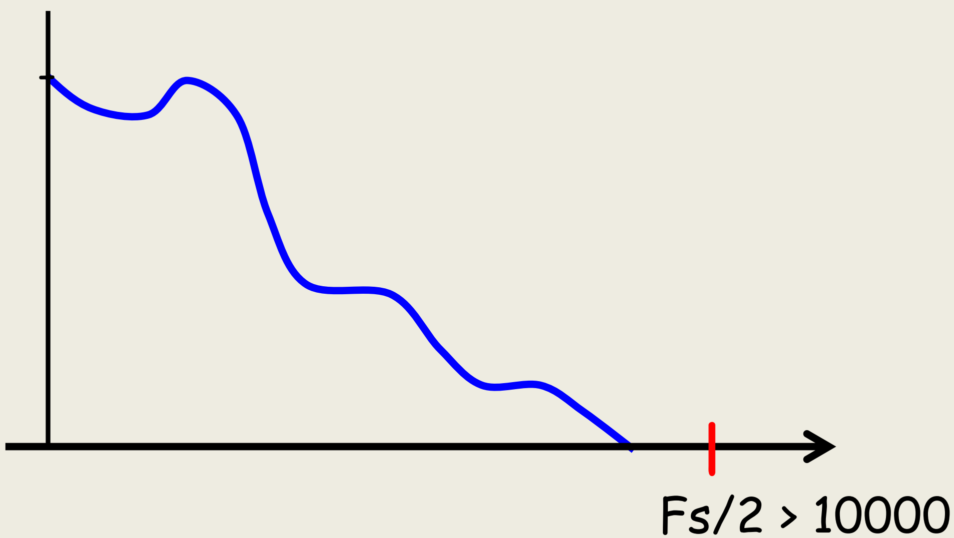
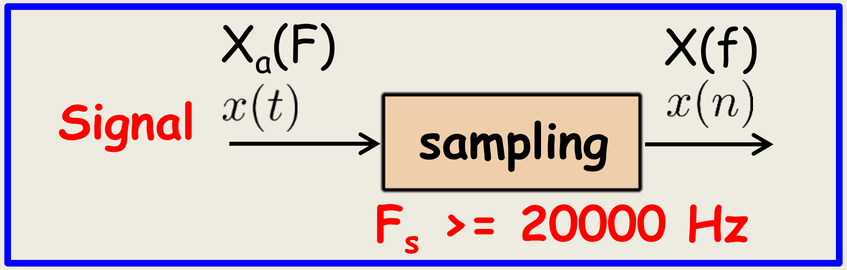




EITF75 Systems and Signals



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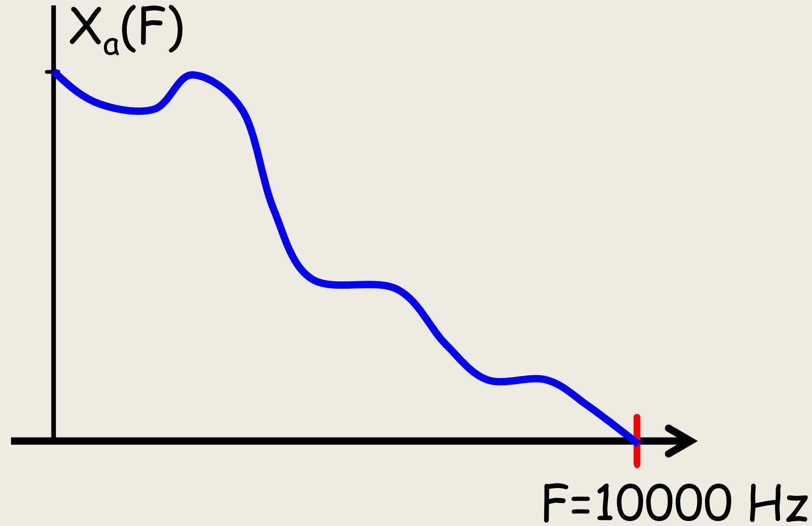
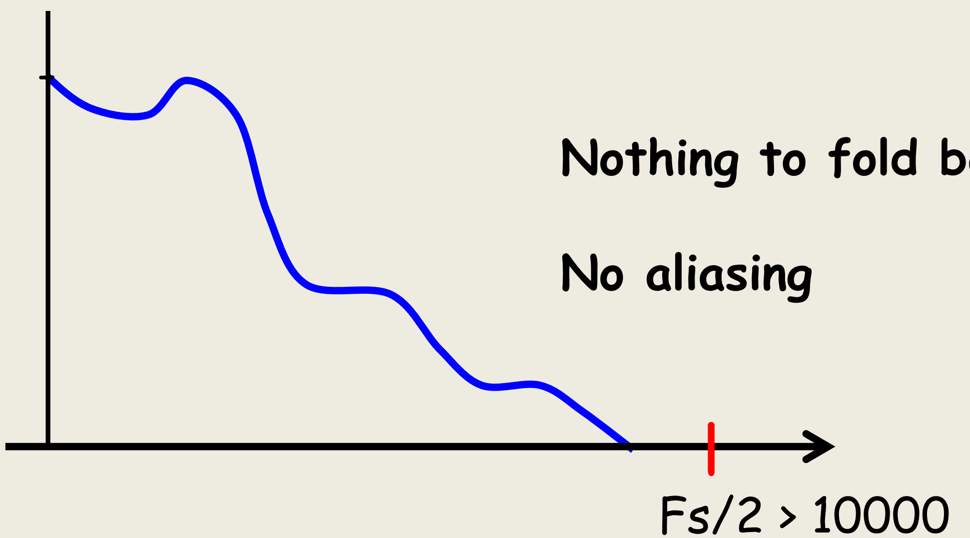




Nothing to fold back

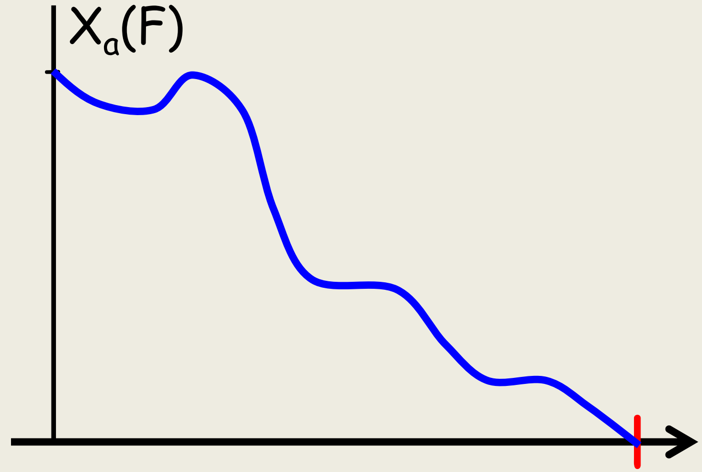
No aliasing

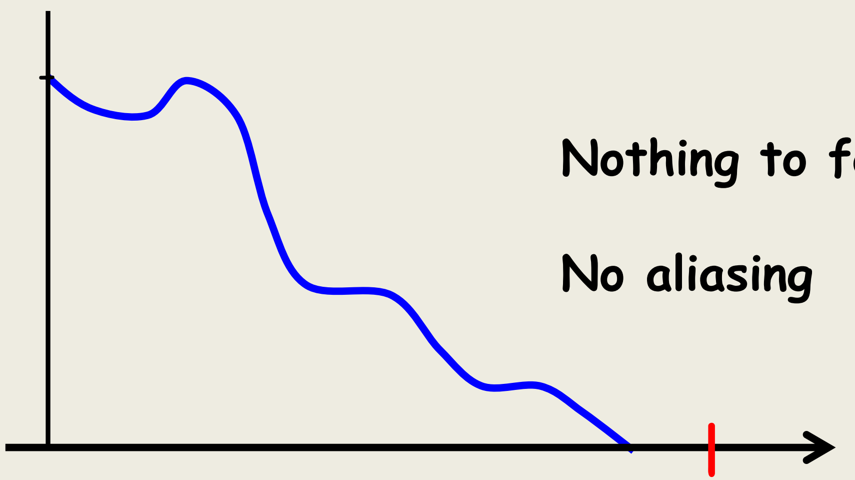
EITF75 Systems and Signals

Signal

F_s >= 20000 Hz

When is sampling lossless?





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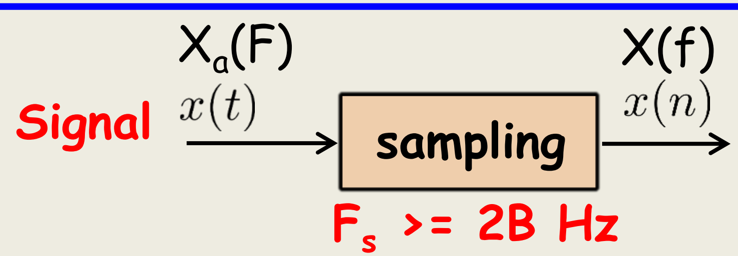
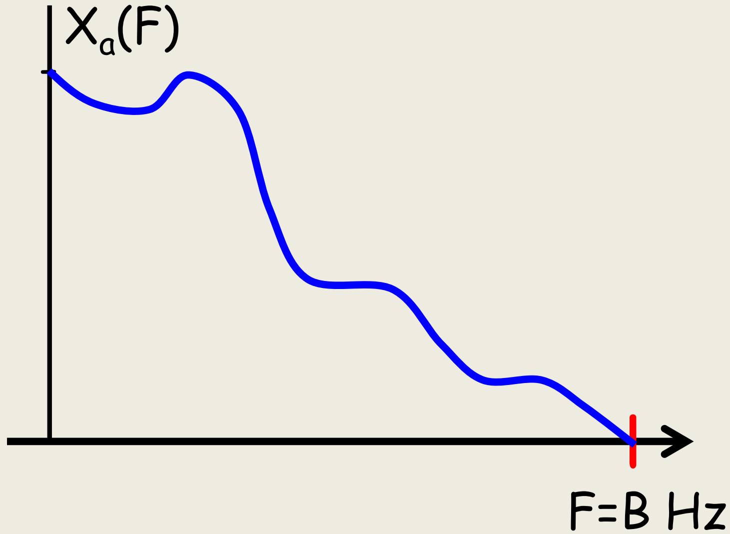
We can recover x(t) from x(n)

EITF75 Systems and Signals



When is sampling lossless?

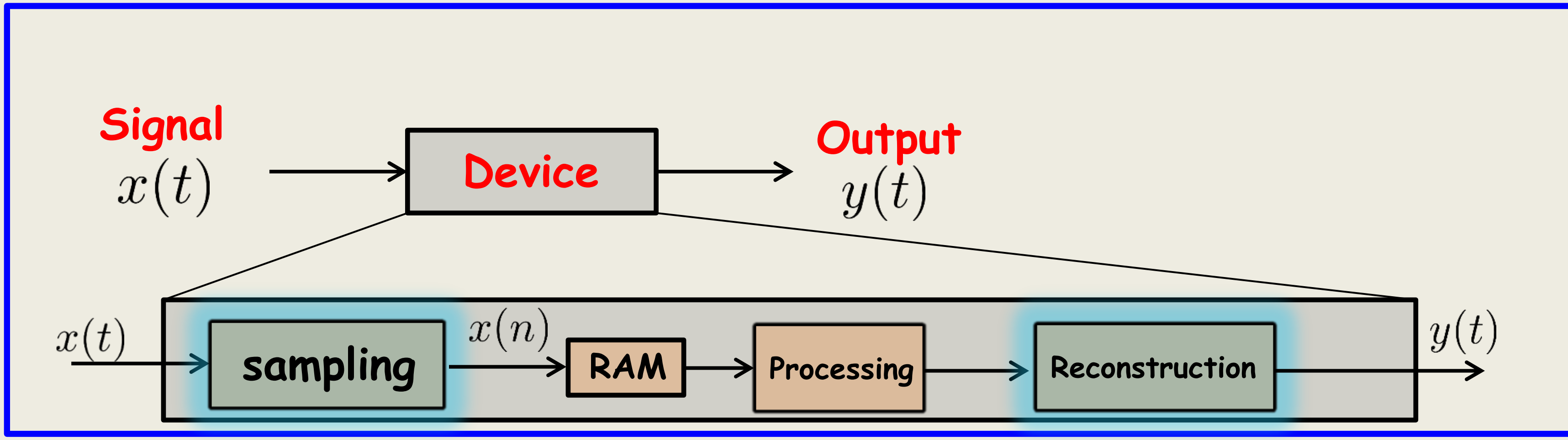
When F_s > 2B Hz



Sampling Theorem (Shannon 1948)

If F_s>2B, where B is the highest frequency of the analog signal, then the analog signal can be recovered from its sampled version

SLIDE 3



An important part of digital signal processing:
Process time-continuous signals digitally

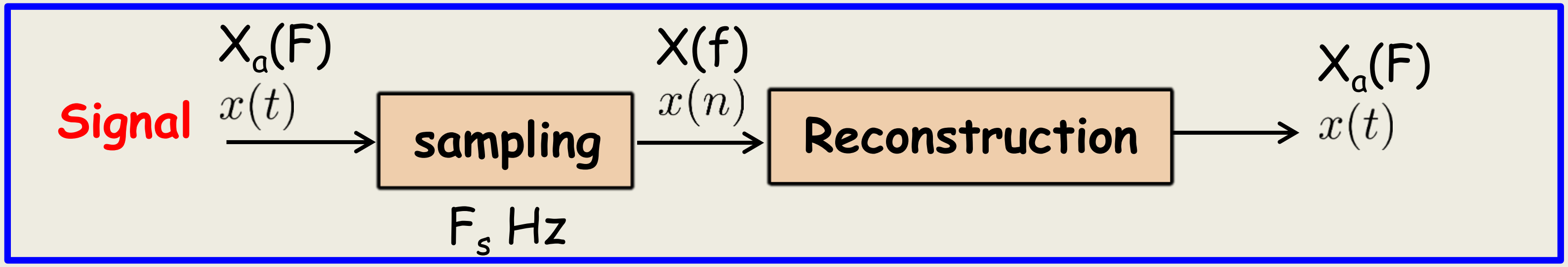
To do so, we need

1. Study effects of sampling (A/D). DONE $X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$

2. Study ways to implement (D/A). Still unclear how to do this

3. Understand when 1&2 are optimally done. DONE $F_s >= 2B$ Hz

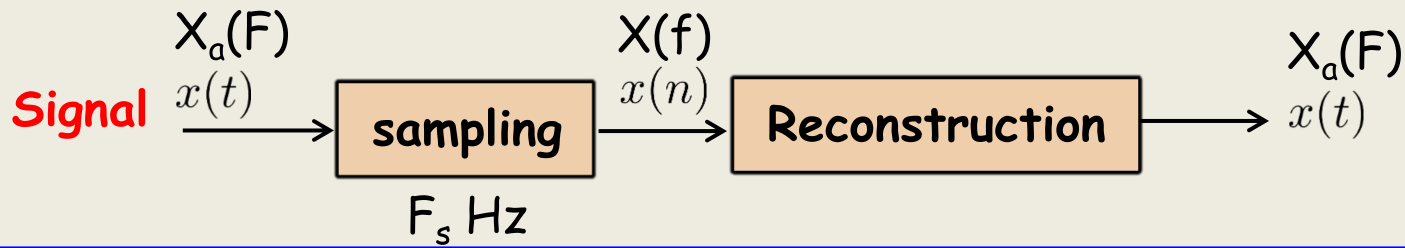
EITF75 Systems and Signals



$$\boxed{X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)}$$

Simplification when no aliasing ?

