EITF75 Systems and Signals

Lecture 5
The discrete-time Fourier transform

Fredrik Rusek

EITF75, Fourier transforms

What is a transform?

An alternative description of an object

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What is a transform?

An alternative description of an object

Example: Difference between (dynamical) pressure in the front of an airplane, and (static) pressure below it. This is measured in Pascal

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It is customary to transform this number into another number, which we call velocity, and present the number in m/s

It is a lossless transformation since we can "go back" and get the pressure difference from the velocity

Here, the presentation assumes that a transformation is always lossless, and if it is not possible to "go back", it is not a transformation

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What is a transform?

An alternative description of an object

In formula: Assume a function $f(x)$ or a vector x

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$$f(x) \longrightarrow \boxed{J[f]} \longrightarrow g(y)$$

J[f] is a functional: maps a function to another function

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$J[f]$ is a **functional**: maps a function to another function

$J[f]$ is a transformation if $g(y)$ contains all information about $f(x)$. It does so if we can "go back"

$$g(y) \longrightarrow \boxed{J^{-1}[g]} \longrightarrow f(x)$$

What is a transform?

An alternative description of an object

In formula: Assume a function f(x) or a vector x

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$$g(y) \longrightarrow \boxed{J^{-1}[g]} \longrightarrow f(x)$$

J[f] is a transformation if and only if $J^{-1}[g]$ exists

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Today's agenda

1. Study the Fourier transformation
   - We do it carefully, and start with continuous signals (which you already studied)
   - We identify that there are **four** different types of signals (continuous and discrete)
   - We establish Parseval's identity
   - We start from scratch, and deal with a lot of equations…
2. Calculate a few examples

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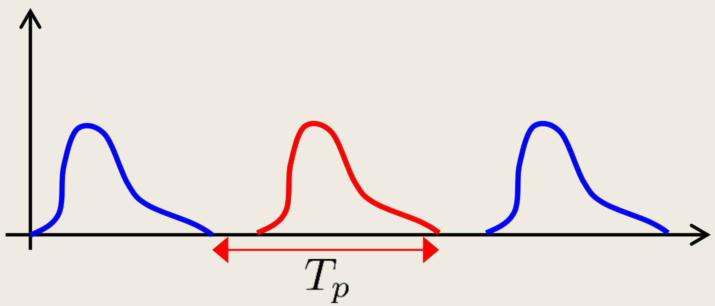
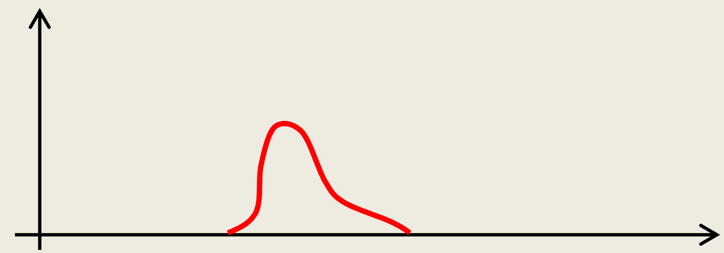
4 different type of signals

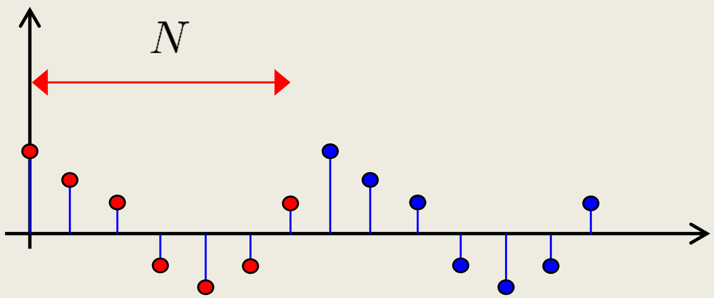
Continuous and periodic



Continuous and aperiodic



Discrete and periodic



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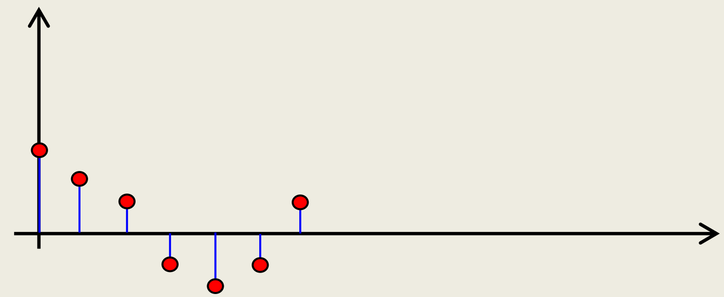

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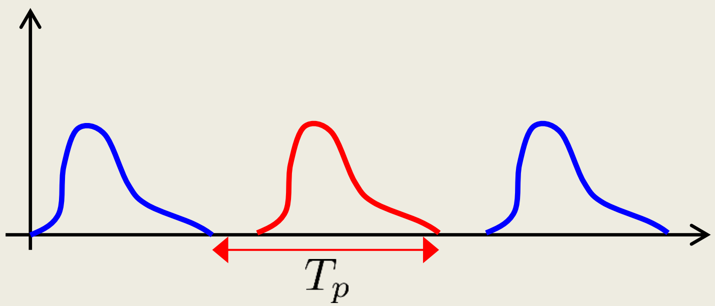

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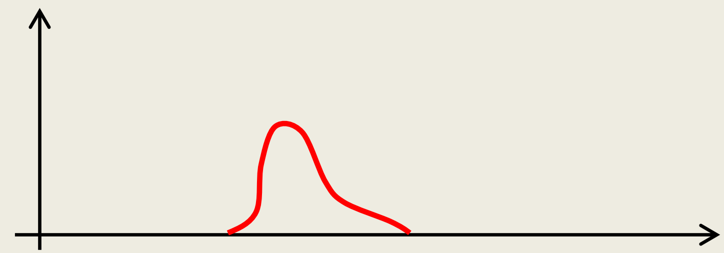

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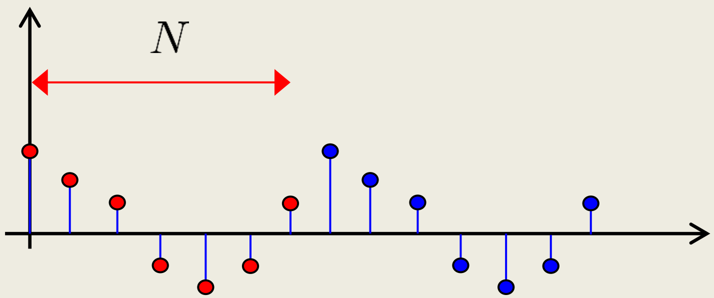
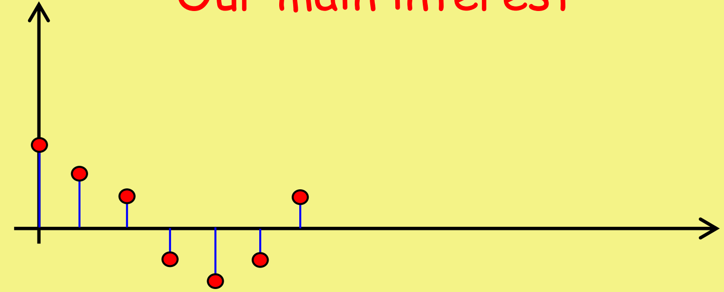


Discrete and aperiodic

Our main interest



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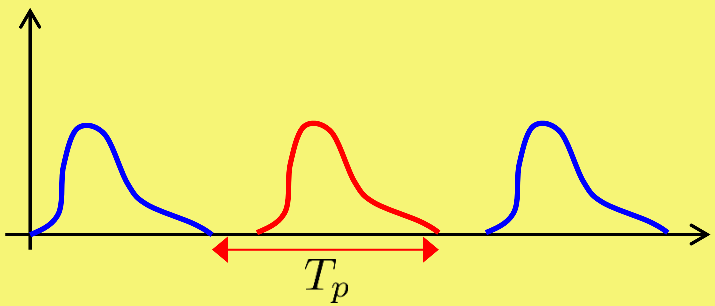
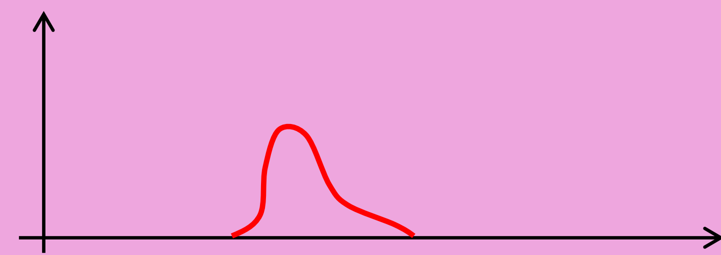
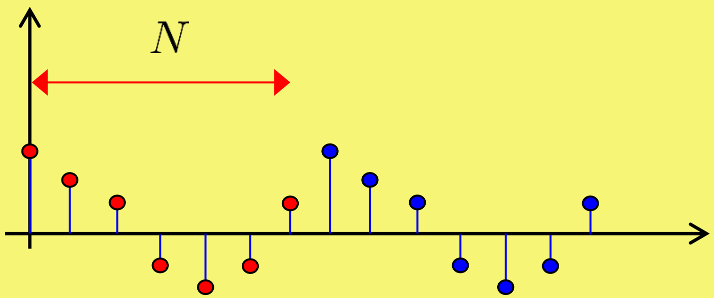


Continuous and aperiodic



Discrete and periodic



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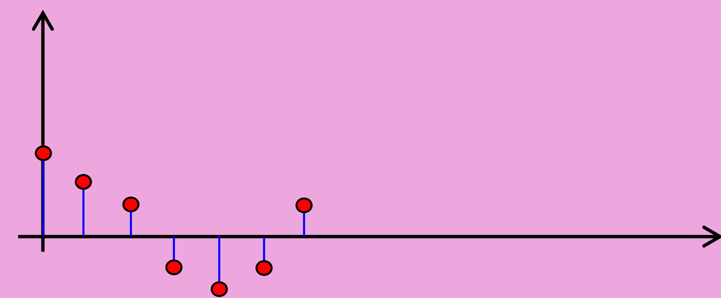

Some common properties

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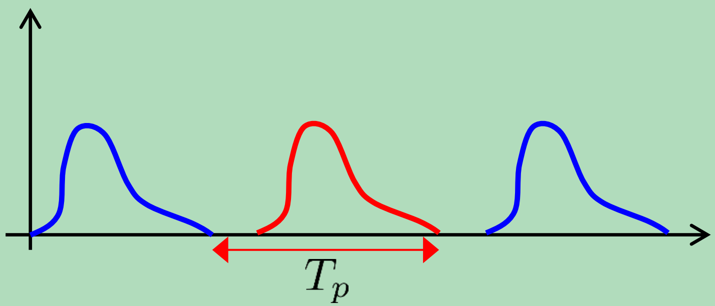
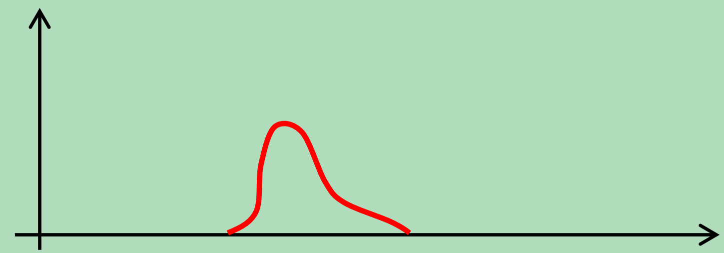
Some common properties

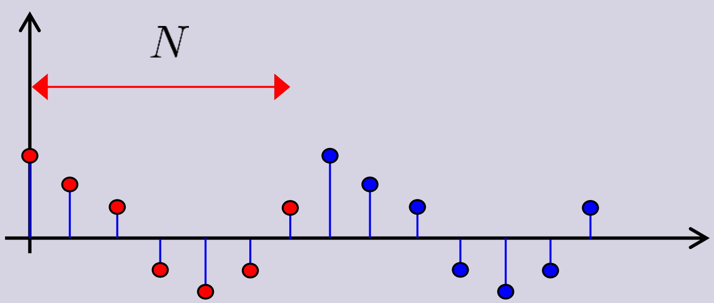
Continuous and periodic



Continuous and aperiodic



Discrete and periodic



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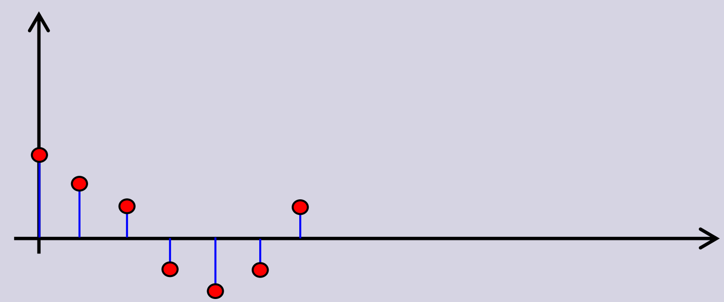

Some common properties

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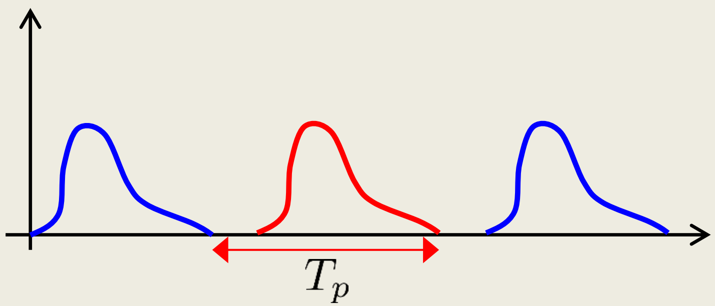
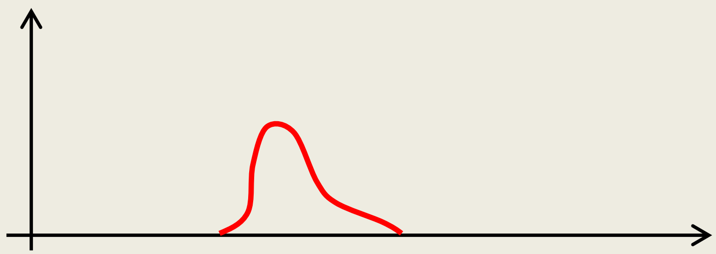
How to get aperiodic transforms:

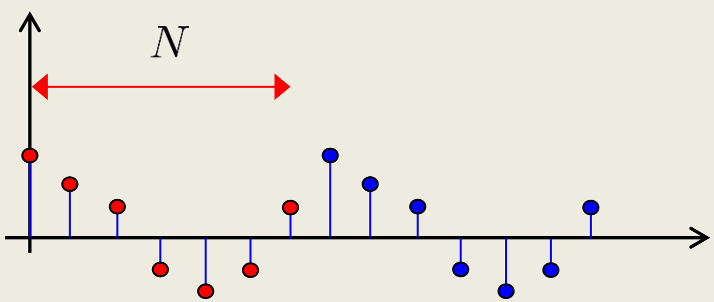
Continuous and periodic



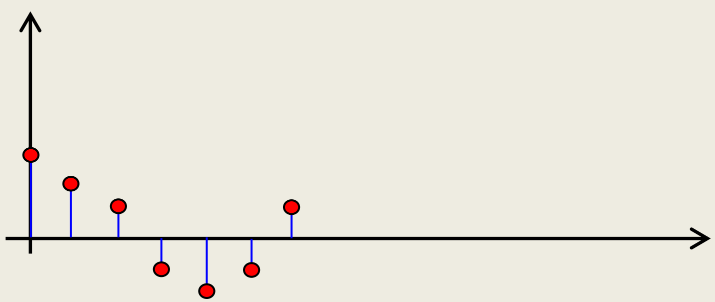
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic



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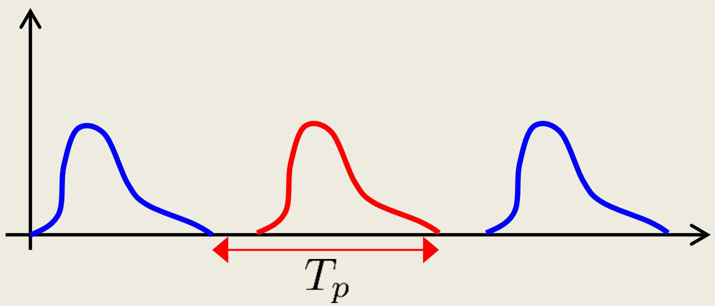
How to get aperiodic transforms:

Continuous and periodic



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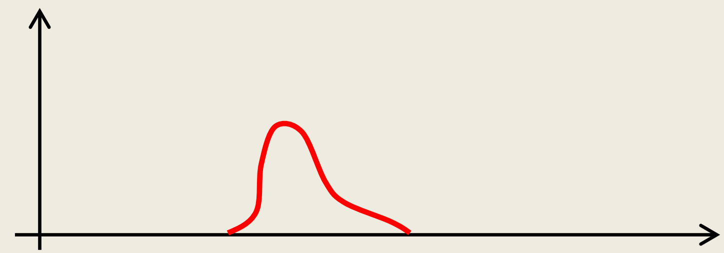
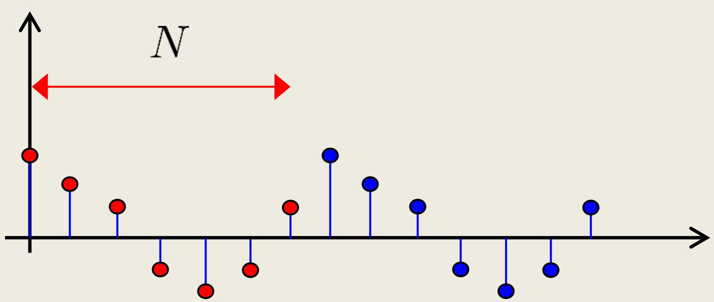
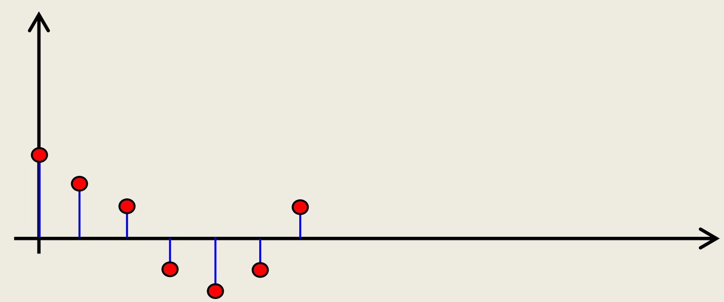


1. Obtain periodic transforms

Discrete and periodic



Discrete and aperiodic



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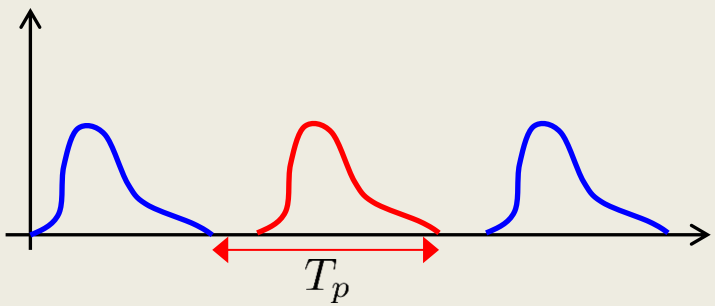
How to get aperiodic transforms:

Continuous and periodic



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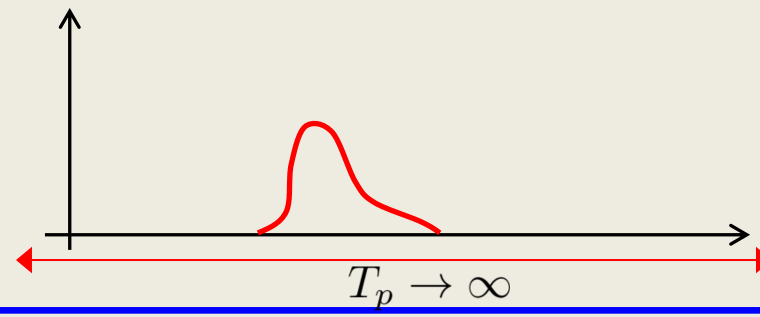
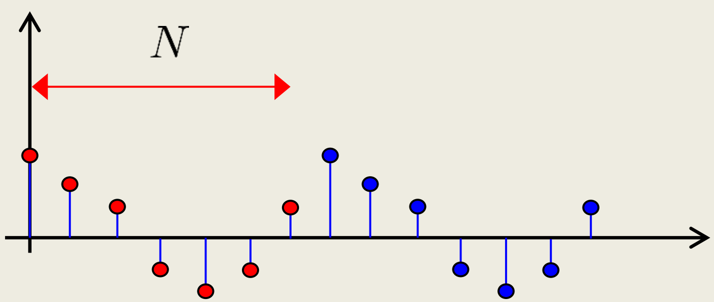


