EITF75 Systems and Signals

Lecture 6
More on the DTFT

Fredrik Rusek

EITF75, DTFT

Agenda:

1. Relation between DTFT and z-transform
2. Some properties of DTFTs
3. Relation between pole-zero diagrams and DTFT
4. Filter characteristics
5. DTFTs of unstable signals

EITF75 Systems and Signals

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

DTFT

(discrete time Fourier transform)

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \exp(-i2\pi n f)$$

$$= X(z|z = \exp(i2\pi f))$$

Important: DTFT is z-transform evaluated at unit circle

EITF75, DTFT

Why do we need the DTFT when we have the z-transform?

Why do we need the DTFT when we have the z-transform?

3 Simple steps to become a mobile phone operator

EITF75, DTFT

Why do we need the DTFT when we have the z-transform?

3 Simple steps to become a mobile phone operator

1. Go to the bank, ask for a big loan

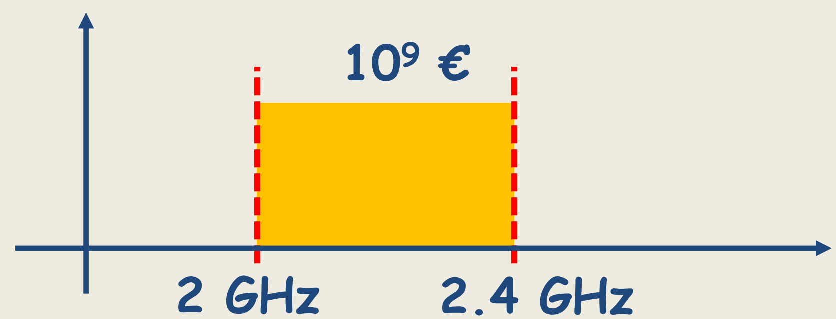
EITF75, DTFT

Why do we need the DTFT when we have the z-transform?

<u>3 Simple steps to become a mobile phone operator</u>

1. Go to the bank, ask for a big loan

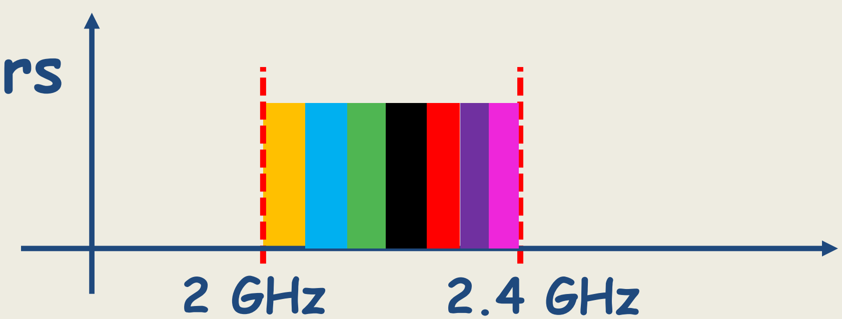
2. Go to PTS, buy some bandwidth



Why do we need the DTFT when we have the z-transform?

3 Simple steps to become a mobile phone operator

1. Go to the bank, ask for a big loan
2. Go to PTS, buy some bandwidth
3. Divide your customers over the band

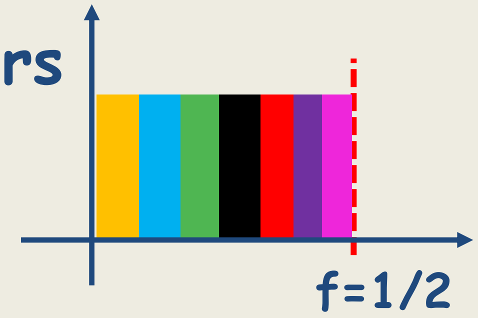


EITF75, DTFT

Why do we need the DTFT when we have the z-transform?

<u>3 Simple steps to become a mobile phone operator</u>

1. Go to the bank, ask for a big loan
2. Go to PTS, buy some bandwidth
3. Divide your customers over the band

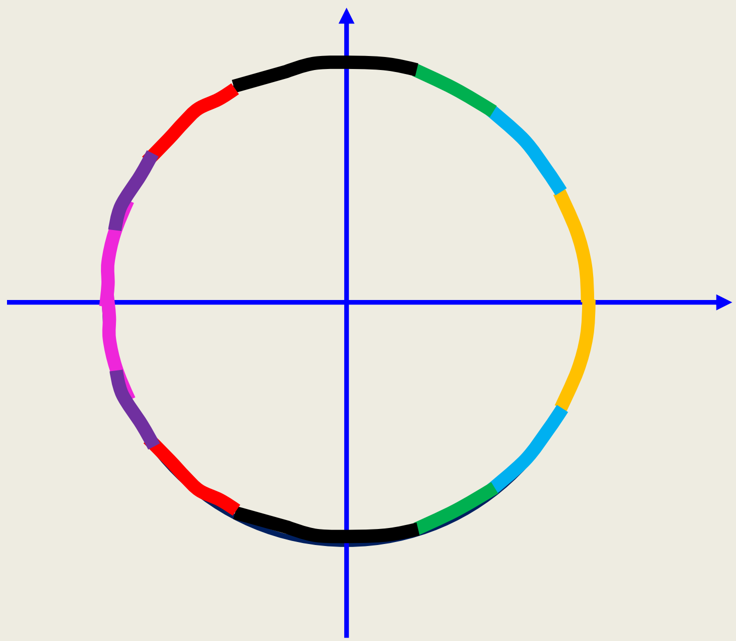


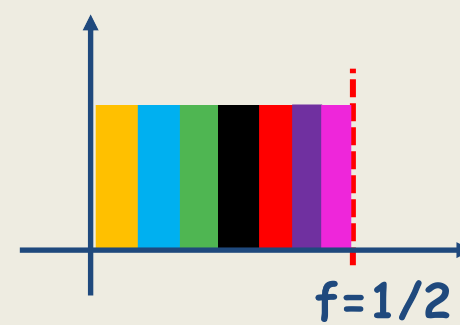
discrete time

EITF75, DTFT

Why do we need the DTFT when we have the z-transform?

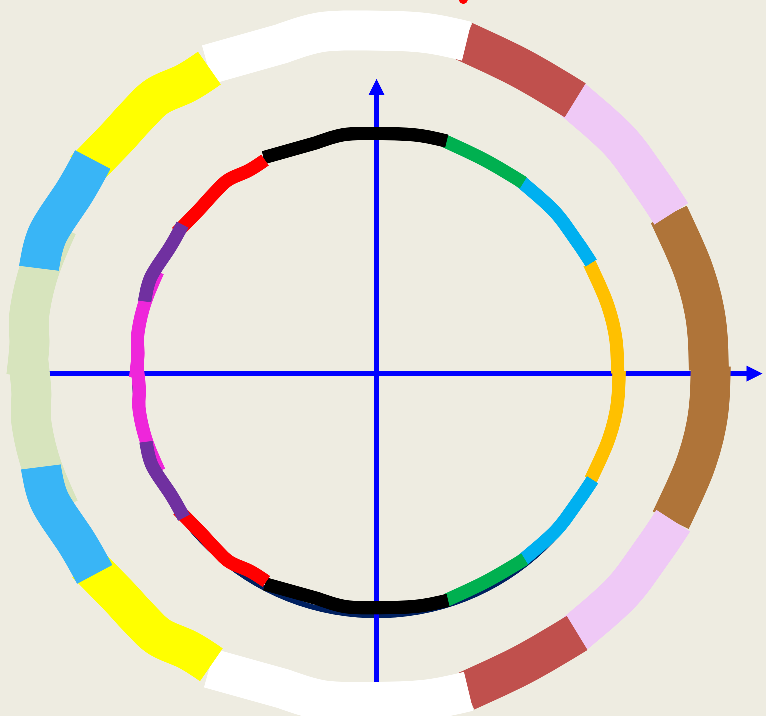
Seen in the z-plane



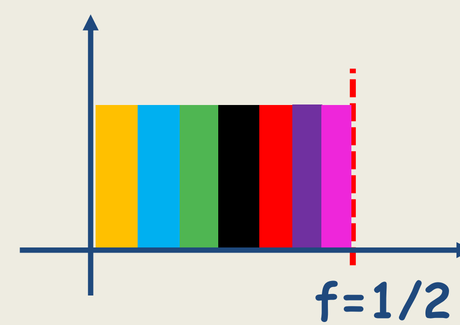


Why do we need the DTFT when we have the z-transform?

Seen in the z-plane



Seems as we can squeeze in more users if we buy a piece of the z-plane



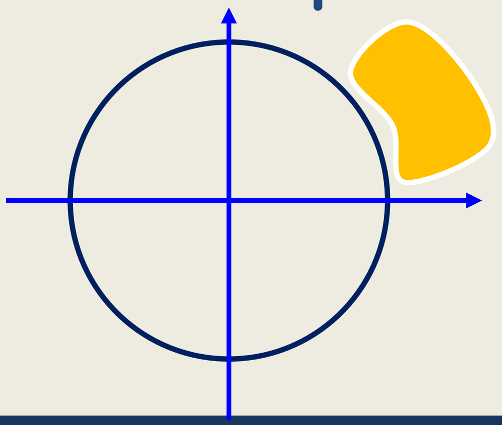
EITF75, DTFT

Why do we need the DTFT when we have the z-transform?

<u>Let us try this (to serve more users)</u>

1. Go to the bank, ask for a big loan

2. Go to PTS, ask for some part of the z-plane



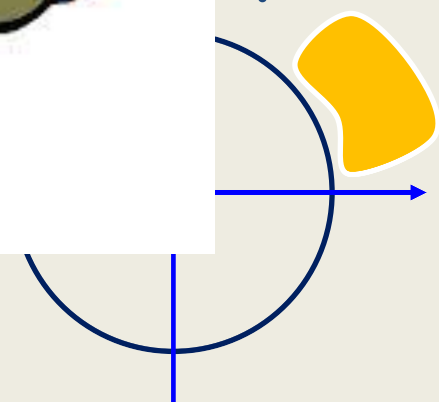
However……

Let us tr

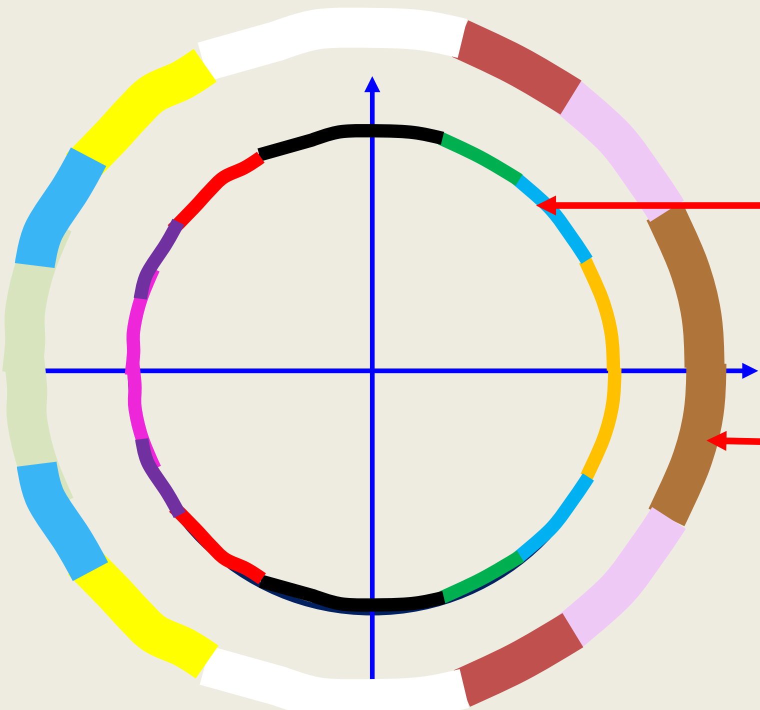
1. Go to

2. Go to

-plane



EITF75, DTFT

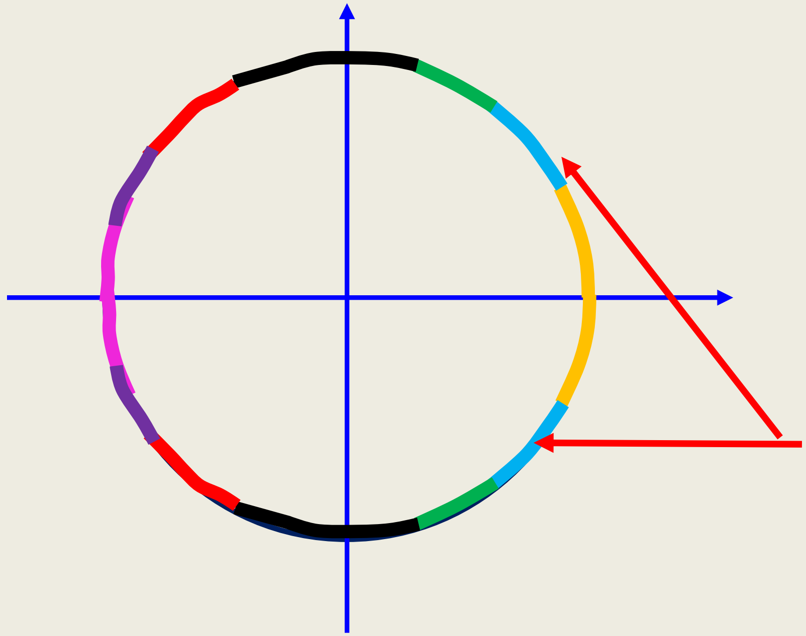


It is a bit strange

We can place users here as we like (this is what Telia does)

But not here

EITF75, DTFT



First

Not true that we can do what we like at the unit circle

We need symmetry at f=0

Same values

So, in the z-plane, there are restrictions on what we can do

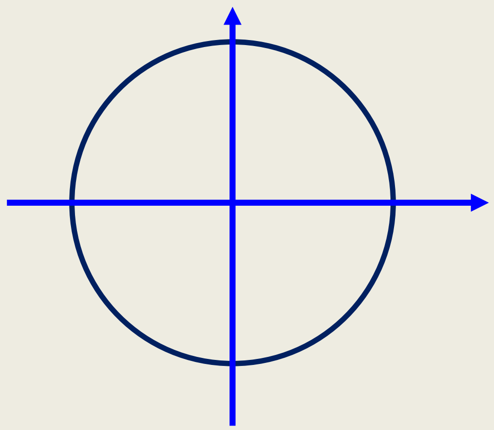
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1}\, dz$$

Where C is any closed curve around the origin, inside the ROC



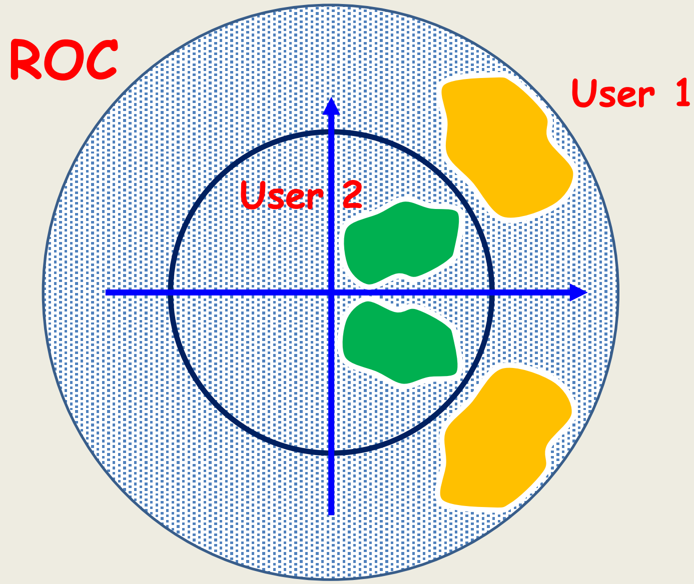
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1} \, dz$$

Where C is any closed curve around the origin, inside the ROC



Transmission to two users.
Different locations in z-space
Zero outside some area.
(so that we can send them at the same time)

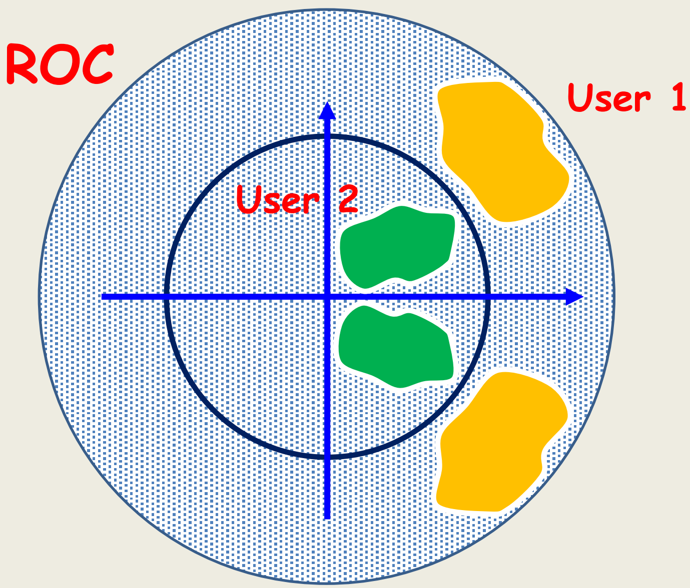
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1}\, dz$$

Where C is any closed curve around the origin, inside the ROC



How do time-signals look?

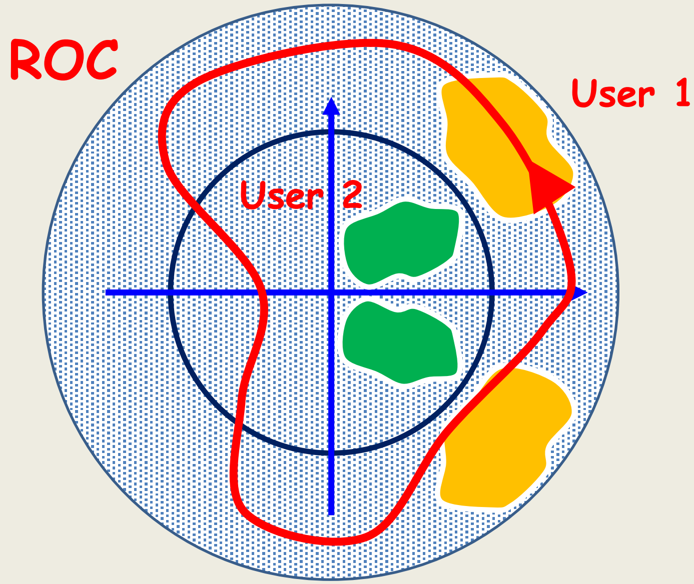
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1}\, dz$$

Where C is any closed curve around the origin, inside the ROC



How do time-signals look?