EITF75 Systems and Signals



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$$X(f) = F_s X_a(f F_s)$$

k=0

EITF75 Systems and Signals

Signal $\xrightarrow[\quad]{X_a(F),\ x(t)}$ **sampling** (F_s Hz) $\xrightarrow[\quad]{X(f),\ x(n)}$ **Reconstruction** $\xrightarrow[\quad]{}$ $X_a(F)$, $x(t)$

$$X(f) = F_{\mathrm{s}} \sum_{k=-\infty}^{\infty} X_a((f-k)F_{\mathrm{s}})$$

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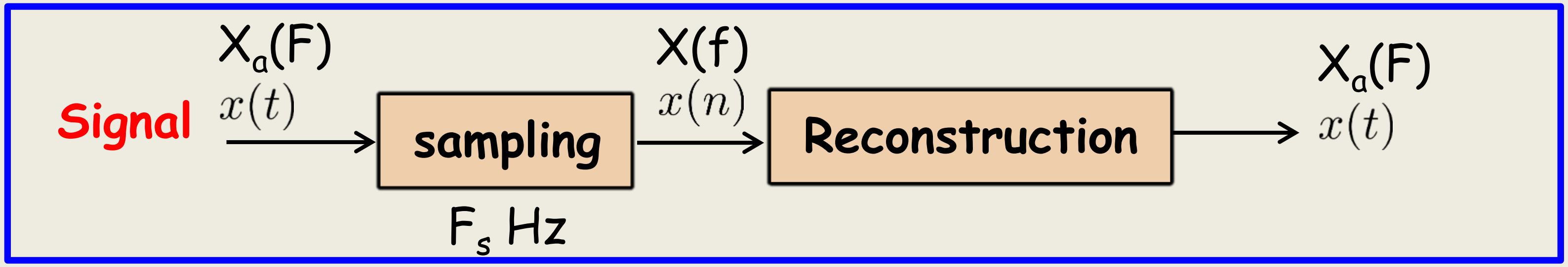
We know that x(t) has a Fourier representation

$$x(t) = \int_{-\infty}^{\infty} X_a(F) \mathrm{e}^{i2\pi F t} \mathrm{d}F$$

EITF75 Systems and Signals



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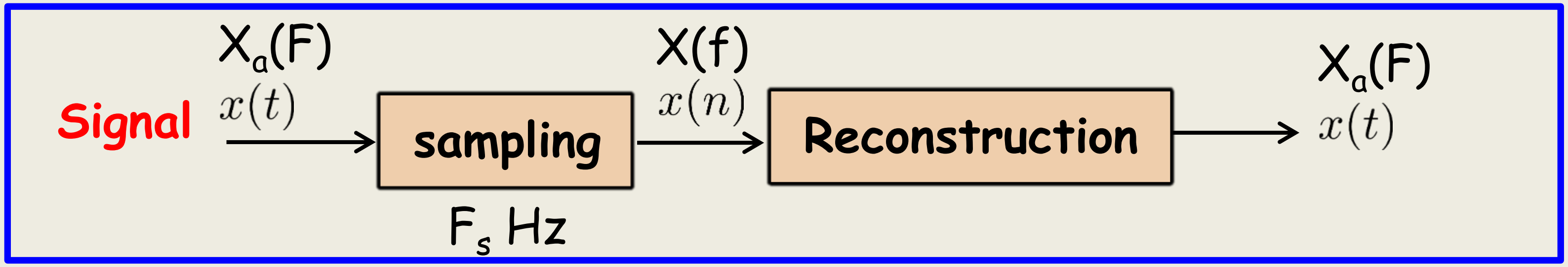
But also that $X_a(F)$ has no support outside $[-F_s/2, F_s/2]$

EITF75 Systems and Signals



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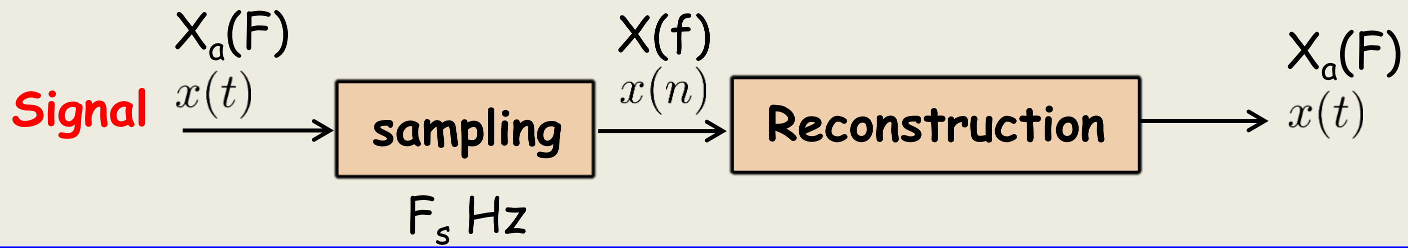
$$f = \frac{F}{F_s} \qquad F = -\frac{F_s}{2} \rightarrow f = -0.5$$

$$dF = F_s df \qquad F = \frac{F_s}{2} \rightarrow f = 0.5$$

EITF75 Systems and Signals



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$$= \int_{-0.5}^{0.5} F_{\mathrm{s}} X_a(f F_{\mathrm{s}})\mathrm{e}^{i2\pi f F_{\mathrm{s}} t}\mathrm{d}f \qquad f = \frac{F}{F_{\mathrm{s}}} \qquad F = -\frac{F_{\mathrm{s}}}{2} \rightarrow f = -0.5$$

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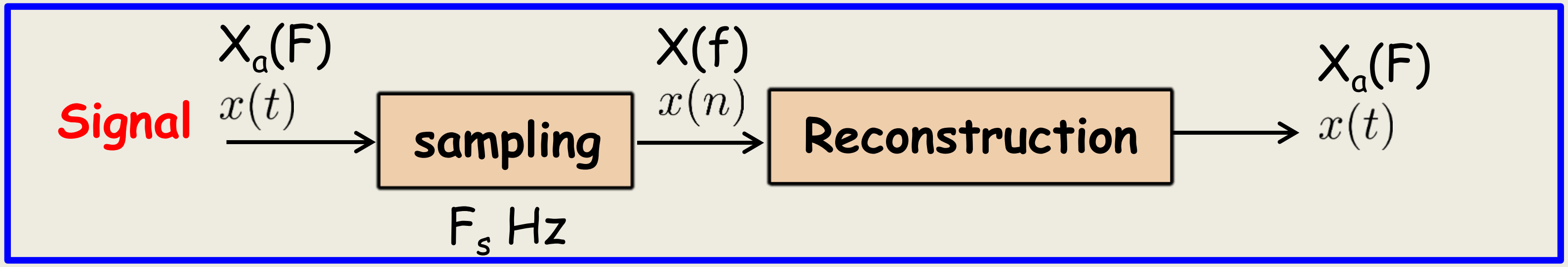
EITF75 Systems and Signals



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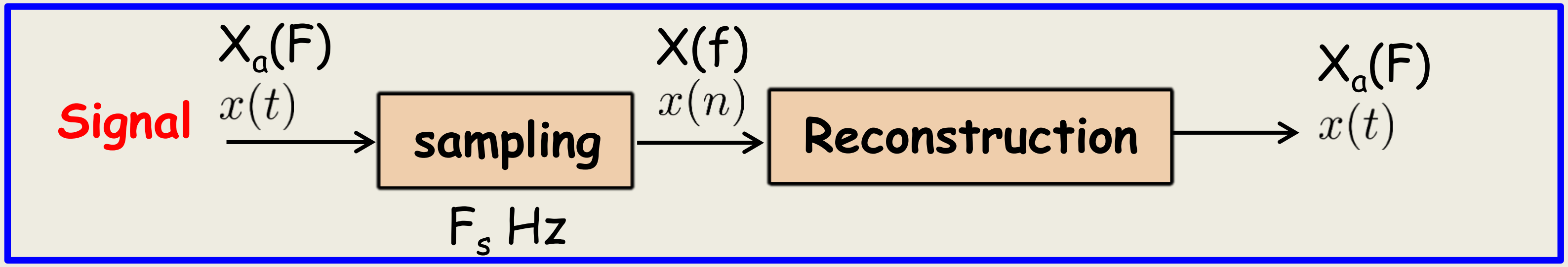
EITF75 Systems and Signals



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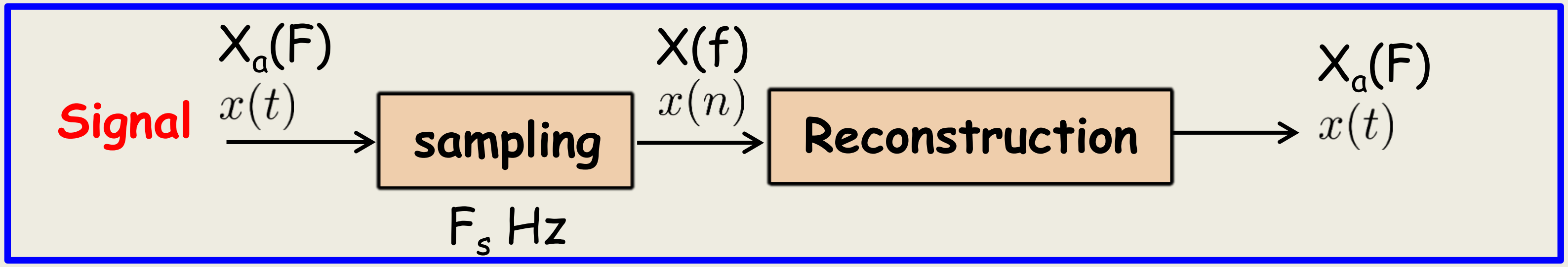

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We know that X(f) is the DTFT of x(n)

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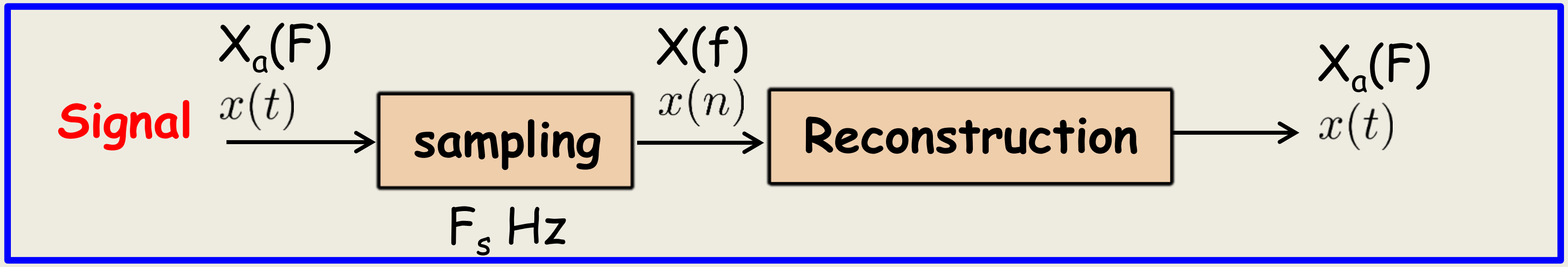
EITF75 Systems and Signals



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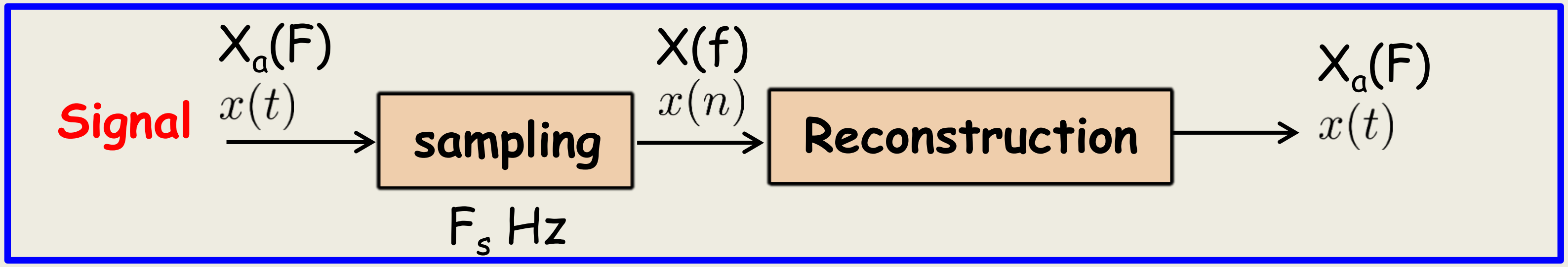

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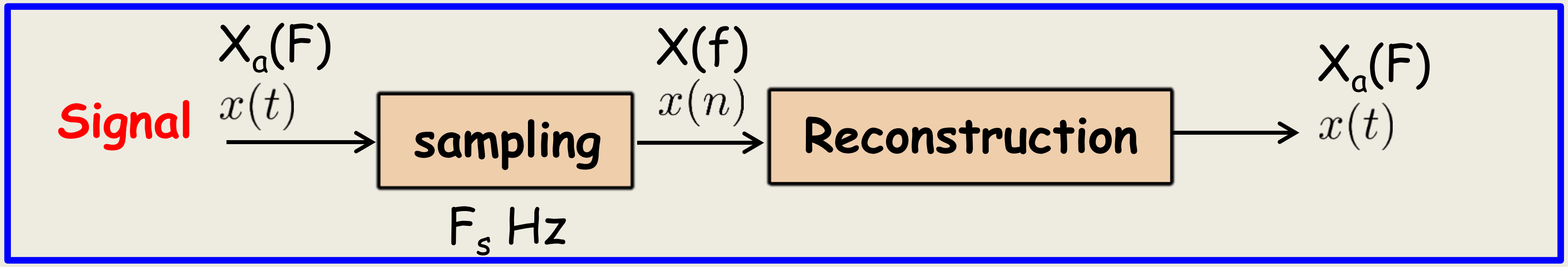

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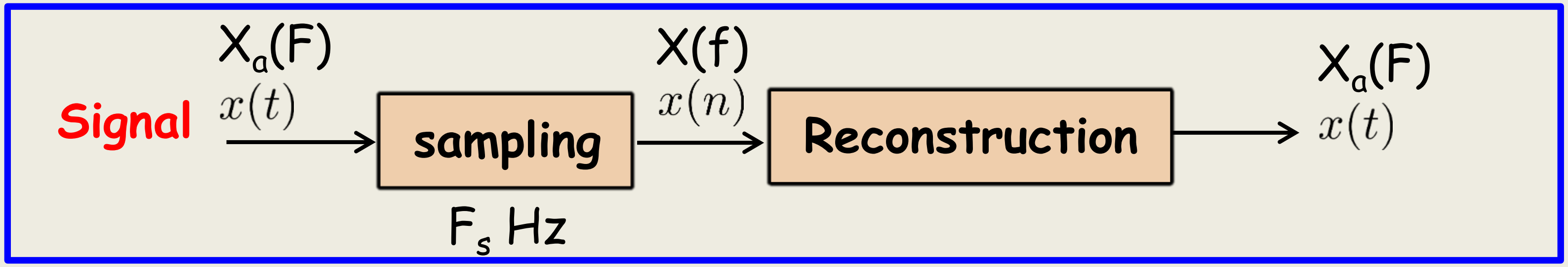
Change order of summation and integration