1. Obtain periodic transforms

2. Send period to infinity

Discrete and periodic



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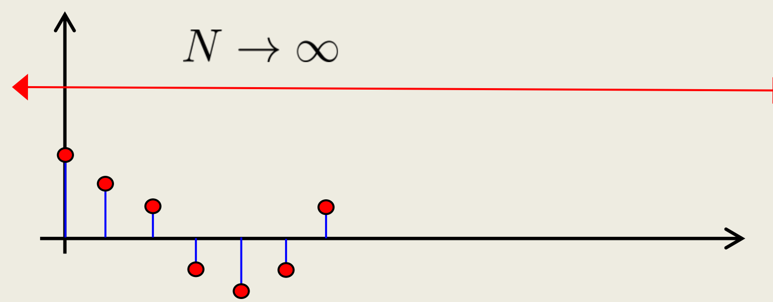

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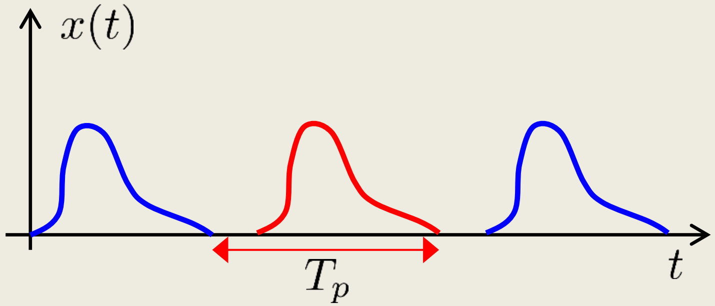
Continuous and periodic



Let us start here

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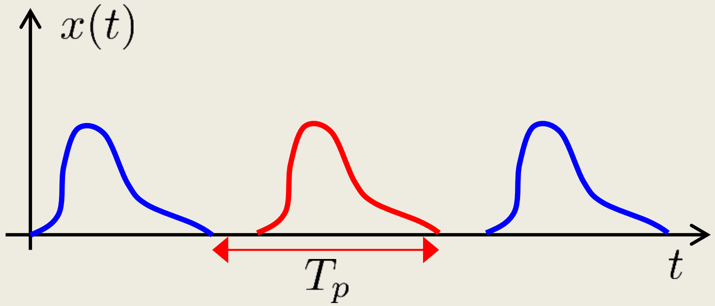
4 different type of signals

Continuous and periodic



For no particular reason, let us calculate

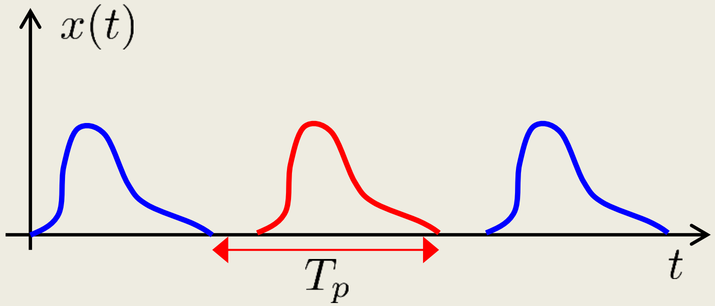
$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

$$F_0 = \frac{1}{T_p}$$

EITF75, Fourier transforms

4 different type of signals

Continuous and periodic



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We now claim that $\{c_k\}_{k=-\infty}^{\infty}$ contains all information about $x(t)$

EITF75, Fourier transforms

4 different type of signals

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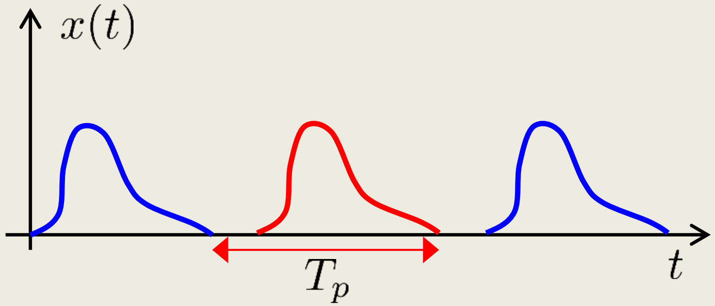

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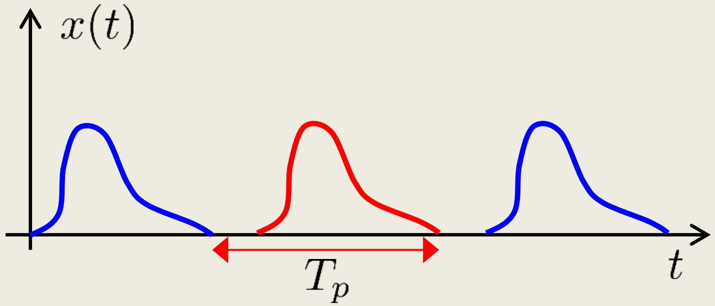

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That if we know $\{c_k\}_{k=-\infty}^{\infty}$ then we can get back $x(t)$ perfectly

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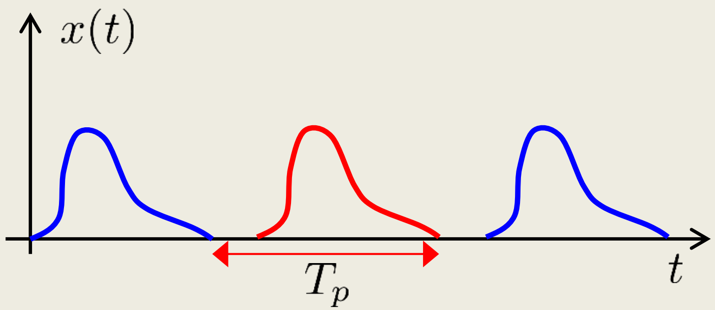

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But what does it mean "that it contains all information" ?

That if we know $\{c_k\}_{k=-\infty}^{\infty}$ then we can get back $x(t)$ perfectly

Remark: A computer would prefer to store $\{c_k\}_{k=-\infty}^{\infty}$ since it is discrete (i.e., can be stored in a normal memory)

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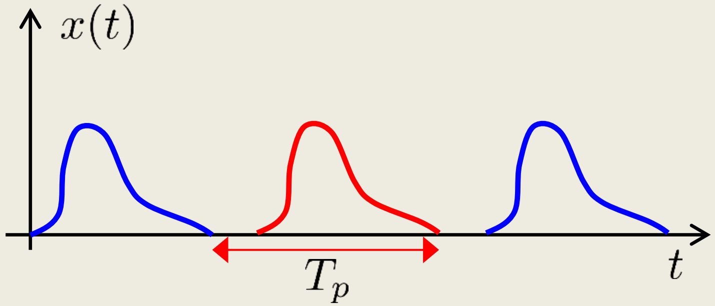
4 different type of signals

Continuous and periodic



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Compute

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

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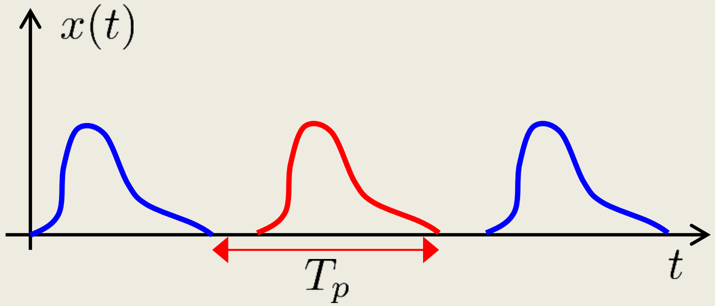
4 different type of signals

Continuous and periodic



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$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

If $\hat{x}(t) = x(t)$, $\{c_k\}_{k=-\infty}^{\infty}$ is an alternative representation of $x(t)$

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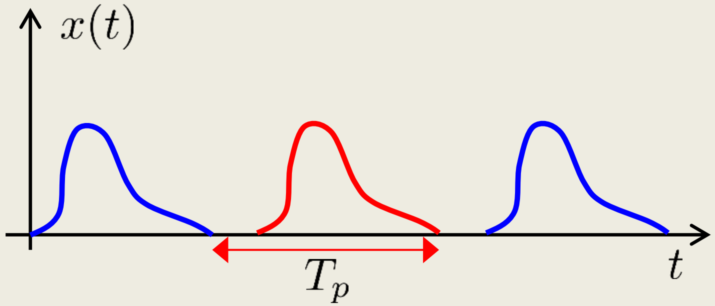
Continuous and periodic



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We now claim that $\{c_k\}_{k=-\infty}^{\infty}$ contains all information about $x(t)$

To be calculated next

$$\boxed{\int_0^{T_p} |\hat{x}(t) - x(t)|^2 \mathrm{d}t}$$

If $\hat{x}(t) = x(t)$, $\{c_k\}_{k=-\infty}^{\infty}$ is an alternative representation of $x(t)$

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

$$F_0 = \frac{1}{T_p}$$

$$\int_0^{T_p} |\hat{x}(t) - x(t)|^2 \mathrm{d}t$$

EITF75, Fourier transforms

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$$F_0 = \frac{1}{T_p}$$

$$\int_0^{T_p} |\hat{x}(t) - x(t)|^2 \mathrm{d}t = \int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) - x(t) \right|^2 \mathrm{d}t$$

EITF75, Fourier transforms

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$$= \int_0^{T_p}\left|\sum_k c_k\exp(i2\pi kF_0t)\right|^2 \mathrm{d}t + \int_0^{T_p}|x(t)|^2\,\mathrm{d}t$$

$$- \int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^*\exp(-i2\pi kF_0t)x(t)\right\}\mathrm{d}t$$

$|a-b|^2 = |a|^2+|b|^2-2\mathrm{Real}(a^*b)$

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

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$$\boxed{- \int_0^{T_p} 2\mathcal{R}\left\{ \sum_k c_k^* \exp(-i2\pi k F_0 t) x(t) \right\} \mathrm{d}t}$$

Let us simplify this one

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

$$F_0 = \frac{1}{T_p}$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi k F_0 t)x(t)\right\}\mathrm{d}t = -2\mathcal{R}\left\{\sum_k c_k^* \int_0^{T_p} \exp(-i2\pi k F_0 t)x(t)\mathrm{d}t\right\}$$

Change order of sum and integration

$$\boxed{-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi k F_0 t)x(t)\right\}\mathrm{d}t}$$

Let us simplify this one

EITF75, Fourier transforms

$$\boxed{c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi kF_0 t)\mathrm{d}t}$$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi kF_0 t)$$

$$F_0 = \frac{1}{T_p}$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi kF_0 t)x(t)\right\}\mathrm{d}t = -2\mathcal{R}\left\{\sum_k c_k^* \boxed{\int_0^{T_p} \exp(-i2\pi kF_0 t)x(t)\mathrm{d}t}\right\}$$

$$\boxed{-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi kF_0 t)x(t)\right\}\mathrm{d}t}$$

Let us simplify this one

EITF75, Fourier transforms

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$$= -2T_p\mathcal{R}\left\{\sum_k c_k^* c_k\right\}$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi kF_0 t)x(t)\right\}\mathrm{d}t$$

Let us simplify this one

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

$$F_0 = \frac{1}{T_p}$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi k F_0 t)x(t)\right\}\mathrm{d}t = -2\mathcal{R}\left\{\sum_k c_k^* \int_0^{T_p} \exp(-i2\pi k F_0 t)x(t)\mathrm{d}t\right\}$$

$$= -2T_p\mathcal{R}\left\{\sum_k c_k^* c_k\right\} = -2T_p\mathcal{R}\left\{\sum_k |c_k|^2\right\}$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi k F_0 t)x(t)\right\}\mathrm{d}t$$

Let us simplify this one

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

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$$= -2T_p\mathcal{R}\left\{\sum_k c_k^* c_k\right\} = -2T_p\mathcal{R}\left\{\sum_k |c_k|^2\right\} = -2T_p\sum_k |c_k|^2$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi k F_0 t)x(t)\right\}\mathrm{d}t$$

Let us simplify this one

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi kF_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi kF_0 t)$$

$$F_0 = \frac{1}{T_p}$$

$$-\int_0^{T_p} 2\mathcal{R}\left\{\sum_k c_k^* \exp(-i2\pi kF_0 t)x(t)\right\}\mathrm{d}t = -2\mathcal{R}\left\{\sum_k c_k^* \int_0^{T_p} \exp(-i2\pi kF_0 t)x(t)\mathrm{d}t\right\}$$

$$= -2T_p\mathcal{R}\left\{\sum_k c_k^* c_k\right\} = -2T_p\mathcal{R}\left\{\sum_k |c_k|^2\right\} = -2T_p\sum_k |c_k|^2$$

$$-2T_p\sum_k |c_k|^2$$

Plug back in

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

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$$= \int_0^{T_p} \left|\sum_k c_k \exp(i2\pi k F_0 t)\right|^2 \mathrm{d}t + \int_0^{T_p} |x(t)|^2 \,\mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi kF_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi kF_0 t)$$

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Let us simplify this one

$$= \boxed{\int_0^{T_p} \left|\sum_k c_k \exp(i2\pi kF_0 t)\right|^2 \mathrm{d}t} + \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t = \int_0^{T_p} \sum_k \sum_\ell c_k \exp(i2\pi k F_0 t) c_\ell^* \exp(-i2\pi \ell F_0 t) \mathrm{d}t$$

Square of a sum = double sum, different indices

$$= \boxed{\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t} + \int_0^{T_p} |x(t)|^2 \, \mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t = \int_0^{T_p} \sum_k \sum_\ell c_k \exp(i2\pi k F_0 t) c_\ell^* \exp(-i2\pi \ell F_0 t) \mathrm{d}t$$

$$= \sum_k \sum_\ell c_k c_\ell^* \int_0^{T_p} \exp(i2\pi (k-\ell) F_0 t) \mathrm{d}t$$

Change order of sum and integration

$$= \boxed{\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t} + \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t = \int_0^{T_p} \sum_k \sum_\ell c_k \exp(i2\pi k F_0 t) c_\ell^* \exp(-i2\pi \ell F_0 t) \mathrm{d}t$$

$$= \sum_k \sum_\ell c_k c_\ell^* \int_0^{T_p} \exp(i2\pi(k-\ell)F_0 t)\mathrm{d}t = \sum_k \sum_\ell c_k c_\ell^* T_p \delta(k-\ell)$$

See book, eq (1.3), p. 233

$$= \boxed{\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t} + \int_0^{T_p} |x(t)|^2 \, \mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$\int_0^{T_p} \left|\sum_k c_k \exp(i2\pi k F_0 t)\right|^2 \mathrm{d}t = \int_0^{T_p} \sum_k \sum_\ell c_k \exp(i2\pi k F_0 t) c_\ell^* \exp(-i2\pi \ell F_0 t)\mathrm{d}t$$

$$= \sum_k \sum_\ell c_k c_\ell^* \int_0^{T_p} \exp(i2\pi(k-\ell)F_0 t)\mathrm{d}t = \sum_k \sum_\ell c_k c_\ell^* T_p \delta(k-\ell)$$

$$= T_p \sum_k |c_k|^2$$

$$= \boxed{\int_0^{T_p} \left|\sum_k c_k \exp(i2\pi k F_0 t)\right|^2 \mathrm{d}t} + \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$\int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) \right|^2 \mathrm{d}t = \int_0^{T_p} \sum_k \sum_\ell c_k \exp(i2\pi k F_0 t) c_\ell^* \exp(-i2\pi \ell F_0 t) \mathrm{d}t$$

$$= \sum_k \sum_\ell c_k c_\ell^* \int_0^{T_p} \exp(i2\pi(k-\ell)F_0 t)\mathrm{d}t = \sum_k \sum_\ell c_k c_\ell^* T_p \delta(k-\ell)$$

Plug back in

$$= \boxed{T_p \sum_k |c_k|^2} + \int_0^{T_p} |x(t)|^2 \, \mathrm{d}t$$

$$-2T_p \sum_k |c_k|^2$$

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

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$$= \boxed{T_p \sum_k |c_k|^2 \quad -2T_p \sum_k |c_k|^2} + \int_0^{T_p} |x(t)|^2 \,\mathrm{d}t$$

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

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$$\int_0^{T_p} |\hat{x}(t) - x(t)|^2 \mathrm{d}t = \int_0^{T_p} \left| \sum_k c_k \exp(i2\pi k F_0 t) - x(t) \right|^2 \mathrm{d}t$$

$$= -\; T_p \sum_k |c_k|^2 \qquad + \int_0^{T_p} |x(t)|^2 \,\mathrm{d}t$$

EITF75, Fourier transforms

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi kF_0 t)\mathrm{d}t \qquad \hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi kF_0 t)$$

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$$= -\,T_p \sum_k |c_k|^2 \qquad + \int_0^{T_p} |x(t)|^2 \,\mathrm{d}t$$

Remains to show $$\int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$$

EITF75, Fourier transforms

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

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$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) x^*(\tau) \exp(i2\pi k F_0 \tau) \mathrm{d}t \mathrm{d}\tau$$

Remains to show $\int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$

EITF75, Fourier transforms

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

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$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \exp(i2\pi k F_0 (\tau - t)) \mathrm{d}t \mathrm{d}\tau$$

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EITF75, Fourier transforms

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

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$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \exp(i2\pi k F_0 (\tau - t)) \mathrm{d}t \mathrm{d}\tau$$

$$= \frac{1}{T_p} \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \sum_{k=-\infty}^{\infty} \exp(i2\pi k F_0 (\tau - t)) \mathrm{d}t \mathrm{d}\tau$$

Remains to show $\int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$

EITF75, Fourier transforms

Wikipedia
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Dirac comb

$$\text{Ш}_T(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{i2\pi n\frac{t}{T}}$$

$$= \frac{1}{T_p}\int_0^{T_p}\int_0^{T_p} x(t)x^*(\tau)\boxed{\sum_{k=-\infty}^{\infty}\exp(i2\pi kF_0(\tau-t))}\mathrm{d}t\mathrm{d}\tau$$

Remains to show $$\int_0^{T_p}|x(t)|^2\mathrm{d}t = T_p\sum_{k=-\infty}^{\infty}|c_k|^2$$

EITF75, Fourier transforms

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Dirac comb

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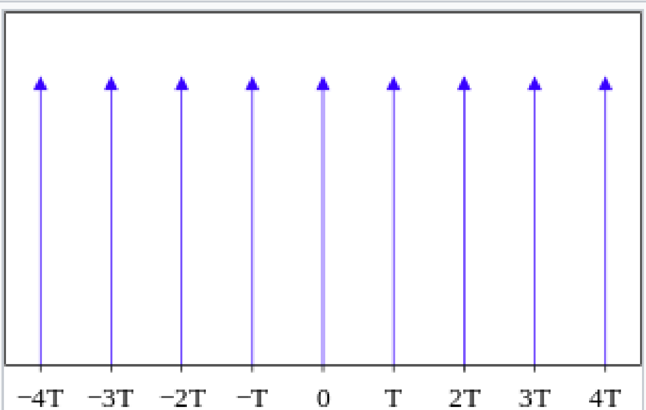
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{i2\pi n\frac{t}{T}}$$

$$= \frac{1}{\ } \int^{T_p} \int^{T_p} x(t)x^*(\tau) \boxed{\sum_{k=-\infty}^{\infty} \exp(i2\pi k F_0(\tau - t))} \mathrm{d}t\mathrm{d}\tau$$

$$x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$$



A Dirac comb is an infinite series of Dirac delta functions spaced at intervals of T

EITF75, Fourier transforms

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Dirac comb

$$\text{Ш}_T(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{i2\pi n \frac{t}{T}}$$

Plug in

$$= \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \text{Ш}_{T_p}(\tau - t) \mathrm{d}t \mathrm{d}\tau$$

Remains to show $\int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$

EITF75, Fourier transforms

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) x^*(\tau) \exp(i2\pi k F_0 \tau) \mathrm{d}t \mathrm{d}\tau$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \exp(i2\pi k F_0 (\tau - t)) \mathrm{d}t \mathrm{d}\tau$$

Diracs spaced by T_p

$$= \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) Ш_{T_p}(\tau - t) \mathrm{d}t \mathrm{d}\tau$$

Remains to show $\int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$

EITF75, Fourier transforms

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) x^*(\tau) \exp(i2\pi k F_0 \tau) \mathrm{d}t \mathrm{d}\tau$$

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Diracs spaced by T_p

$$= \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \text{Ш}_{T_p}(\tau - t) \mathrm{d}t \mathrm{d}\tau \quad = \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

Remains to show $$\int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$$

EITF75, Fourier transforms

$$\boxed{T_p \sum_{k=-\infty}^{\infty} |c_k|^2 =} T_p \sum_{k=-\infty}^{\infty} \left| \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t \right|^2$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) x^*(\tau) \exp(i2\pi k F_0 \tau) \mathrm{d}t\mathrm{d}\tau$$

$$= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) \exp(i2\pi k F_0 (\tau - t)) \mathrm{d}t\mathrm{d}\tau$$

Diracs spaced by T_p

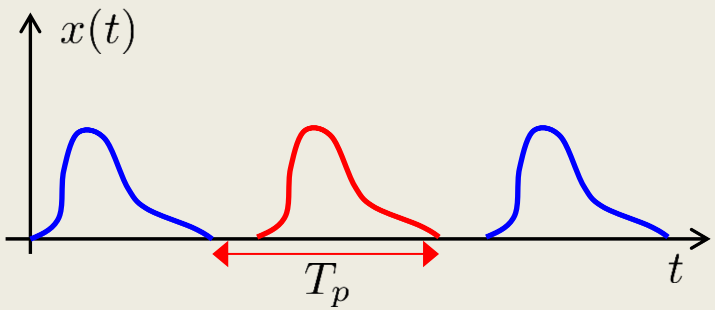
$$= \int_0^{T_p} \int_0^{T_p} x(t) x^*(\tau) Ш_{T_p}(\tau - t) \mathrm{d}t\mathrm{d}\tau \quad \boxed{= \int_0^{T_p} |x(t)|^2 \mathrm{d}t}$$