To find the signal to user 1:
Integrate, e.g., along this curve

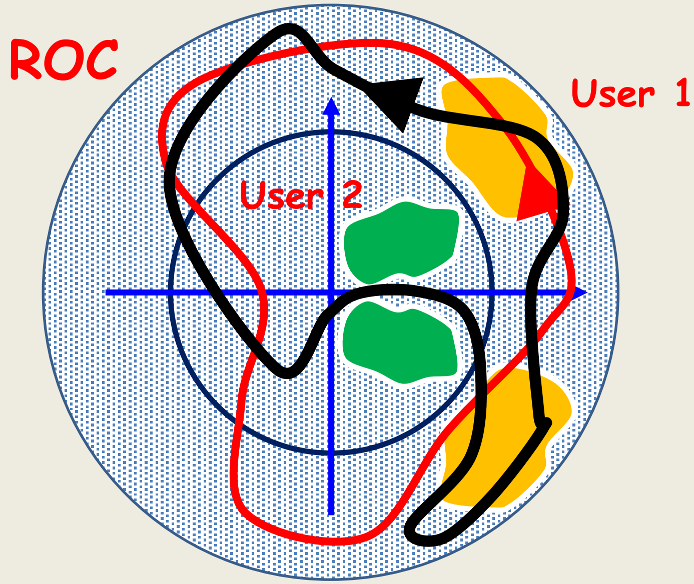
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1} \, dz$$

Where C is any closed curve around the origin, inside the ROC



How do time-signals look?

To find the signal to user 1:
Integrate, e.g., along this curve
….or this….

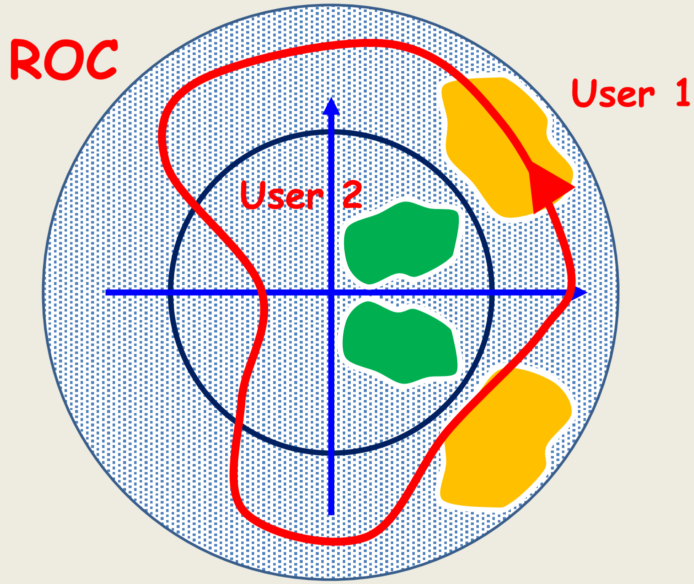
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1}\, dz$$

Where C is any closed curve around the origin, inside the ROC



How do time-signals look?

To find the signal to user 1:
Integrate, e.g., along this curve

We now have $y_1(n)$. But this is

1. Determined by $Y_1(z)$ along the line
2. Not determined by $Y_1(z)$ in green area

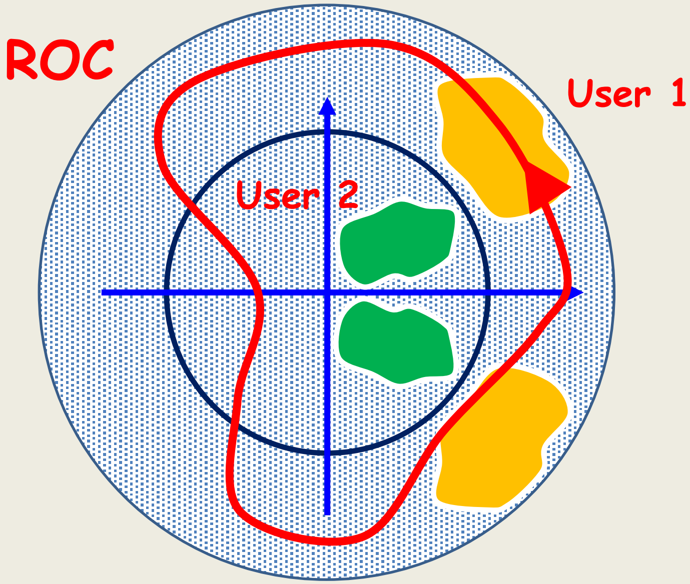
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1} \, dz$$

Where C is any closed curve around the origin, inside the ROC



How do time-signals look?

To find the signal to user 1:
Integrate, e.g., along this curve

We now have $y_1(n)$.
We can now compute $Y_1(z)$
But this will not be zero in the green part
(since this information was never used in the construction)

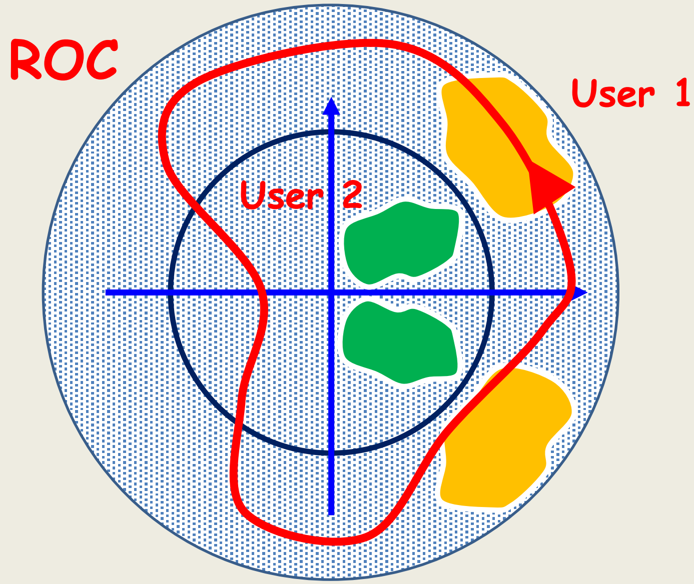
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1}\, dz$$

Where C is any closed curve around the origin, inside the ROC



Summary:

We cannot select $Y_1(z)$ arbitrarily
Because in general it will not be a valid transform.

That is,
Choose $Y_1(z)$ arbitrarily
Compute $y_1(n)$ by integration along a curve
Compute z-transform. Not same as $Y_1(z)$

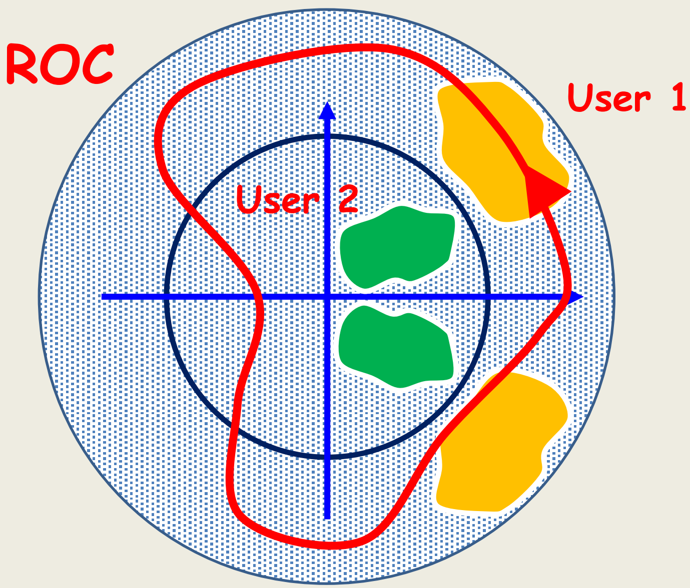
EITF75, DTFT

Secondly

Inverse z-transform (not mentioned before. Difficult to use)

$$y(n) = \frac{1}{2\pi j} \cdot \oint_C Y(z) z^{n-1}\, dz$$

Where C is any closed curve around the origin, inside the ROC



Summary:

In fact, entire $Y_1(z)$ is determined from its values along any closed curve

We can select this as the unit circle

So the Fourier transform (z-transf. on unit circle) determines the entire z-transform

EITF75, DTFT

Agenda:

1. Relation between DTFT and z-transform
2. Some properties of DTFTs
3. Relation between pole-zero diagrams and DTFT
4. Filter characteristics
5. DTFTs of unstable signals

EITF75, DTFT

Time delays

$$x(n) \leftrightarrow X(f)$$

$$x(n - n_0) \leftrightarrow \exp(-i2\pi f n_0) X(f)$$

Convolutions

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

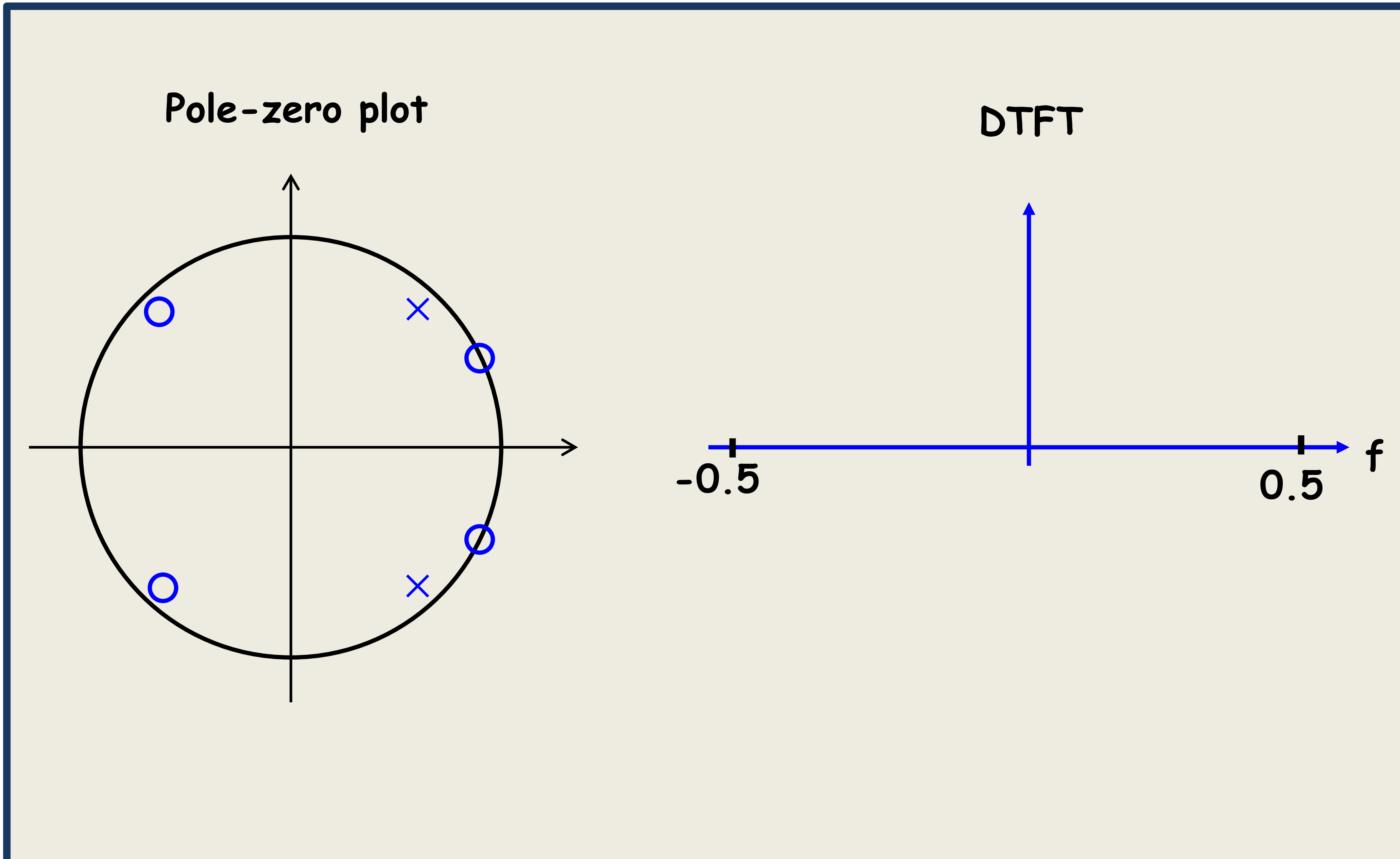
Easy to prove using z-transform. Do this at home

EITF75, DTFT

Agenda:

1. Relation between DTFT and z-transform
2. Some properties of DTFTs
3. Relation between pole-zero diagrams and DTFT
4. Filter characteristics
5. DTFTs of unstable signals

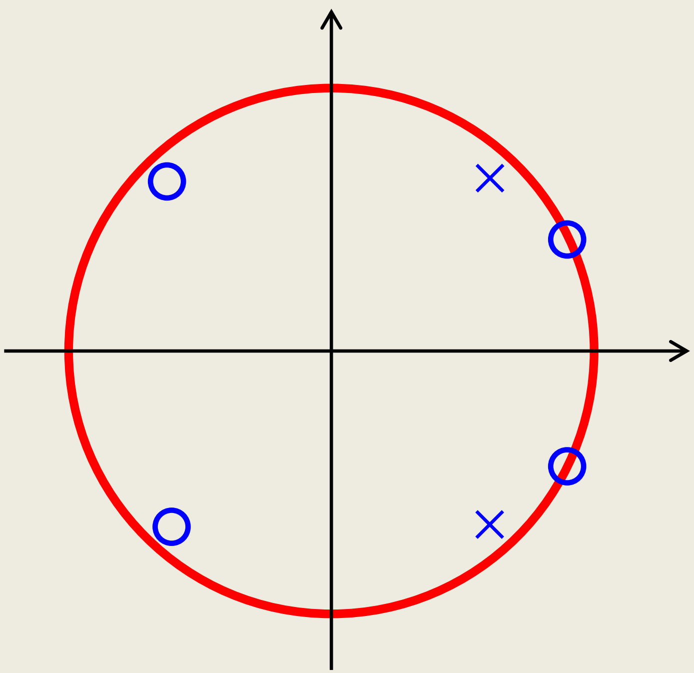
EITF75, DTFT



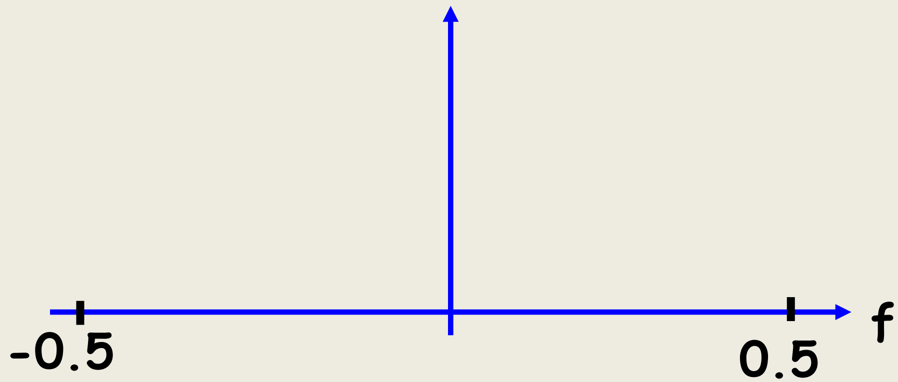
Book makes a big deal out of this. But quite easy….

EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle

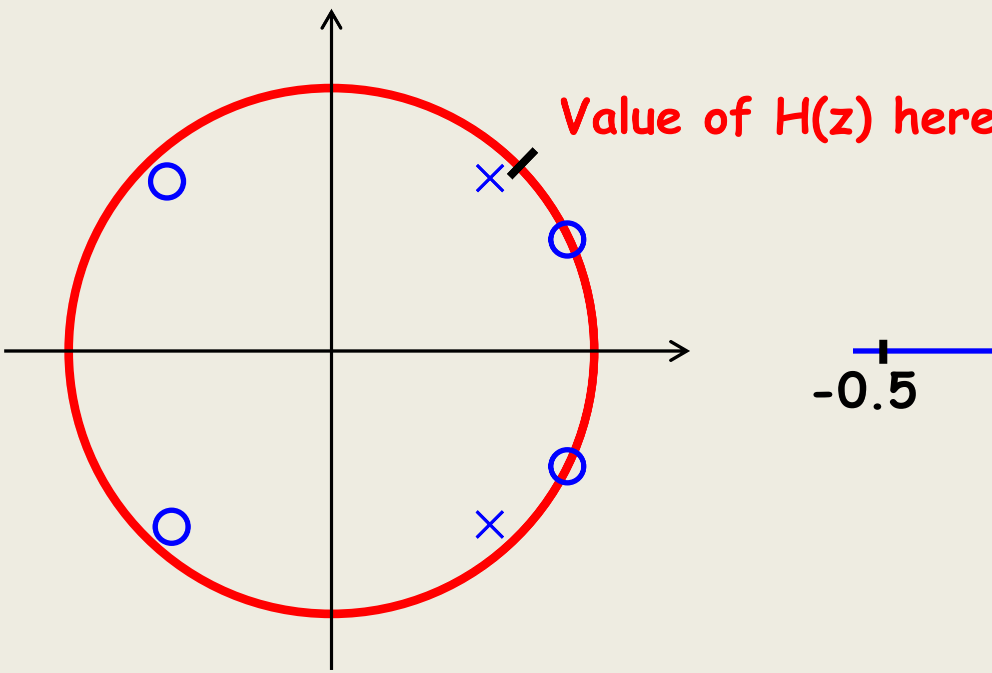


Recall

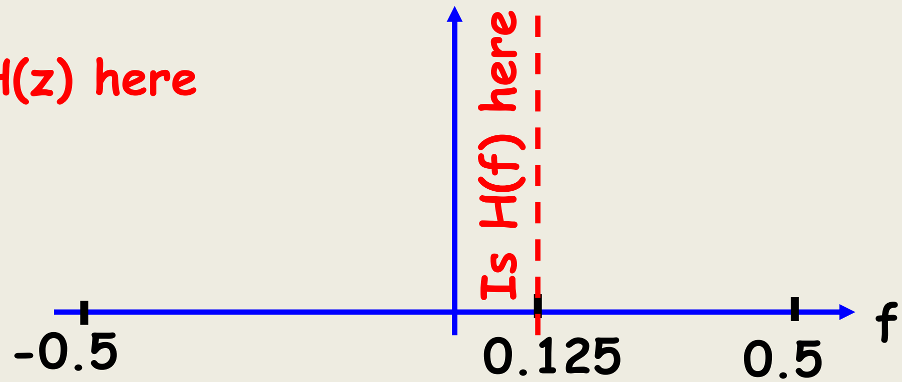
$$\boxed{H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}}$$

EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle

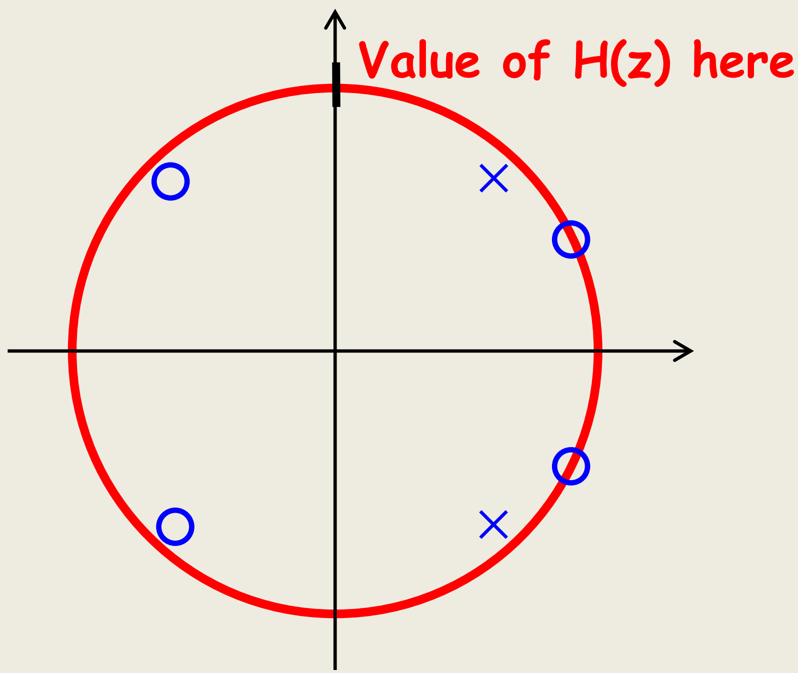


Recall

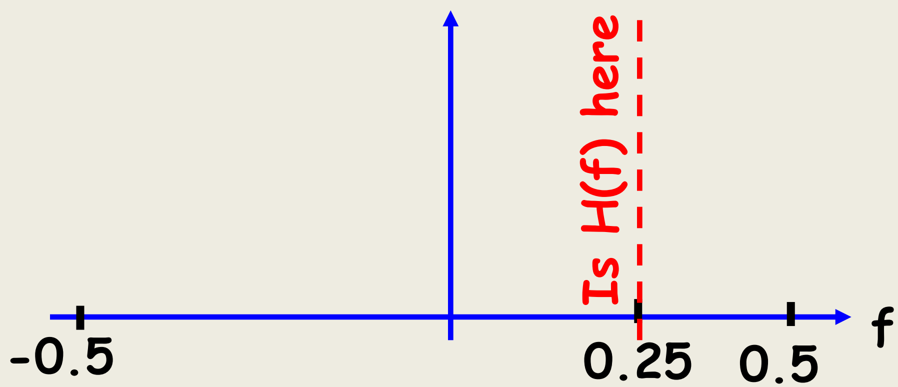
$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

EITF75, DTFT

Pole-zero plot



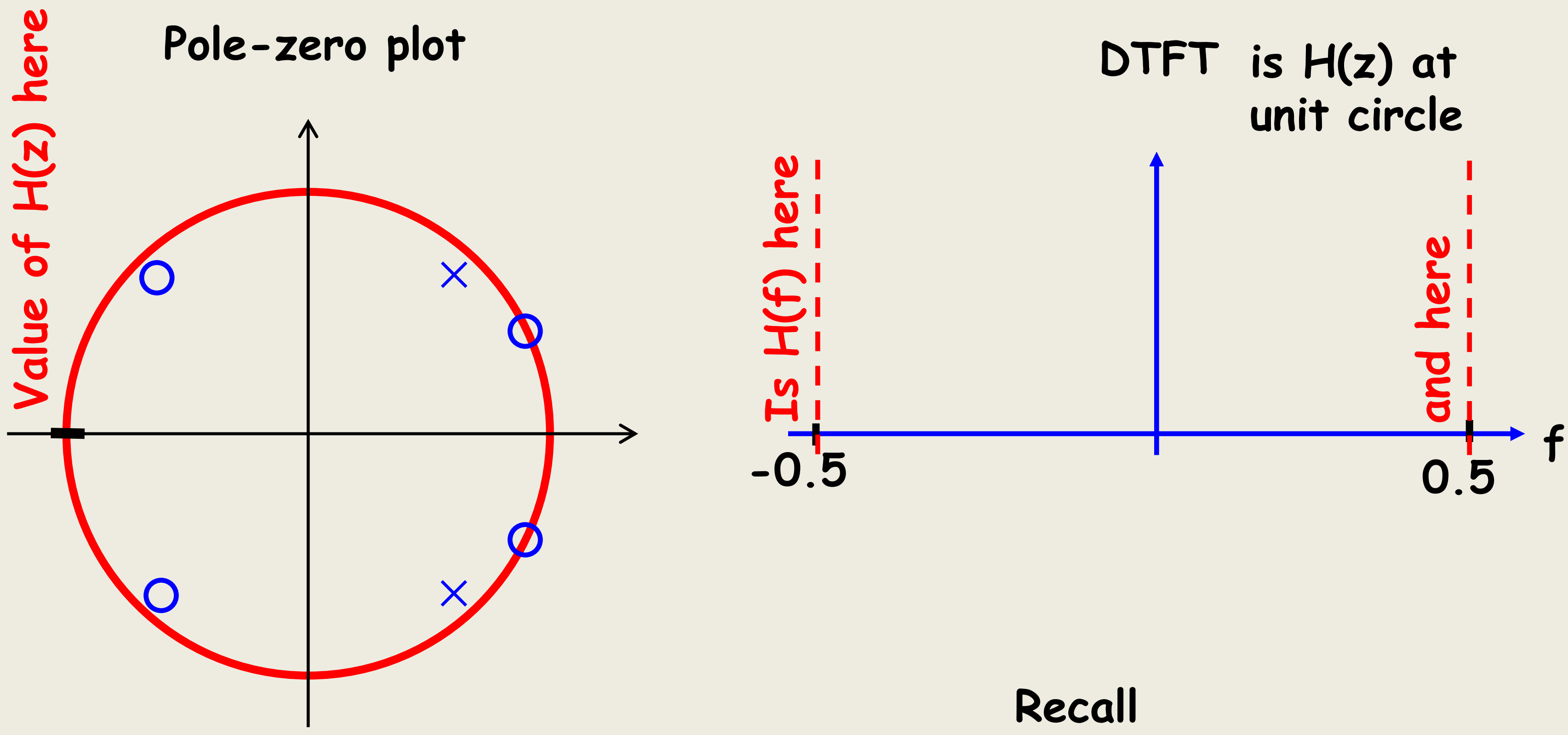
DTFT is H(z) at unit circle



Recall

$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

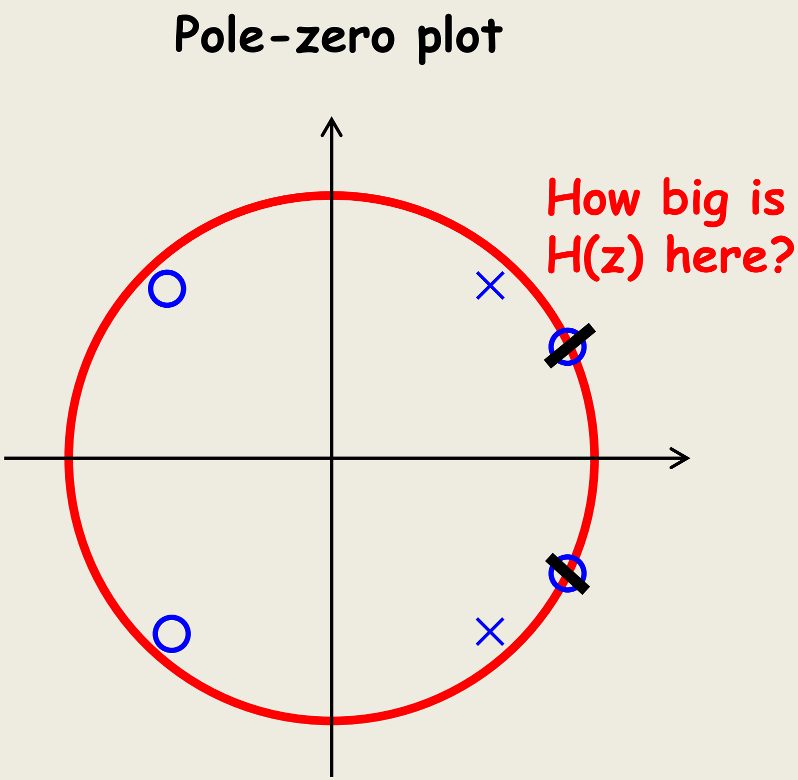
EITF75, DTFT

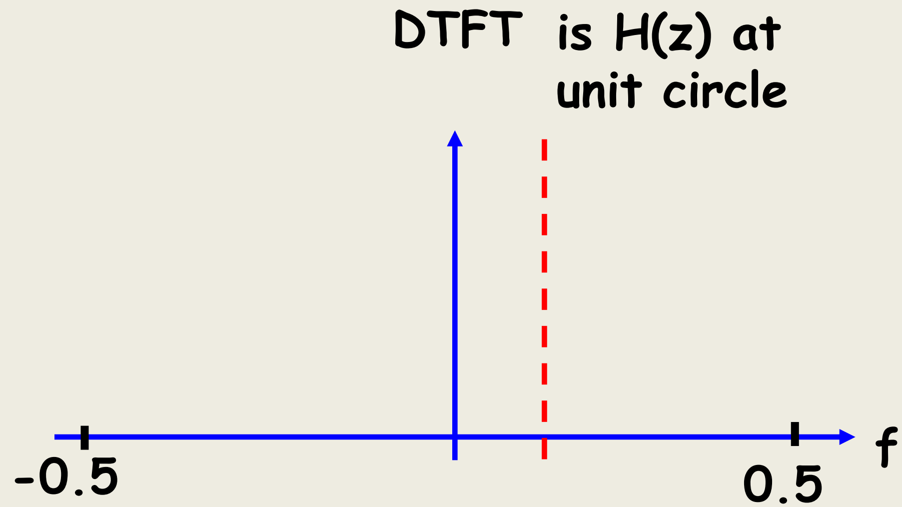


Recall

$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

EITF75, DTFT

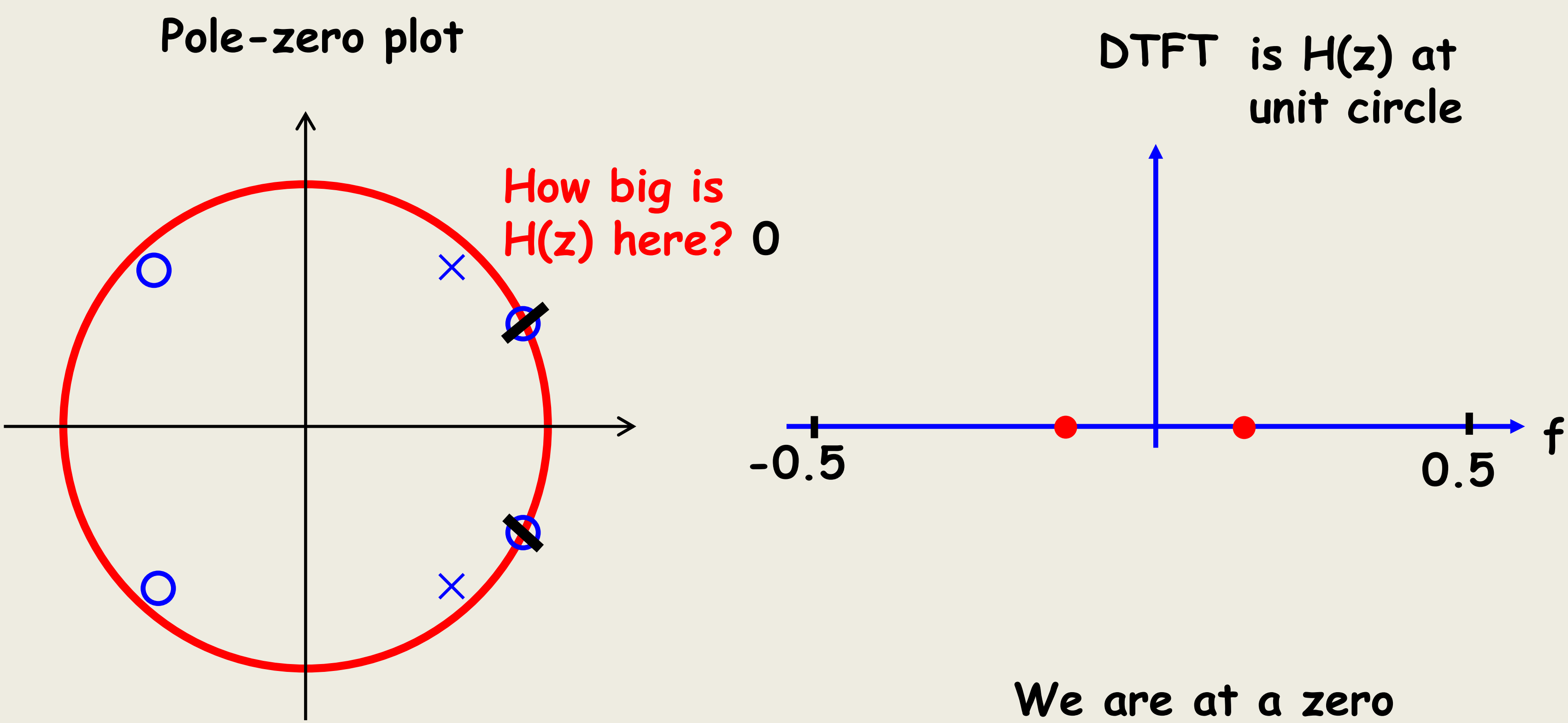




Recall

$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

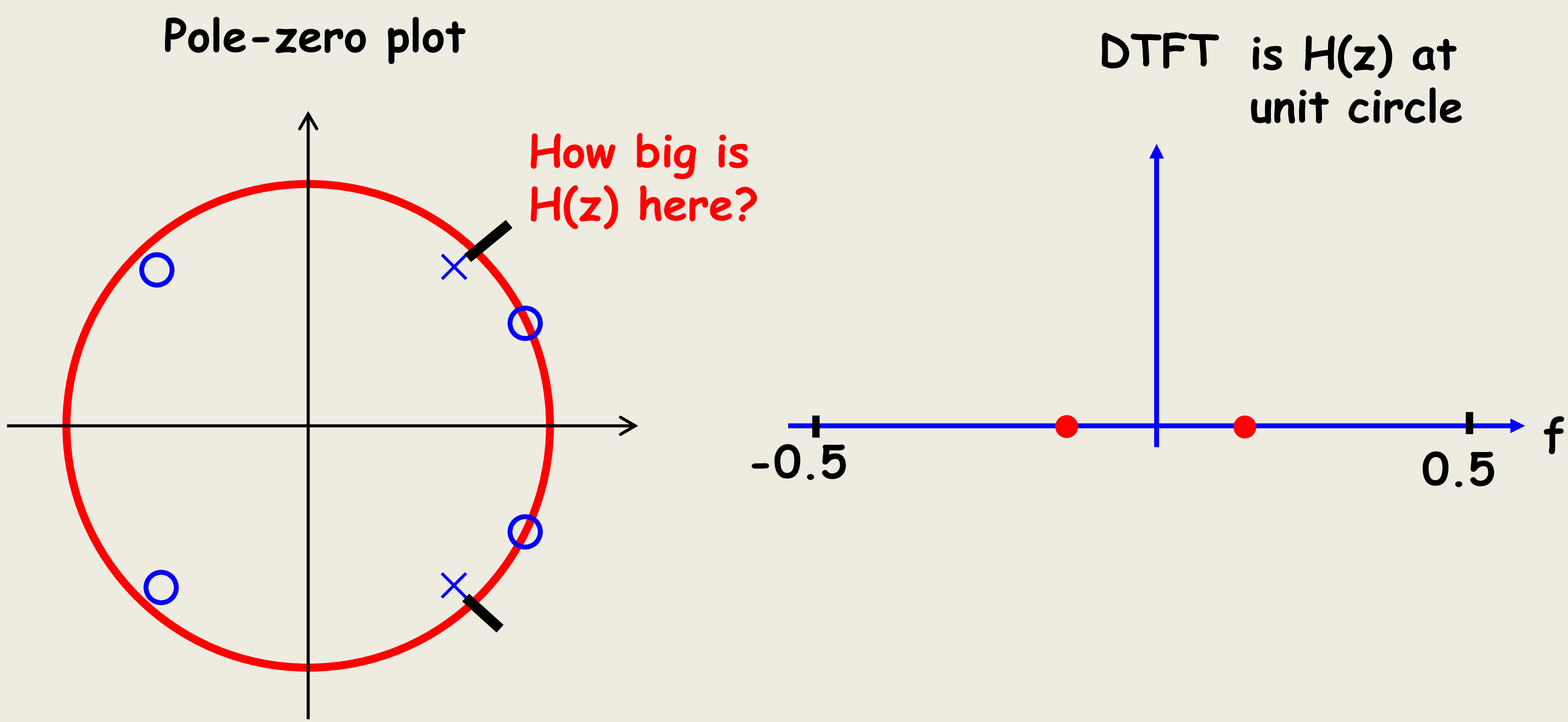
EITF75, DTFT



We are at a zero

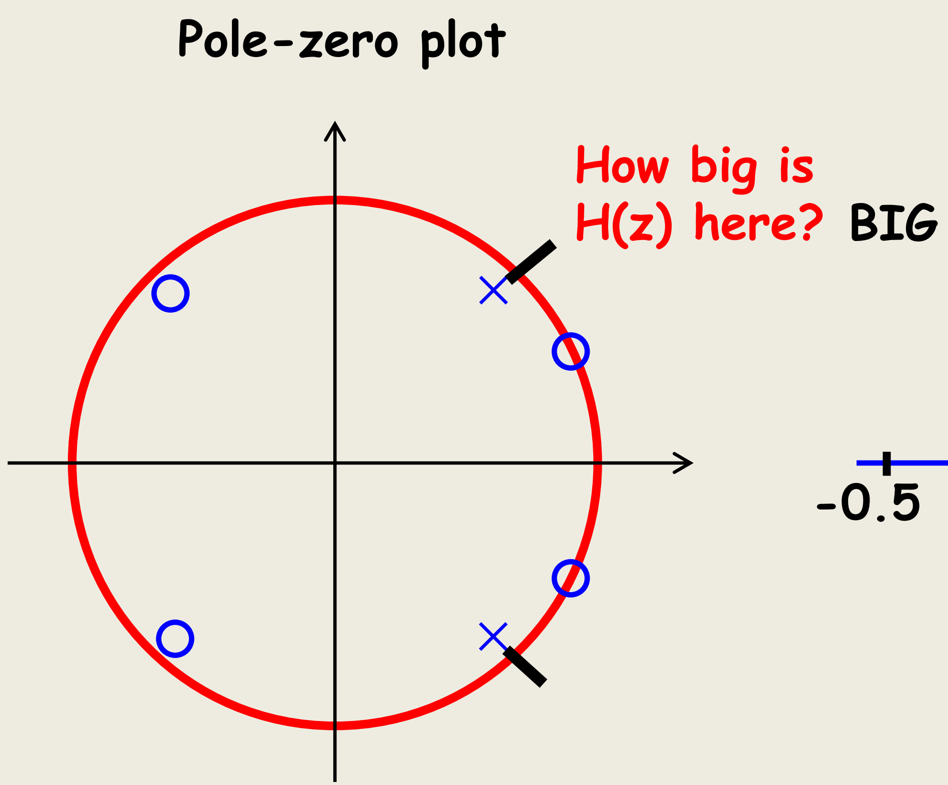
$$H(z) = \frac{\boxed{(z - z_1)}\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