EITF75 Systems and Signals



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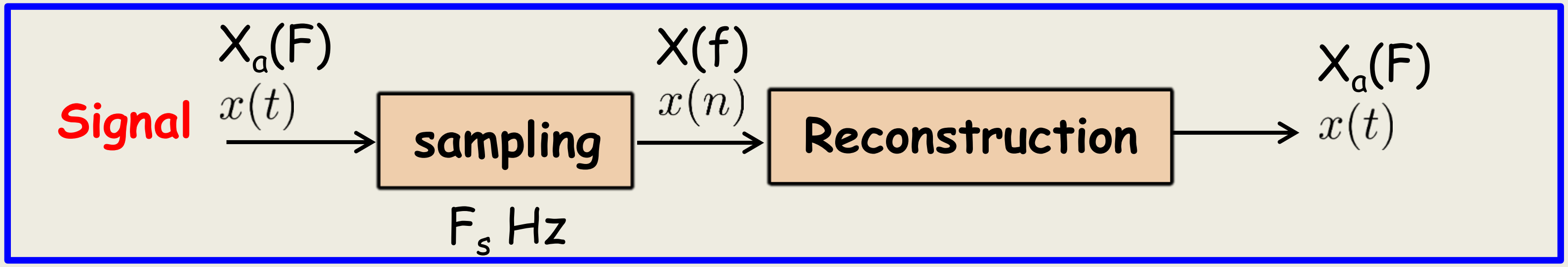
EITF75 Systems and Signals



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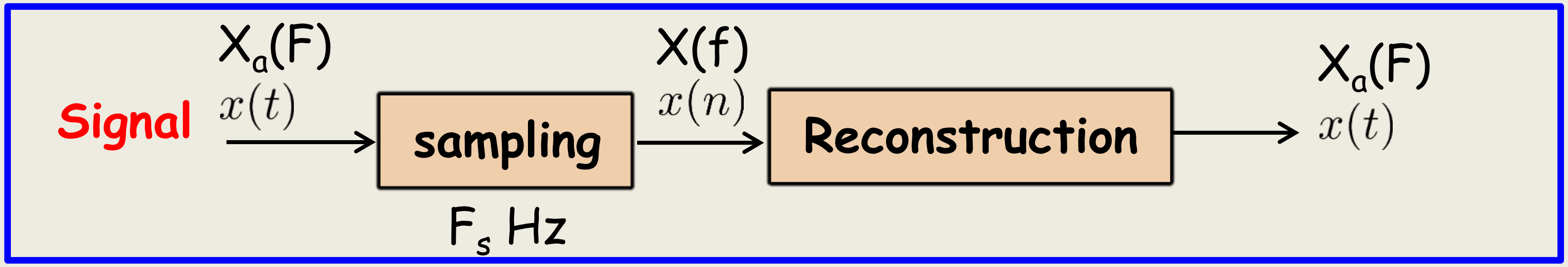
Function of t, n, F_s

Can be pre-computed

No dependency on x(n)

EITF75 Systems and Signals



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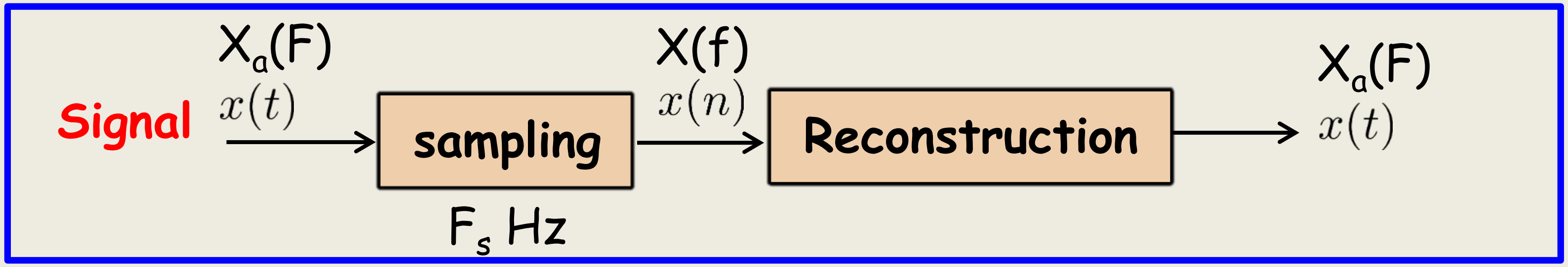
EITF75 Systems and Signals



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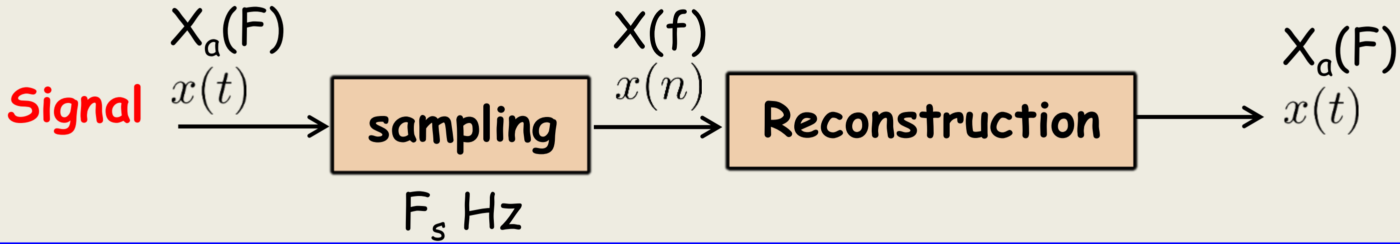
EITF75 Systems and Signals



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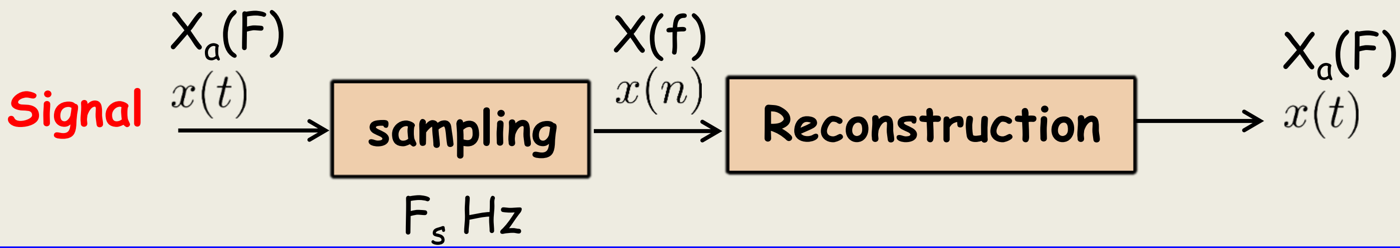
Complete formula for reconstruction

EITF75 Systems and Signals



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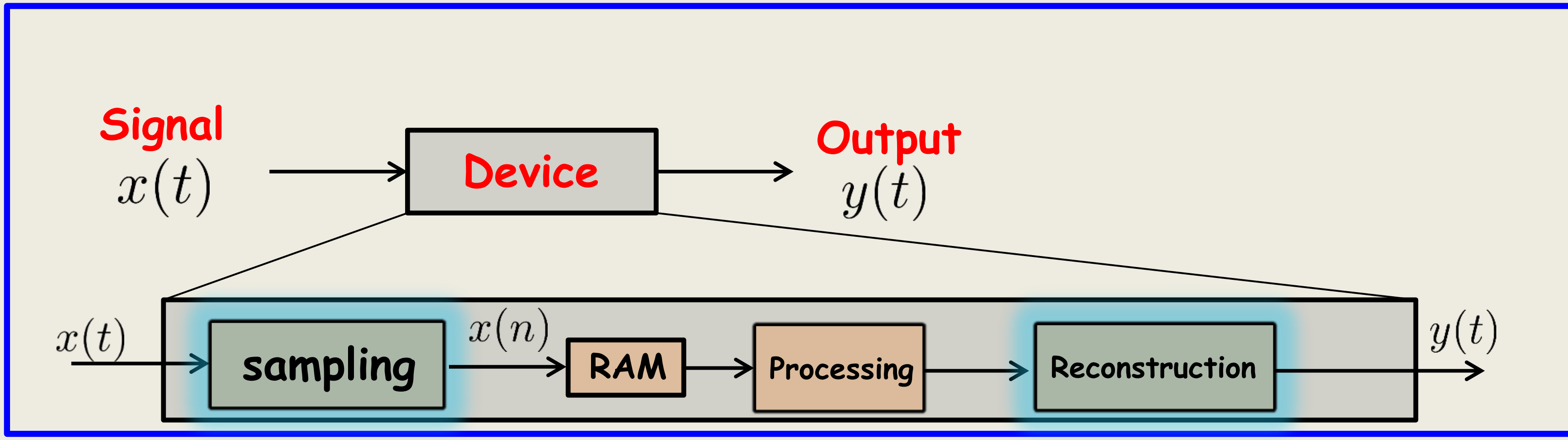
$$x(n) = x(n/F_s)$$

SLIDE 3



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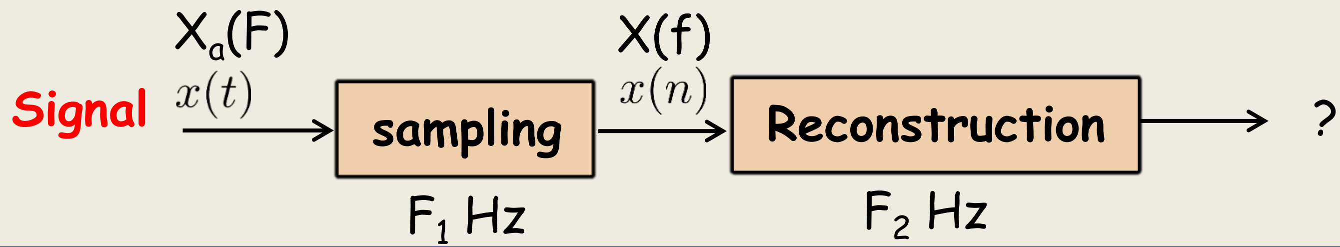
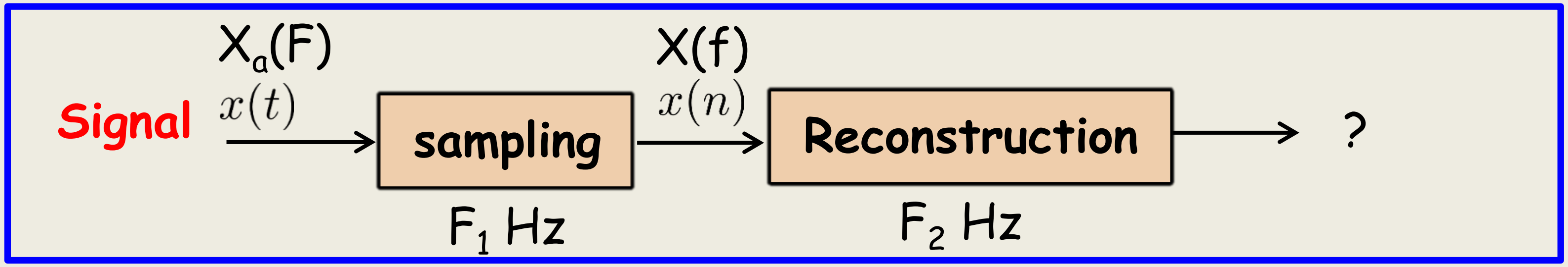
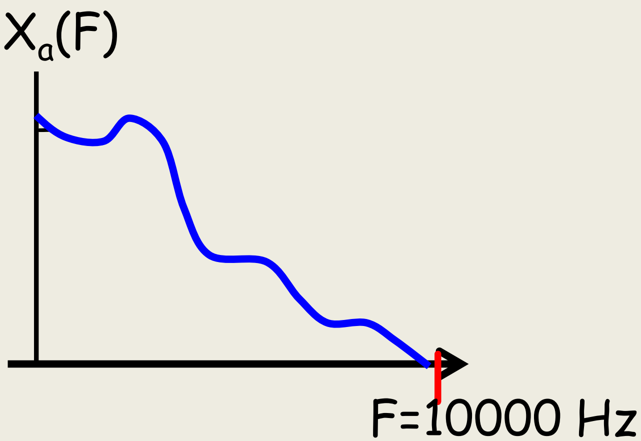
EITF75 Systems and Signals