Remains to show $\displaystyle \int_0^{T_p} |x(t)|^2 \mathrm{d}t = T_p \sum_{k=-\infty}^{\infty} |c_k|^2$ Done

EITF75, Fourier transforms

4 different type of signals

Continuous and periodic



Compute

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

For no particular reason, let us calculate

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

$$F_0 = \frac{1}{T_p}$$

We now claim that $\{c_k\}_{k=-\infty}^{\infty}$ contains all information about $x(t)$

To be calculated next

$$\boxed{\int_0^{T_p} |\hat{x}(t) - x(t)|^2 \mathrm{d}t}$$

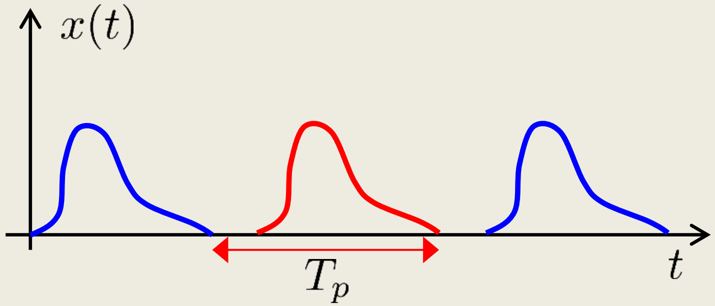
If $\hat{x}(t) = x(t)$, $\{c_k\}_{k=-\infty}^{\infty}$ is an alternative representation of $x(t)$

RECALL

EITF75, Fourier transforms

4 different type of signals

Continuous and periodic



Compute

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

For no particular reason, let us calculate

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

$$F_0 = \frac{1}{T_p}$$

We now claim that $\{c_k\}_{k=-\infty}^{\infty}$ contains all information about $x(t)$

To be calculated next

$$\boxed{\int_0^{T_p} |\hat{x}(t) - x(t)|^2 \mathrm{d}t = 0}$$

If $\hat{x}(t) = x(t)$, $\{c_k\}_{k=-\infty}^{\infty}$ is an alternative representation of $x(t)$

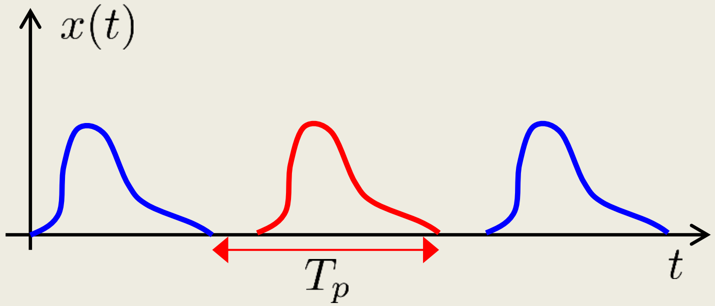
RECALL

EITF75, Fourier transforms

Summary

Continuous and periodic

$$F_0 = \frac{1}{T_p}$$



Fourier series representation

Analysis equation

$$c_k = \frac{1}{T_p}\int_0^{T_p} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

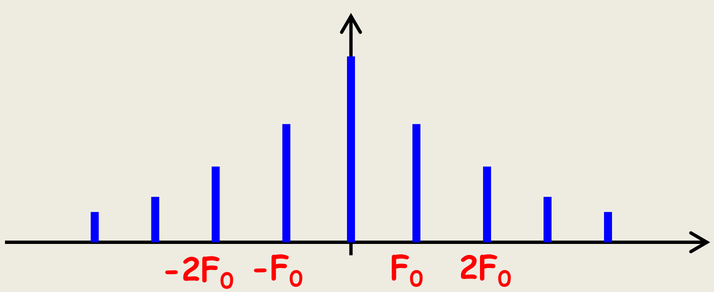
Synthesis equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

Parseval's identity

$$T_p \sum_{k=-\infty}^{\infty} |c_k|^2 = \int_0^{T_p} |x(t)|^2 \mathrm{d}t$$

Power spectrum

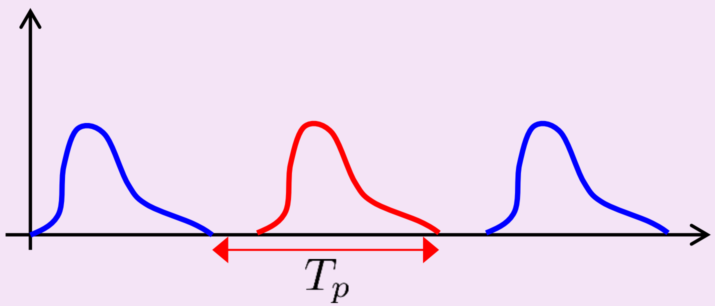


4 different type of signals

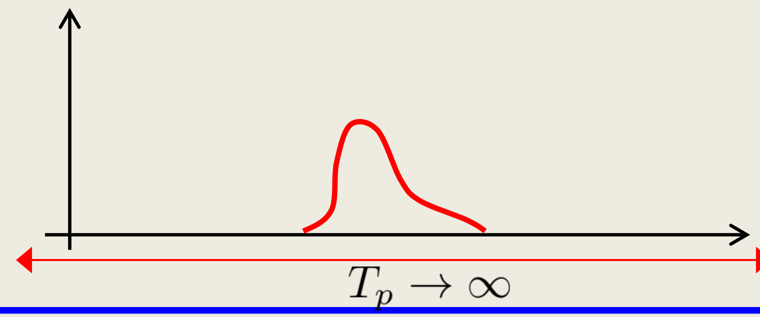
Next case

Continuous and periodic

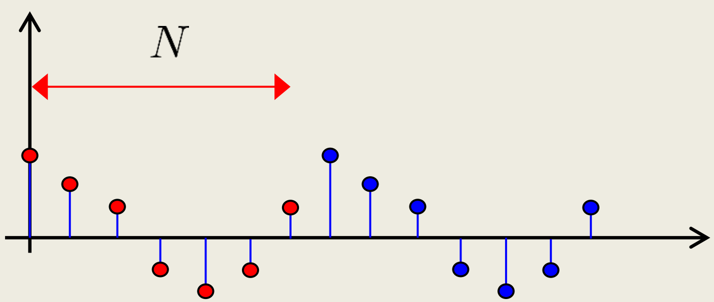
Done



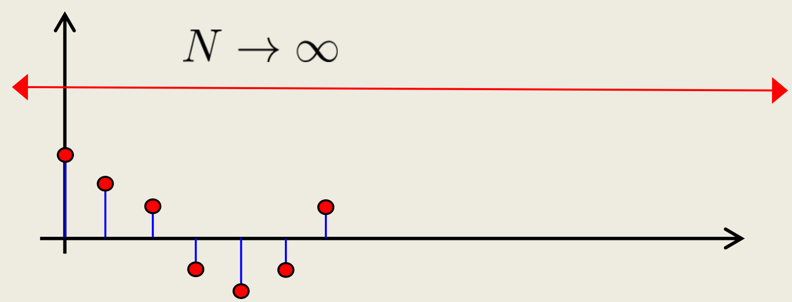
Continuous and aperiodic



Discrete and periodic



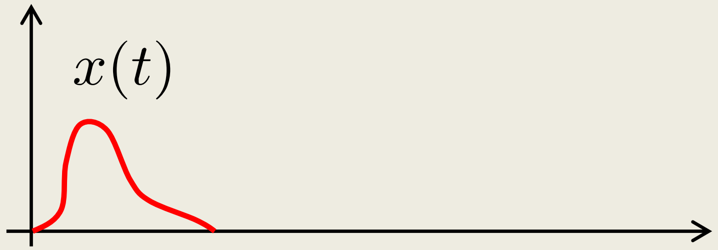
Discrete and aperiodic



EITF75, Fourier transforms

4 different type of signals

Continuous and aperiodic



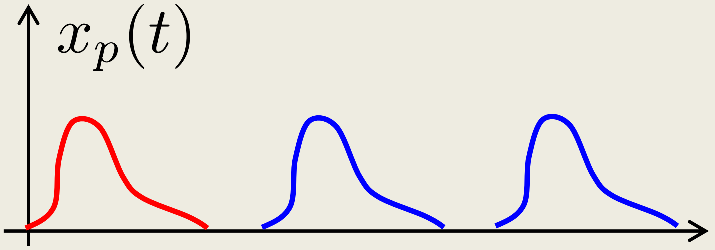
EITF75, Fourier transforms

4 different type of signals

Let us define another signal $x_p(t)$

We know how to handle $x_p(t)$

Continuous and aperiodic



EITF75, Fourier transforms

4 different type of signals

Let us define another signal $x_p(t)$

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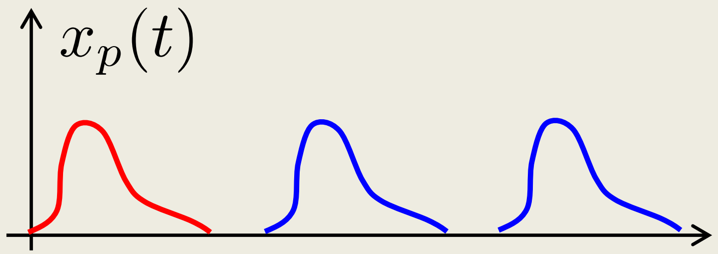
The one of interest is

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

Continuous and aperiodic



EITF75, Fourier transforms

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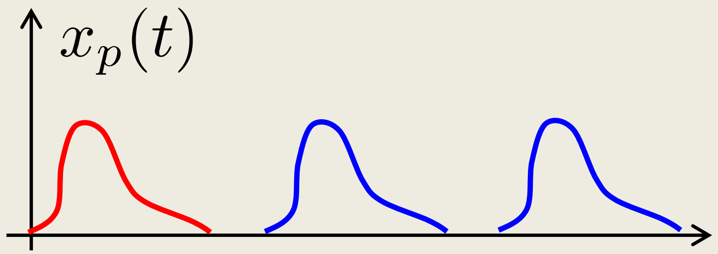
$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

We have from before

$$c_k = \frac{1}{T_p} \int_0^{T_p} x_p(t) \exp(-i2\pi k F_0 t) \mathrm{d}t$$

Continuous and aperiodic



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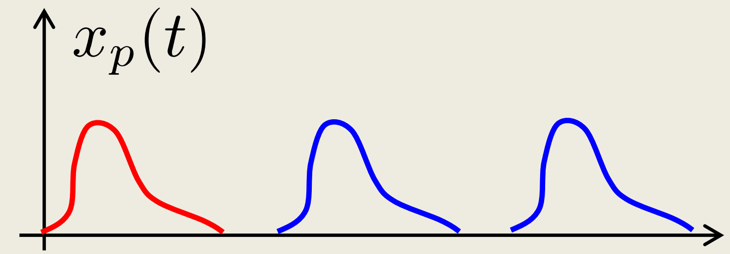
Problematic...

Continuous and aperiodic



EITF75, Fourier transforms

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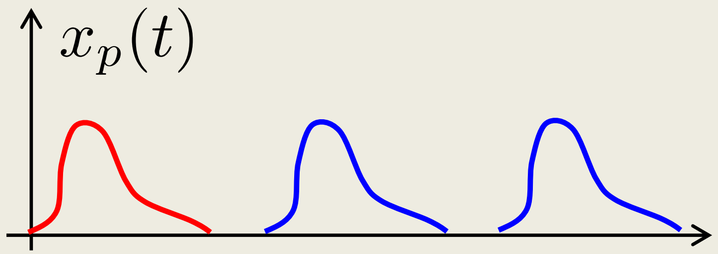
Continuous and aperiodic



We have from before

$$c_k = \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t$$

Within one period, we can replace $x_p(t)$ with $x(t)$

EITF75, Fourier transforms

4 different type of signals

Let us define another signal $x_p(t)$

We know how to handle $x_p(t)$

The one of interest is

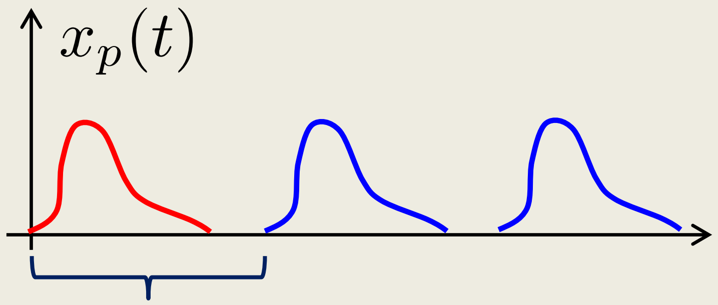
$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

We have from before

$$c_k = \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t$$

Within one period, we can replace $x_p(t)$ with $x(t)$

Continuous and aperiodic



$x_p(t) = x(t)$

EITF75, Fourier transforms

4 different type of signals

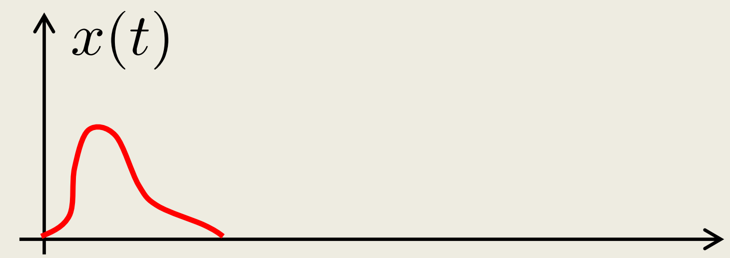
Let us define another signal $x_p(t)$

We know how to handle $x_p(t)$

The one of interest is

$$x(t) = \lim_{T_p \to \infty} x_p(t)$$

Continuous and aperiodic



We have from before

$$c_k = \frac{1}{T_p} \int_0^{T_p} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t$$

$$c_k = \frac{1}{T_p} \int_{-\infty}^{\infty} x(t) \exp(-i2\pi k F_0 t) \mathrm{d}t$$

…and change limits

EITF75, Fourier transforms

4 different type of signals

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

EITF75, Fourier transforms

4 different type of signals

Let us define the following function

$$X(F) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi Ft)\mathrm{d}t$$

X(F) **not dependent on T_p**

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi kF_0t)\mathrm{d}t$$

EITF75, Fourier transforms

4 different type of signals

Let us define the following function

$$X(F) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi F t)\mathrm{d}t$$

X(F) not dependent on T_p

Goal: Show that we can recover x(t) from X(F)

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

EITF75, Fourier transforms

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Let us define the following function

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X(F) **not dependent on T_p**

We have $\frac{1}{T_p}X(kF_0) = c_k$

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

EITF75, Fourier transforms

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X(F) **not** **dependent on** T_p

$$\frac{1}{T_p}X\left(\frac{k}{T_p}\right) = \frac{1}{T_p}X(kF_0) = c_k$$

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi kF_0 t)\mathrm{d}t$$

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$$\frac{1}{T_p}X\left(\frac{k}{T_p}\right) = \frac{1}{T_p}X(kF_0) = c_k$$

$$x_p(t) = \sum_k c_k \exp(i2\pi kF_0 t)$$

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi kF_0 t)\mathrm{d}t$$

EITF75, Fourier transforms

4 different type of signals

Let us define the following function

$$\boxed{X(F) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi F t)\mathrm{d}t}$$

X(F) **not** dependent on T_p

$$\frac{1}{T_p} X\left(\frac{k}{T_p}\right) = \frac{1}{T_p} X(kF_0) = c_k$$

$$\boxed{x_p(t) = \sum_k c_k \exp(i2\pi k F_0 t) = \frac{1}{T_p}\sum_k X\left(\frac{k}{T_p}\right)\exp(i2\pi k F_0 t)}$$

$$c_k = \frac{1}{T_p}\int_{-\infty}^{\infty} x(t)\exp(-i2\pi k F_0 t)\mathrm{d}t$$

EITF75, Fourier transforms

4 different type of signals

Let us define the following function

$$X(F)=\int_{-\infty}^{\infty} x(t)\exp(-i2\pi Ft)\mathrm{d}t$$

X(F) **not dependent on T_p**

$$\frac{1}{T_p}X\left(\frac{k}{T_p}\right)=\frac{1}{T_p}X(kF_0)=c_k$$

$$x_p(t)=\sum_k c_k\exp(i2\pi kF_0t)=\frac{1}{T_p}\sum_k X\left(\frac{k}{T_p}\right)\exp(i2\pi kF_0t)$$

Define $\Delta F=\frac{1}{T_p}$

EITF75, Fourier transforms

4 different type of signals

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$$x_p(t) = \sum_k X(k\Delta F)\exp(i2\pi k\Delta F t)\Delta F$$

EITF75, Fourier transforms

4 different type of signals

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EITF75, Fourier transforms

4 different type of signals

Let us define the following function

$$X(F) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi Ft)\mathrm{d}t$$

X(F) **not** dependent on T_p

Very fundamental question: What is this ?

$$x(t) = \lim_{T_p\to\infty} x_p(t) = \lim_{\Delta F\to 0} \sum_k X(k\Delta F)\exp(i2\pi k\Delta F t)\Delta F$$

EITF75, Fourier transforms

4 different type of signals

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X(F) **not** dependent on T_p

Very fundamental question: What is this ?

The definition of a (Riemann) integral

$$x(t) = \lim_{T_p \to \infty} x_p(t) = \lim_{\Delta F \to 0} \sum_k X(k\Delta F) \exp(i2\pi k \Delta F t) \Delta F$$

EITF75, Fourier transforms

4 different type of signals

Fourier transform

$$X(F) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi Ft)\mathrm{d}t$$

Fourier transform

Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(F)\exp(i2\pi Ft)\mathrm{d}F$$

EITF75, Fourier transforms

4 different type of signals

Fourier transform

$$X(F) = \int_{-\infty}^{\infty} x(t)\exp(-i2\pi F t)\mathrm{d}t$$

Parseval's identity

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |x(F)|^2 \mathrm{d}F$$

Energy in time, must be present in frequency as well

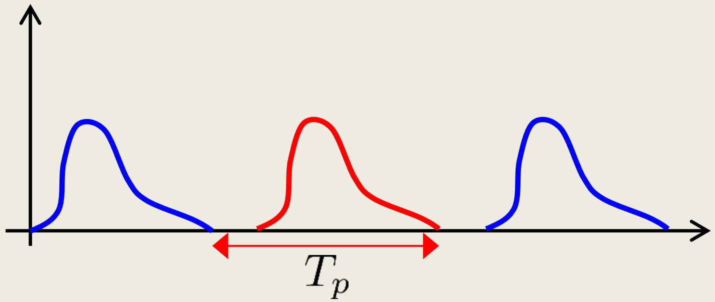
Inverse Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(F)\exp(i2\pi F t)\mathrm{d}F$$

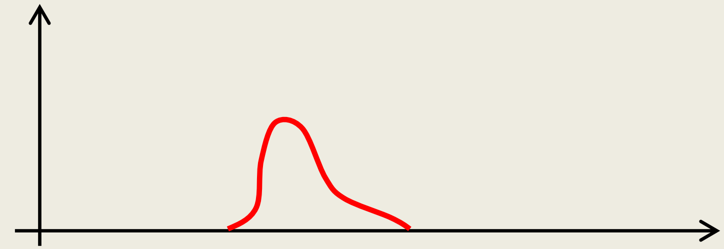
EITF75, Fourier transforms

So far, we did these two

Continuous and periodic



Continuous and aperiodic



EITF75, Fourier transforms

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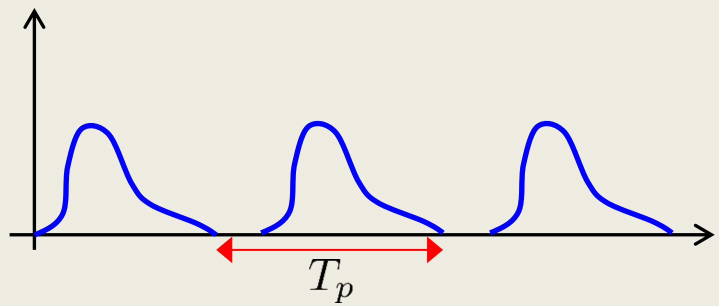
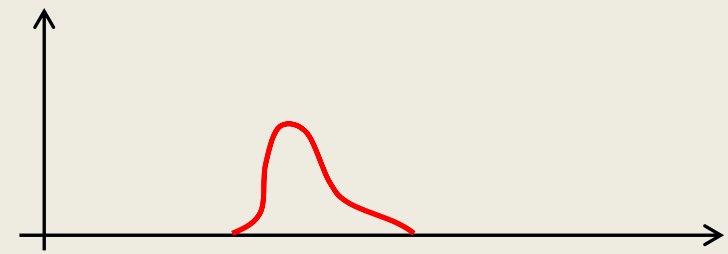
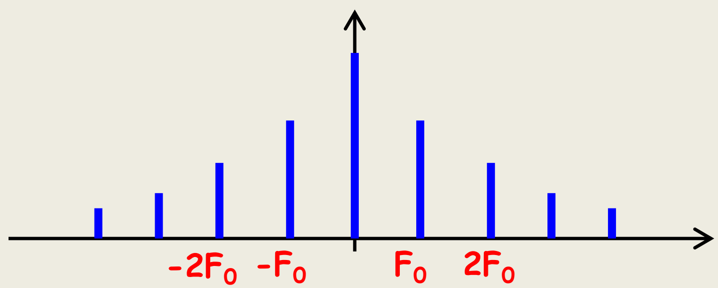