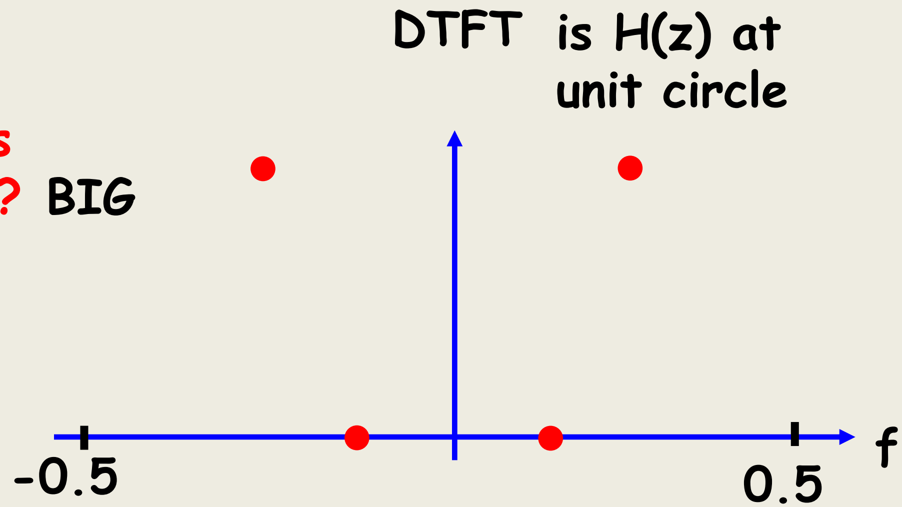
EITF75, DTFT



$$H(z) = \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

EITF75, DTFT

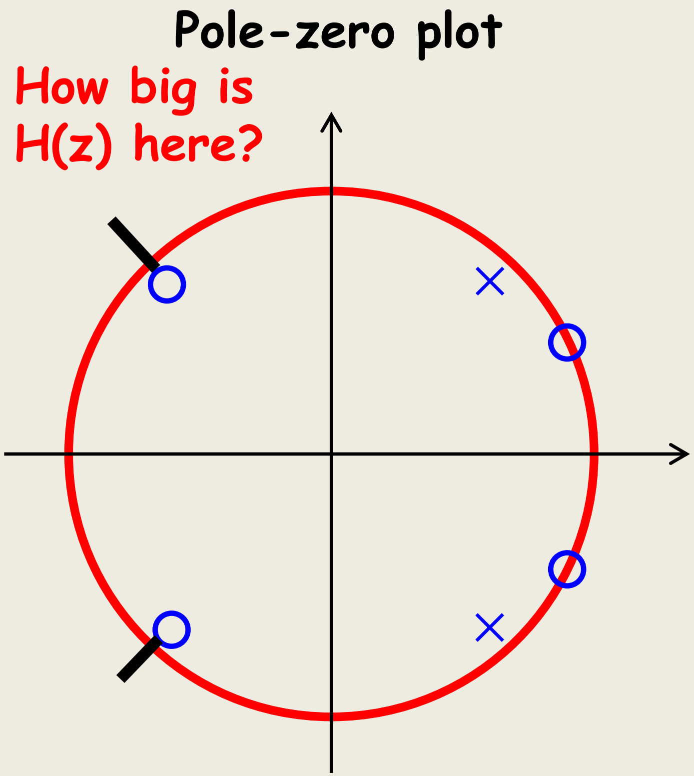


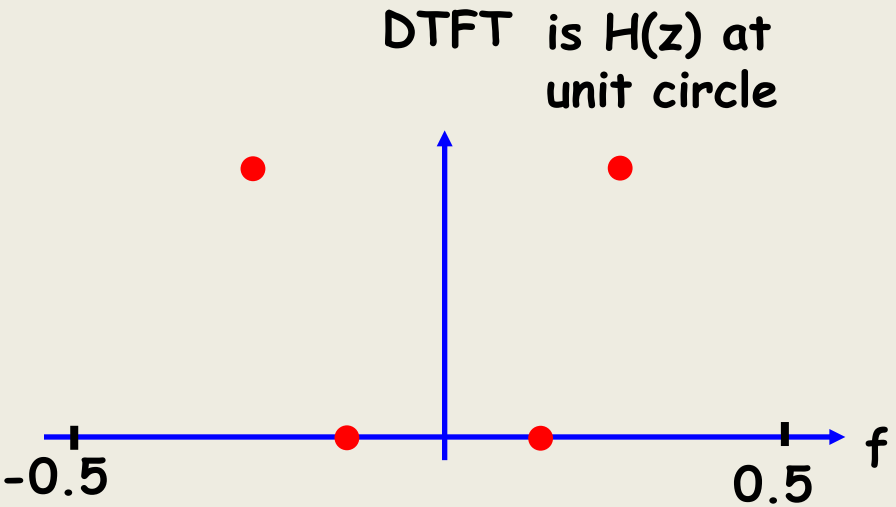


We are close to a pole

$$H(z) = \frac{(z-z_1)\cdots(z-z_M)}{\boxed{(z-p_1)}\cdots(z-p_N)}$$

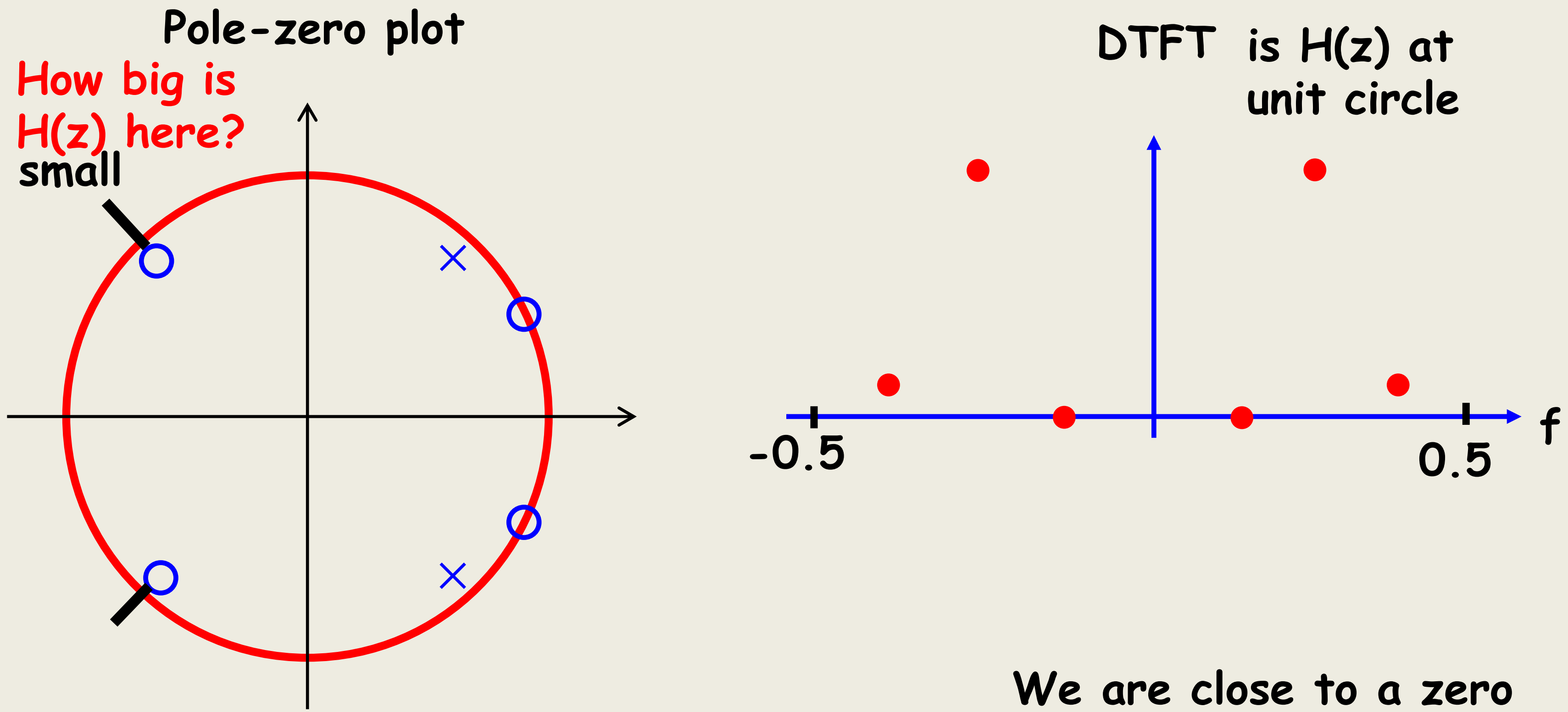
EITF75, DTFT





$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

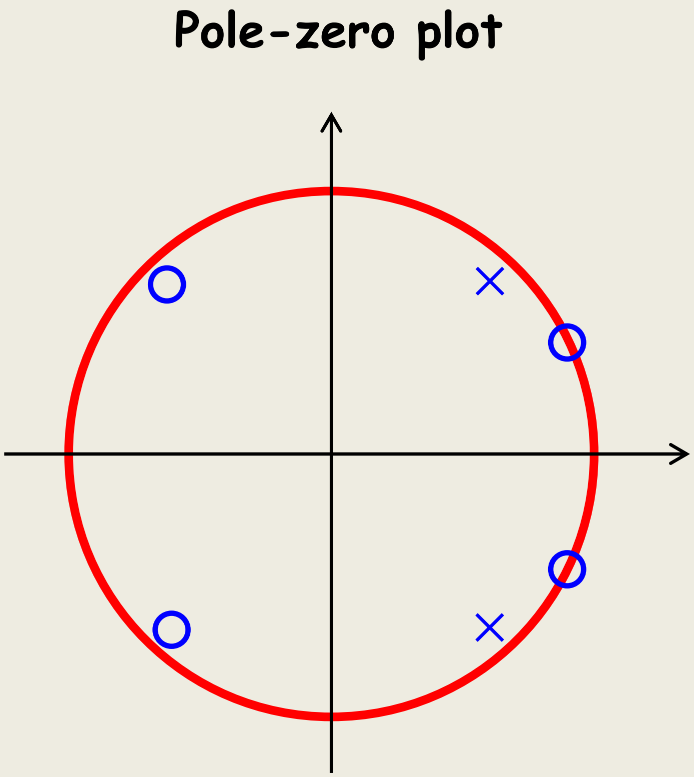
EITF75, DTFT

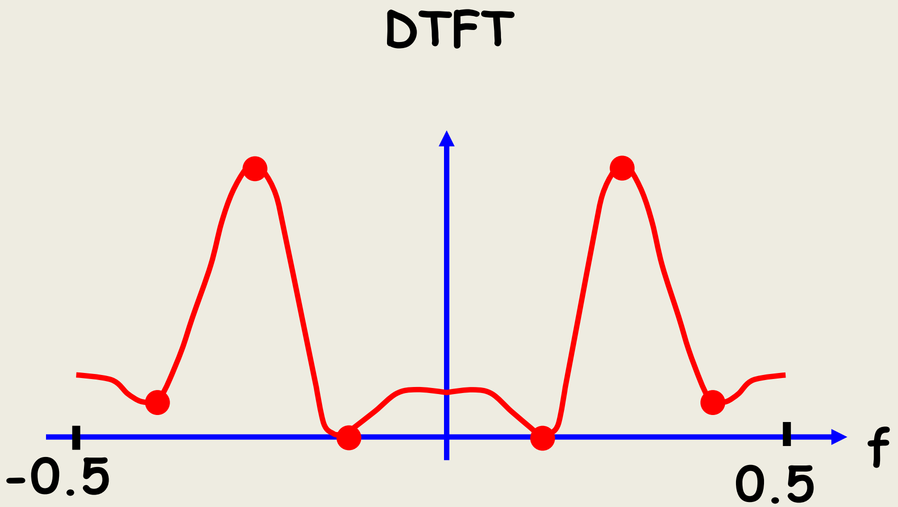


We are close to a zero

$$H(z) = \frac{(z-z_1)\cdots\boxed{(z-z_M}}{(z-p_1)\cdots(z-p_N)}$$

EITF75, DTFT

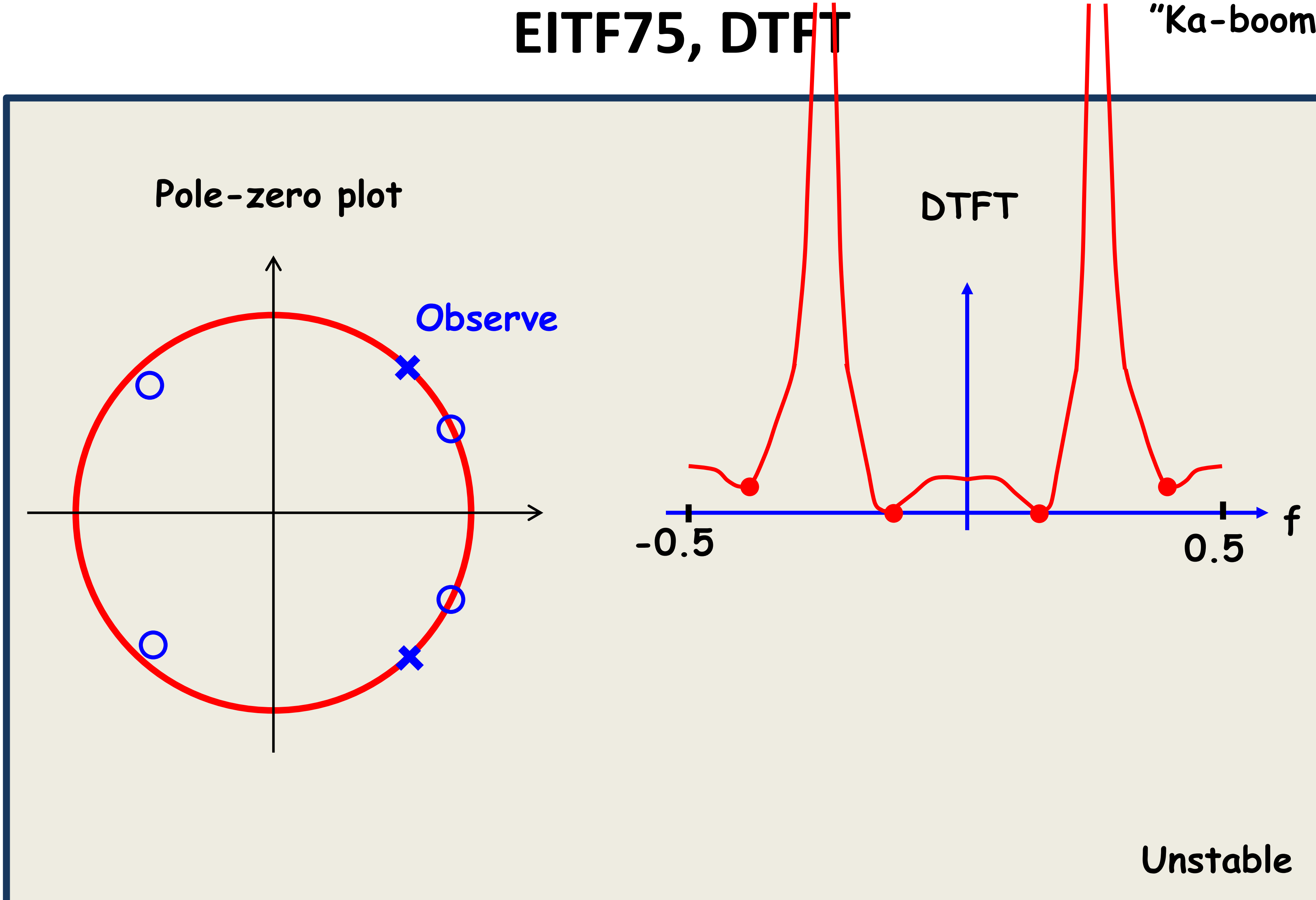




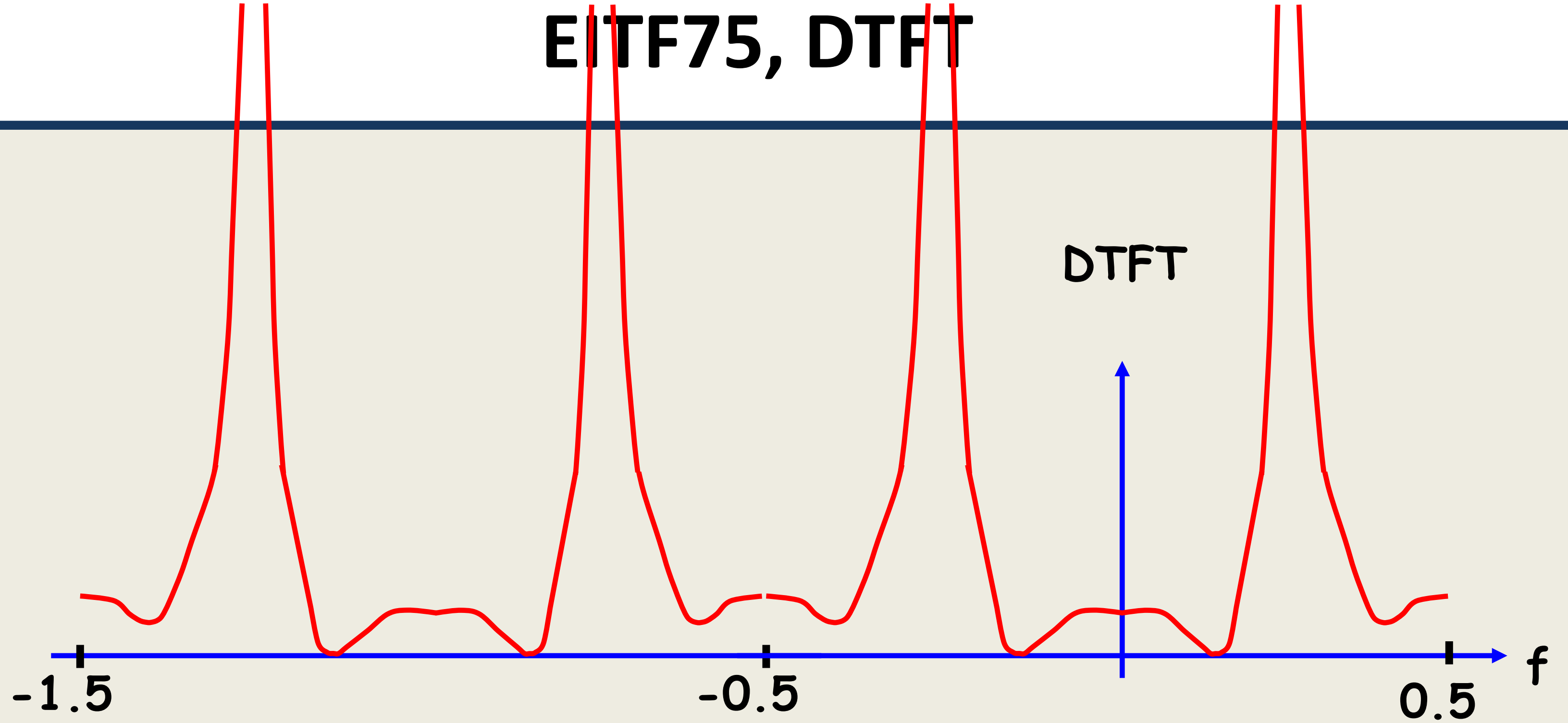
Non-zero everywhere else, since no further zeros at unit circle