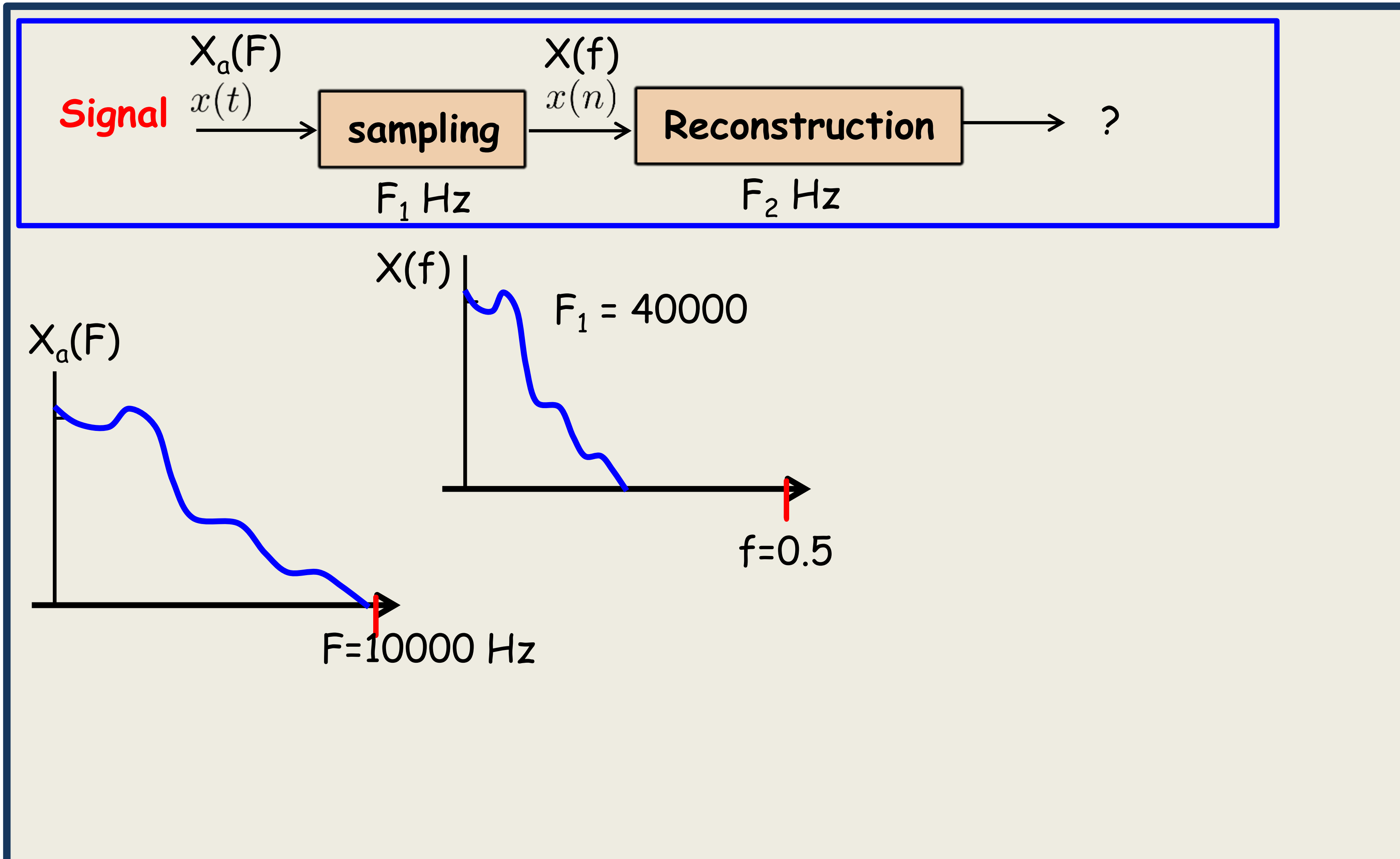


EITF75 Systems and Signals

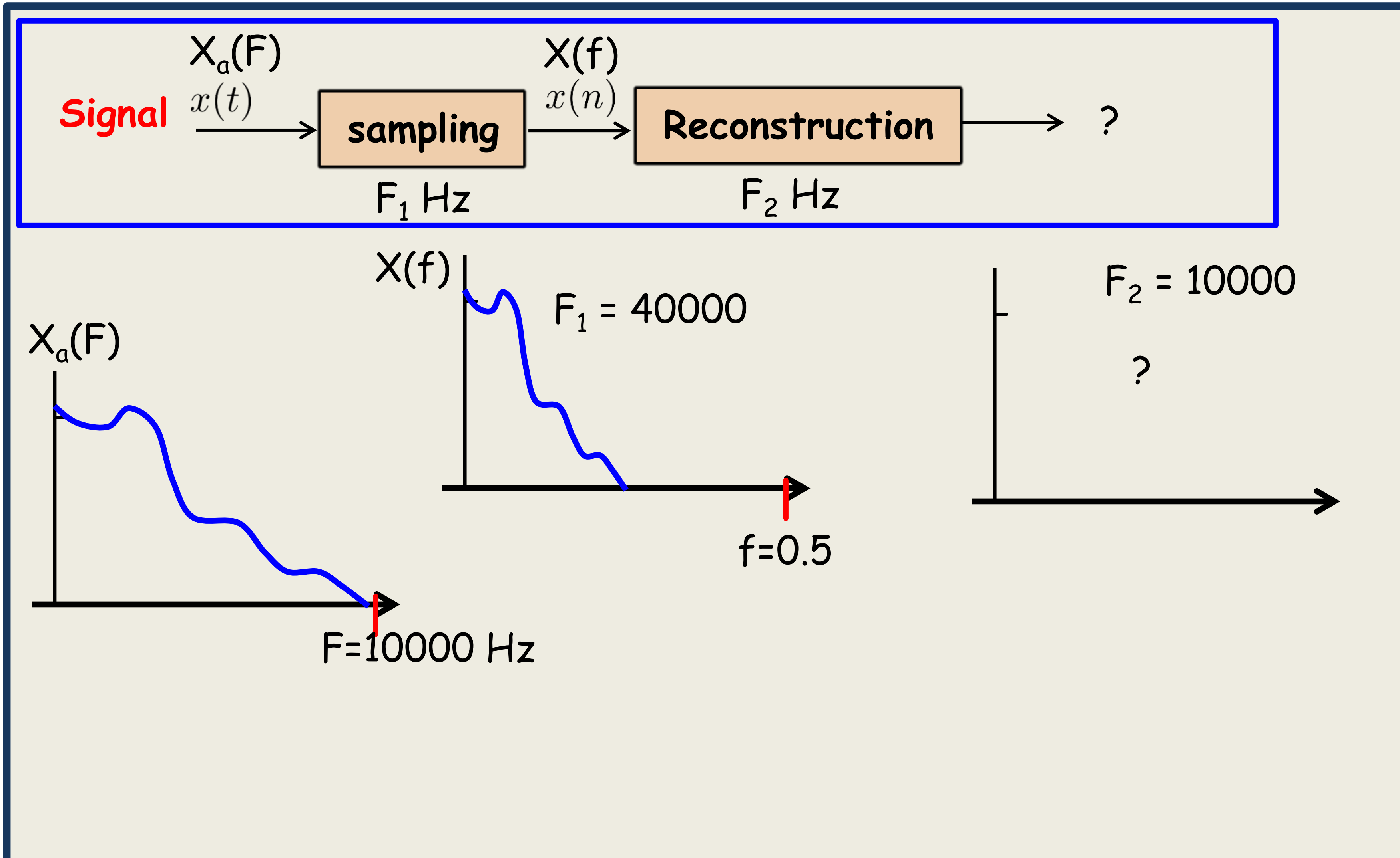




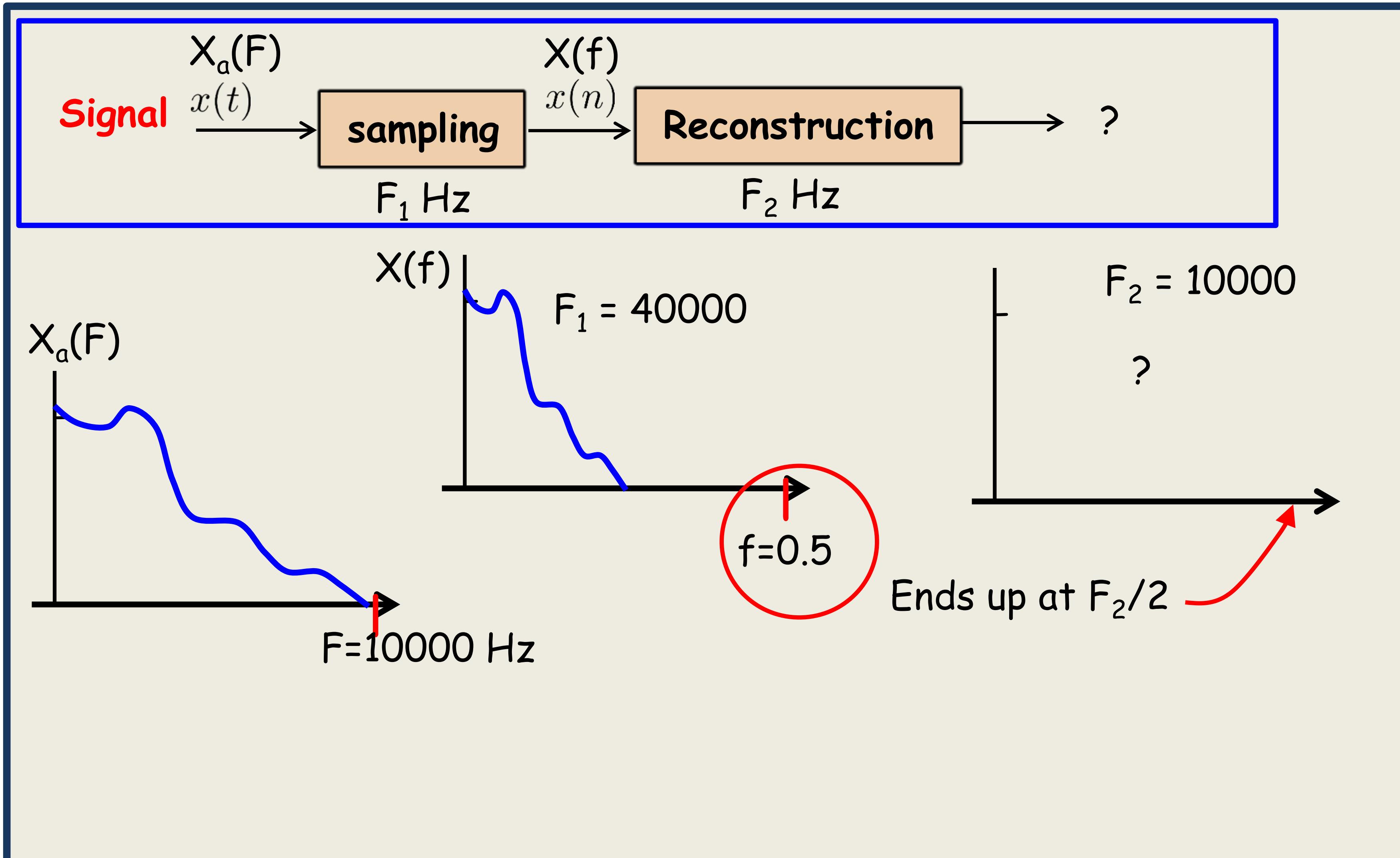
EITF75 Systems and Signals



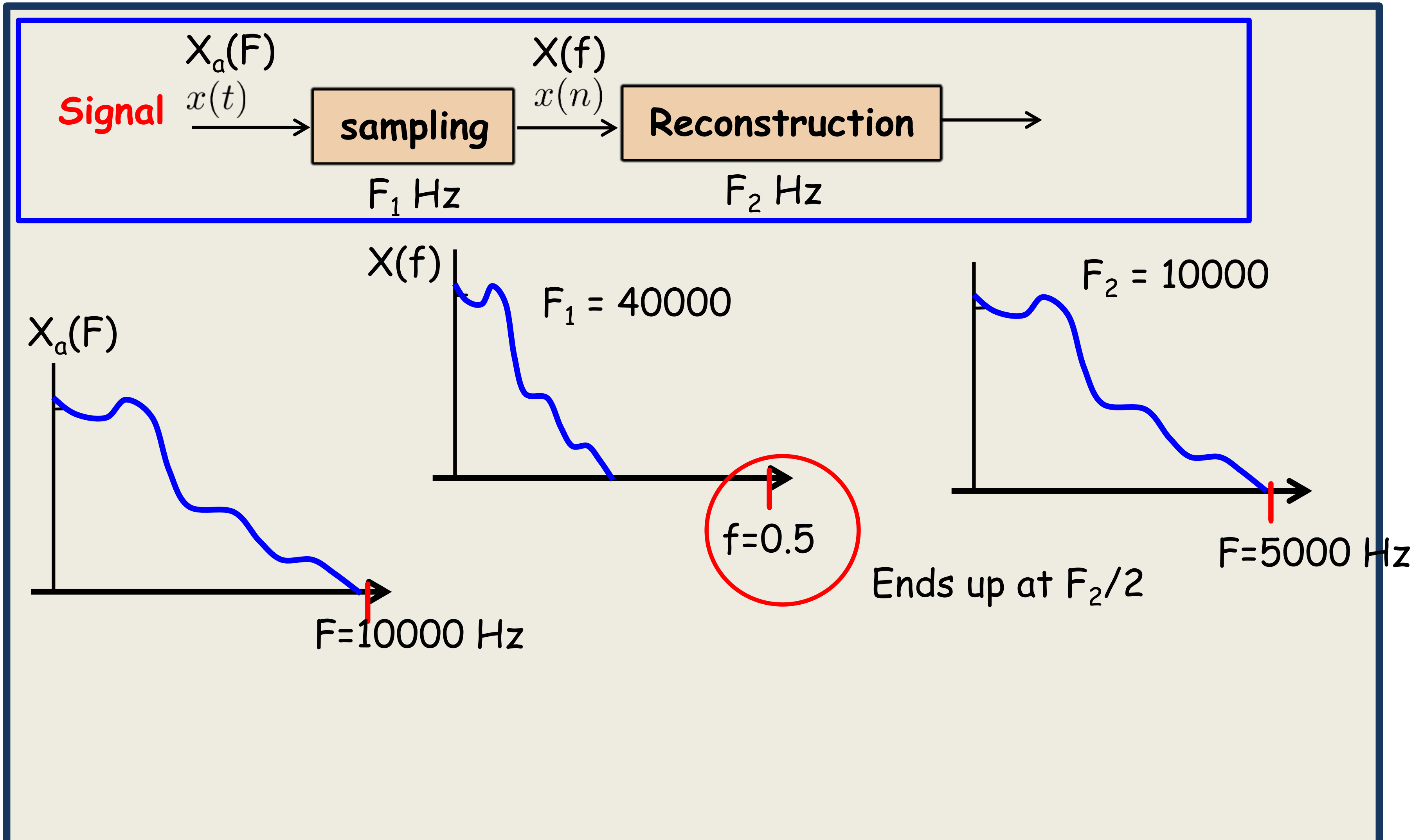
EITF75 Systems and Signals



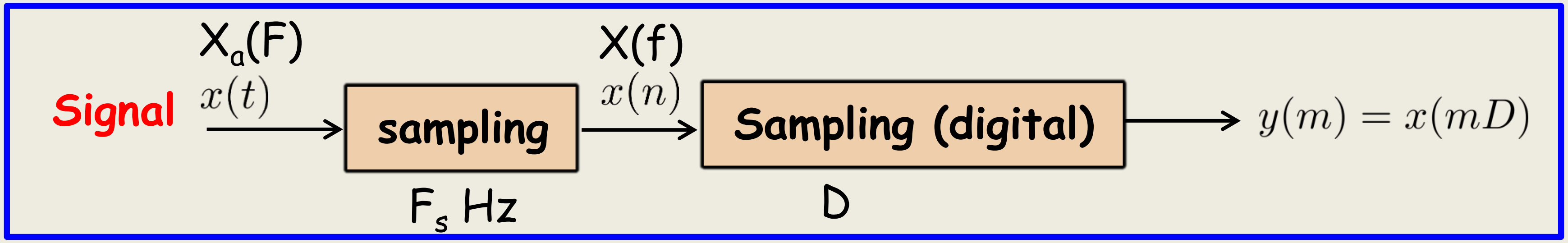
EITF75 Systems and Signals



EITF75 Systems and Signals

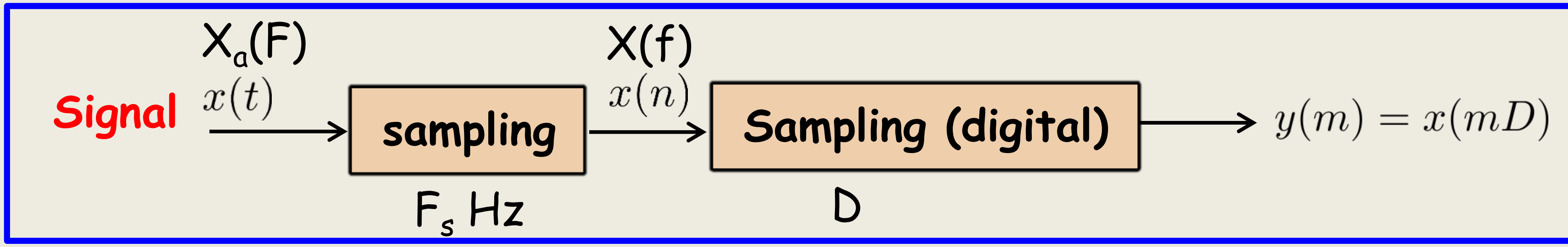


EITF75 Systems and Signals



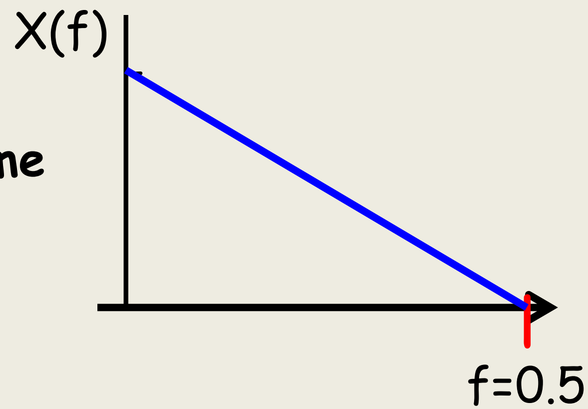
Decimation (Used for inter connecting systems with different sampling rates)