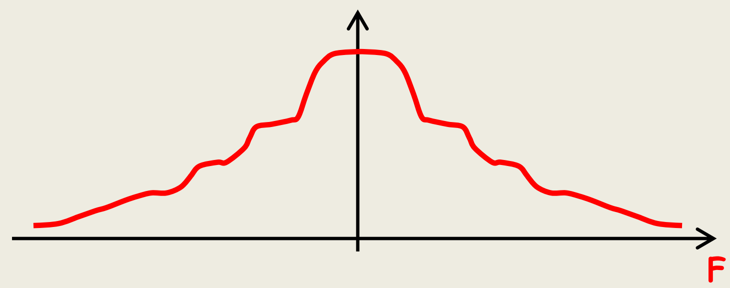
Continuous and aperiodic



Power spectrum



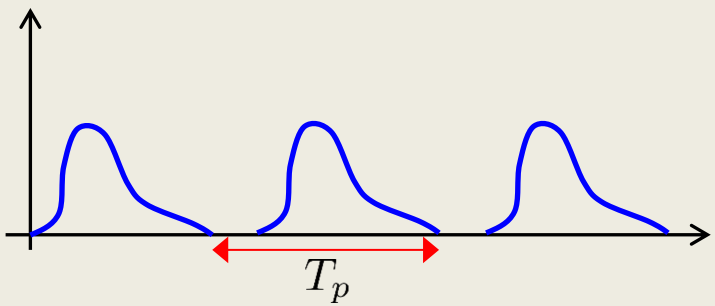
Power spectrum



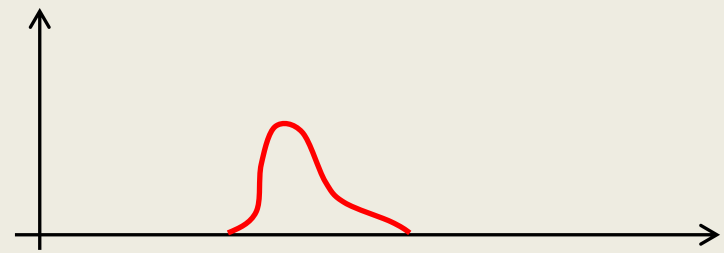
EITF75, Fourier transforms

Question: Is this true? (The left spectrum is samples from the right)

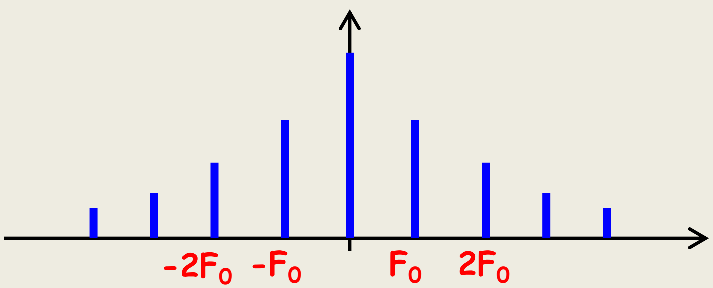
Continuous and periodic



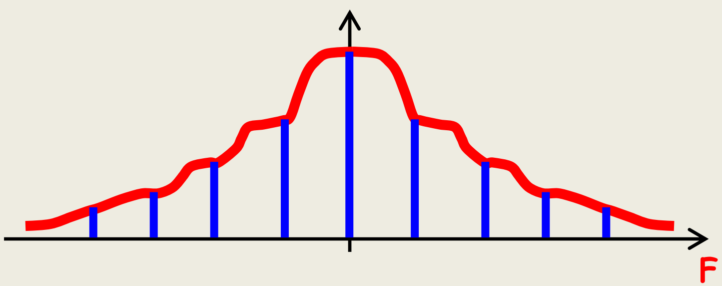
Continuous and aperiodic



Power spectrum

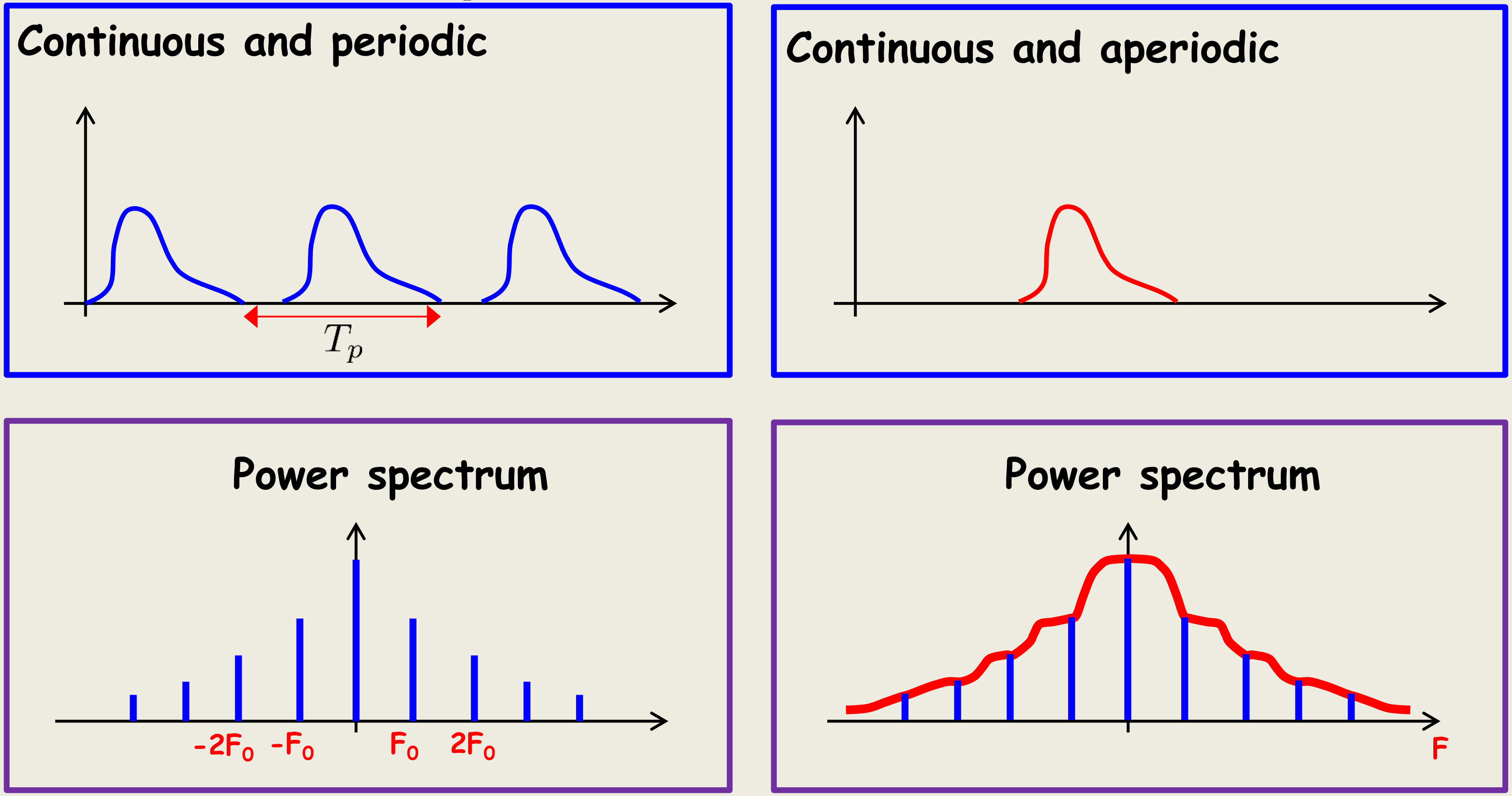


Power spectrum



EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p}X(kF_0) = \; c_k$



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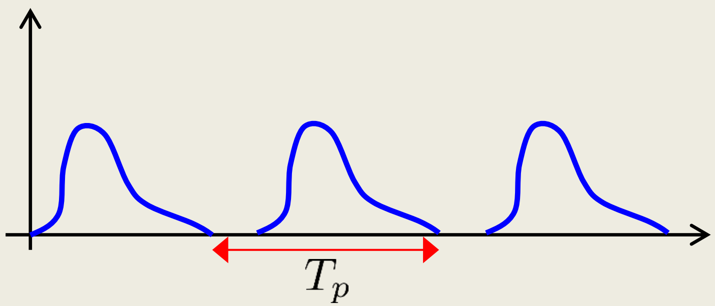

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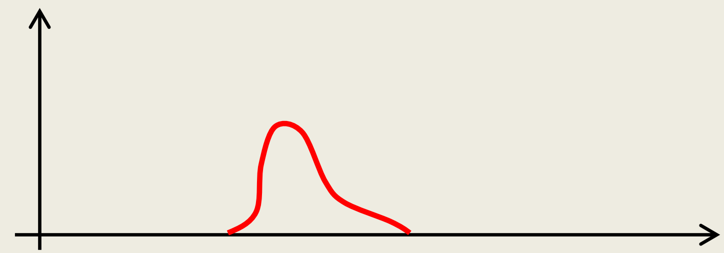
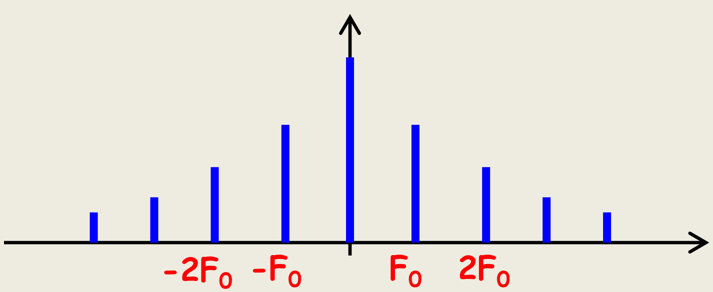
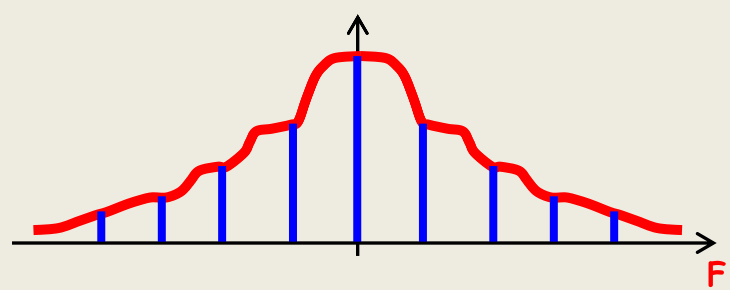


Power spectrum



Power spectrum

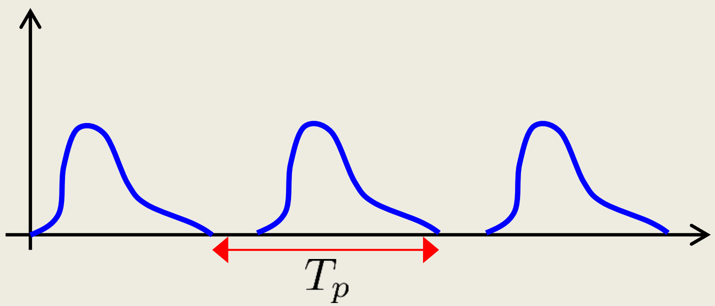


True if $T_p=1$

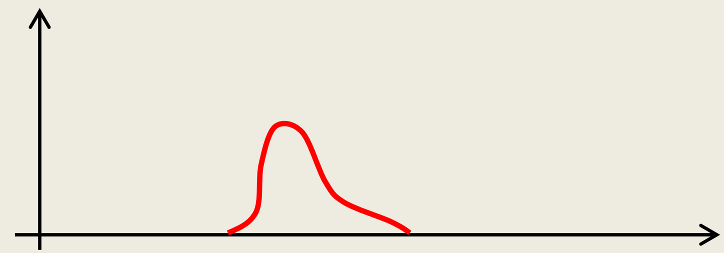
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p}X(kF_0) = c_k$

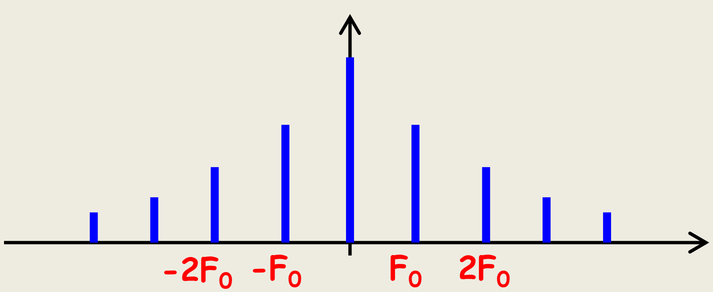
Continuous and periodic



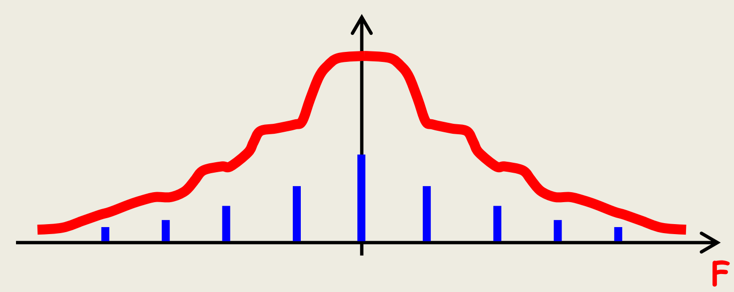
Continuous and aperiodic



Power spectrum



Power spectrum

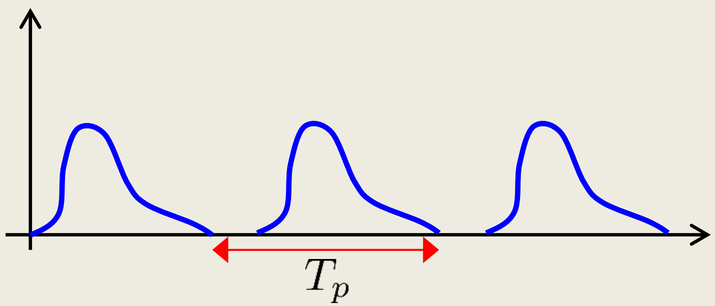


True if T_p=2 ???

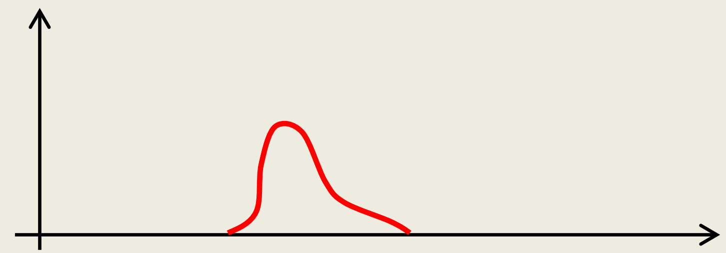
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

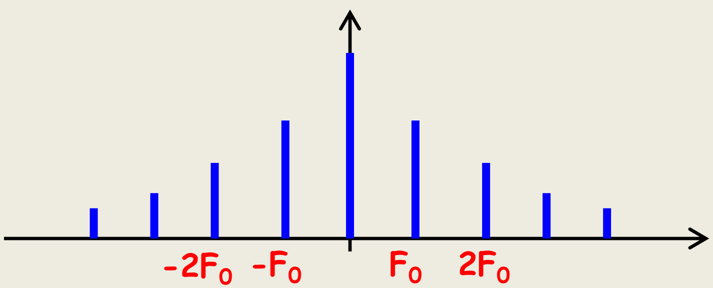
Continuous and periodic



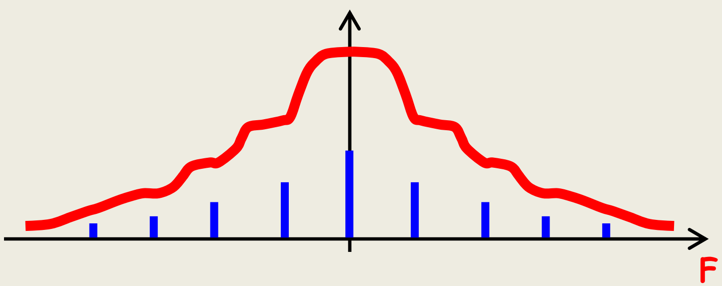
Continuous and aperiodic



Power spectrum



Power spectrum

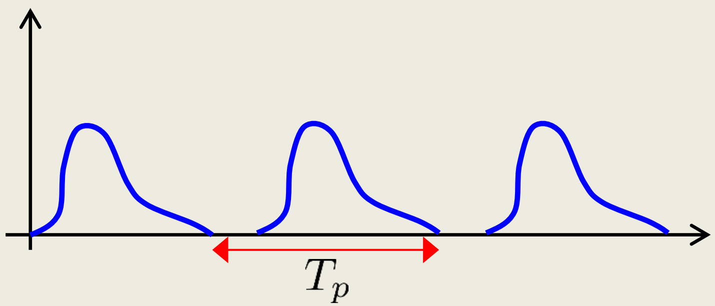


True if $T_p=2$??? NO

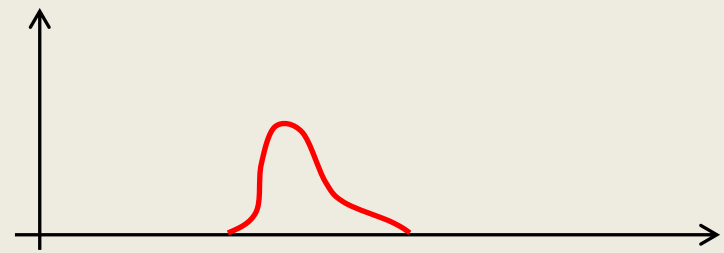
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p}X(kF_0) = c_k$

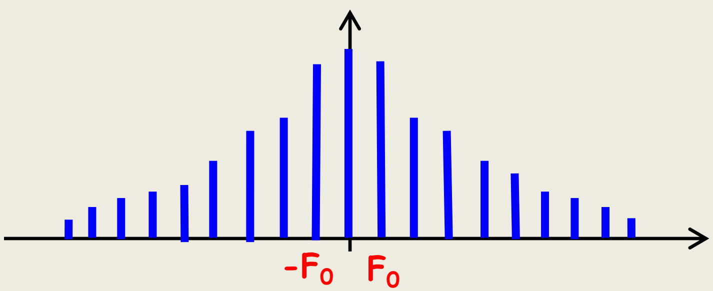
Continuous and periodic



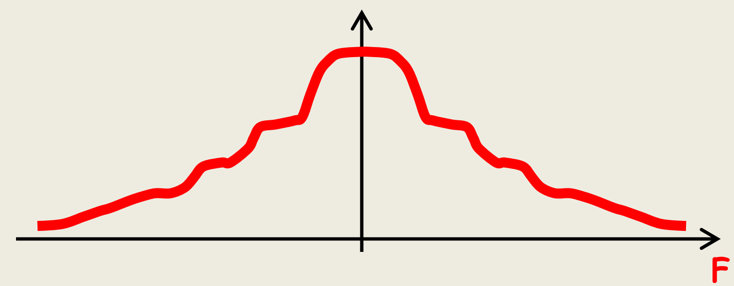
Continuous and aperiodic



Power spectrum



Power spectrum



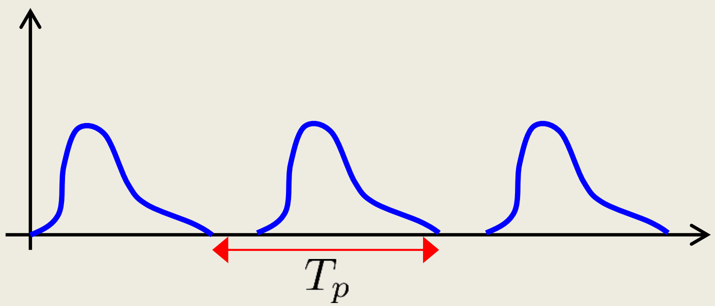
Effect 1: Denser sampling

$T_p=2$

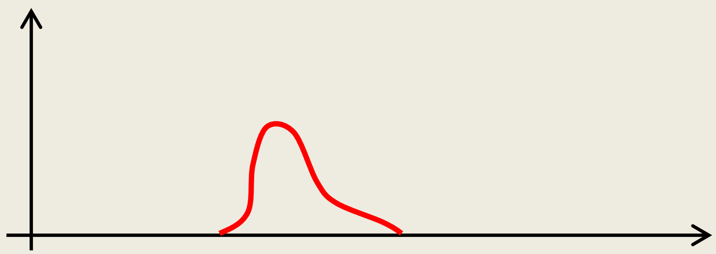
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