EITF75, DTFT

“Ka-boom”



EITF75, DTFT



Final remark: X(f) is periodic

EITF75, DTFT

Agenda:

1. Relation between DTFT and z-transform
2. Some properties of DTFTs
3. Relation between pole-zero diagrams and DTFT
4. Filter characteristics
5. DTFTs of unstable signals

EITF75, Fourier transforms

Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

Seen in the Fourier-plane

EITF75, Fourier transforms

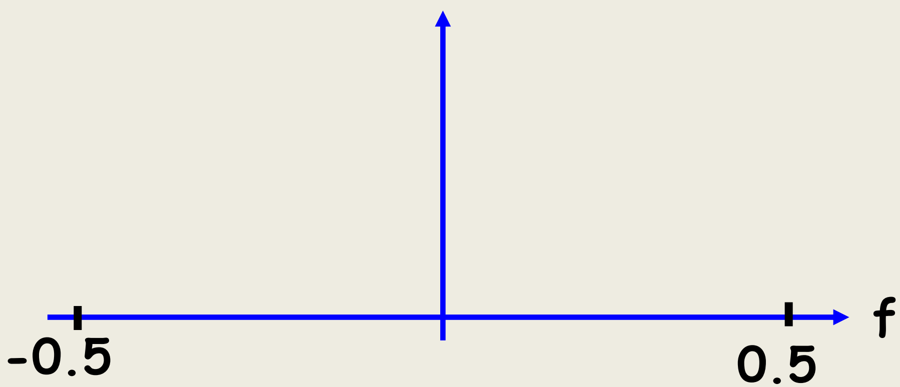
Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

Seen in the Fourier-plane

Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter



EITF75, Fourier transforms

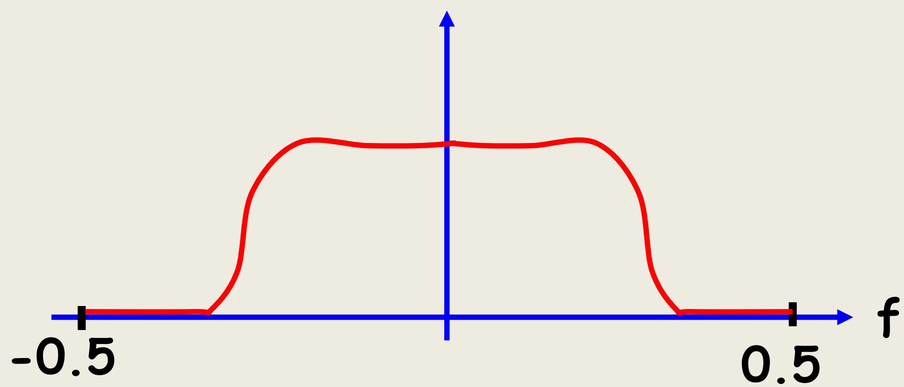
Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter

Allows low frequncies to pass



EITF75, Fourier transforms

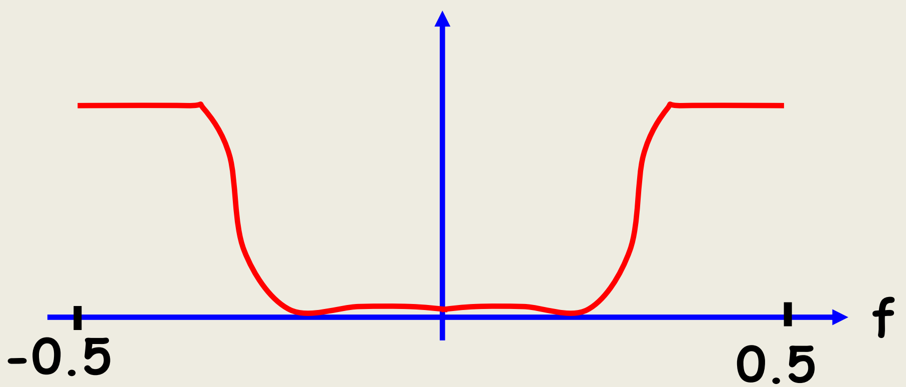
Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter

Allows high frequncies to pass



EITF75, Fourier transforms

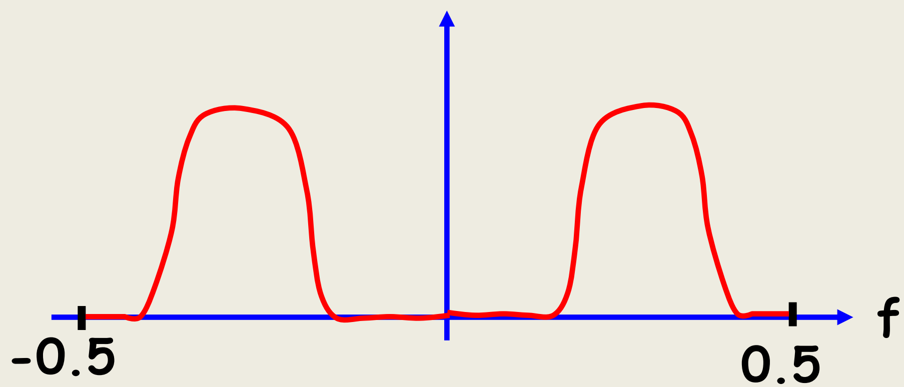
Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter

Allows some frequncies to pass



EITF75, Fourier transforms

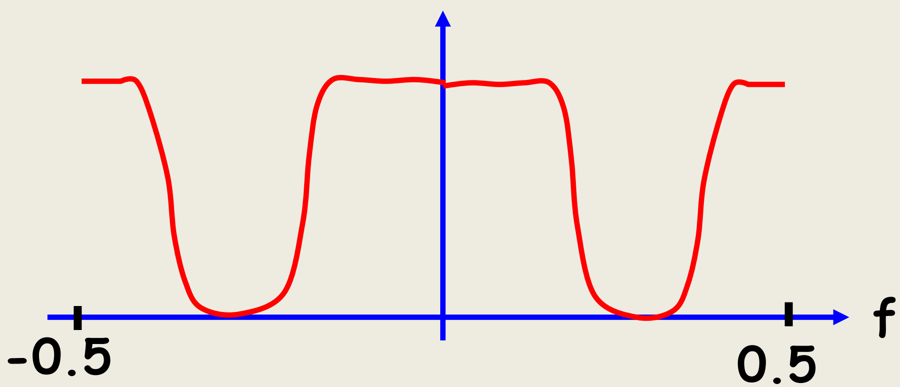
Input-Output relation of LTI system

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) * h(n)$$

$$X(f) \longrightarrow \boxed{H(f)} \longrightarrow Y(f) = X(f)H(f)$$

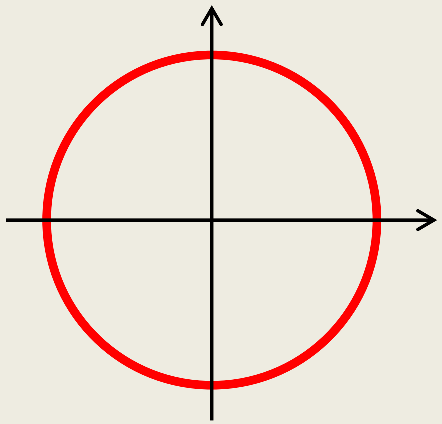
Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter

Blocks some frequncies

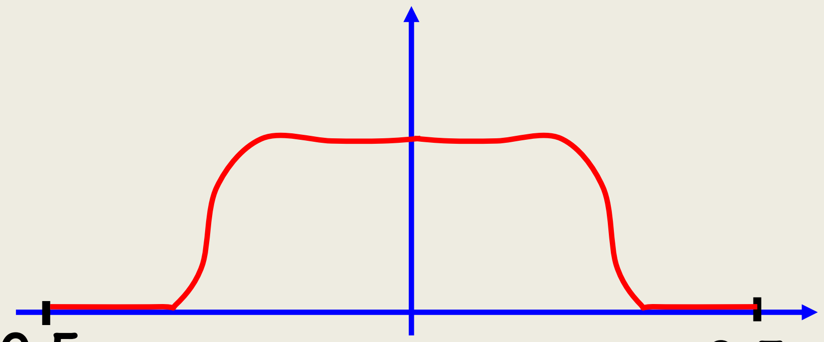


EITF75, Fourier transforms

Pole-zero plot ??

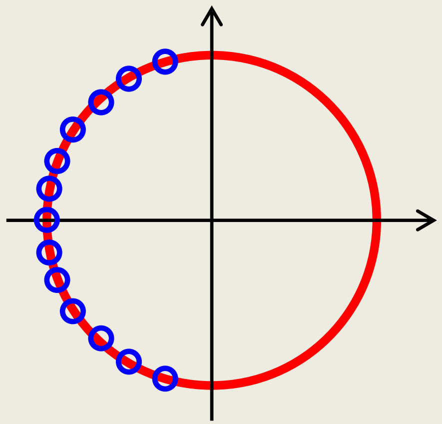


Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter



EITF75, Fourier transforms

Pole-zero plot

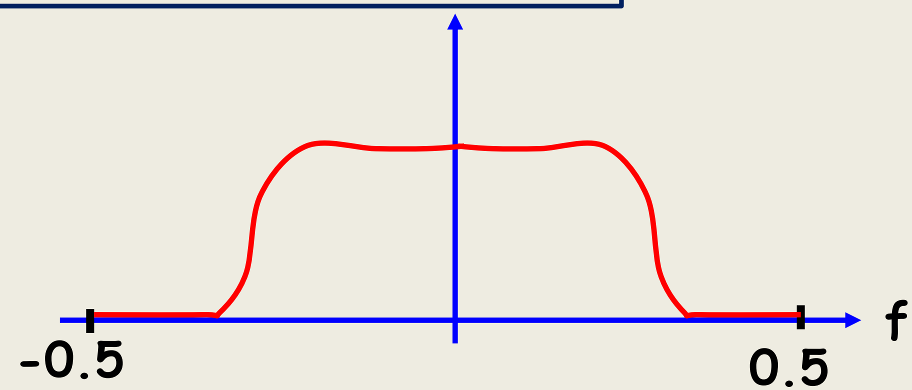


Filter type
Low pass filter
High pass filter
Band pass filter
Band stop filter

Hard to say (we will get back later).

But zeros located in the stop-band

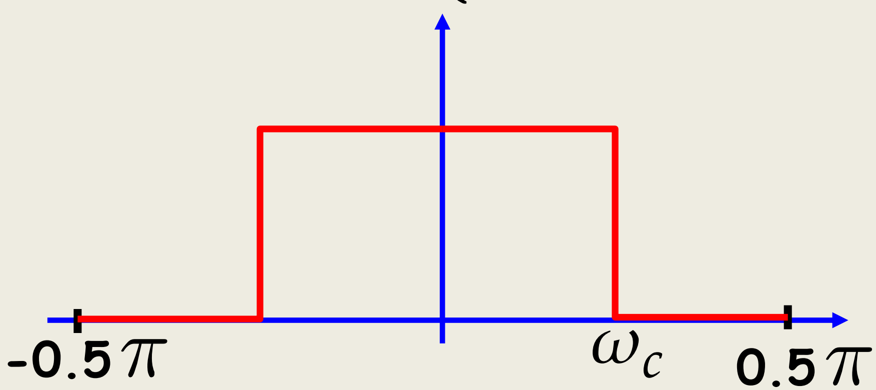
EITF75 7.5 ECTS before
EITF75 6.0 ECTS now
Filter design removed



EITF75, Fourier transforms

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$\omega = 2\pi f$$

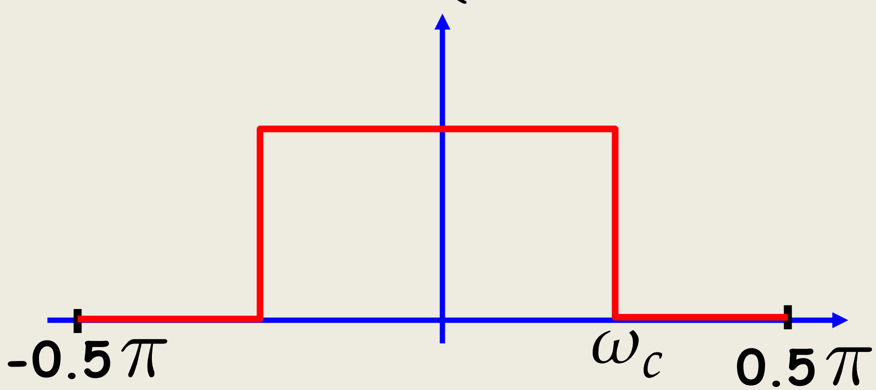
EITF75, Fourier transforms

Let us find the impulse response in the time-domain

$$h(n) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \, d\omega$$

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$\omega = 2\pi f$$

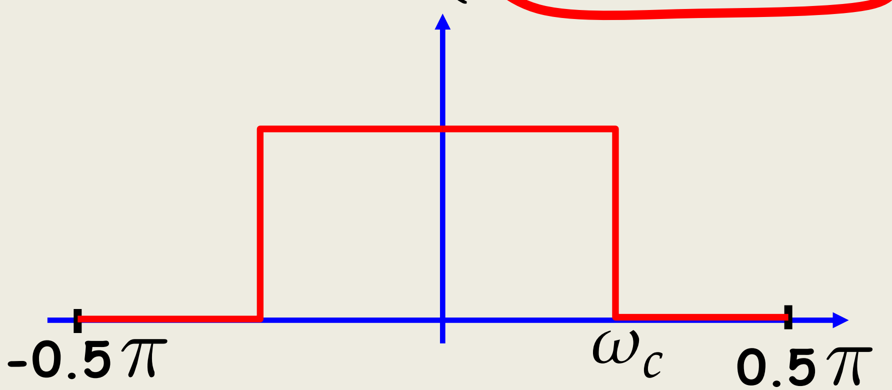
EITF75, Fourier transforms

Let us find the impulse response in the time-domain

$$h(n) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega) e^{j\omega n} \, d\omega = \frac{1}{2\pi} \cdot \int_{-\omega_c}^{\omega_c} H(\omega) e^{j\omega n} \, d\omega$$

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$\omega = 2\pi f$$

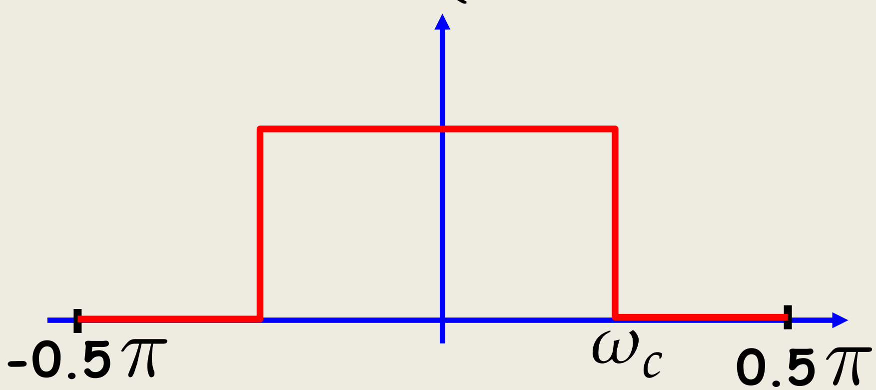
EITF75, Fourier transforms

Let us find the impulse response in the time-domain

$$h(n) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} \mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega$$

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$\omega = 2\pi f$$

EITF75, Fourier transforms

Let us find the impulse response in the time-domain

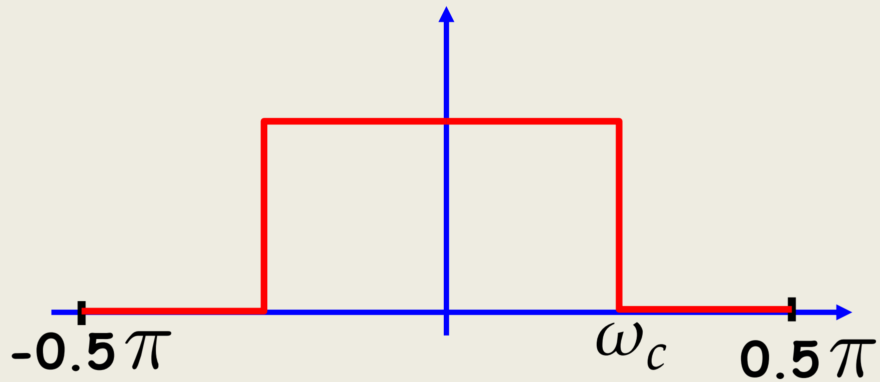
$$h(n) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} \mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\cdot\frac{\mathrm{e}^{\mathrm{j}\omega_c n} - \mathrm{e}^{-\mathrm{j}\omega_c n}}{\mathrm{j}n}$$

Integral of exponential function

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$H(\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$\omega = 2\pi f$$

EITF75, Fourier transforms

Let us find the impulse response in the time-domain

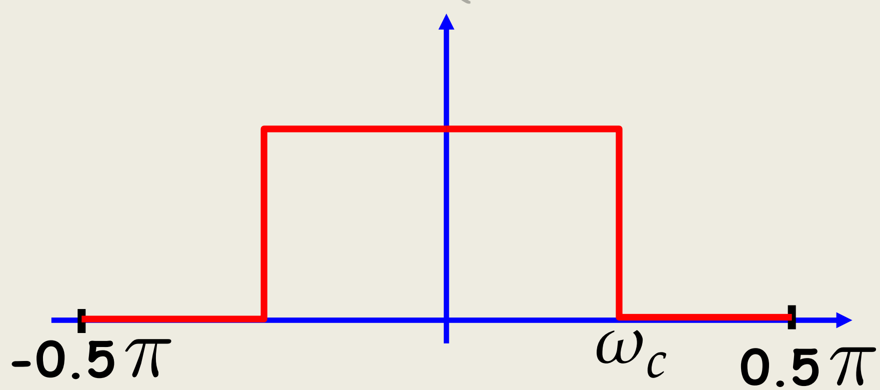
$$h(n) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} \mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\cdot\frac{\mathrm{e}^{\mathrm{j}\omega_c n}-\mathrm{e}^{-\mathrm{j}\omega_c n}}{\mathrm{j}n} = \frac{\omega_c}{\pi}\cdot\frac{\sin(\omega_c n)}{\omega_c n}$$