EITF75 Systems and Signals

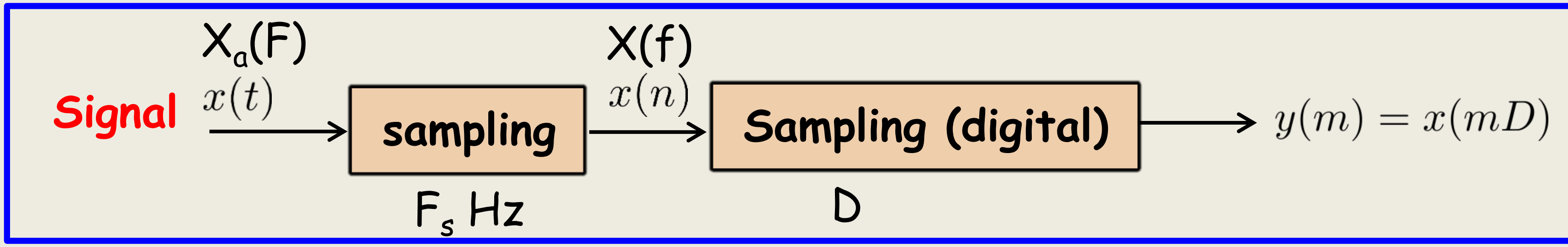


Decimation (Used for inter connecting systems with different sampling rates)

Assume



EITF75 Systems and Signals



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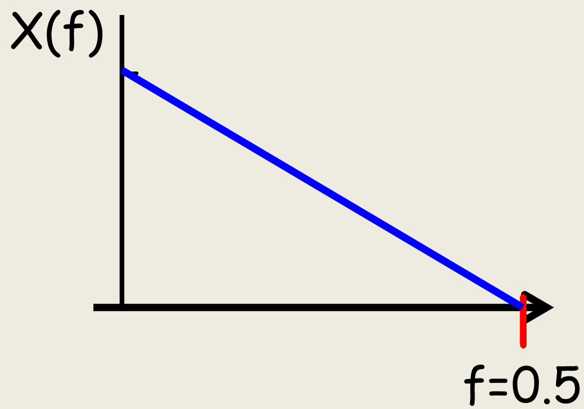
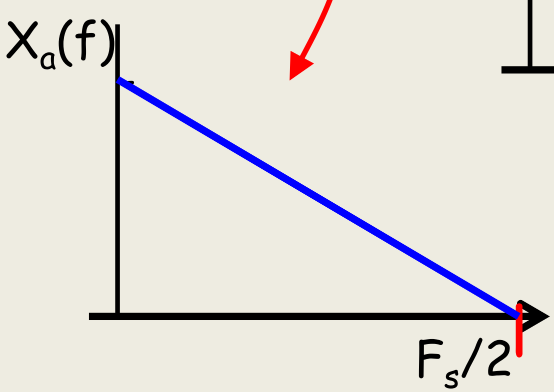


Would result from



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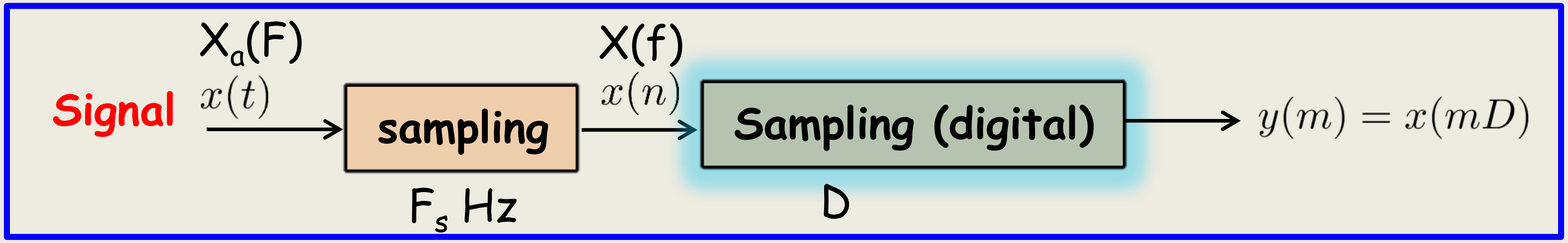

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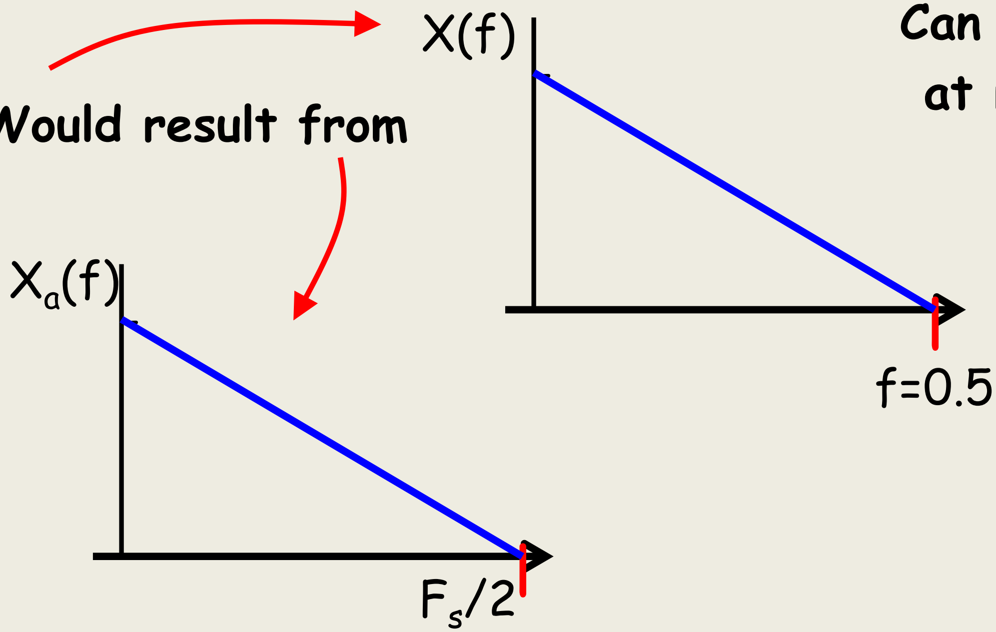
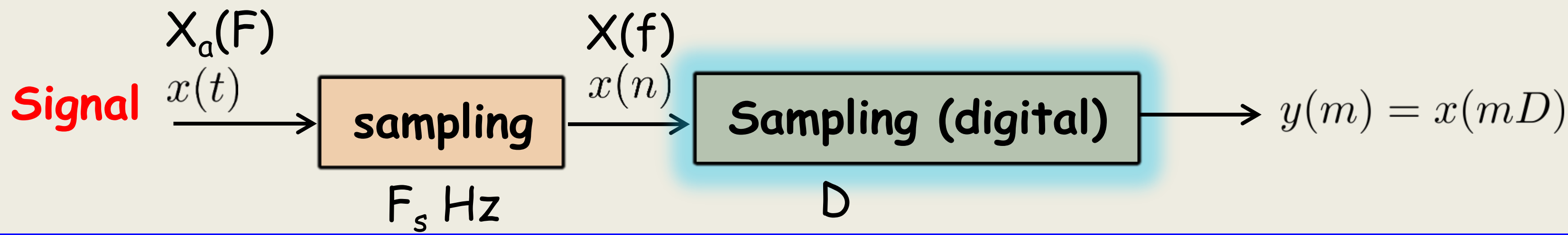


Can be seen as sampling $x(t)$ at rate F_s/D

EITF75 Systems and Signals



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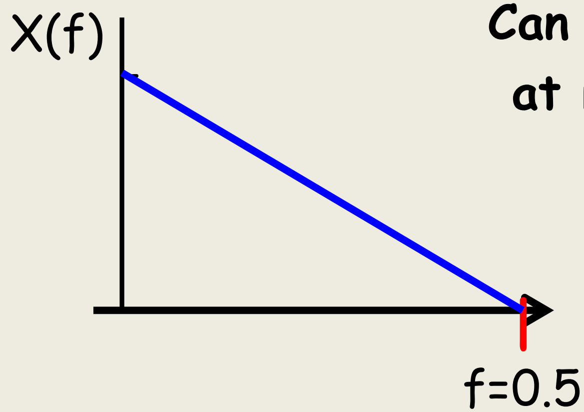
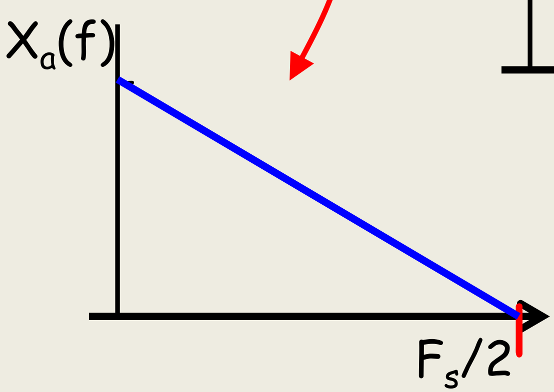
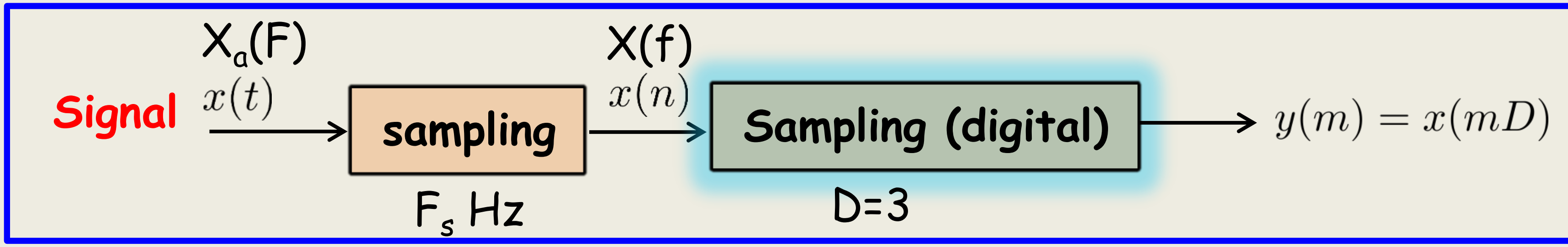




Can be seen as sampling $x(t)$ at rate F_s/D

Aliasing !
Fold at $F_s/2D$

EITF75 Systems and Signals



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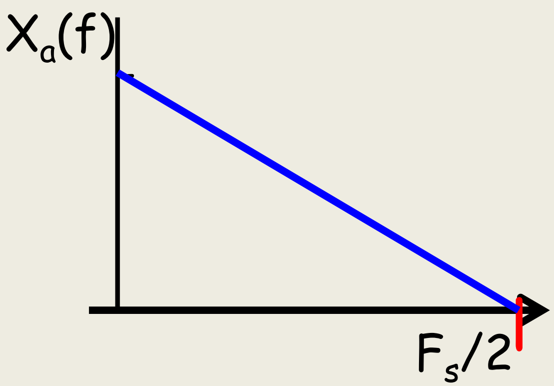
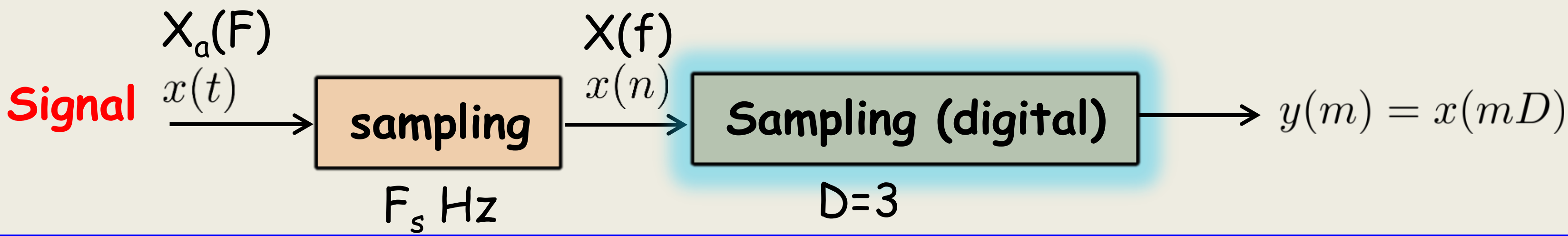


Assume D=3

EITF75 Systems and Signals



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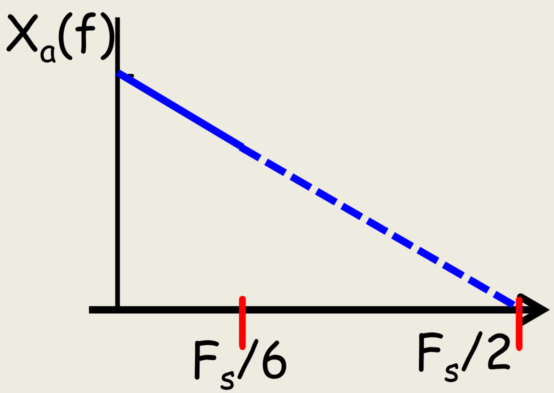

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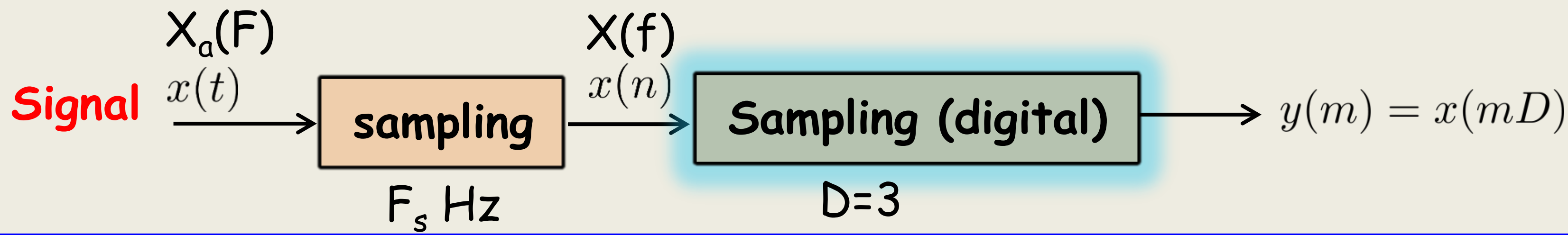
EITF75 Systems and Signals