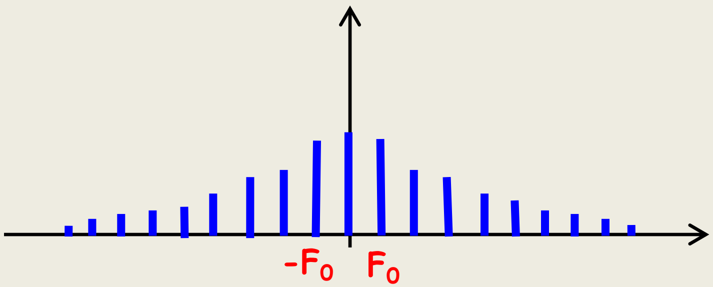
Continuous and periodic



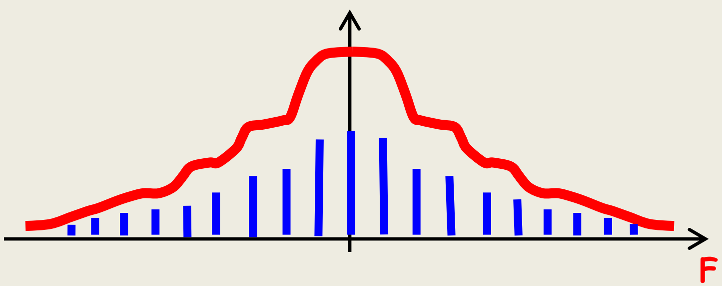
Continuous and aperiodic



Power spectrum



Power spectrum



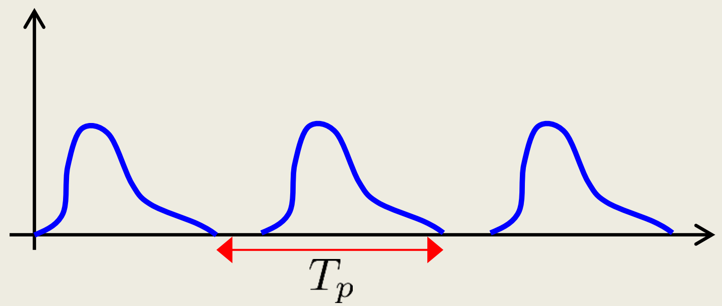
Effect 2: Scaled amplitude

$T_p=2$

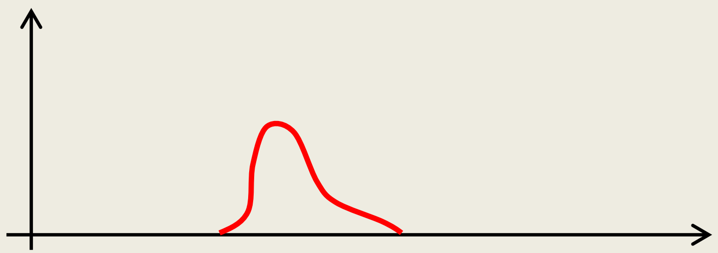
EITF75, Fourier transforms

Yes and no. Recall $\frac{1}{T_p} X(kF_0) = c_k$

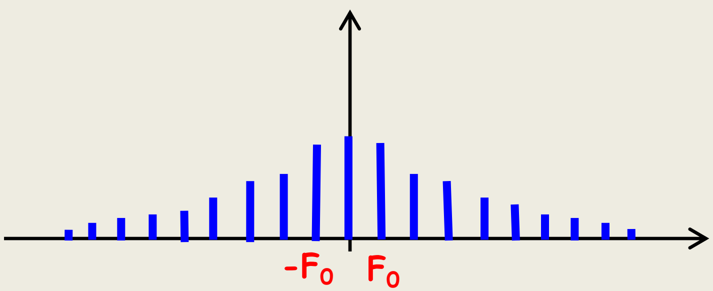
Continuous and periodic



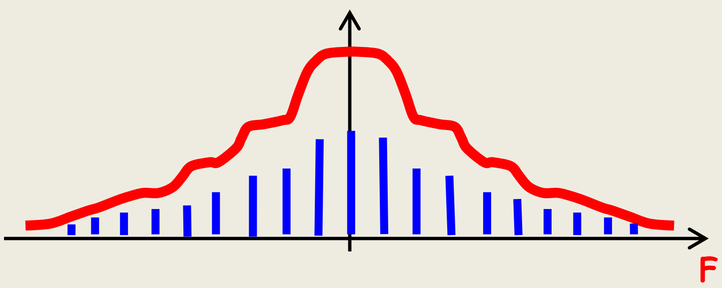
Continuous and aperiodic



Power spectrum



Power spectrum

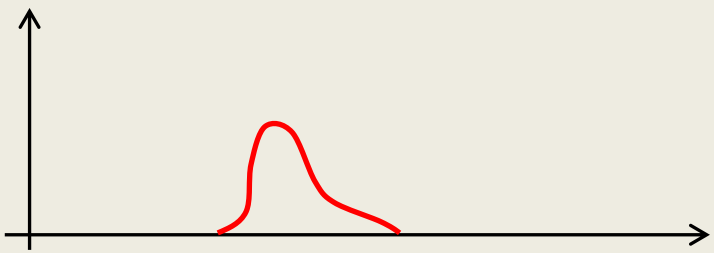


Homework: Read about symmetries, p.237 and 245

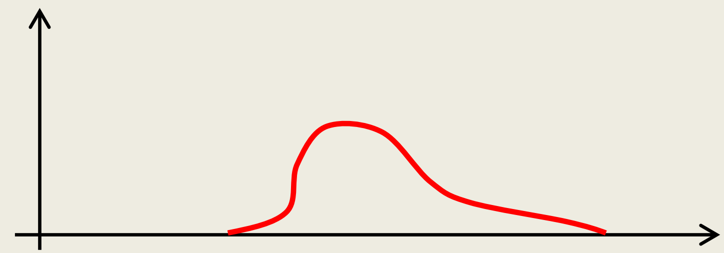
EITF75, Fourier transforms

Fundamental engineering knowledge

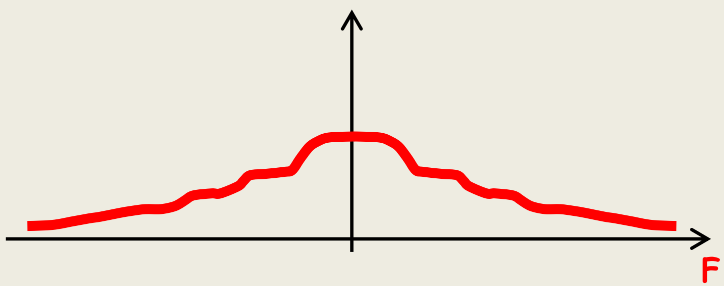
Continuous and aperiodic



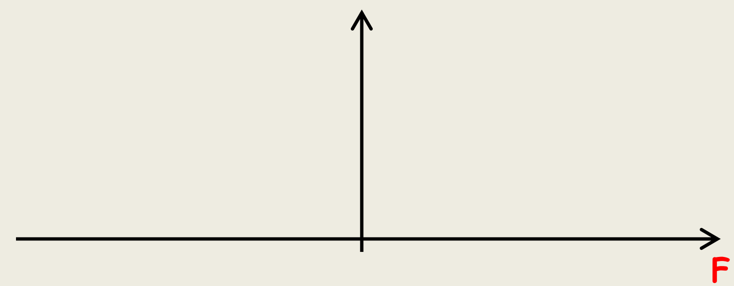
Same shape, twice the length



Power spectrum



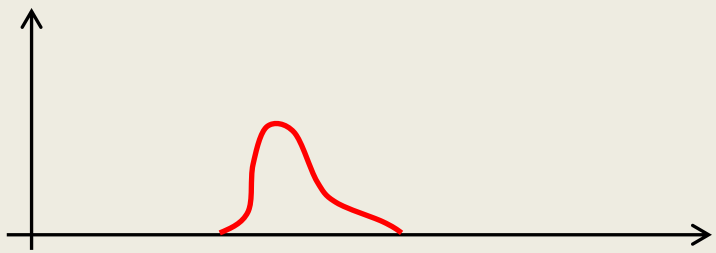
Power spectrum ?



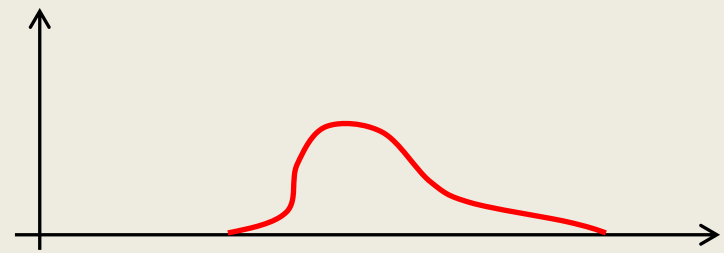
EITF75, Fourier transforms

Fundamental engineering knowledge

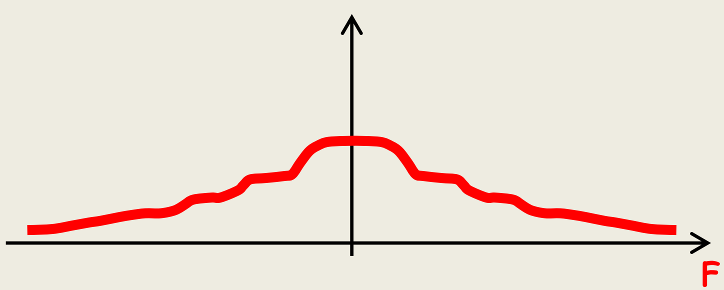
Continuous and aperiodic



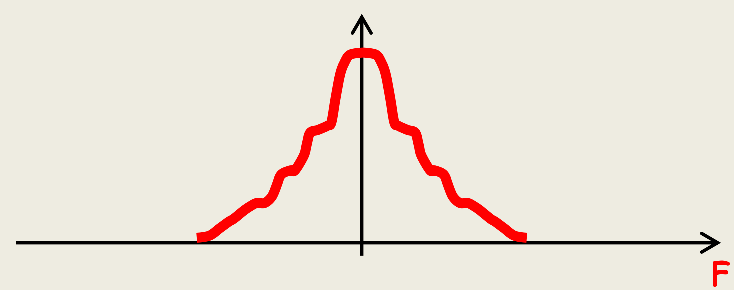
Same shape, twice the length



Power spectrum



Power spectrum ?

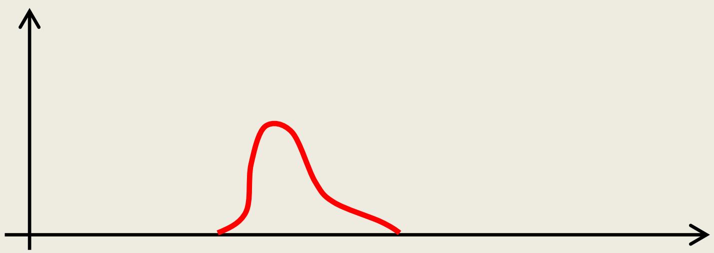


Strect in time is compression in frequency (and vica versa)

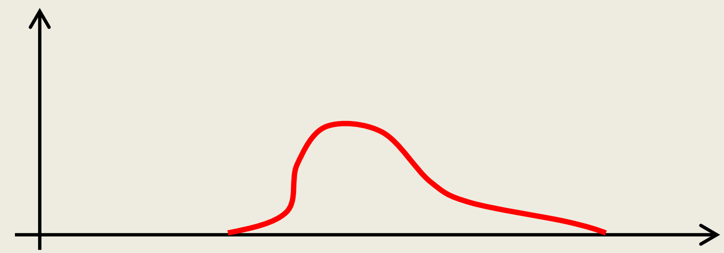
EITF75, Fourier transforms

Fundamental engineering knowledge

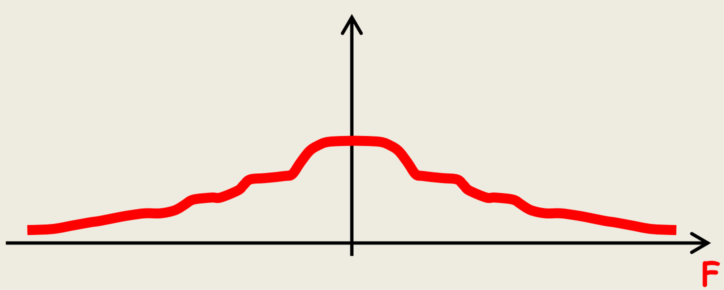
Continuous and aperiodic



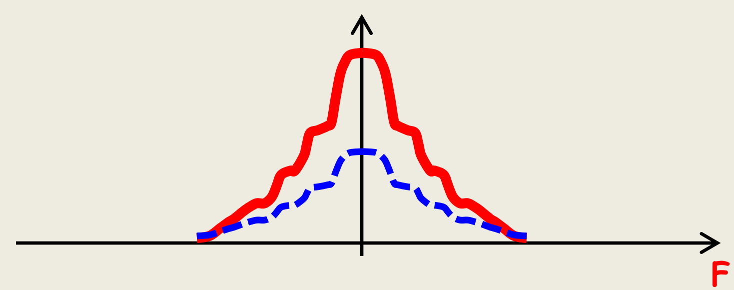
Same shape, twice the length



Power spectrum



Power spectrum ?

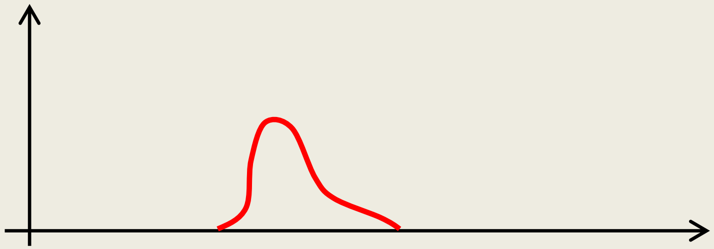


Simplest way to understand the amplitude change?

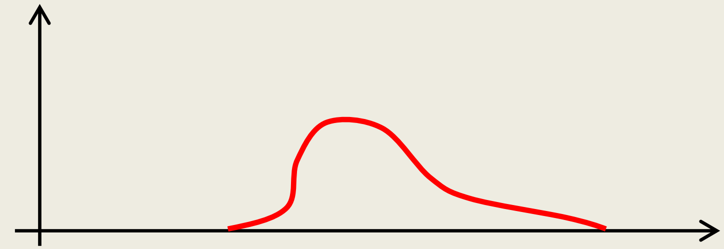
EITF75, Fourier transforms

Fundamental engineering knowledge

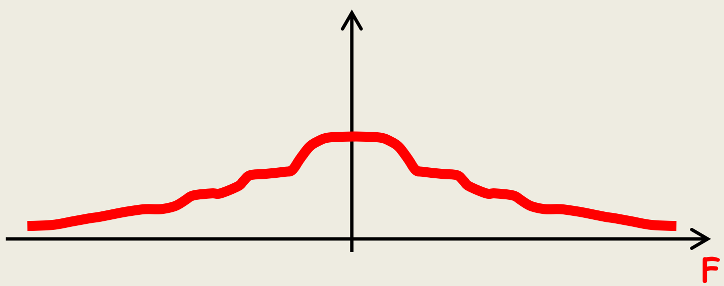
Continuous and aperiodic



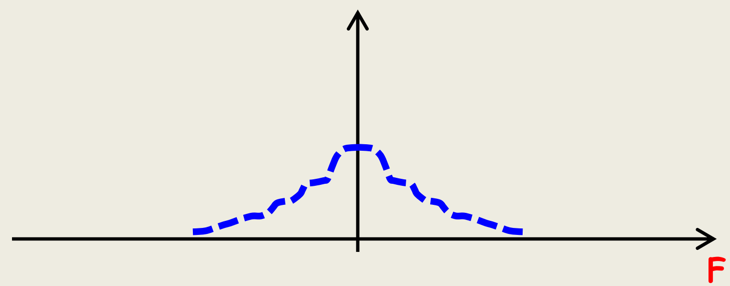
Same shape, twice the length



Power spectrum



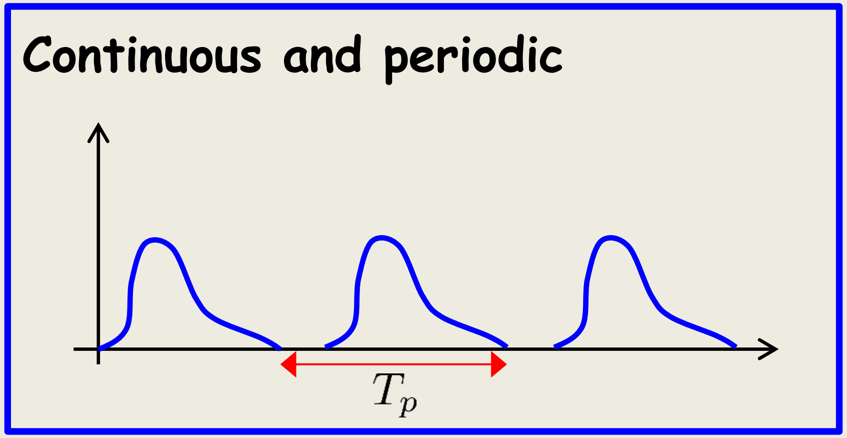
Power spectrum ?

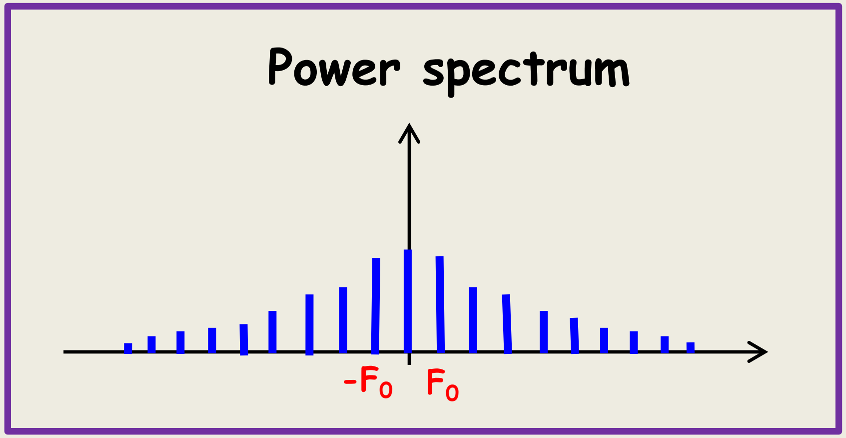


Simplest way to understand the amplitude change?

Parseval's identity: No way the integral-of-the-right-plot-squared equals the integral-of-the-left-plot-squared

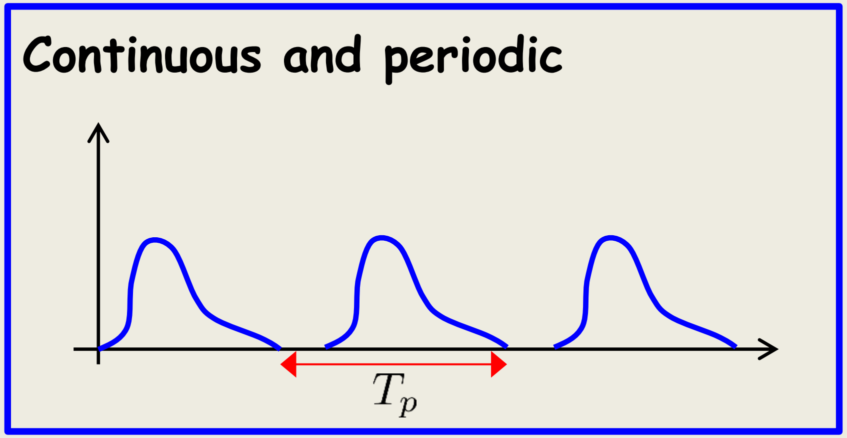
EITF75, Fourier transforms

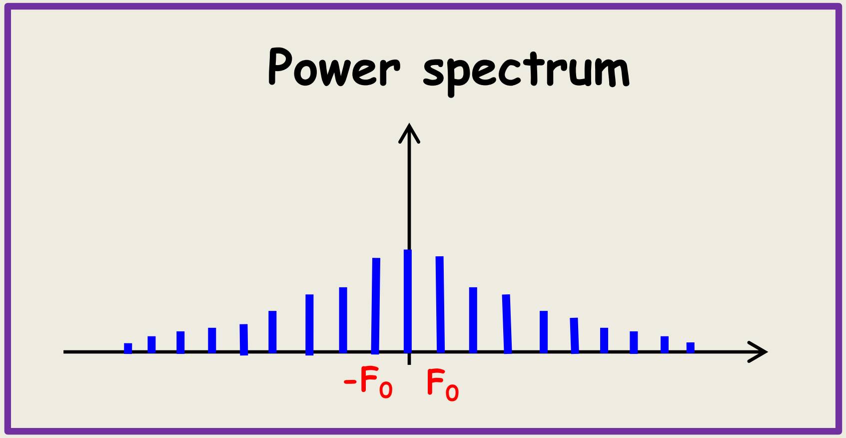




Why is the spectrum discrete?

EITF75, Fourier transforms





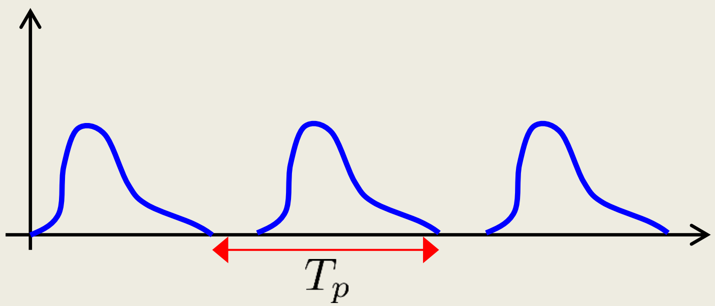
Why is the spectrum discrete?