Euler

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$H(\omega) = \begin{cases} 1, & |\omega| \leq \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$



$$\omega = 2\pi f$$

EITF75, Fourier transforms

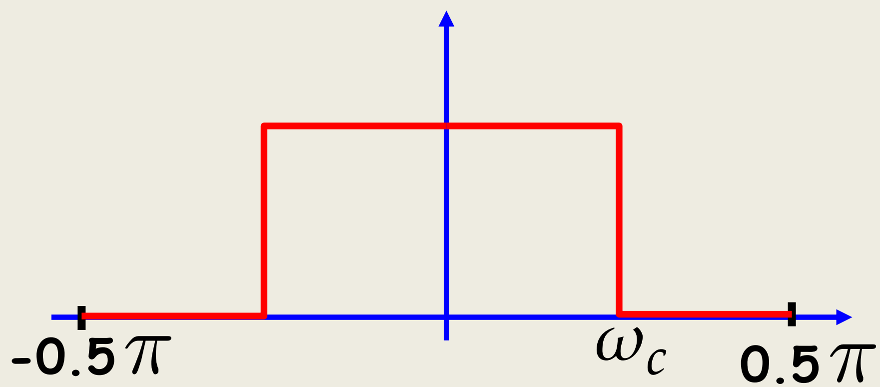
Let us find the impulse response in the time-domain

$$h(n) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} \mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\cdot\frac{\mathrm{e}^{\mathrm{j}\omega_c n} - \mathrm{e}^{-\mathrm{j}\omega_c n}}{\mathrm{j}n} = \frac{\omega_c}{\pi}\cdot\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi}\cdot\operatorname{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

Filter type
Ideal Low pass filter
High pass filter
Band pass filter
Band stop filter

$$\operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



$$\omega = 2\pi f$$

EITF75, Fourier transforms

Let us find the impulse response in the time-domain

$$h(n) = \frac{1}{2\pi} \cdot \int_{-\pi}^{\pi} H(\omega) e^{j\omega n}\, d\omega = \frac{1}{2\pi} \cdot \int_{-\omega_c}^{\omega_c} H(\omega) e^{j\omega n}\, d\omega = \frac{1}{2\pi} \cdot \int_{-\omega_c}^{\omega_c} e^{j\omega n}\, d\omega$$

$$= \frac{1}{2\pi} \cdot \frac{e^{j\omega_c n} - e^{-j\omega_c n}}{jn} = \frac{\omega_c}{\pi} \cdot \frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi} \cdot \mathrm{sinc}\left(\frac{\omega_c}{\pi} n\right)$$

Problems:

1. Not finite length
2. Starts at $n = -\infty$
3. Unstable $\sum_n |\mathrm{sinc}(n)| = \infty$

EITF75, Fourier transforms

Let us find the impulse response in the time-domain

$$h(n) = \frac{1}{2\pi}\cdot\int_{-\pi}^{\pi} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} H(\omega)\mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega = \frac{1}{2\pi}\cdot\int_{-\omega_c}^{\omega_c} \mathrm{e}^{\mathrm{j}\omega n}\,\mathrm{d}\omega$$

$$= \frac{1}{2\pi}\cdot\frac{\mathrm{e}^{\mathrm{j}\omega_c n} - \mathrm{e}^{-\mathrm{j}\omega_c n}}{\mathrm{j}n} = \frac{\omega_c}{\pi}\cdot\frac{\sin(\omega_c n)}{\omega_c n} = \frac{\omega_c}{\pi}\cdot\mathrm{sinc}\left(\frac{\omega_c}{\pi}n\right)$$

Problems:

1. Not finite length
2. Starts at $n = -\infty$
3. Unstable $\sum_n |\mathrm{sinc}(n)| = \infty$

Attempt of solution:

1. Truncate to N taps
2. Delay (N-1)/2 makes it causal
3. Truncation fixes this

EITF75, Fourier transforms

$$h(n) = \frac{\omega_c}{\pi} \cdot \operatorname{sinc}\left(\frac{\omega_c}{\pi} n\right)$$ **Ideal filter**

$$h(n) = \frac{\omega_c}{\pi} \cdot \frac{\sin\left(\omega_c\left(n - \frac{N-1}{2}\right)\right)}{\omega_c\left(n - \frac{N-1}{2}\right)} \quad \text{for } 0 \le n < N$$ **Truncated filter**

Problems:

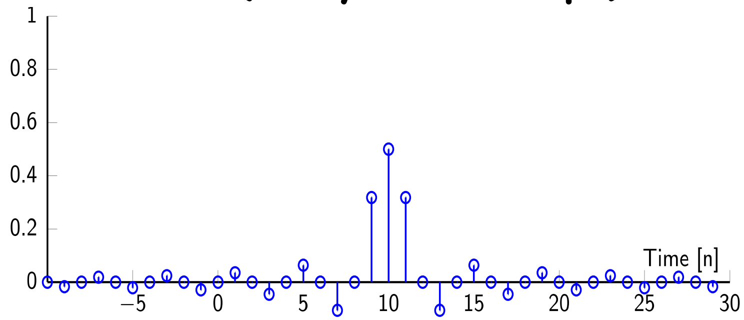
1. Not finite length
2. Starts at $n = -\infty$
3. Unstable $\sum_n |\operatorname{sinc}(n)| = \infty$

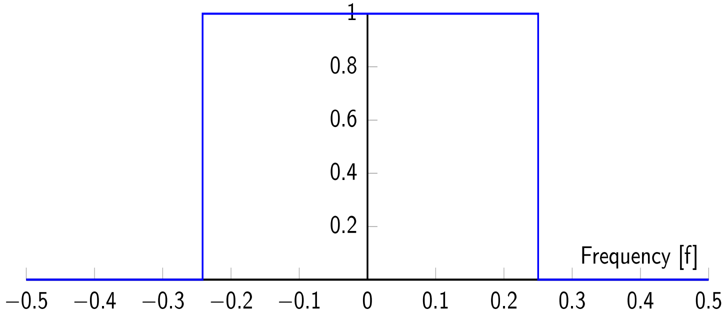
Attempt of solution:

1. Truncate to N taps
2. Delay (N-1)/2 makes it causal
3. Truncation fixes this

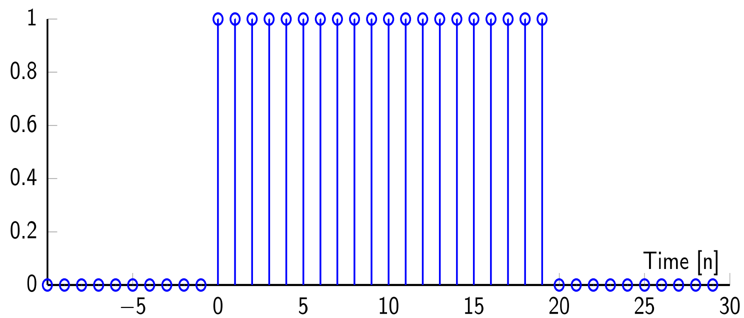
N=21

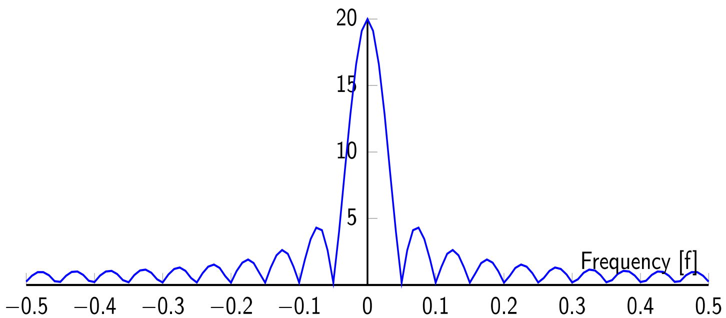
Ideal filter (delayed 10 steps)



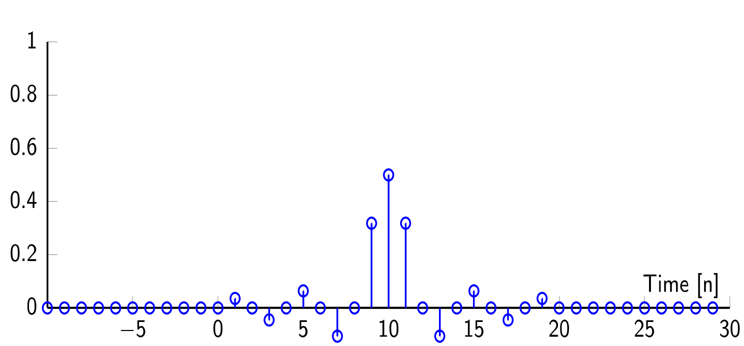


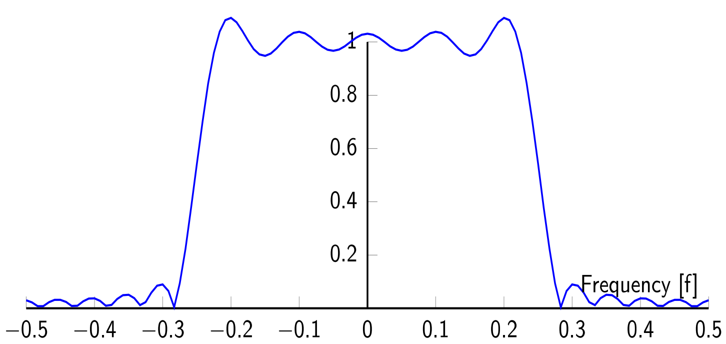
Truncation: multiply with this





Truncated filter

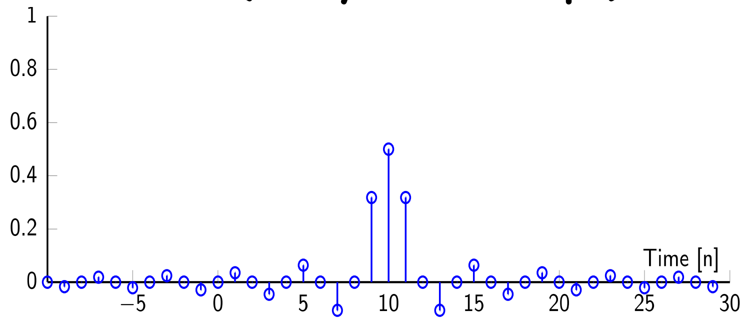


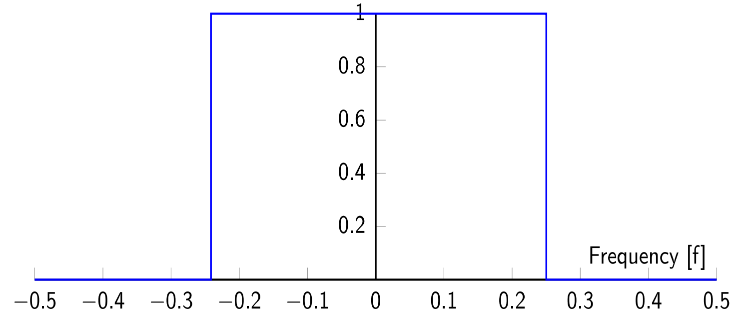


Quite bad. Lots of ripple
in both pass band and stop band

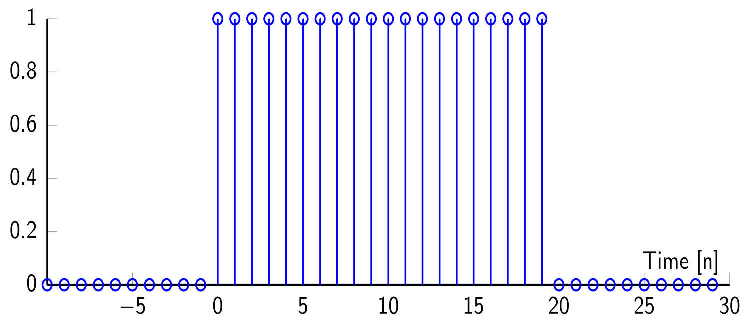
N=21

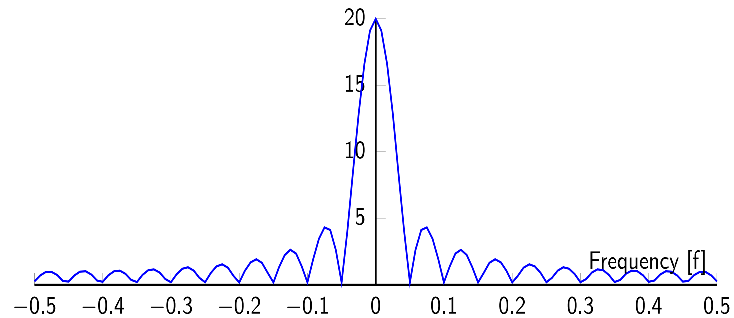
Ideal filter (delayed 10 steps)



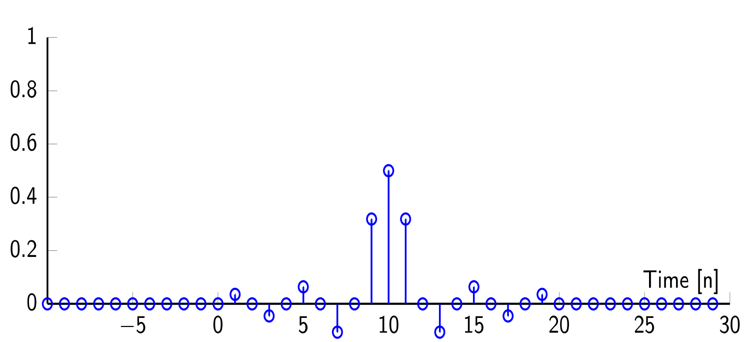


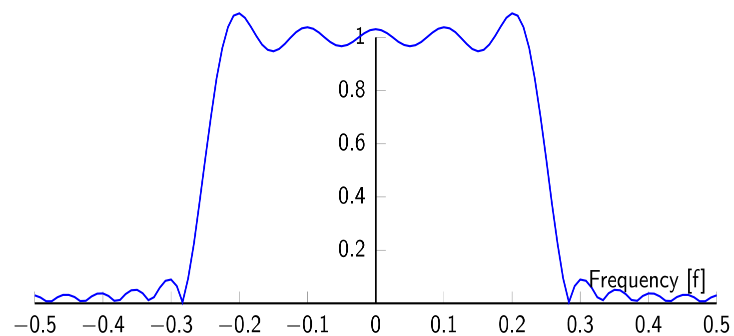
Truncation: multiply with this





Truncated filter

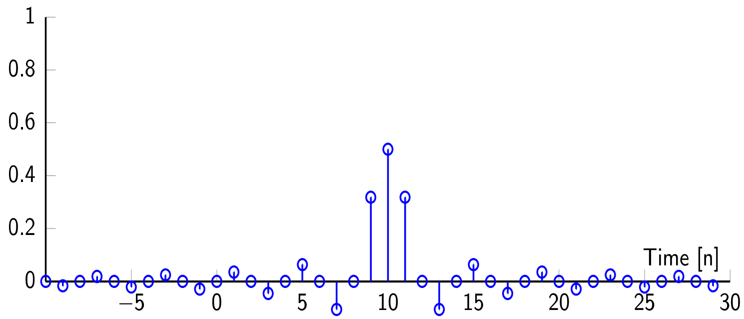


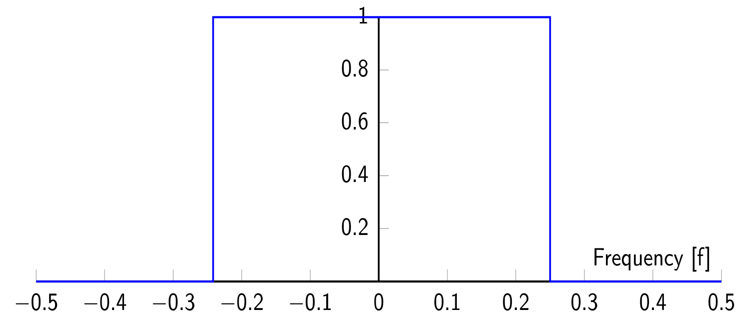


CAN WE DO BETTER

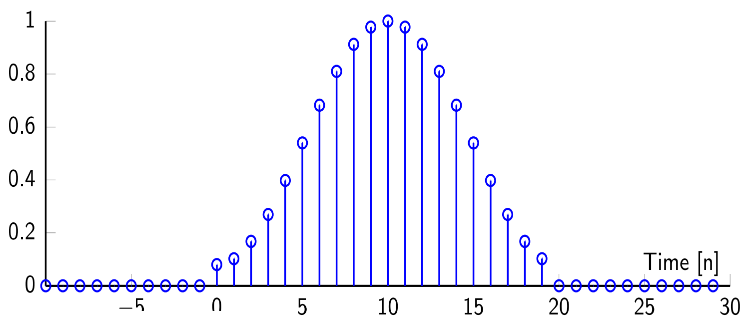
N=21

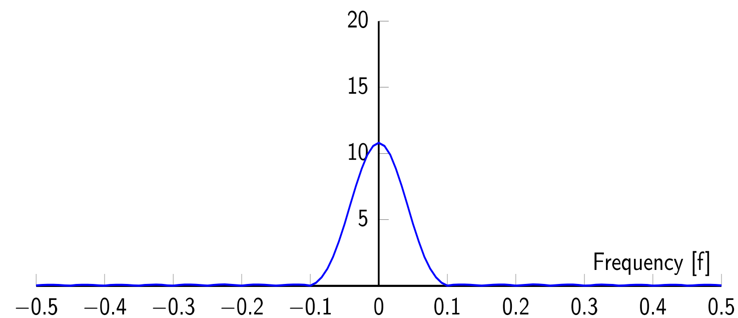
Ideal filter (delayed 10 steps)