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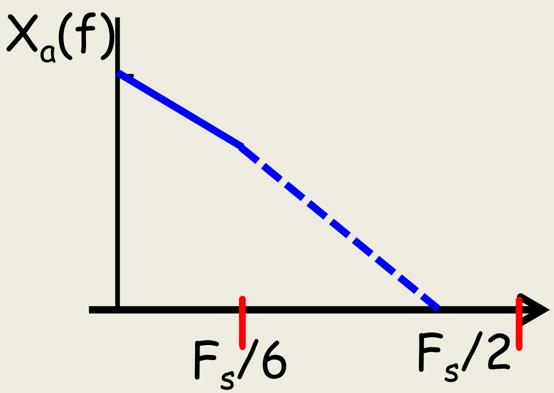

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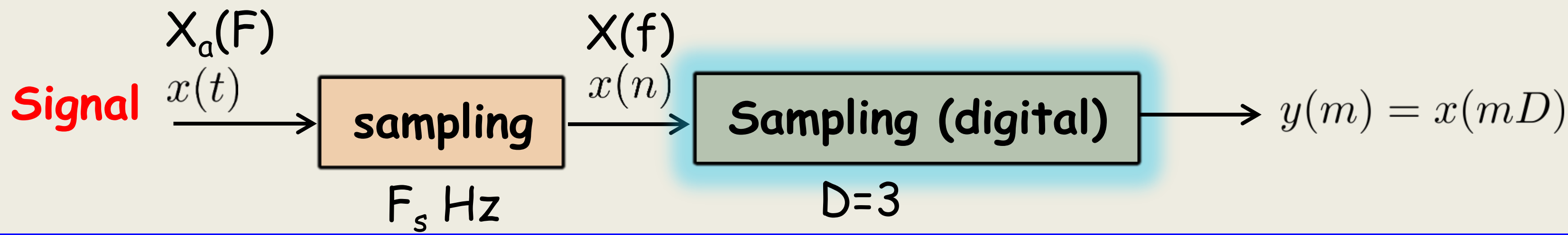

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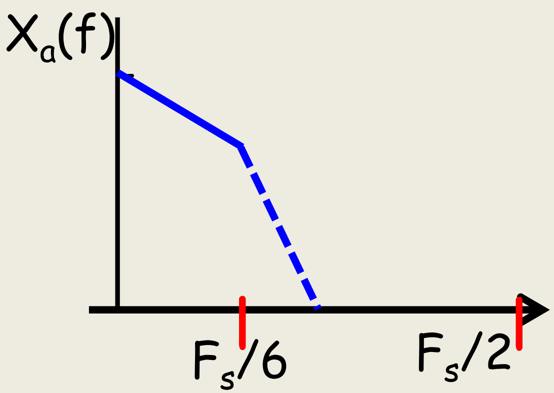

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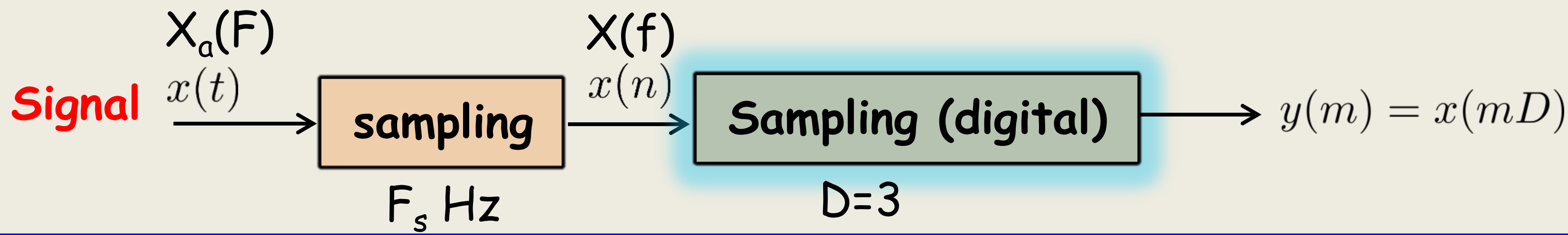

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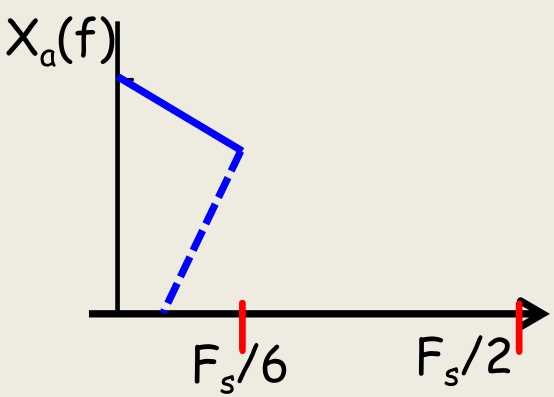

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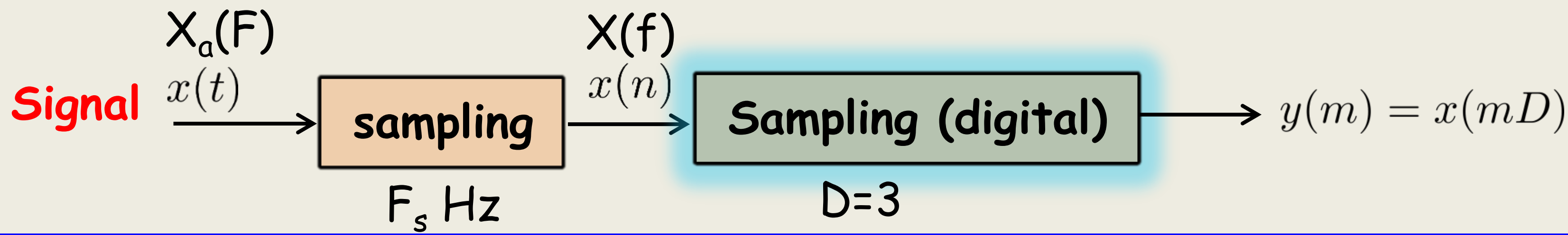

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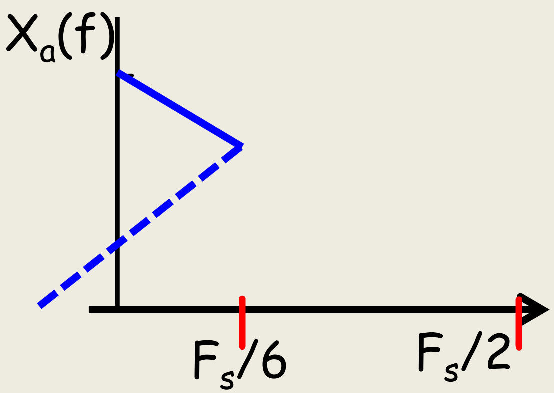
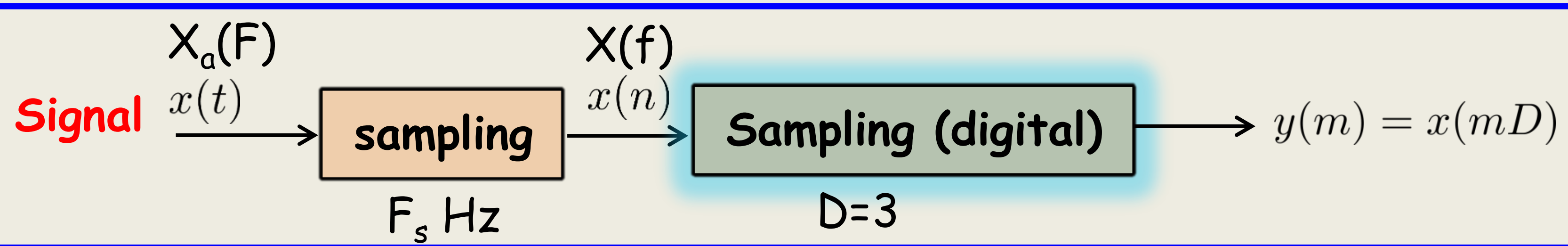


EITF75 Systems and Signals



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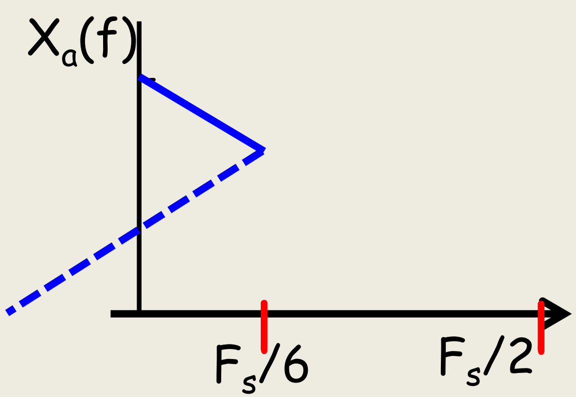

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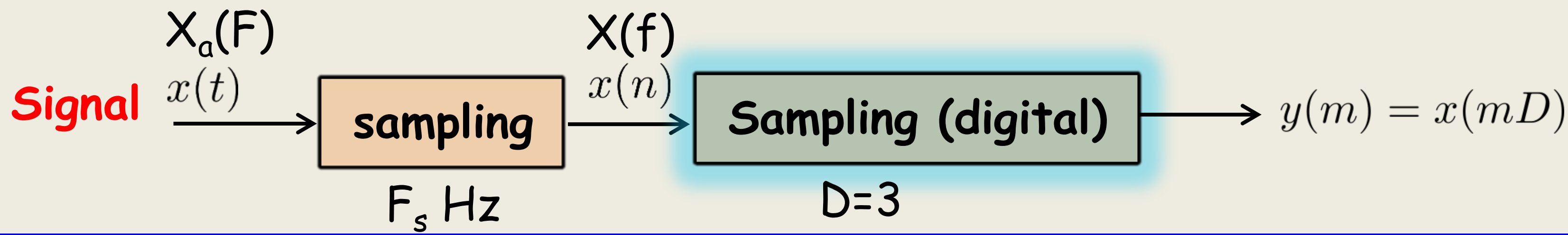
EITF75 Systems and Signals



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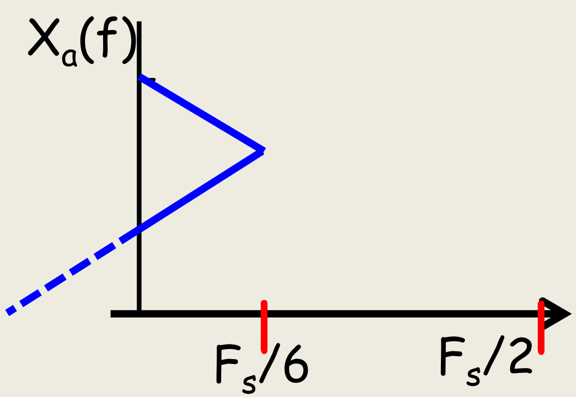

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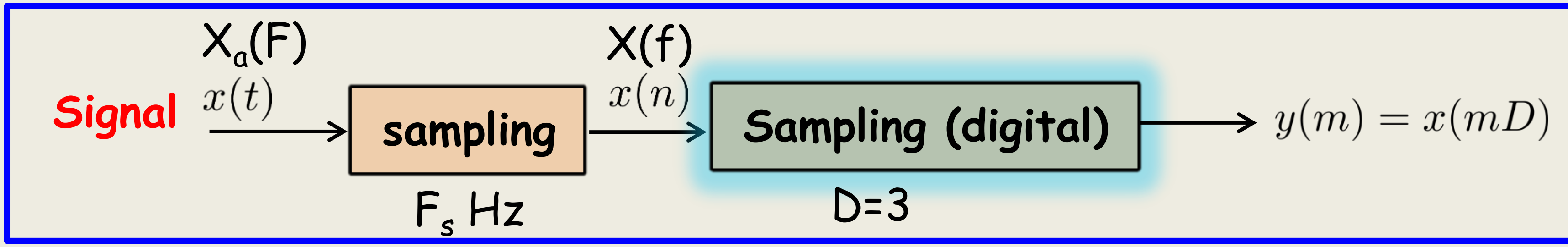
EITF75 Systems and Signals



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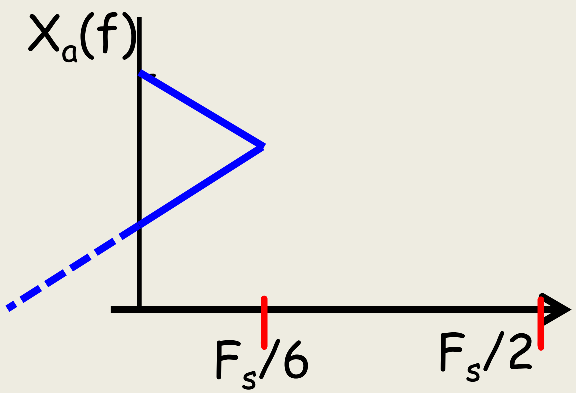
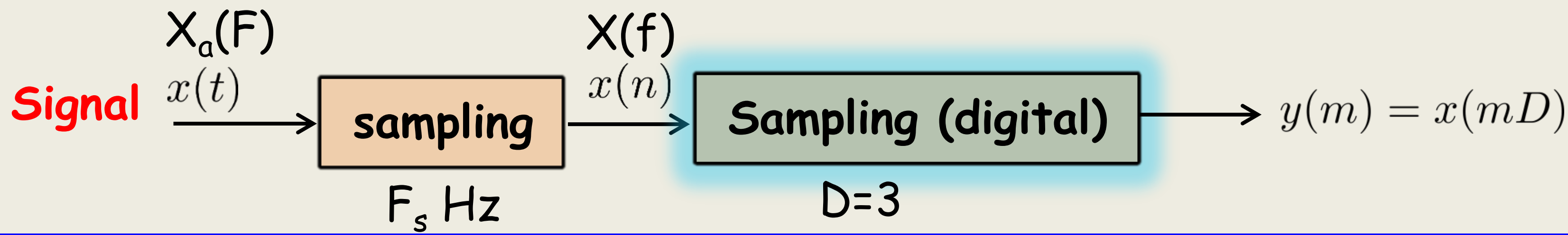


Assume D=3

EITF75 Systems and Signals



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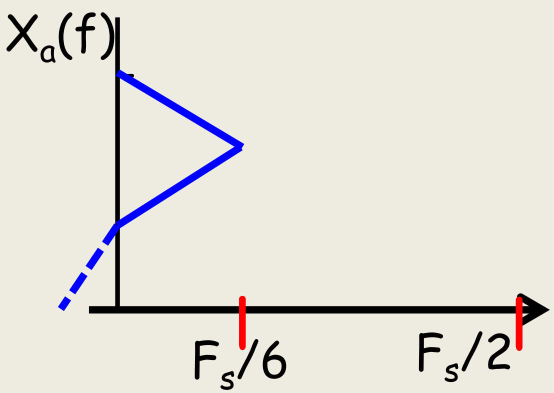