We can write the signal as

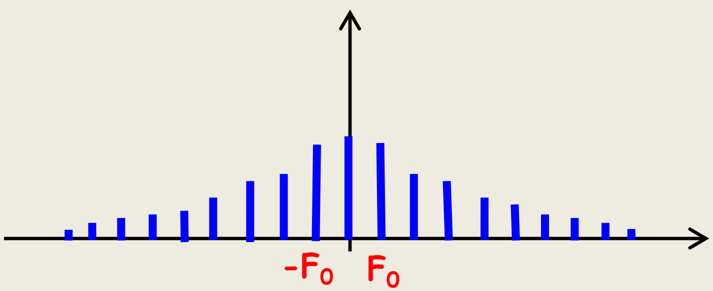
$$x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

EITF75, Fourier transforms

Continuous and periodic



Power spectrum



Why is the spectrum discrete?

We can write the signal as

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \exp(i2\pi k F_0 t)$$

However, $\exp(i2\pi k F_0 t)$

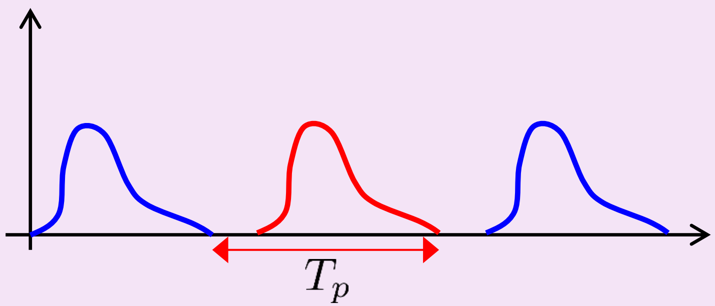
is not periodic with period Tp unless k is an integer.

Thus, there can be no non-integer components in the spectrum

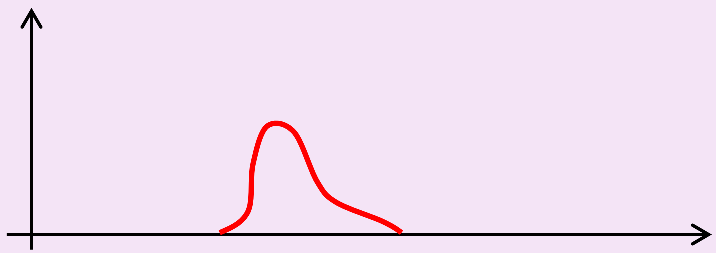
EITF75, Fourier transforms

4 different type of signals

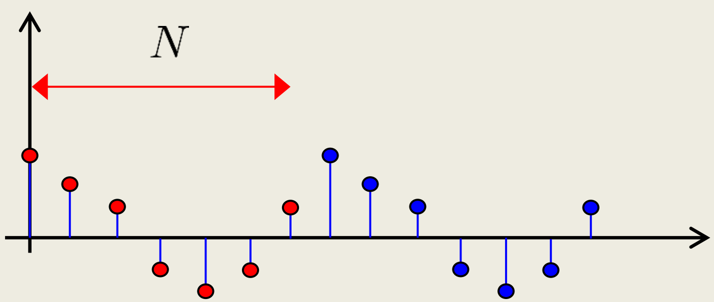
Continuous and periodic — Done



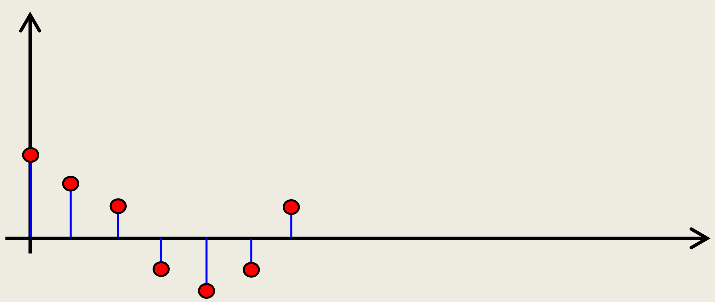
Continuous and aperiodic — Done



Discrete and periodic — Next case



Discrete and aperiodic



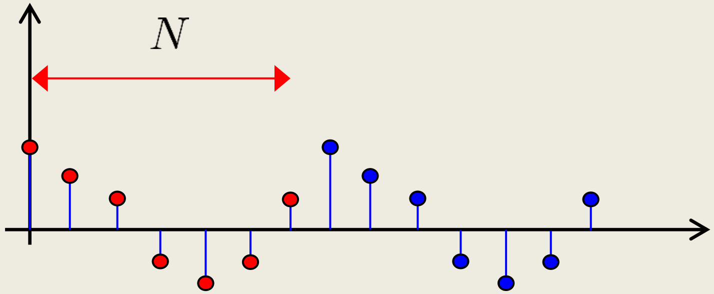
EITF75, Fourier transforms

Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

Discrete and periodic

Next case



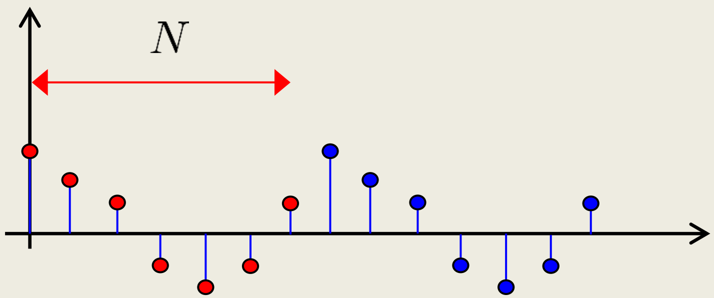
EITF75, Fourier transforms

Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

- A set of coefficients c_k representing x(n)

Discrete and periodic   Next case



EITF75, Fourier transforms

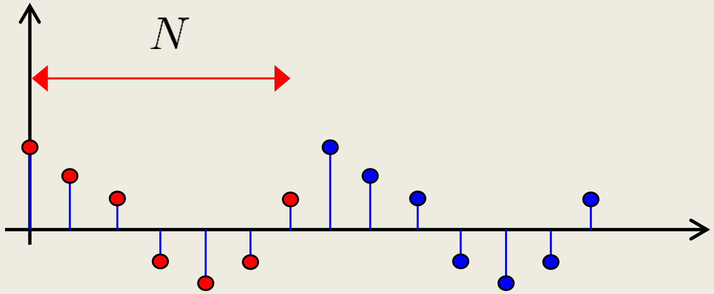
Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

- A set of coefficients c_k representing x(n)
- A continuous function X(F)

Discrete and periodic

Next case



EITF75, Fourier transforms

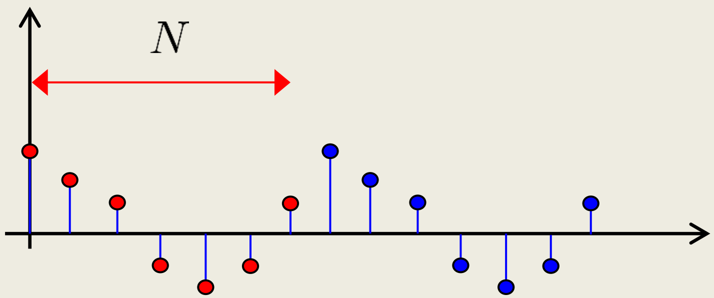
Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

- A set of coefficients c_k representing x(n)
- A continuous function X(F)
- Something else

Discrete and periodic

Next case



EITF75, Fourier transforms

Recall what our goal is right now:

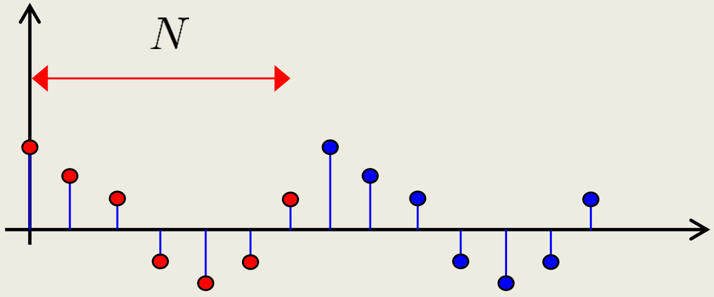
Given a periodic x(n), find either (we dont know which one yet)

- A set of coefficients c_k representing x(n)
- A continuous function X(F)
- Something else

F meant Hz before. But in discrete world, frequency is dimensionless

Discrete and periodic

Next case



EITF75, Fourier transforms

Recall what our goal is right now:

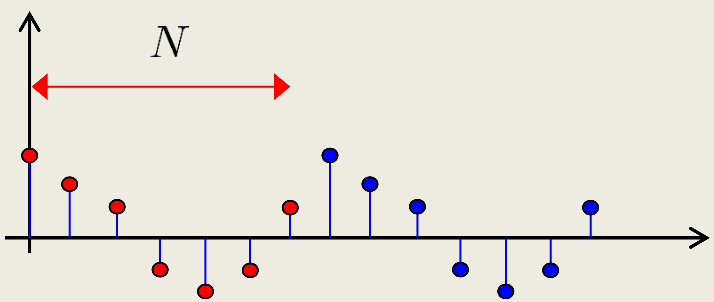
Given a periodic x(n), find either (we dont know which one yet)

- A set of coefficients c_k representing x(n)
- A continuous function X(F)
- Something else

We used an infinite number of coefficients before. But now the signal is of finite length. So, inefficient to use an infinite number of coefficients.

Discrete and periodic

Next case



EITF75, Fourier transforms

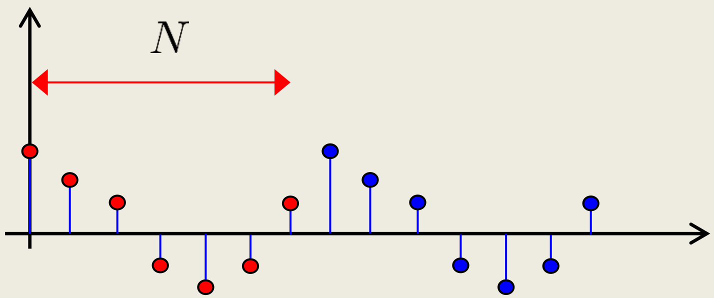
Recall what our goal is right now:

Given a periodic $x(n)$, find either (we dont know which one yet)

- A set of coefficients c_k representing $x(n)$
- A continuous function $X(F)$
- Something else

Discrete and periodic

Next case



In fact, the signal can be described by N numbers.

EITF75, Fourier transforms

Recall what our goal is right now:

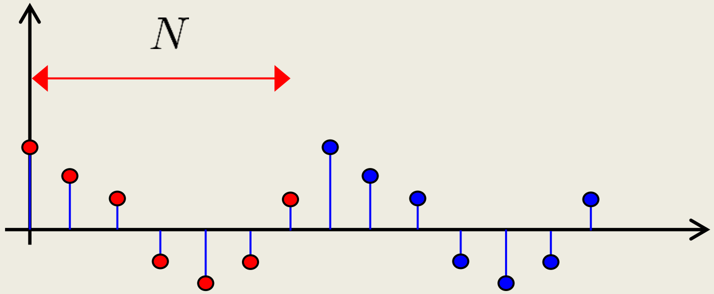
Given a periodic x(n), find either (we dont know which one yet)

- A set of N coefficients c_k representing x(n)

  Likely this should be the result

Discrete and periodic

Next case



EITF75, Fourier transforms

Recall what our goal is right now:

Given a periodic x(n), find either (we dont know which one yet)

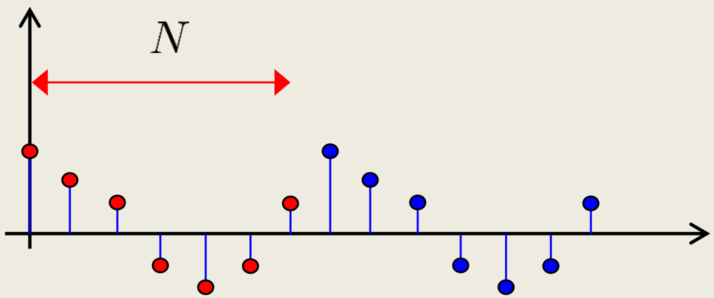
- A set of N coefficients c_k representing x(n)

  Likely this should be the result

Don't forget (Lecture 1): For discrete signals, there is no difference between normalized frequency f=0.4 and f=...-0.6,1.4, 2.4,...

Therefore: Likely that the coefficients c_k are periodically extended.

Discrete and periodic **Next case**



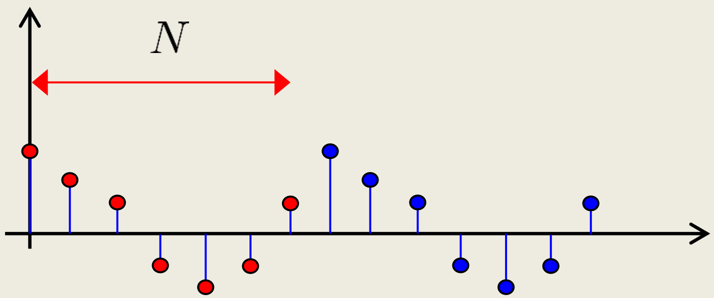
EITF75, Fourier transforms

Construct *x(n)* as

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Note the different approach compared with continuous and periodic

Discrete and periodic **Next case**



EITF75, Fourier transforms

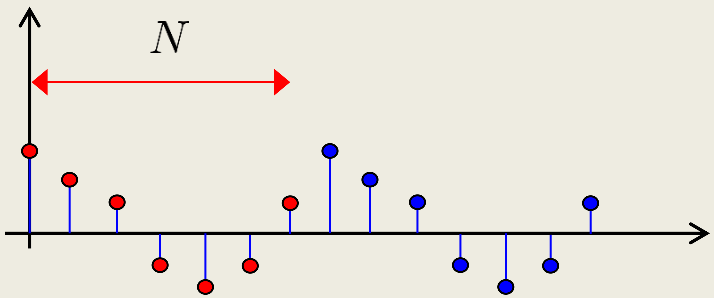
Construct *x(n)* **as**

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Now compute

$$\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N)$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

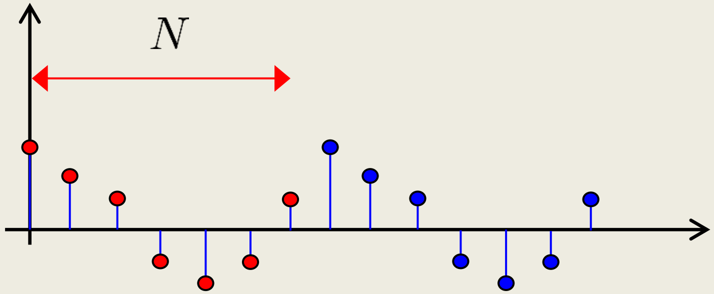
Construct *x(n)* **as**

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Now compute (depends on ℓ**)**

$$\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N)$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

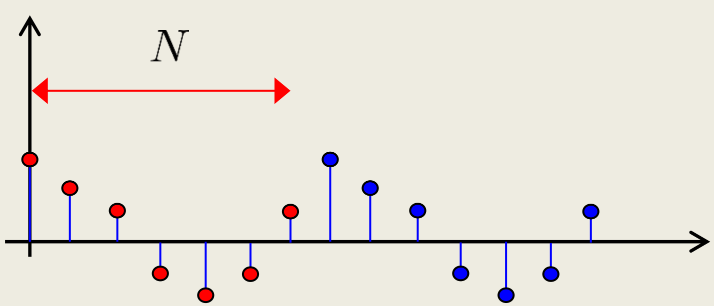
Construct *x(n)* **as**

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Now compute

$$\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N) = \sum_{n=0}^{N-1} \sum_{k=0}^{N-1} c_k \exp(i2\pi (k-\ell)n/N)$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

Construct *x(n)* **as**

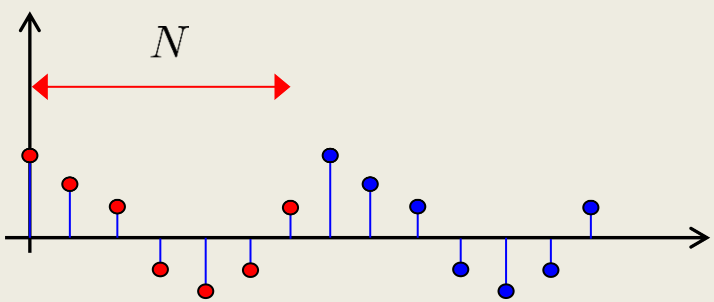
$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Now compute **Switch order**

$$\sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k \exp(i2\pi(k-\ell)n/N)$$

$$= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N)$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

Fact (a geometric sum, so easy to verify)

$$\sum_{n=0}^{N-1} \exp(i2\pi kn/N) = \begin{cases} N, & k = 0,\ \pm N,\ \pm 2N,\ \ldots \\ 0, & \text{otherwise} \end{cases}$$

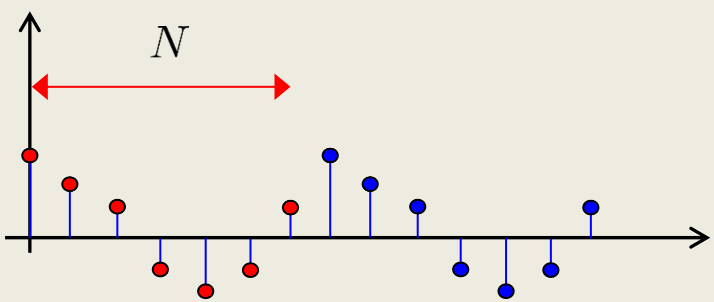
Now compute **Switch order**

$$\sum_{n=0}^{N-1} x(n)\exp(-i2\pi \ell n/N) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k \exp(i2\pi(k-\ell)n/N)$$

$$= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N)$$

Solve this sum

Discrete and periodic **Next case**



EITF75, Fourier transforms

Change k to k-ℓ

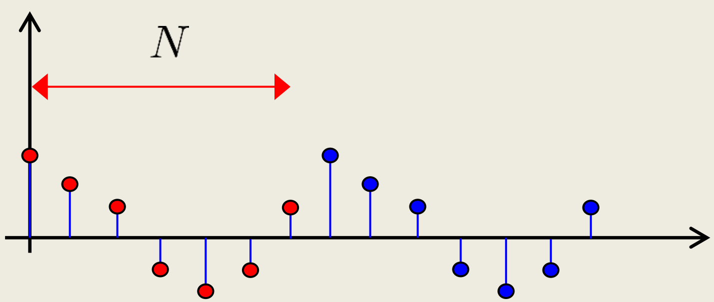
$$\sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N) = \begin{cases} N, & k-\ell = 0,\ \pm N,\ \pm 2N,\ \ldots \\ 0, & \text{otherwise} \end{cases}$$

Now compute **Switch order**

$$\sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k \exp(i2\pi(k-\ell)n/N)$$

$$= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N)$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

Change k to k-ℓ

$$\sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N) = \begin{cases} N, & k-\ell = 0,\ \pm N,\ \pm 2N,\ \ldots \\ 0, & \text{otherwise} \end{cases}$$

Now compute

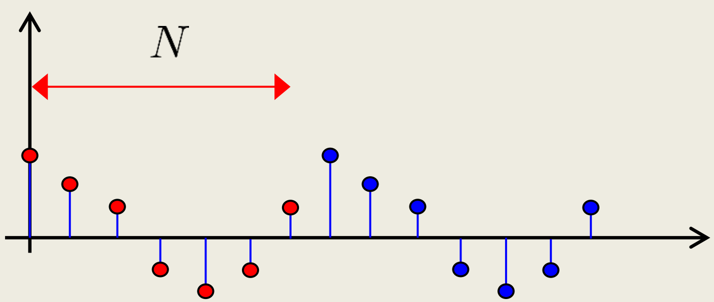
$$\sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k \exp(i2\pi(k-\ell)n/N)$$

$$= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N)$$

Zero if k is not ℓ

Discrete and periodic

Next case



EITF75, Fourier transforms

Change k to k-ℓ

$$\sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N) = \begin{cases} N, & k-\ell = 0,\ \pm N,\ \pm 2N,\ \ldots \\ 0, & \text{otherwise} \end{cases}$$