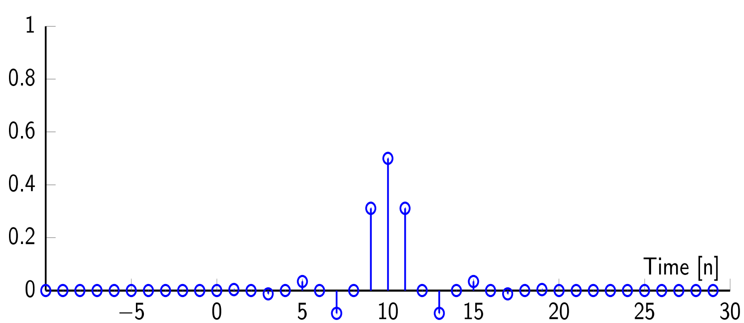


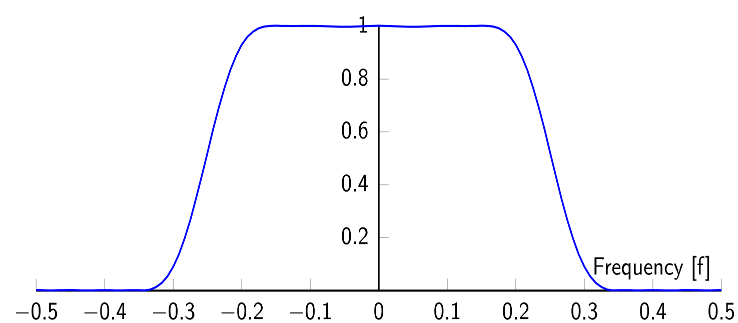
MAKE A SMOOTH TRUNCATION: Hamming window





Truncated filter

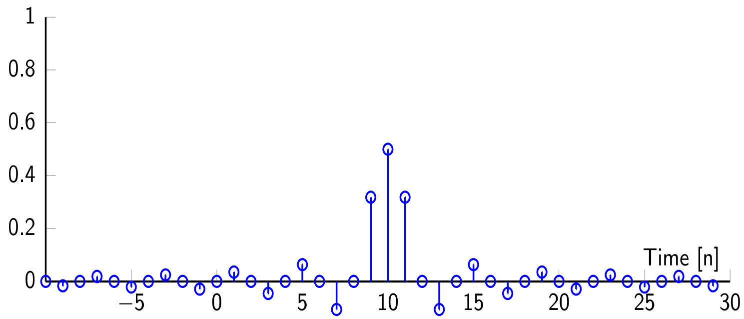


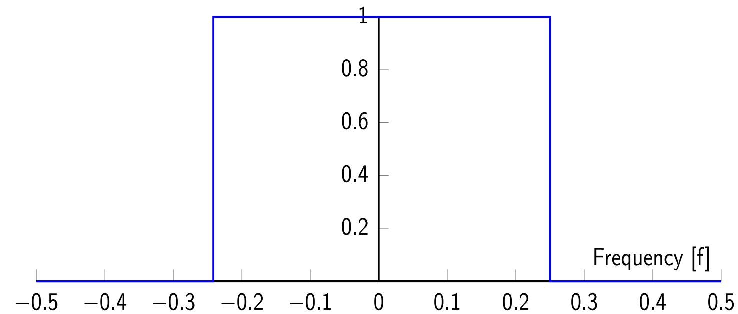


Yes

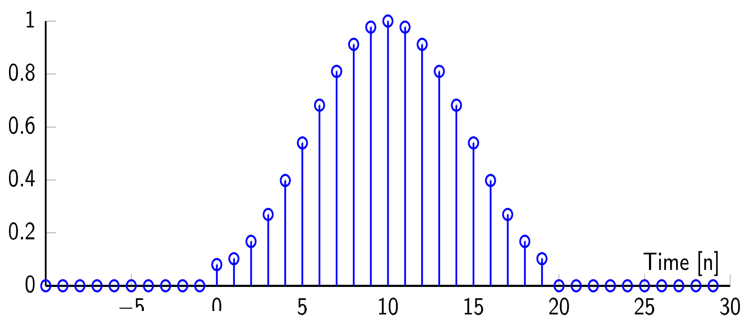
N=21

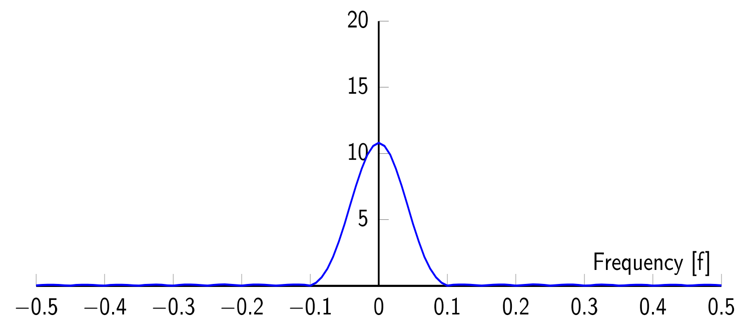
Ideal filter (delayed 10 steps)



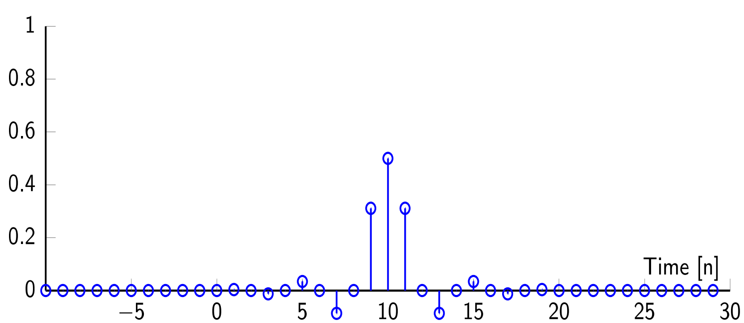


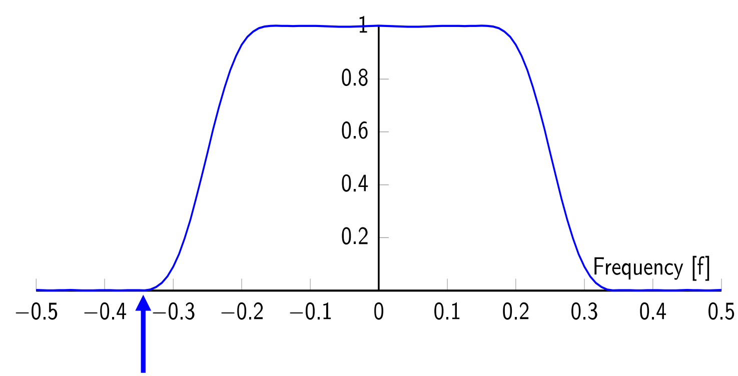
MAKE A SMOOTH TRUNCATION: Hamming window





Truncated filter

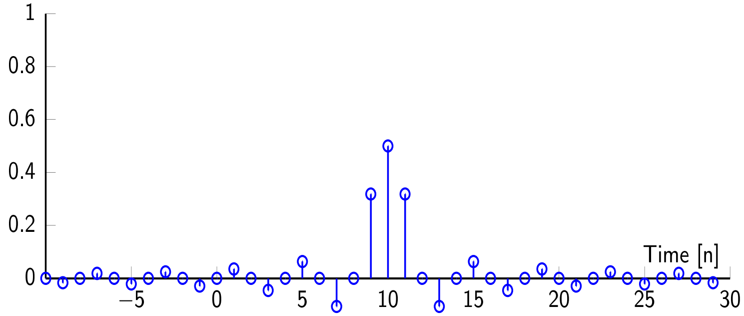


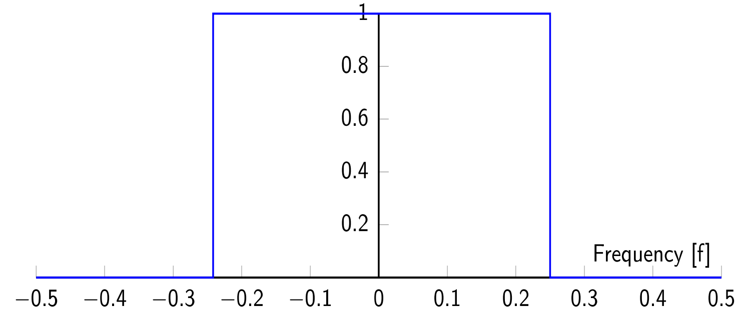


Homework 1: Make sure you understand why this is at f ≈ -0.35

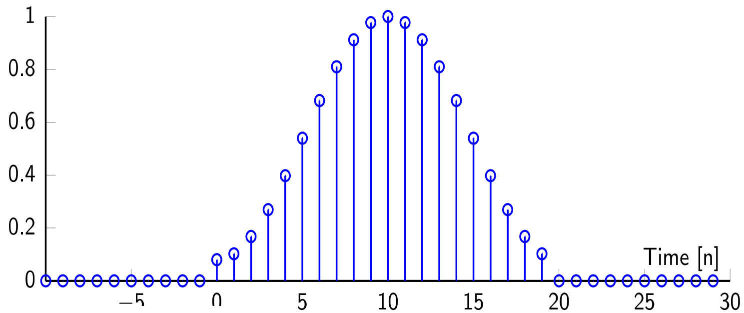
N=21

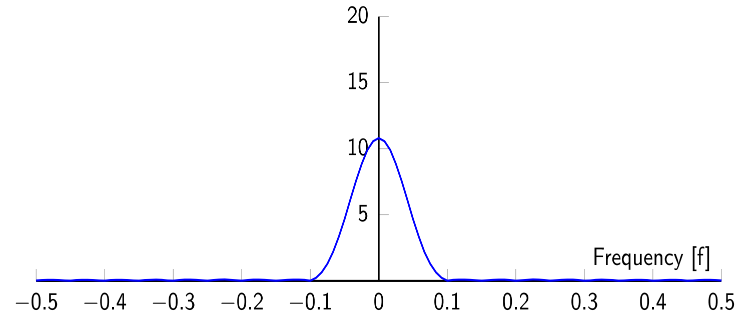
Ideal filter (delayed 10 steps)



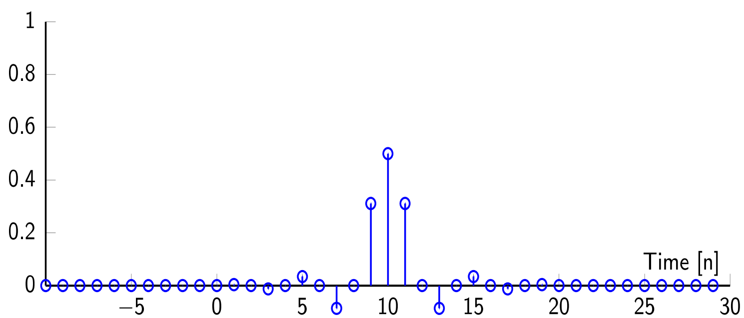


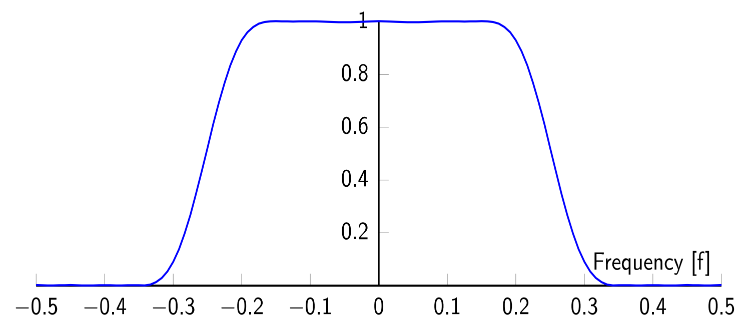
MAKE A SMOOTH TRUNCATION: Hamming window





Truncated filter





Homework 2: Make a pole-zero plot of filter for the rectangular case (Matlab's roots function)

EITF75, Fourier transforms

Interlude.

We know this

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

EITF75, Fourier transforms

Interlude.

We know this

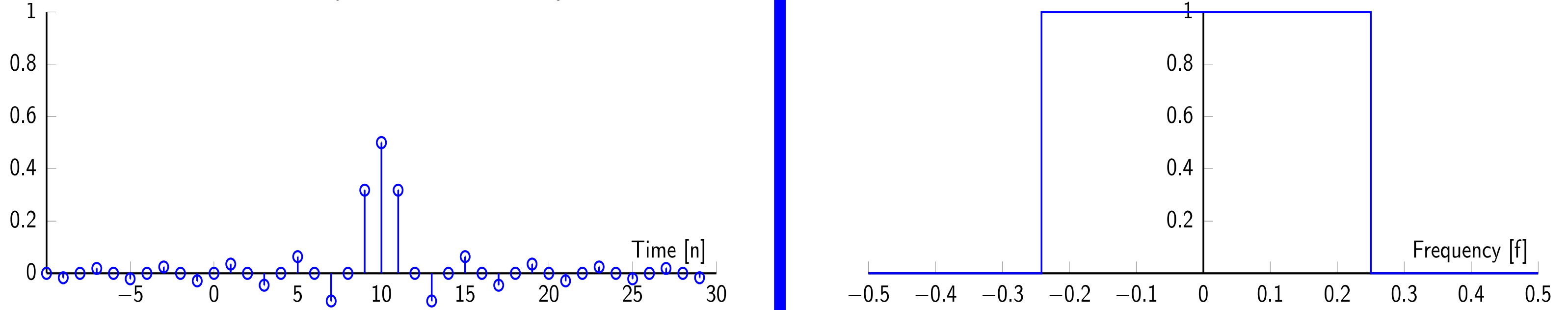
$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is the following true?

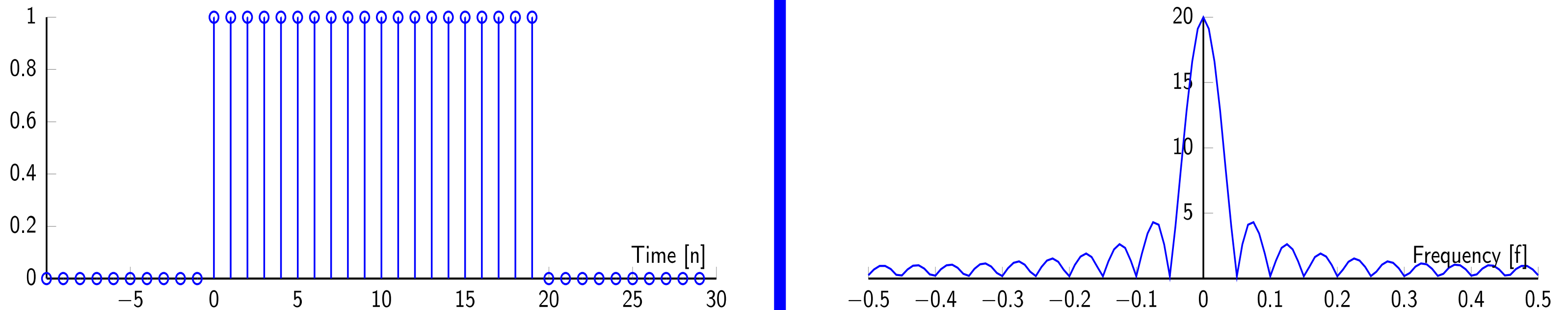
$$y(n) = x(n) \cdot h(n) \leftrightarrow Y(f) = X(f) \star H(f)$$

N=21

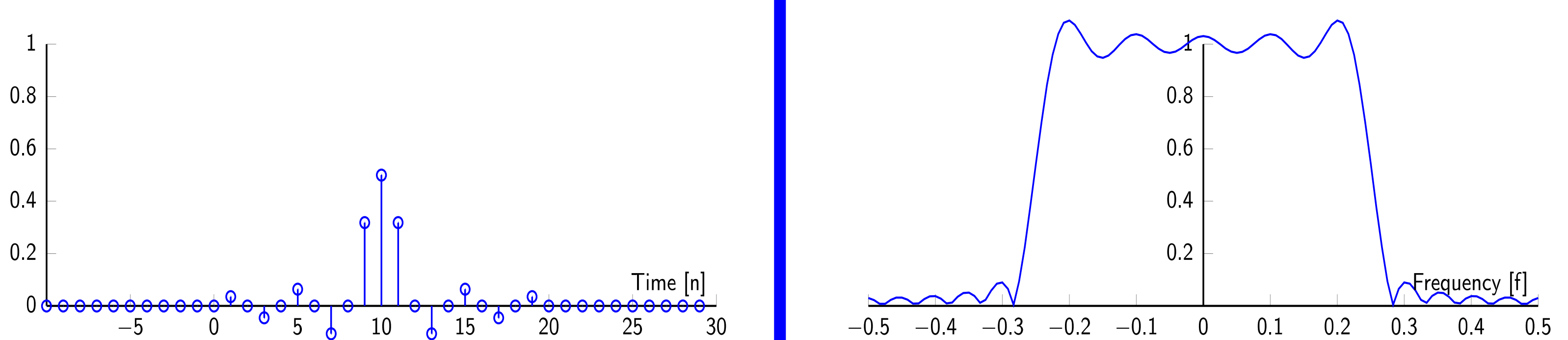
Ideal filter (delayed 10 steps)



Truncation: multiply with this



Truncated filter



In other words: Is the bottom Fourier transform the convolution of the two above ?

EITF75, Fourier transforms

WIKIPEDIA
The Free Encyclopedia

Discrete-time Fourier transform

From Wikipedia, the free encyclopedia

Properties of discrete-time Fourier transforms [edit]

Convolution in time / Multiplication in frequency	$x[n] * y[n]$	$X_{2\pi}(\omega) \cdot Y_{2\pi}(\omega)$	
Multiplication in time / Convolution in frequency	$x[n] \cdot y[n]$	$\frac{1}{2\pi} \int_{-\pi}^{\pi} X_{2\pi}(\nu) \cdot Y_{2\pi}(\omega - \nu) d\nu$	Periodic convolution

A convolution lengthens the signals

Y(f) defined in -0.5 to 0.5

One must do periodic convolution (wrap the end to the begining)

EITF75, DTFT

Agenda:

1. Relation between DTFT and z-transform
2. Some properties of DTFTs
3. Relation between pole-zero diagrams and DTFT
4. Filter characteristics
5. DTFTs of unstable signals

EITF75, DTFT

We know

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is this always true?

EITF75, DTFT

We know

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is this always true? It cannot be true if X(f) does not exist

EITF75, DTFT

We know

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is this always true? It cannot be true if X(f) does not exist

X(f) does not exist if x(n) is unstable (not absolutely summable)

EITF75, DTFT

We know

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is this always true? It cannot be true if X(f) does not exist

X(f) does not exist if x(n) is unstable (not absolutely summable)

We cannot claim the above for those signals

EITF75, DTFT

We know

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is this always true? It cannot be true if X(f) does not exist

X(f) does not exist if x(n) is unstable (not absolutely summable)

We cannot claim the above for those signals

Ok, so X(f) does not exist. Does that mean that x(n) does not exist ?

EITF75, DTFT

We know

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

But is this always true? It cannot be true if X(f) does not exist

X(f) does not exist if x(n) is unstable (not absolutely summable)

We cannot claim the above for those signals

Ok, so X(f) does not exist. Does that mean that y(n) does not exist ?

No, it exists, in general.