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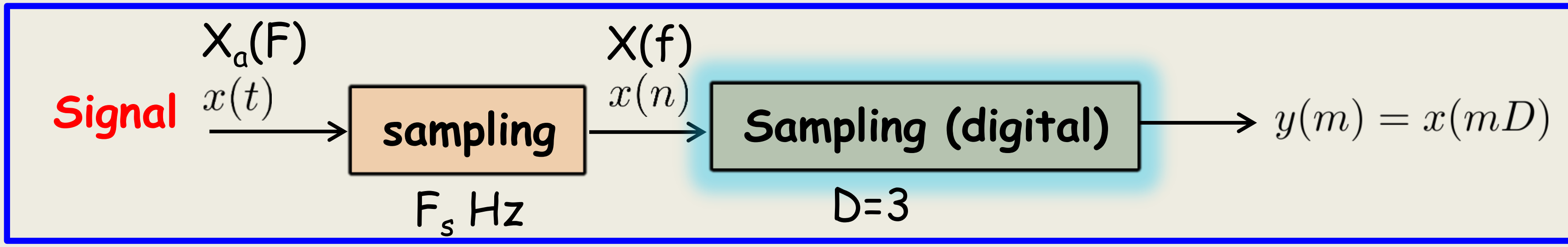
EITF75 Systems and Signals



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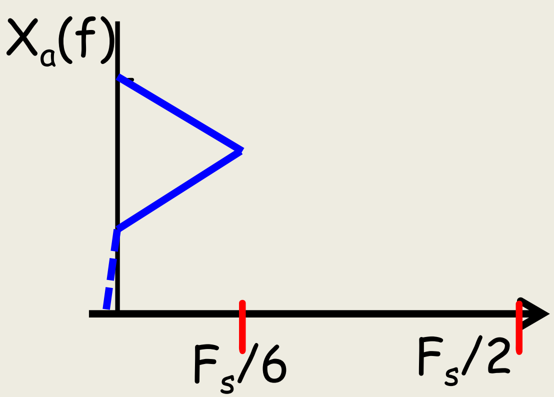


Assume D=3

EITF75 Systems and Signals

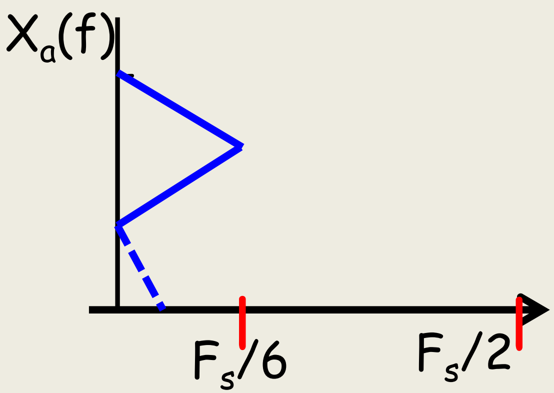
Signal $x(t)$ → sampling (F_s Hz) → $x(n)$ → Sampling (digital) (D=3) → $y(m) = x(mD)$

$X_a(F)$ $X(f)$

Decimation (Used for inter connecting systems with different sampling rates)

Can be seen as sampling $x(t)$ at rate F_s/D

Aliasing !
Fold at $F_s/2D$



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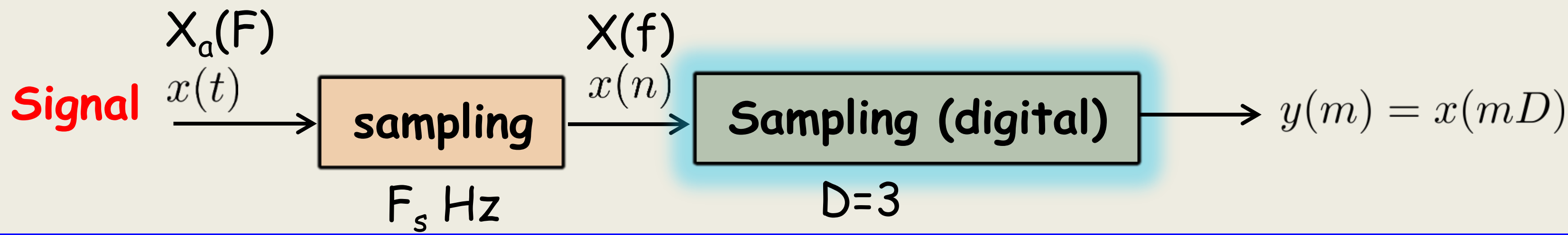
EITF75 Systems and Signals



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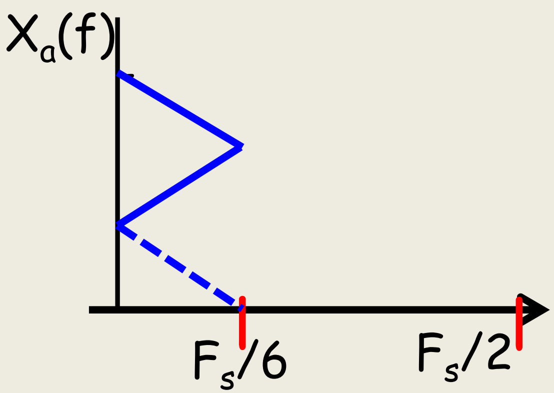

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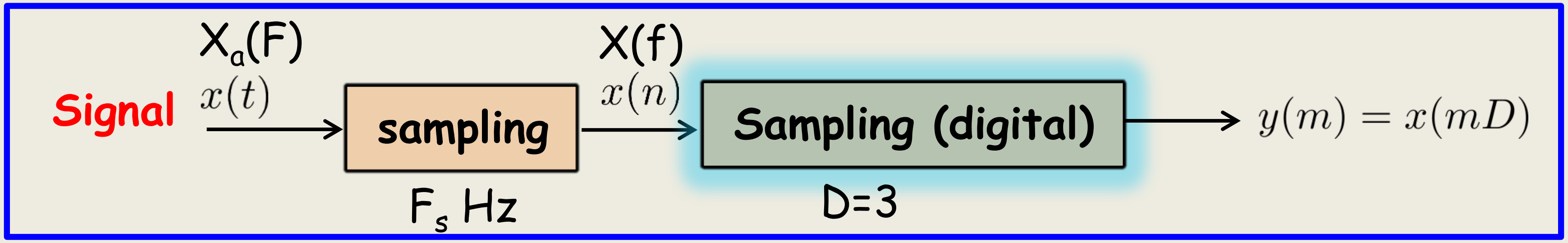
EITF75 Systems and Signals



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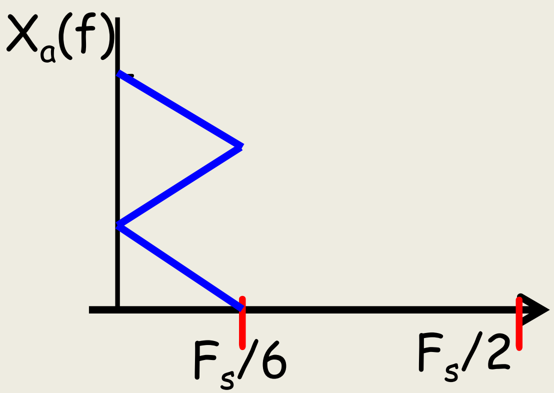
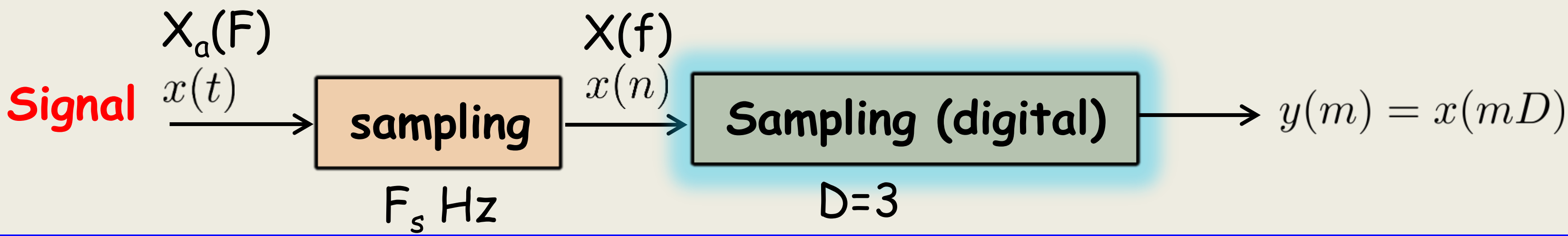


EITF75 Systems and Signals



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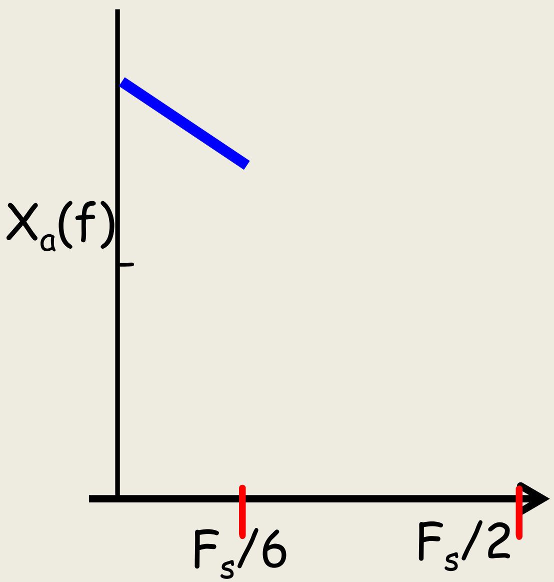
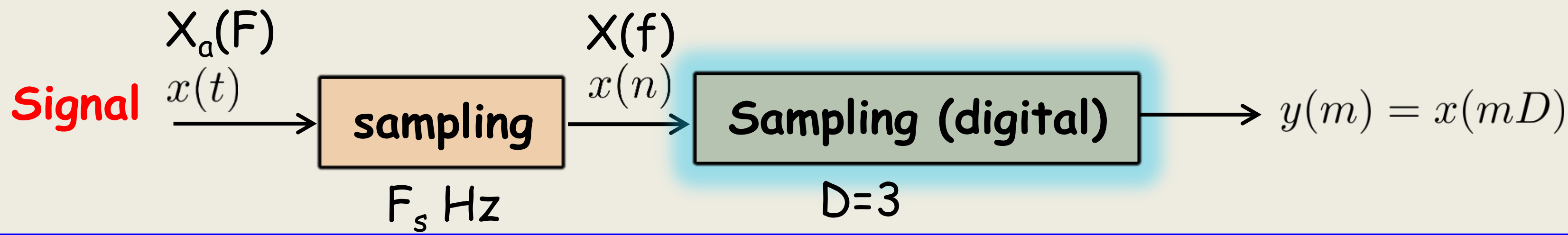


EITF75 Systems and Signals



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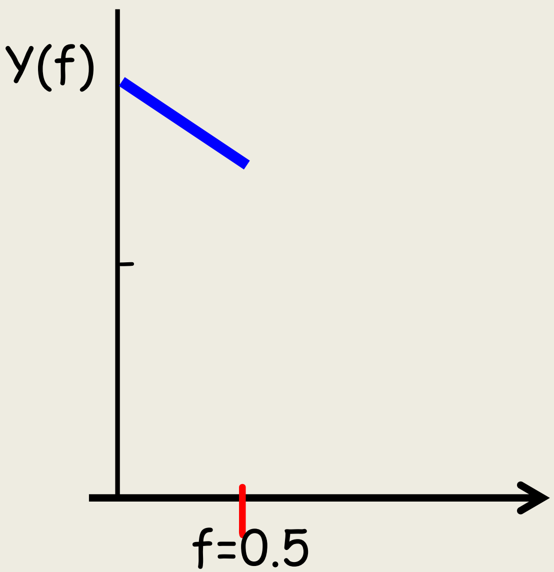

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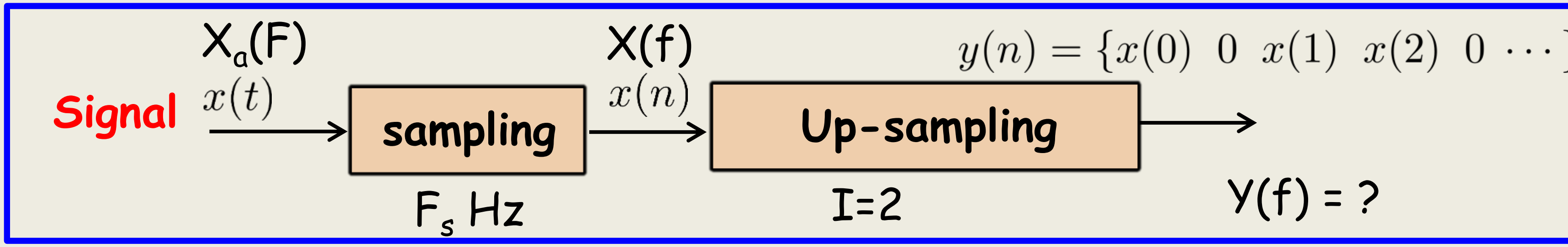
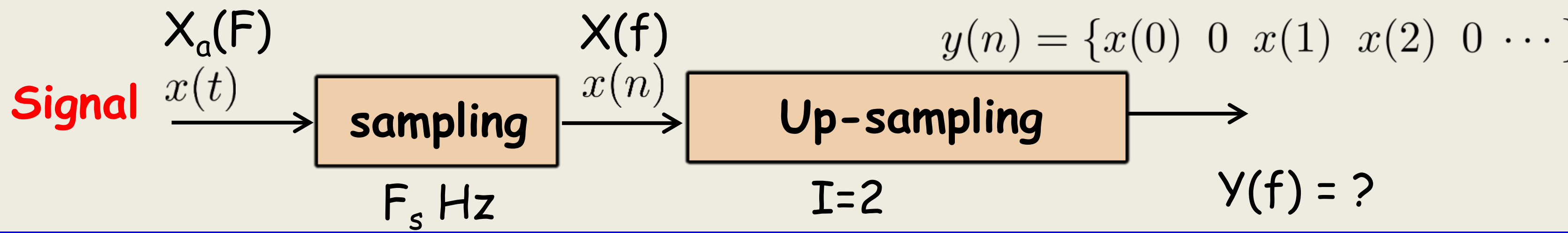
EITF75 Systems and Signals



Interpolation (Used for inter connecting systems with different sampling rates)

Page 778 (note: 773-778 included in course)