Now compute

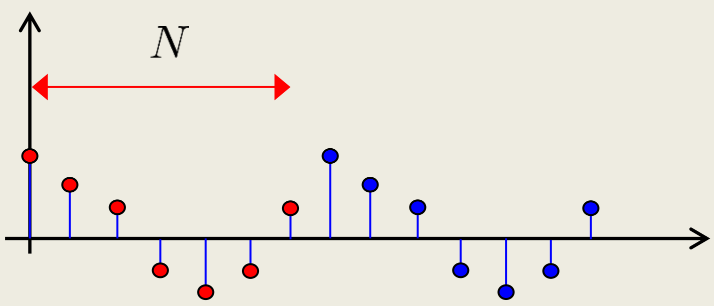
$$\sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k \exp(i2\pi(k-\ell)n/N)$$

$$= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N)$$

Zero if k is not ℓ

$$= c_\ell N$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

Construct *x(n)* as

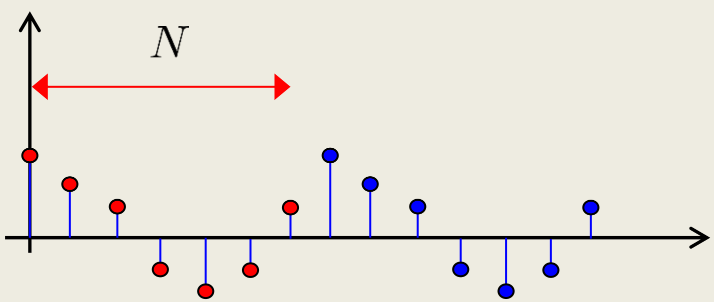
$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Now compute

$$c_\ell N = \sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N) = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k \exp(i2\pi(k-\ell)n/N)$$

$$= \sum_{k=0}^{N-1} c_k \sum_{n=0}^{N-1} \exp(i2\pi(k-\ell)n/N)$$

Discrete and periodic **Next case**



EITF75, Fourier transforms

Synthesis

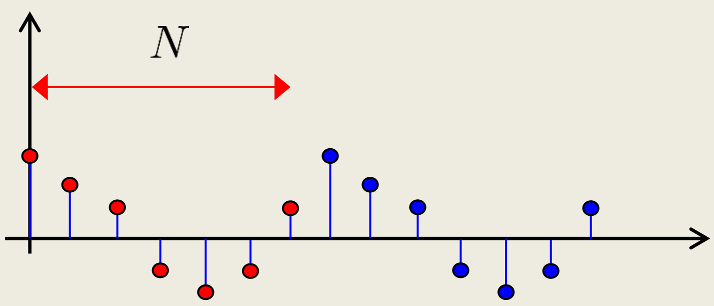
$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi \ell n/N)$$

Discrete and periodic

Next case



Note:

- c_k is a periodic sequence
- Period is N

EITF75, Fourier transforms

Synthesis

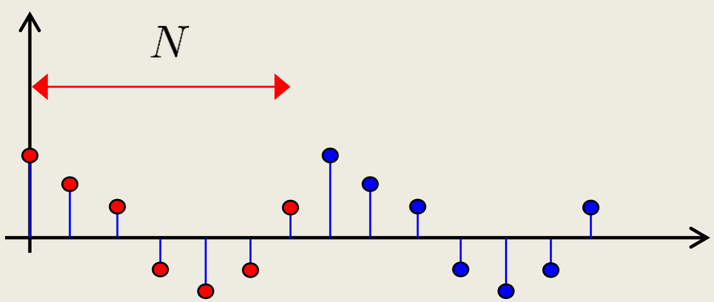
$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N)$$

Alternative view of

Discrete and periodic Next case



EITF75, Fourier transforms

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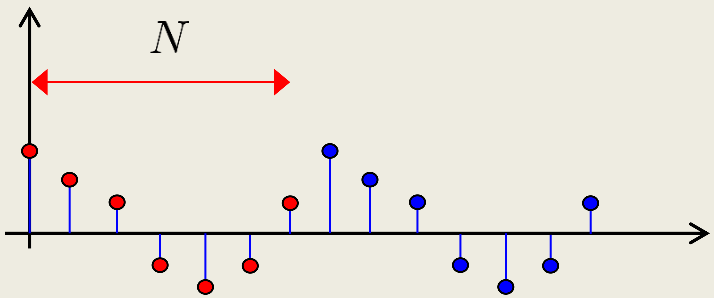
Alternative view of

There are N different values in *x(n)*

Discrete and periodic

Next case



EITF75, Fourier transforms

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Analysis

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Alternative view of

There are N different values in *x(n)*

Put them in a vector

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi \ell n/N)$$

Alternative view of

There are N different values in *x(n)*

Compute c_0

$$\begin{bmatrix} c_0 \end{bmatrix} = \frac{1}{N}\begin{bmatrix} e^{-i2\pi 0\cdot 0/N} & e^{-i2\pi 0\cdot 1/N} & \cdots & e^{-i2\pi 0\cdot (N-1)/N} \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi \ell n/N)$$

Alternative view of

There are N different values in *x(n)*

$$\begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \mathrm{e}^{-i2\pi 0\cdot 0/N} & \mathrm{e}^{-i2\pi 0\cdot 1/N} & \ldots & \mathrm{e}^{-i2\pi 0\cdot (N-1)/N} \\ \mathrm{e}^{-i2\pi 1\cdot 0/N} & \mathrm{e}^{-i2\pi 1\cdot 1/N} & \ldots & \mathrm{e}^{-i2\pi 1\cdot (N-1)/N} \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

Compute c_1

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N)$$

Alternative view of

There are N different values in *x(n)*

Compute all c_k

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} e^{-i2\pi 0\cdot 0/N} & e^{-i2\pi 0\cdot 1/N} & \ldots & e^{-i2\pi 0\cdot (N-1)/N} \\ e^{-i2\pi 1\cdot 0/N} & e^{-i2\pi 1\cdot 1/N} & \ldots & e^{-i2\pi 1\cdot (N-1)/N} \\ & & \vdots & \\ e^{-i2\pi (N-1)\cdot 0/N} & & \ldots & e^{-i2\pi (N-1)\cdot (N-1)/N} \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N)$$

Alternative view of

There are N different values in *x(n)*

Lossless transformation whenever matrix is invertible

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = \frac{1}{N}\begin{bmatrix} \mathrm{e}^{-i2\pi 0\cdot 0/N} & \mathrm{e}^{-i2\pi 0\cdot 1/N} & \ldots & \mathrm{e}^{-i2\pi 0\cdot (N-1)/N} \\ \mathrm{e}^{-i2\pi 1\cdot 0/N} & \mathrm{e}^{-i2\pi 1\cdot 1/N} & \ldots & \mathrm{e}^{-i2\pi 1\cdot (N-1)/N} \\ & & \vdots & \\ \mathrm{e}^{-i2\pi (N-1)\cdot 0/N} & & \ldots & \mathrm{e}^{-i2\pi (N-1)\cdot (N-1)/N} \end{bmatrix}\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N)$$

Alternative view of

There are N different values in *x(n)*

Means that the inverse is like this

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \begin{bmatrix} e^{i2\pi 0\cdot 0/N} & e^{i2\pi 0\cdot 1/N} & \dots & e^{i2\pi 0\cdot (N-1)/N} \\ e^{i2\pi 1\cdot 0/N} & e^{i2\pi 1\cdot 1/N} & \dots & e^{i2\pi 1\cdot (N-1)/N} \\ & & \vdots & \\ e^{i2\pi (N-1)\cdot 0/N} & \dots & & e^{i2\pi (N-1)\cdot (N-1)/N} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n) \exp(-i2\pi \ell n/N)$$

Alternative view of

There are N different values in *x(n)*

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} = \frac{1}{N}\begin{bmatrix} e^{-i2\pi 0\cdot 0/N} & e^{-i2\pi 0\cdot 1/N} & \dots & e^{-i2\pi 0\cdot (N-1)/N} \\ e^{-i2\pi 1\cdot 0/N} & e^{-i2\pi 1\cdot 1/N} & \dots & e^{-i2\pi 1\cdot (N-1)/N} \\ & \vdots & & \\ e^{-i2\pi (N-1)\cdot 0/N} & & \dots & e^{-i2\pi (N-1)\cdot (N-1)/N} \end{bmatrix}\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix}$$

Verify (in matlab) that the inverse of the above is to remove "-" in exponents, and the (1/N) factor

EITF75, Fourier transforms

Synthesis

$$x(n) = \sum_{k=0}^{N-1} c_k \exp(i2\pi kn/N)$$

Analysis

$$c_\ell = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi\ell n/N)$$

Alternative view of

There are N different values in *x(n)*

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{N-1} \end{bmatrix} = A \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

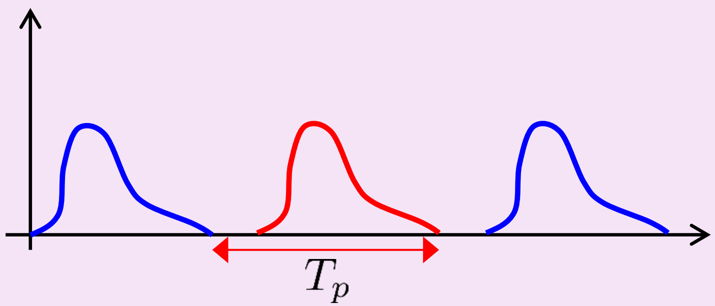
Any invertible matrix A specifies a transformation. In general, an arbitrary A provides no insight into the structure of x(n).

The Fourier A does.

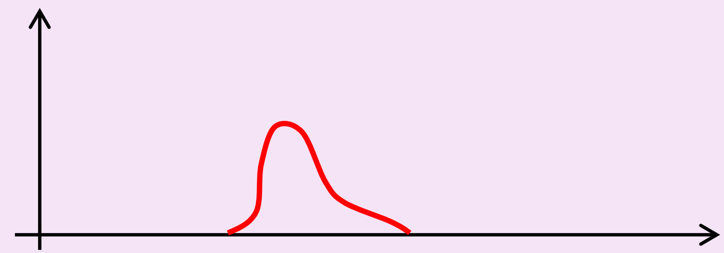
EITF75, Fourier transforms

4 different type of signals

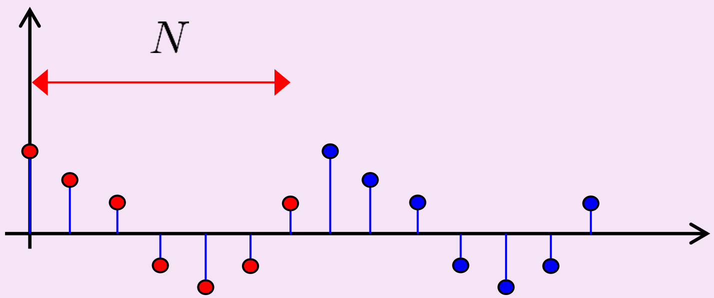
Continuous and periodic — Done



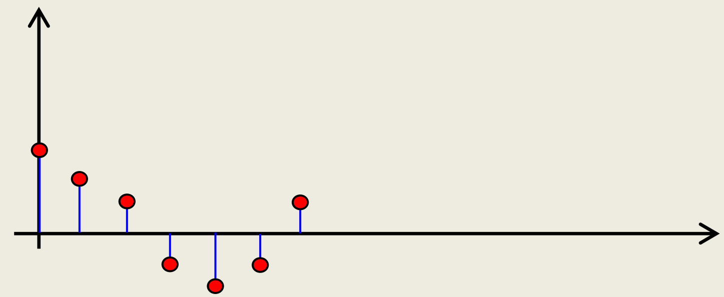
Continuous and aperiodic — Done



Discrete and periodic — Done



Discrete and aperiodic — Next case

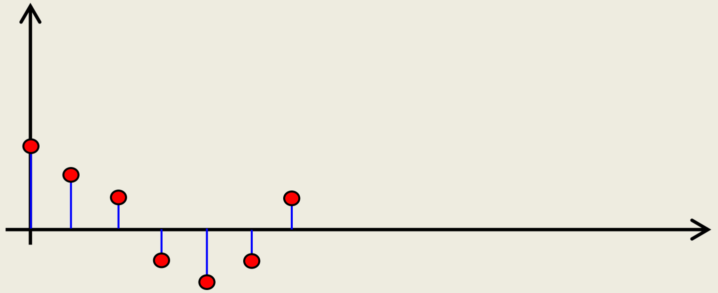


EITF75, Fourier transforms

Claim $$X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi nf)$$

To show: Possible to get x(n) back from X(f)

Discrete and aperiodic



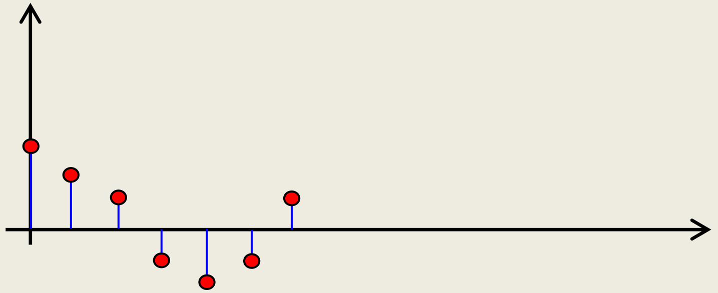
EITF75, Fourier transforms

Claim $X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi nf)$

Discrete and periodic

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp(-i2\pi nk/N)$$

Discrete and aperiodic



EITF75, Fourier transforms

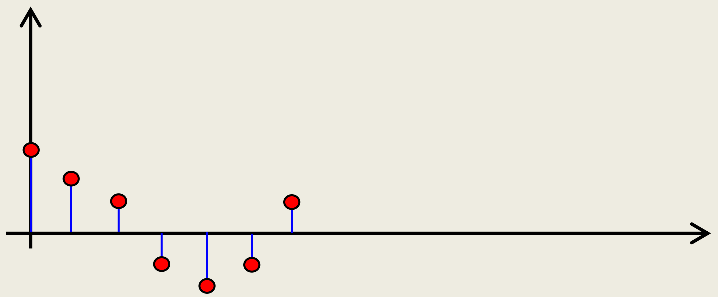
Claim $X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi nf)$

$$Nc_k = X(k/N)$$

Discrete and periodic

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp(-i2\pi nk/N)$$

Discrete and aperiodic



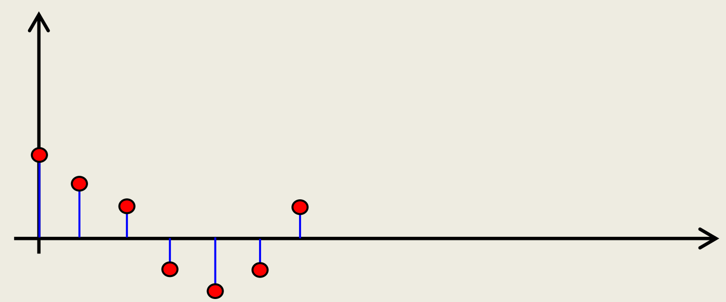
EITF75, Fourier transforms

Claim $$X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi n f)$$

$$N c_k = X(k/N)$$

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k/N) \exp(i2\pi nk/N)$$ Discrete and periodic

Discrete and aperiodic



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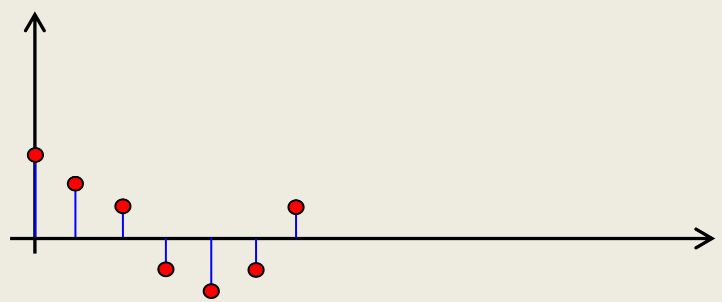
Discrete and aperiodic



$$\Delta f = \frac{1}{N}$$

EITF75, Fourier transforms

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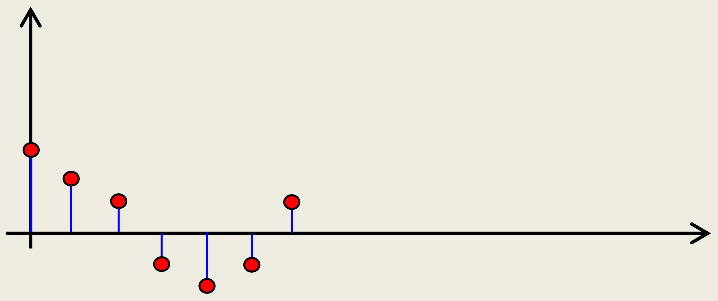
$$N c_k = X(k/N)$$

$$x_p(n) = \quad \Delta f \sum_{k=0}^{N-1} X(k\Delta f) \exp(i2\pi n k \Delta f)$$

Discrete and aperiodic



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EITF75, Fourier transforms

Claim $$X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi n f)$$

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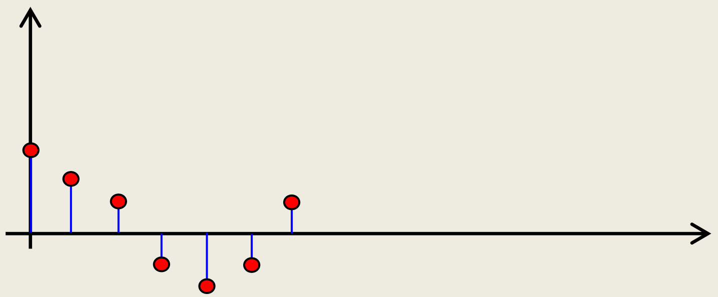
Discrete and aperiodic



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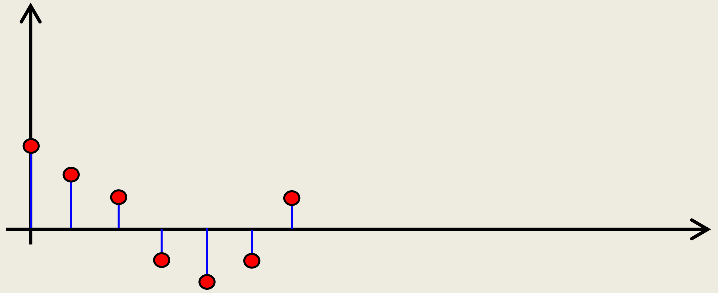
$$= \int_0^1 X(f)\exp(i2\pi n f)\mathrm{d}f$$

Riemann integral

Discrete and aperiodic



EITF75, Fourier transforms

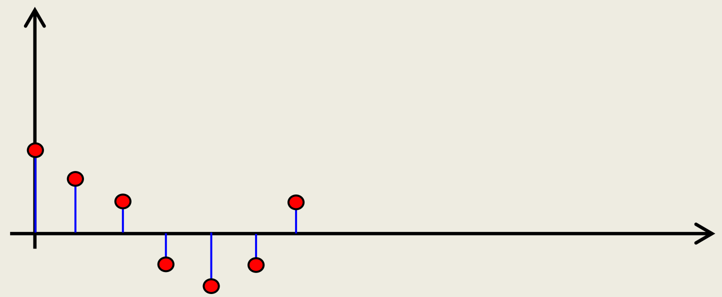
Claim $$X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi n f)$$ **Periodic function, Period =1**

$$Nc_k = X(k/N)$$

$$x(n) = \lim_{N\to\infty} x_p(n) = \lim_{\Delta f\to 0} \Delta f \sum_{k=0}^{N-1} X(k\Delta f) \exp(i2\pi n k \Delta f)$$

$$= \int_0^1 X(f) \exp(i2\pi n f) \mathrm{d}f$$

Discrete and aperiodic



EITF75, Fourier transforms

Claim $$X(f) = \sum_{n=0}^{N-1} x(n)\exp(-i2\pi nf)$$ Periodic function, Period =1

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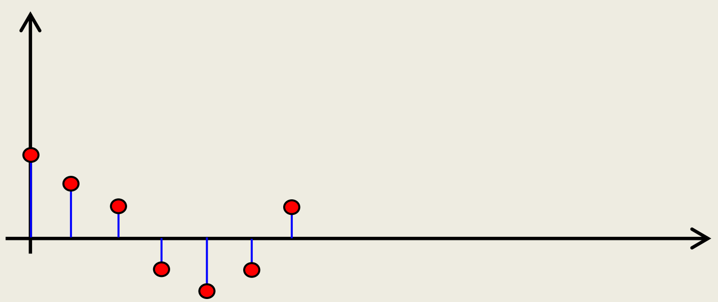
$$= \int_0^1 X(f)\exp(i2\pi nf)\mathrm{d}f$$

Any interval of length 1 gives the same result

Discrete and aperiodic



EITF75, Fourier transforms

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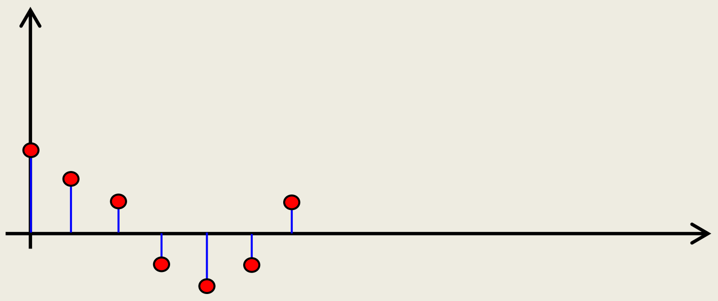
$$x(n) = \lim_{N\to\infty} x_p(n) = \lim_{\Delta f\to 0} \Delta f \sum_{k=0}^{N-1} X(k\Delta f)\exp(i2\pi n k\Delta f)$$

$$= \int_0^1 X(f)\exp(i2\pi n f)\mathrm{d}f$$

$$= \int_{-0.5}^{0.5} X(f)\exp(i2\pi n f)\mathrm{d}f$$

Discrete and aperiodic



Convention: Fundamental period of X(f) between -0.5 < f < 0.5

EITF75, Fourier transforms

Convergence

Consider

$$X(f) = \sum_{n=0}^{N-1} x(n)\exp(-i2\pi n f)$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi nk/N)$$

$$\hat{X}(f) = \sum_{n=0}^{N-1} c_n \exp(i2\pi n f$$

EITF75, Fourier transforms

Convergence

Consider

$$X(f) = \sum_{n=0}^{N-1} x(n)\exp(-i2\pi nf)$$

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\exp(-i2\pi nk/N)$$

$$\hat{X}(f) = \sum_{n=0}^{N-1} c_n \exp(i2\pi nf)$$

Uniform convergence

$$\hat{X}(f) = X(f)$$

if, absolutely summable

$$\sum |x(n)| < \infty$$

Mean square sense convergence

$$\int_{-0.5}^{0.5} |\hat{X}(f) - X(f)|^2 \mathrm{d}f = 0$$

if, square summable

$$\sum |x(n)|^2 < \infty$$

EITF75, Fourier transforms

Convergence

Consider

$$X(f) = \sum_{n=0}^{N-1} x(n) \exp(-i2\pi n f)$$

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp(-i2\pi n k/N)$$

$$\hat{X}(f) = \sum_{n=0}^{N-1} c_n \exp(i2\pi n f)$$

What can we say about this one?