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

Let us solve

$$y_0(n) = x_0(n) * h(n)$$

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

Let us solve

$$y_0(n) = x_0(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k) x_0(n-k)$$

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

Let us solve

$$y_0(n) = x_0(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k) x_0(n-k)$$

$$= \sum_k h(k) \exp(i2\pi f_0 (n-k))$$

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

Let us solve

$$y_0(n) = x_0(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k) x_0(n-k)$$

$$= \sum_k h(k) \exp(i2\pi f_0 (n-k))$$

$$= \sum_k h(k) \exp(-i2\pi f_0 k) \exp(i2\pi f_0 n)$$

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

Let us solve

$$y_0(n) = x_0(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k) x_0(n-k)$$

$$= \sum_k h(k) \exp(i2\pi f_0 (n-k))$$

$$= \boxed{\sum_k h(k) \exp(-i2\pi f_0 k)} \exp(i2\pi f_0 n)$$

Definition of DTFT

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

Let us solve

$$y_0(n) = x_0(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k) x_0(n-k)$$

$$= \sum_k h(k) \exp(i2\pi f_0 (n-k))$$

$$= \sum_k h(k) \exp(-i2\pi f_0 k) \exp(i2\pi f_0 n)$$

$$= H(f_0) \exp(i2\pi f_0 n)$$

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

So, y(n) very much exists

$$y_0(n) = H(f_0)\exp(i2\pi f_0 n)$$

Remember the matrix-view of LTI systems from Lecture 2 ?
From the above, something very interesting can be said about that matrix' eigenvectors and eigenvalues !!

(Homework to think about this) (This is called Szegö's theorem)

EITF75, DTFT

Assume $x_0(n) = \exp(i2\pi f_0 n)$ (unstable, non-causal, $X_0(f)$ does not exist)

So, y(n) very much exists

$$y_0(n) = H(f_0)\exp(i2\pi f_0 n)$$

Cannot be normal function

Would be nice to extend the Fourier transforms so that we have

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

EITF75, DTFT

$$x_0(n) = \exp(i2\pi f_0 n)$$

$$y_0(n) = H(f_0)\exp(i2\pi f_0 n)$$

Educated guess: $X(f) = \delta(f - f_0)$ Dirac (distribution)

EITF75, DTFT

$$x_0(n) = \exp(i2\pi f_0 n)$$

$$y_0(n) = H(f_0)\exp(i2\pi f_0 n)$$

Educated guess: $X(f) = \delta(f - f_0)$ Dirac (distribution)

How to test? Compute inverse Fourier transform

$$y_0(n) \stackrel{?}{=} \int_{-0.5}^{0.5} H(f)X(f)\exp(i2\pi f n)\mathrm{d}f$$

EITF75, DTFT

$$x_0(n) = \exp(i2\pi f_0 n)$$

$$y_0(n) = H(f_0)\exp(i2\pi f_0 n)$$

Educated guess: $X(f) = \delta(f - f_0)$ Dirac (distribution)

How to test? Compute inverse Fourier transform

$$\begin{aligned} y_0(n) &\stackrel{?}{=} \int_{-0.5}^{0.5} H(f)X(f)\exp(i2\pi f n)\mathrm{d}f \\ &= \int_{-0.5}^{0.5} H(f)\delta(f - f_0)\exp(i2\pi f n)\mathrm{d}f \end{aligned}$$

EITF75, DTFT

$$x_0(n) = \exp(i2\pi f_0 n)$$

$$y_0(n) = H(f_0)\exp(i2\pi f_0 n)$$

Educated guess: $X(f) = \delta(f - f_0)$ Dirac (distribution)

How to test? Compute inverse Fourier transform

$$\begin{aligned} y_0(n) &\overset{!}{=} \int_{-0.5}^{0.5} H(f)X(f)\exp(i2\pi f n)\mathrm{d}f \\ &= \int_{-0.5}^{0.5} H(f)\delta(f - f_0)\exp(i2\pi f n)\mathrm{d}f \\ &= H(f_0)\exp(i2\pi f_0 n) \end{aligned}$$

Sifting property of Dirac

Correct!

Summary:

By allowing for diracs in Fourier transforms (NOT LONGER NORMAL FUNCTIONS) we have furnished for the valididty of

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

Summary:

By allowing for diracs in Fourier transforms (NOT LONGER NORMAL FUNCTIONS) we have furnished for the valididty of

$$y(n) = x(n) \star h(n) \leftrightarrow Y(f) = H(f)X(f)$$

By using Euler:

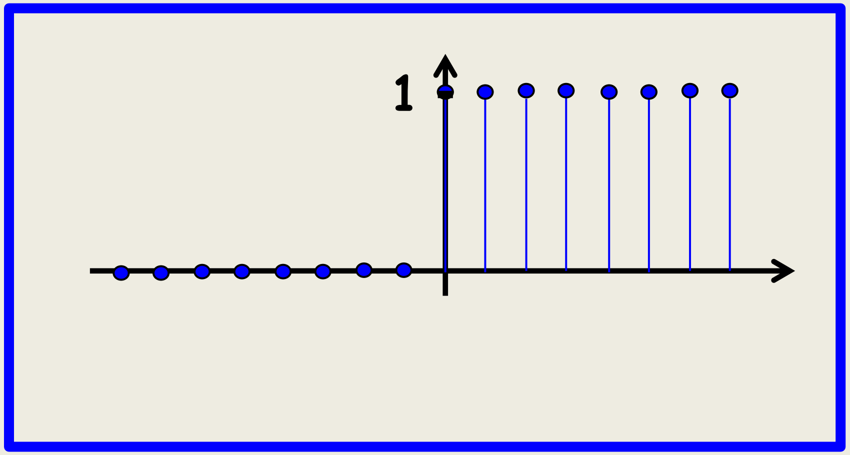
$$x(n) = \cos(2\pi f_0 n) = \frac{1}{2}\left(\mathrm{e}^{i2\pi f n} + \mathrm{e}^{-i2\pi f n}\right) \qquad X(f) = \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

$$x(n) = \sin(2\pi f_0 n) = \frac{1}{2i}\left(\mathrm{e}^{i2\pi f n} - \mathrm{e}^{-i2\pi f n}\right) \qquad X(f) = \frac{1}{2i}\left[\delta(f - f_0) - \delta(f + f_0)\right]$$

Infinite power at one frequency (and its negative counterpart)

EITF75, DTFT

Step function u(n)

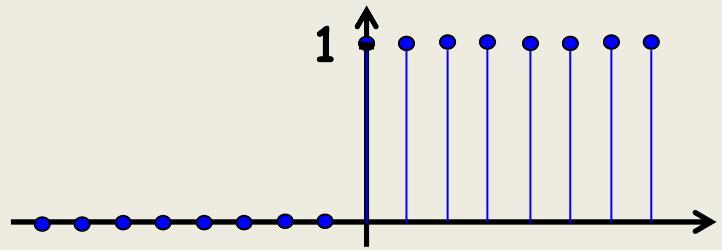


EITF75, DTFT

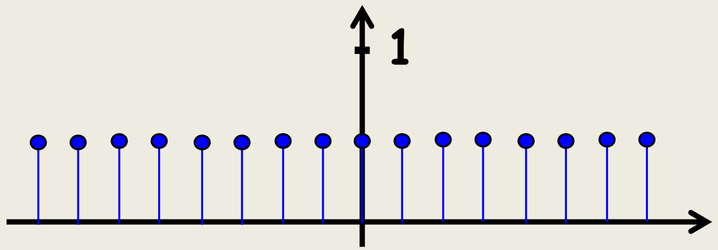
Step function u(n)

Trick !

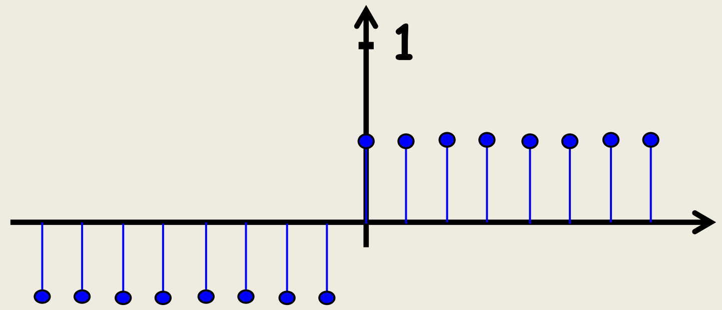
$$u(n) = g(n) + f(n)$$



$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



EITF75, DTFT

Step function u(n)

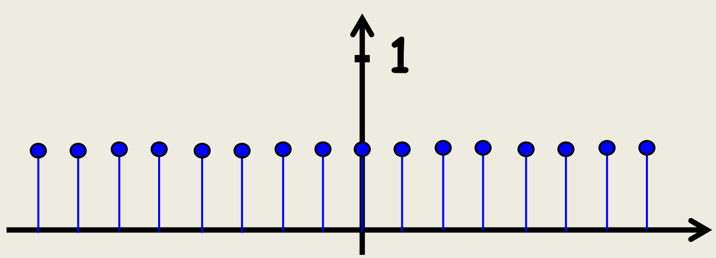
Trick !

$$u(n) = g(n) + f(n)$$

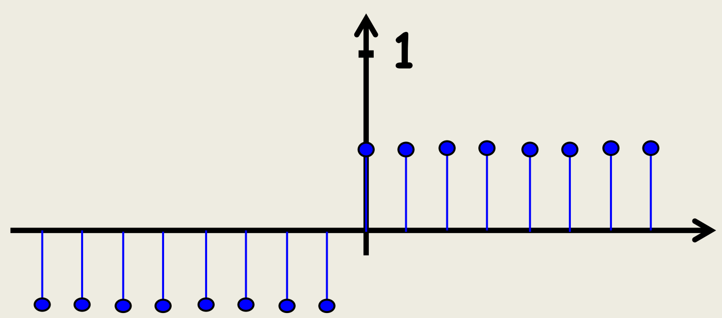
Another Trick !

$$g(n) - g(n-1) = \delta(n)$$

$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



EITF75, DTFT

Step function u(n)

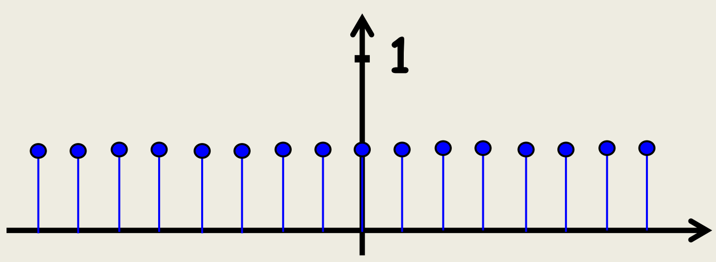
Trick !

$$u(n) = g(n) + f(n)$$

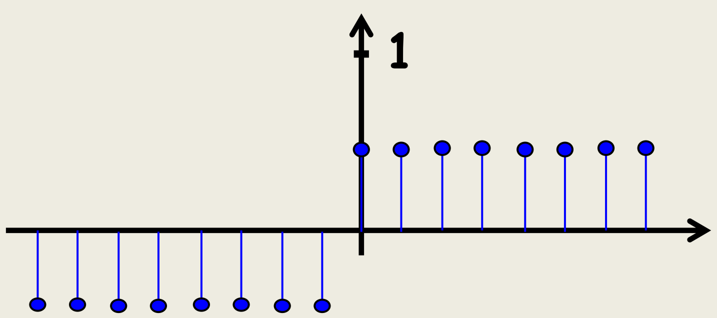
From definition of DTFT $\delta(n) \leftrightarrow 1$

$$g(n) - g(n-1) = \delta(n)$$

$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



EITF75, DTFT

Step function u(n)

Trick !

$$u(n) = g(n) + f(n)$$

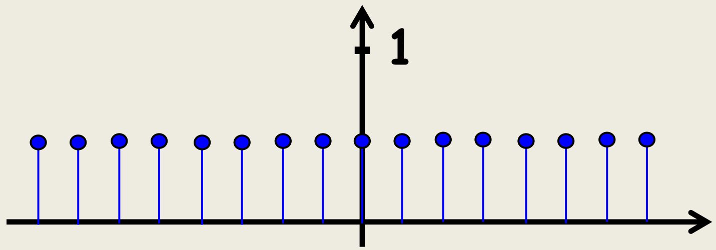
Time-shift property

$$G(\omega) - \exp(-i\omega)G(\omega) = 1$$

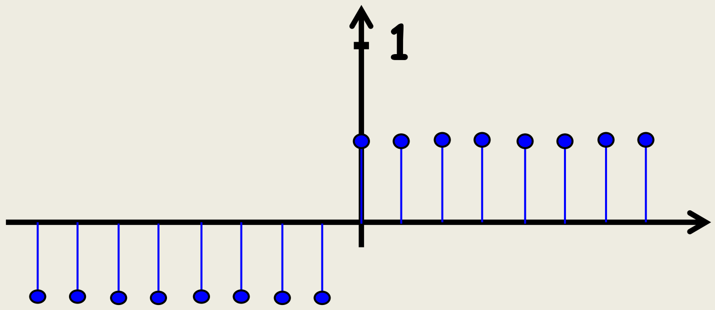
From definition of DTFT $\delta(n) \leftrightarrow 1$

$$g(n) - g(n-1) = \delta(n)$$

$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



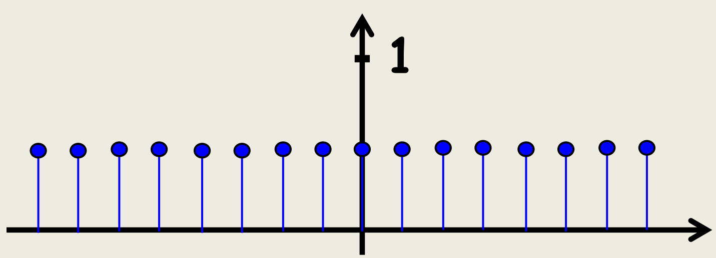
EITF75, DTFT

Step function u(n)

Trick !

$$u(n) = g(n) + f(n)$$

$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



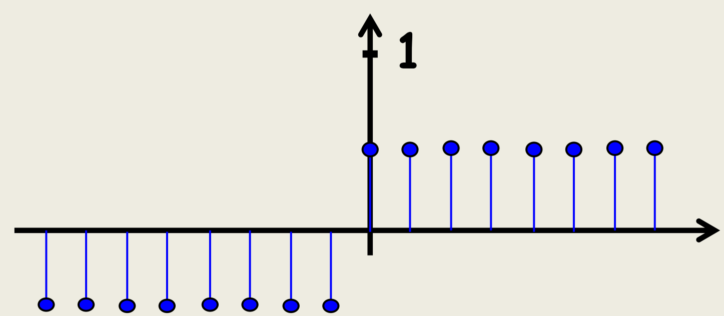
Time-shift property

$$G(\omega) - \exp(-i\omega)G(\omega) = 1$$

Conclusion

$$G(\omega) = \frac{1}{1 - \exp(-i\omega)}$$

$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



EITF75, DTFT

Step function u(n)

Trick !

$$u(n) = g(n) + f(n)$$

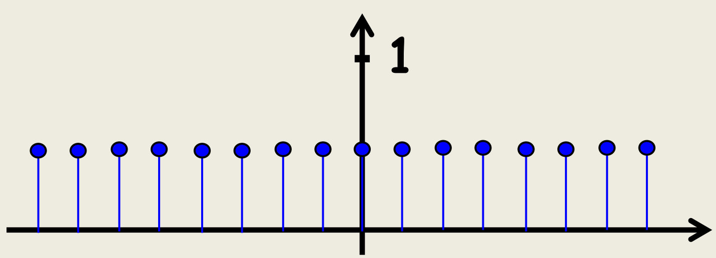
Time-shift property

$$G(\omega) - \exp(-i\omega)G(\omega) = 1$$

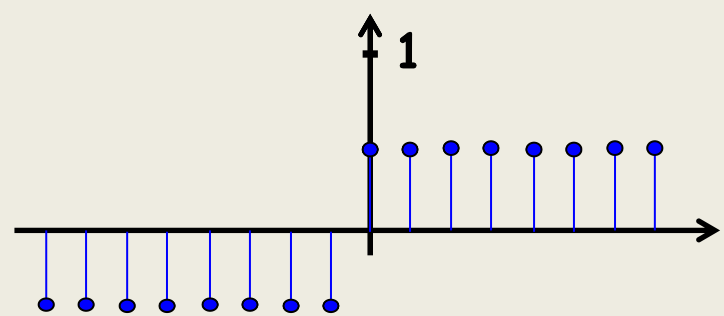
Conclusion

$$G(\omega) = \frac{1}{1 - \exp(-i\omega)}$$

$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



RECALL $\quad x_0(n) = \mathrm{e}^{\mathrm{j}\omega_0 n} \qquad X(\omega) = \delta(\omega - \omega_0)$

EITF75, DTFT

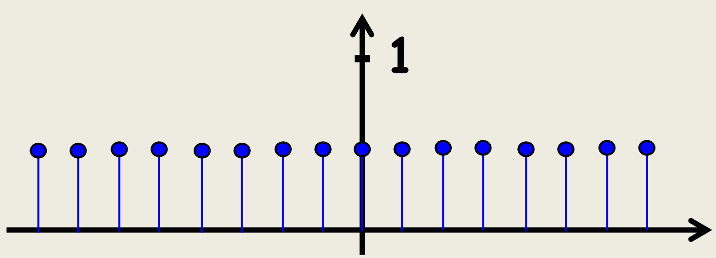
Step function u(n)

Trick !

$$u(n) = g(n) + f(n)$$

Set w_0**=0** $f(n) \leftrightarrow \frac{1}{2}\delta(\omega)$

$$f(n) = \frac{1}{2}, \quad -\infty < n < \infty$$



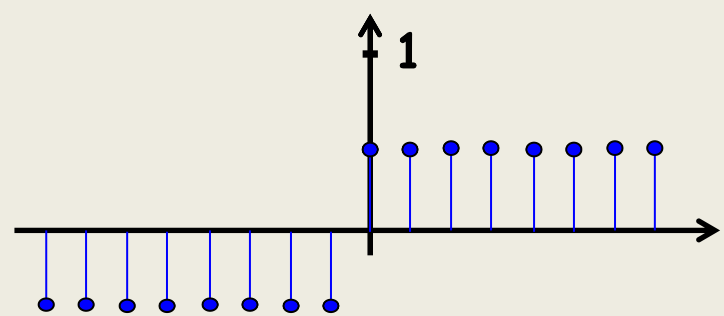
Time-shift property

$$G(\omega) - \exp(-i\omega)G(\omega) = 1$$

Conclusion

$$G(\omega) = \frac{1}{1 - \exp(-i\omega)}$$

$$g(n) = \begin{cases} 0.5, & n \geq 0 \\ -0.5 & n < 0 \end{cases}$$



RECALL $x_0(n) = \mathrm{e}^{\mathrm{j}\omega_0 n}$ $X(\omega) = \delta(\omega - \omega_0)$

EITF75, DTFT

Step function u(n)

Trick !

$$u(n) = g(n) + f(n)$$

Set w_0**=0** $f(n) \leftrightarrow \frac{1}{2}\delta(\omega)$

Conclusion

$$G(\omega) = \frac{1}{1-\exp(-i\omega)}$$

Summary (a piece of art):

$$u(n) \leftrightarrow \frac{1}{1-\exp(-i\omega)} + \frac{1}{2}\delta(\omega)$$