EITF75 Systems and Signals



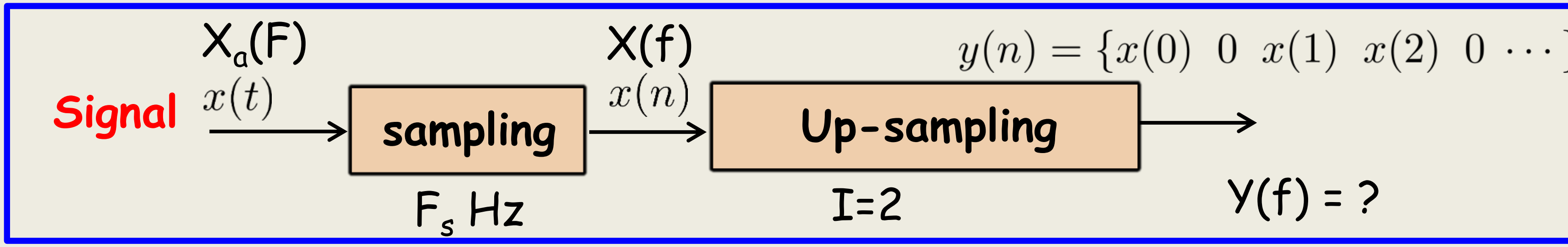
Interpolation (Used for inter connecting systems with different sampling rates)

Page 778 (note: 773-778 included in course)

$$Y(z) = \sum_{m=-\infty}^{\infty} y(m) z^{-m}$$

$$Y(f) = Y\left(z | z = \mathrm{e}^{i2\pi f}\right)$$

EITF75 Systems and Signals



Interpolation (Used for inter connecting systems with different sampling rates)

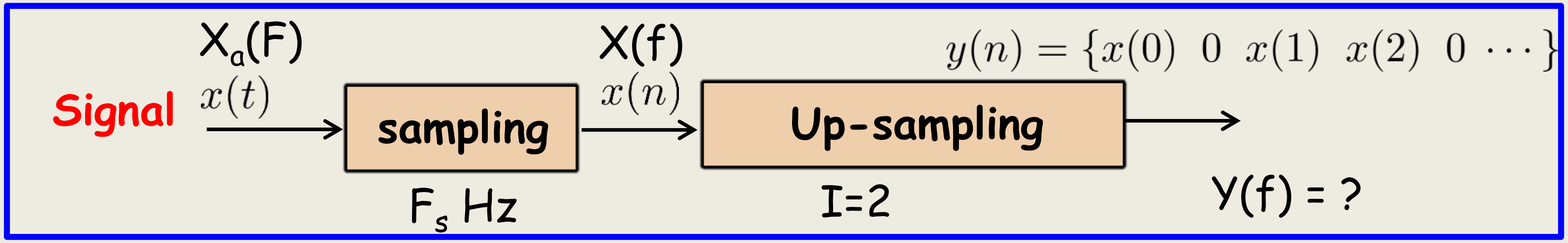
Page 778 (note: 773-778 included in course)

$$Y(z) = \sum_{m=-\infty}^{\infty} y(m) z^{-m}$$

$$= \sum_{m=-\infty}^{\infty} x(m) z^{-Im}$$

$$Y(f) = Y\left(z | z = \mathrm{e}^{i2\pi f}\right)$$

EITF75 Systems and Signals



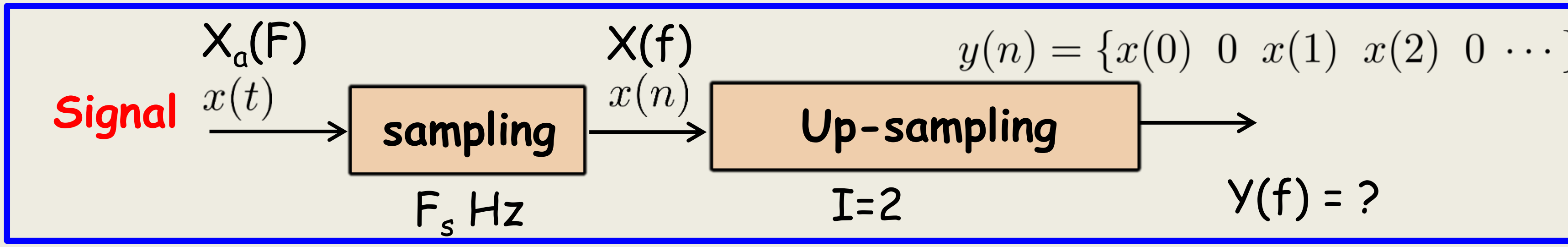
Interpolation (Used for inter connecting systems with different sampling rates)

Page 778 (note: 773-778 included in course)

$$
\begin{aligned}
Y(z) &= \sum_{m=-\infty}^{\infty} y(m) z^{-m} \\
&= \sum_{m=-\infty}^{\infty} x(m) z^{-Im} \\
&= X(z^I)
\end{aligned}
$$

$$Y(f) = Y\left(z | z = \mathrm{e}^{i2\pi f}\right)$$

EITF75 Systems and Signals



Interpolation (Used for inter connecting systems with different sampling rates)

Page 778 (note: 773-778 included in course)

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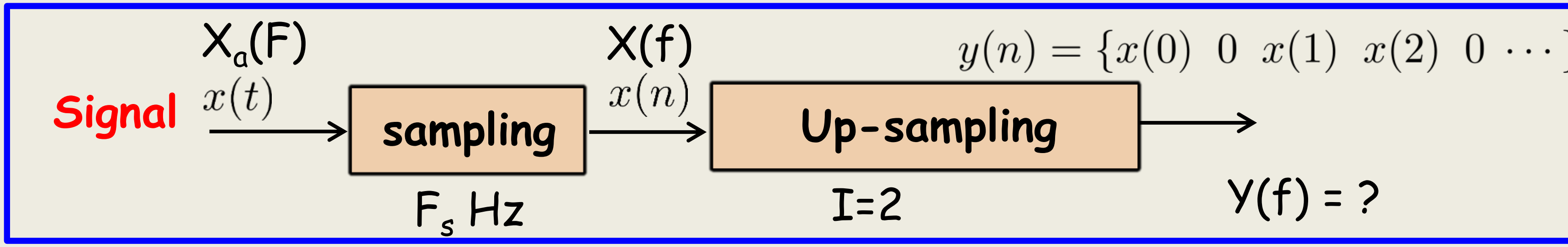
$$= \sum_{m=-\infty}^{\infty} x(m) z^{-Im}$$

$$= X(z^I)$$

$$Y(f) = X(If)$$

EITF75 Systems and Signals



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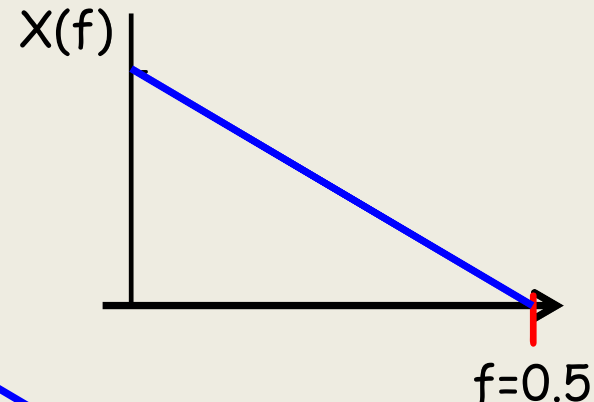

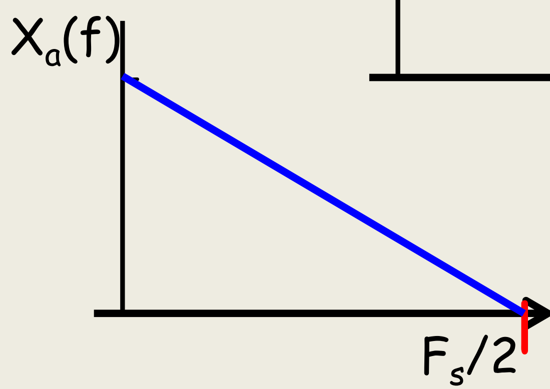

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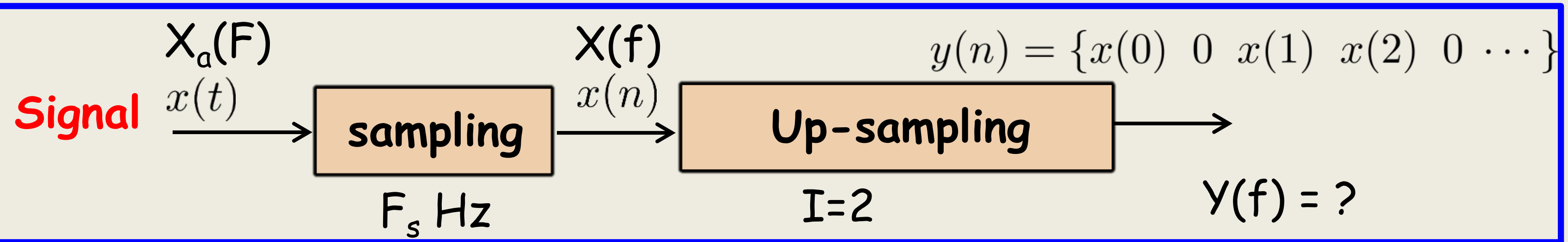
EITF75 Systems and Signals



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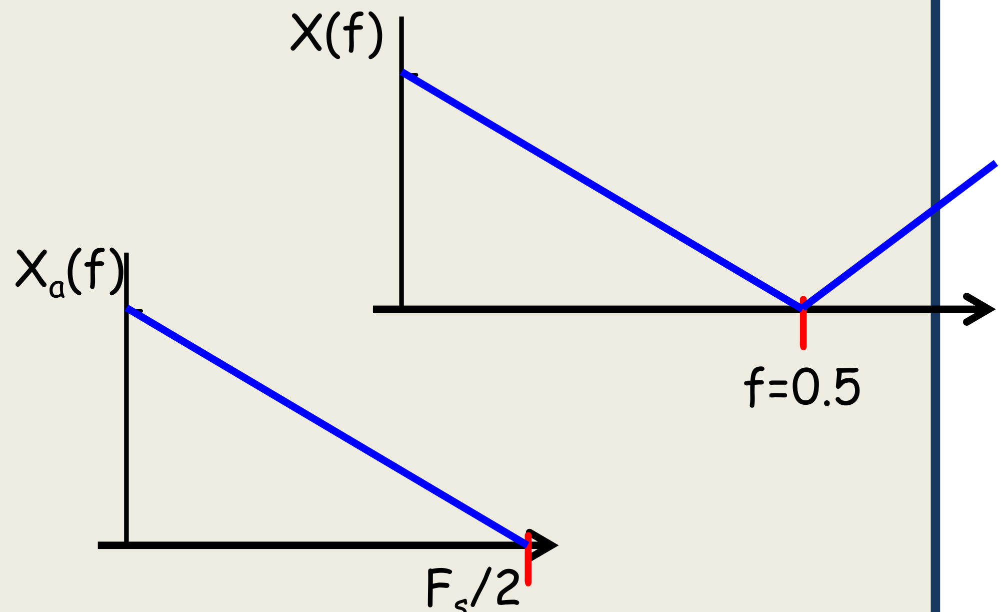
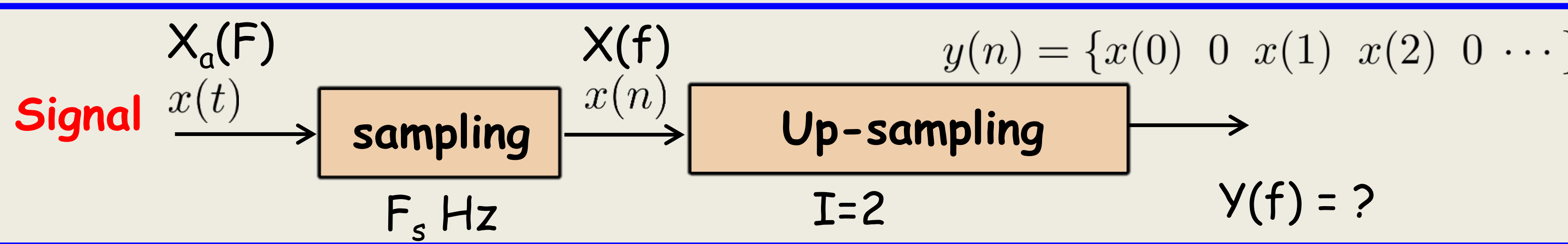


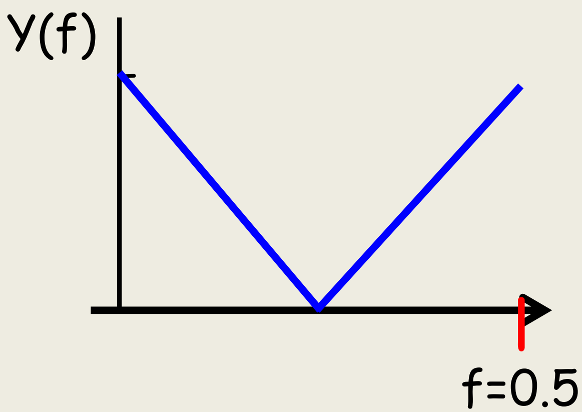
$$Y(f) = X(If)$$

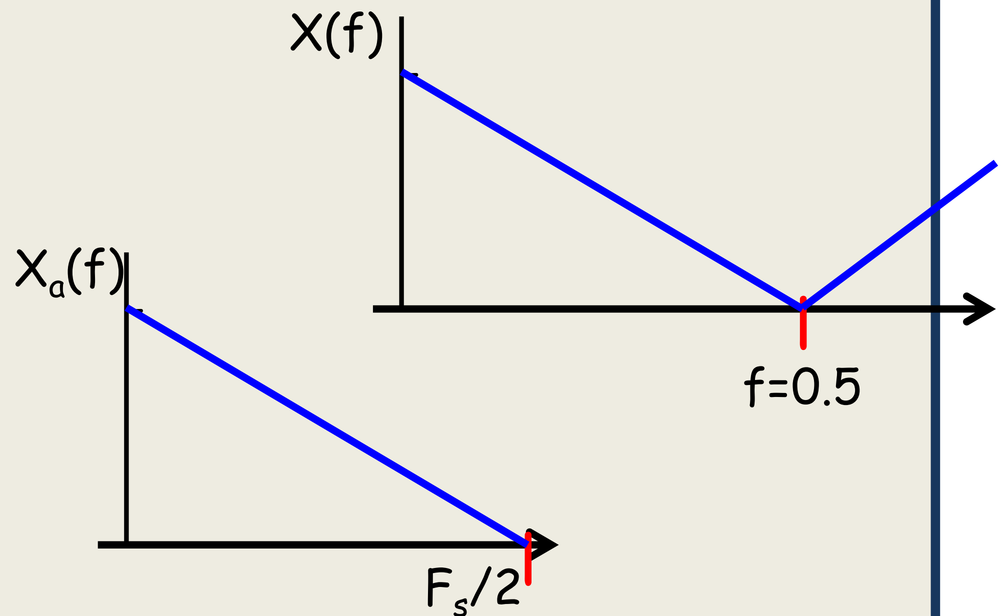
EITF75 Systems and Signals



Interpolation (Used for inter connecting systems with different sampling rates)

Page 778 (note: 773-778 included in course)





$$Y(f) = X(If)$$