$$x(n) = \frac{1}{n+1} u(n)$$

Uniform convergence

$$\hat{X}(f) = X(f)$$

if, absolutely summable

$$\sum |x(n)| < \infty$$

Mean square sense convergence

$$\int_{-0.5}^{0.5} |\hat{X}(f) - X(f)|^2 \mathrm{d}f = 0$$

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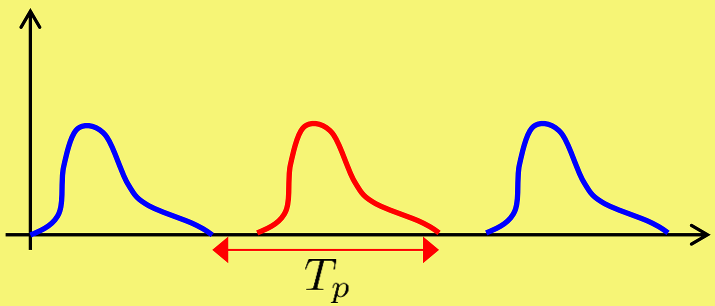
$$\sum |x(n)|^2 < \infty$$

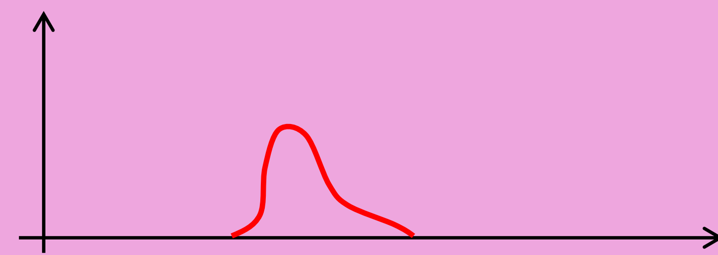
EITF75, Fourier transforms

4 different type of signals

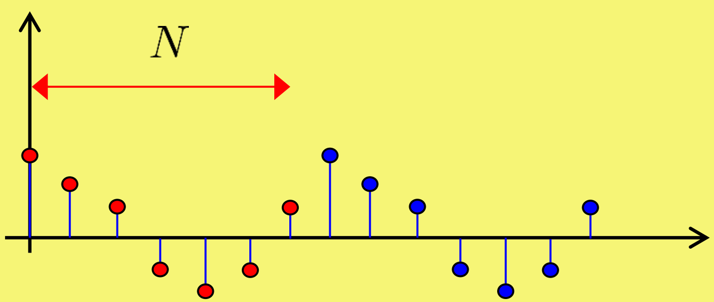
Continuous and periodic



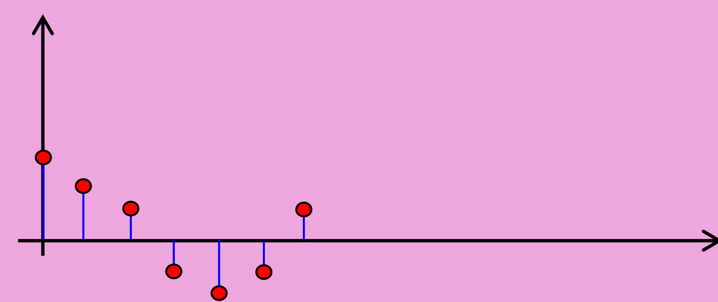
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic



Discrete spectra

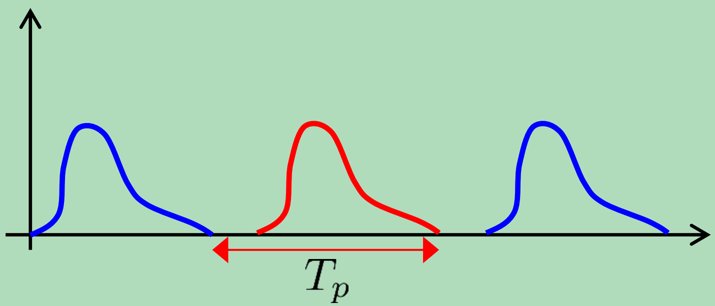
Continuous spectra

EITF75, Fourier transforms

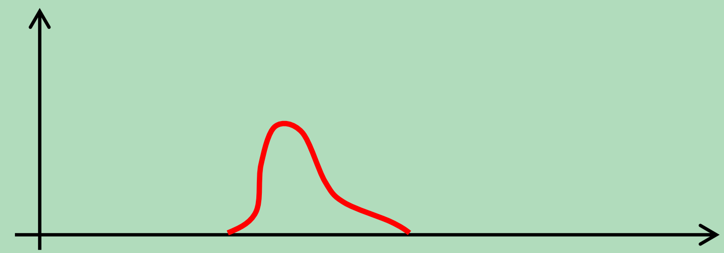
4 different type of signals

Aperiodic spectra

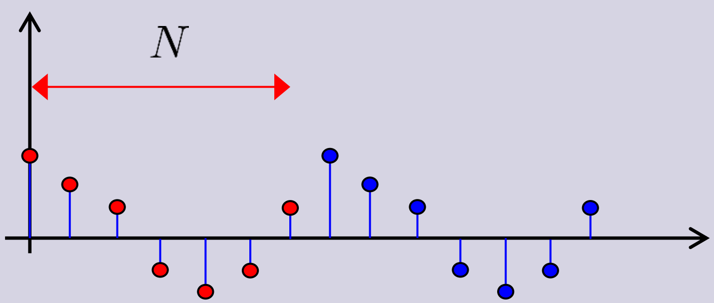
Continuous and periodic



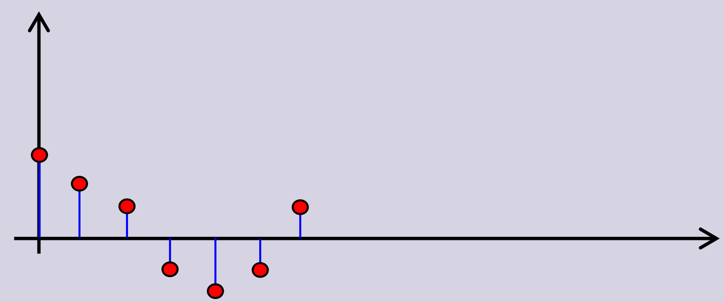
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic



Periodic spectra

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic

Transform is continuous and aperiodic

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}t \quad = \int_{0}^{T} 1 \cdot \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}t$$

By definition

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi F t} \, dt \quad = \int_0^T 1 \cdot e^{-j2\pi F t} \, dt$$

$$= \frac{e^{-j2\pi F T} - 1}{-j2\pi F}$$

Elementary integral

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}t \quad = \int_{0}^{T} 1 \cdot \mathrm{e}^{-\mathrm{j}2\pi F t}\, \mathrm{d}t$$

$$= \frac{\mathrm{e}^{-\mathrm{j}2\pi F T} - 1}{-\mathrm{j}2\pi F} \quad = \frac{\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}\left(\mathrm{e}^{\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F} - \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}\right)}{\mathrm{j}2\pi F}$$

Minor manipulation

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t) \mathrm{e}^{-\mathrm{j}2\pi F t}\,\mathrm{d}t \quad = \int_{0}^{T} 1 \cdot \mathrm{e}^{-\mathrm{j}2\pi F t}\,\mathrm{d}t$$

$$= \frac{\mathrm{e}^{-\mathrm{j}2\pi F T} - 1}{-\mathrm{j}2\pi F} \quad = \frac{\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}\left(\mathrm{e}^{\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F} - \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}\right)}{\mathrm{j}2\pi F}$$

$$= T \cdot \frac{\sin\left(2\pi \cdot \frac{T}{2} \cdot F\right)}{2\pi \cdot \frac{T}{2} \cdot F} \cdot \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{T}{2} \cdot F}$$

Euler’s formula

EITF75, Fourier transforms

Example Find the Fourier transform of a square pulse

$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-\mathrm{j}2\pi Ft}\,\mathrm{d}t = \int_{0}^{T} 1\cdot \mathrm{e}^{-\mathrm{j}2\pi Ft}\,\mathrm{d}t$$

$$= \frac{\mathrm{e}^{-\mathrm{j}2\pi FT}-1}{-\mathrm{j}2\pi F} = \frac{\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}\left(\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}-\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}\right)}{\mathrm{j}2\pi F}$$

$$= T\cdot\frac{\sin\left(2\pi\cdot\frac{T}{2}\cdot F\right)}{2\pi\cdot\frac{T}{2}\cdot F}\cdot \mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}$$

$$= T\cdot \mathrm{sinc}\left(2\cdot\frac{T}{2}\cdot F\right)\cdot \mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{T}{2}\cdot F}$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$\lim_{x\to 0}\frac{\sin(\pi x)}{\pi x} = 1$$

Definition of a sinc-pulse

In 1948, Shannon used this pulse to derive the ultimate limit, in bits/sec, of communication. Super-important pulse in EE

Emre Telatar:
(In brief: Superstar)

"What Shannon's 48 paper has done for communication engineering has no parallel in any engineering field"

EITF75, Fourier transforms

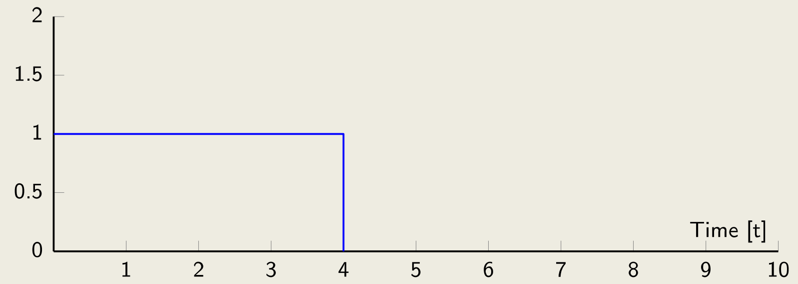
Example Find the Fourier transform of a square pulse

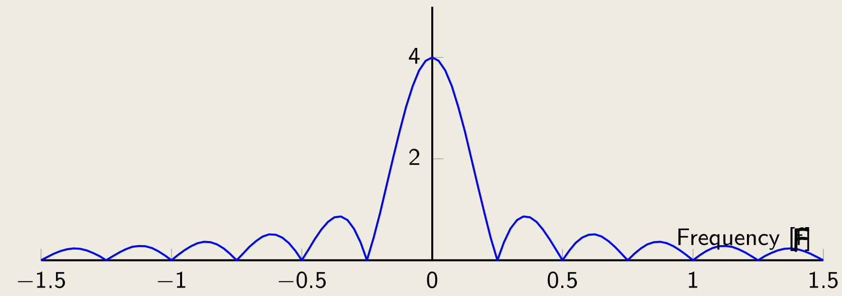
$$x(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$$

Signal is continuous and aperiodic
Transform is continuous and aperiodic

$$X(F) = T \cdot \operatorname{sinc}\left(2 \cdot \frac{T}{2} \cdot F\right) \cdot e^{-j2\pi \cdot \frac{T}{2} \cdot F} \qquad \operatorname{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$x(t) \quad T = 4$ $\qquad |X(F)|$





EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

Note that the periodicity rules out the sinc-shape from the continuous case

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

By definition

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

$$= \frac{1-\mathrm{e}^{-\mathrm{j}\omega N}}{1-\mathrm{e}^{-\mathrm{j}\omega}}$$

Geometric series

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) \mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

$$= \frac{1 - \mathrm{e}^{-\mathrm{j}\omega N}}{1 - \mathrm{e}^{-\mathrm{j}\omega}}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\omega N/2}\left(\mathrm{e}^{\mathrm{j}\omega \cdot \frac{N}{2}} - \mathrm{e}^{-\mathrm{j}\omega \cdot \frac{N}{2}}\right)}{\mathrm{e}^{\mathrm{j}\omega/2}\left(\mathrm{e}^{\mathrm{j}\omega \cdot \frac{1}{2}} - \mathrm{e}^{-\mathrm{j}\omega \cdot \frac{1}{2}}\right)}$$

Manipulation to reach Euler's

EITF75, Fourier transforms

Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} = \sum_{n=0}^{N-1} 1 \cdot \mathrm{e}^{-\mathrm{j}\omega n}$$

$$= \frac{1-\mathrm{e}^{-\mathrm{j}\omega N}}{1-\mathrm{e}^{-\mathrm{j}\omega}}$$

$$= \frac{\mathrm{e}^{\mathrm{j}\omega N/2}\left(\mathrm{e}^{\mathrm{j}\omega\cdot\frac{N}{2}} - \mathrm{e}^{-\mathrm{j}\omega\cdot\frac{N}{2}}\right)}{\mathrm{e}^{\mathrm{j}\omega/2}\left(\mathrm{e}^{\mathrm{j}\omega\cdot\frac{1}{2}} - \mathrm{e}^{-\mathrm{j}\omega\cdot\frac{1}{2}}\right)}$$

$$= N \cdot \frac{\sin\left(\omega\cdot\frac{N}{2}\right)}{N\sin\left(\omega\cdot\frac{1}{2}\right)} \cdot \mathrm{e}^{-\mathrm{j}\omega(N-1)/2}$$

Period = 1

EITF75, Fourier transforms

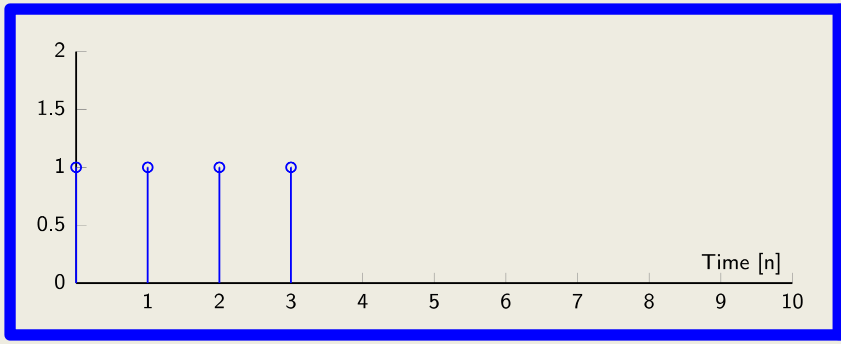
Example Find the Fourier transform of a discrete square pulse

$$x(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}$$

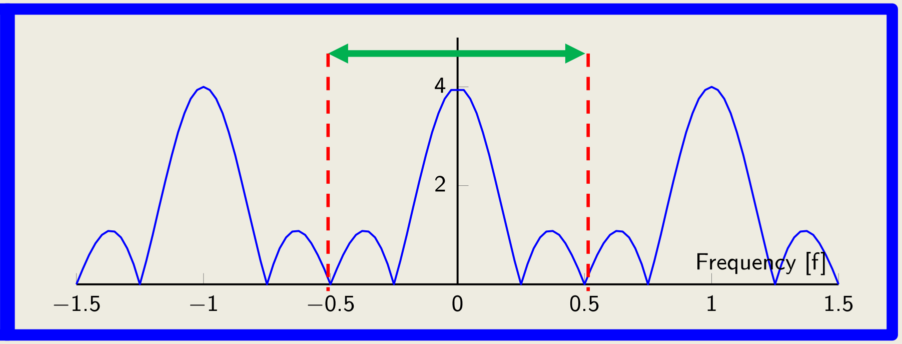
Signal is discrete and aperiodic
Transform is continuous and periodic

$$X(\omega) = N \cdot \frac{\sin\left(\omega \cdot \frac{N}{2}\right)}{N \sin\left(\omega \cdot \frac{1}{2}\right)} \cdot e^{-j\omega(N-1)/2}$$

$x(n)$ $N = 4$



$|X(f)|$



Period = 1

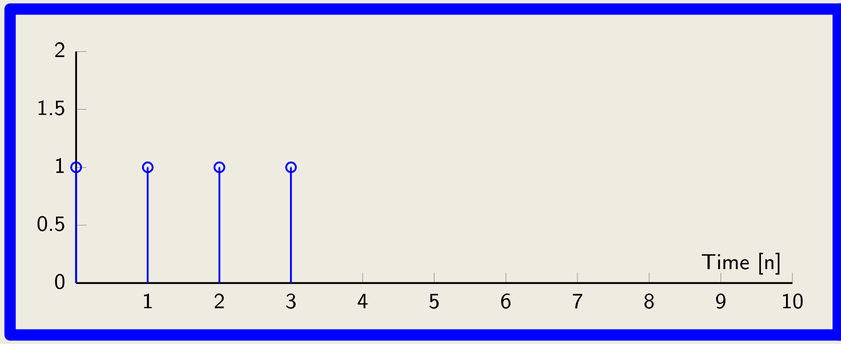
EITF75, Fourier transforms

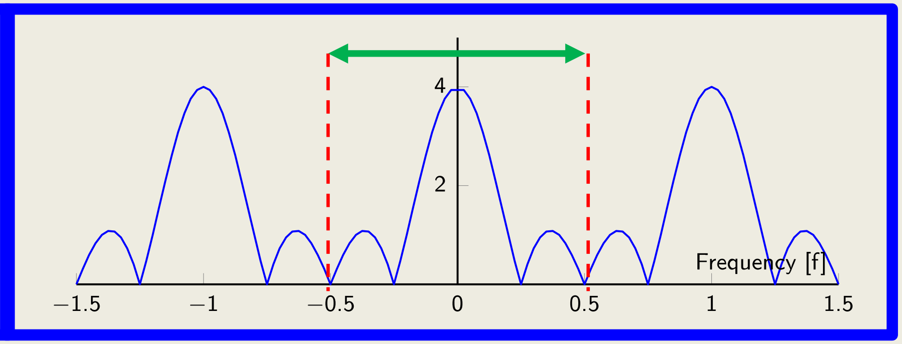
Remember?

$$\int_0^1 X(f)\exp(i2\pi nf)\mathrm{d}f \;=\int_{-0.5}^{0.5} X(f)\exp(i2\pi nf)\mathrm{d}f$$

$x(n)\;\; N=4$

$|X(f)|$





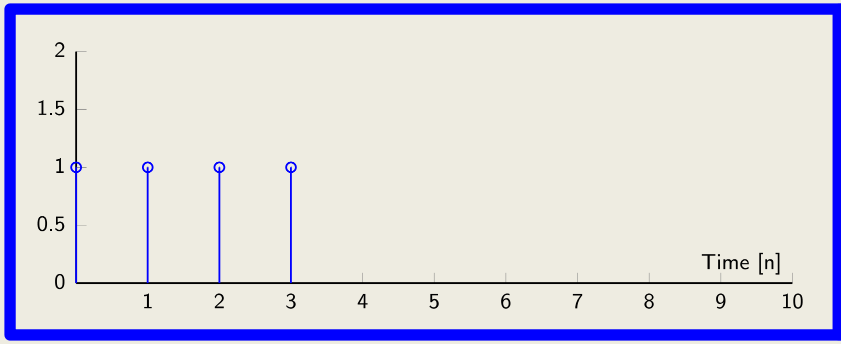
Period = 1

EITF75, Fourier transforms

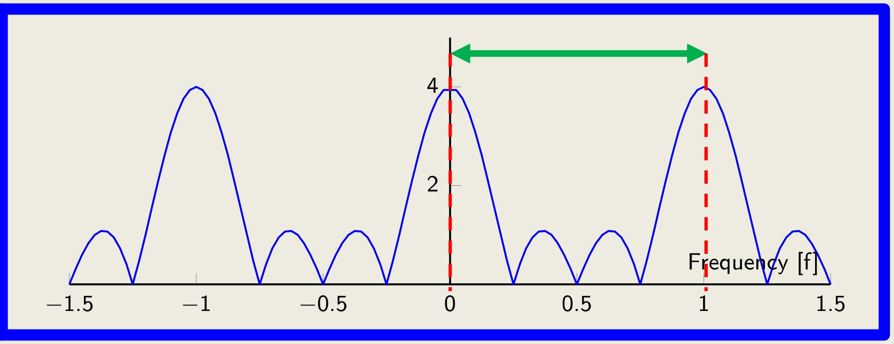
Remember?

$$\int_0^1 X(f)\exp(i2\pi nf)\mathrm{d}f = \int_{-0.5}^{0.5} X(f)\exp(i2\pi nf)\mathrm{d}f$$

$x(n)\ \ N=4$



$|X(f)|$



Same content, different order