EITF75 Systems and Signals

Lecture 3
The z-transform

Fredrik Rusek

EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

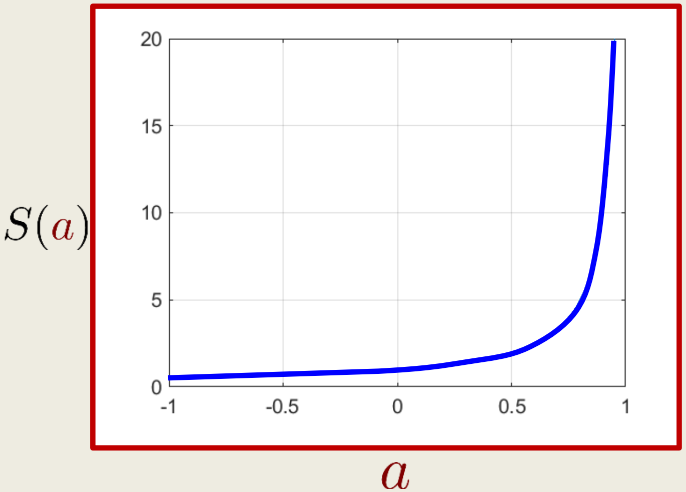
$$\boxed{S(a) = \sum_{n=0}^{\infty} a^n}$$

EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$S(a) = \sum_{n=0}^{\infty} a^n$$



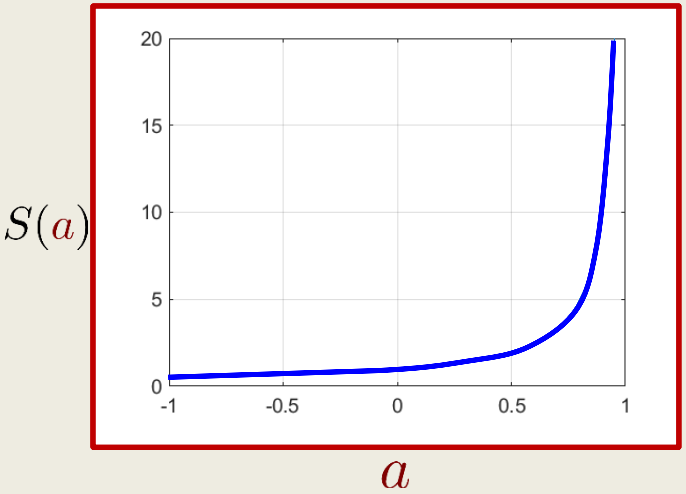
If $|a| \geq 1$ the sum is not convergent

EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1$$



If $|a| \geq 1$ the sum is not convergent

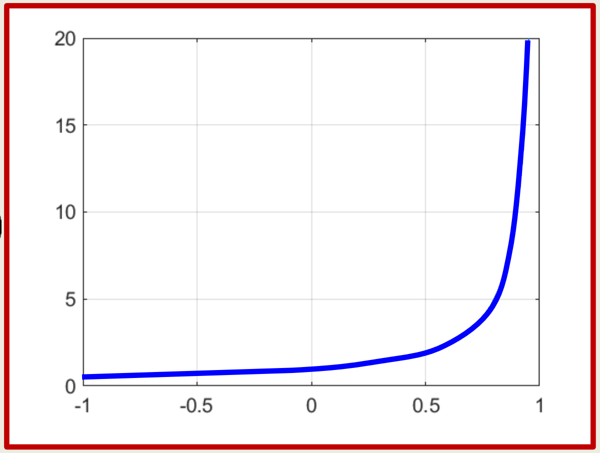
EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1$$

$S(a)$



a

Logics and a Question:

- We know that

$$S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1$$

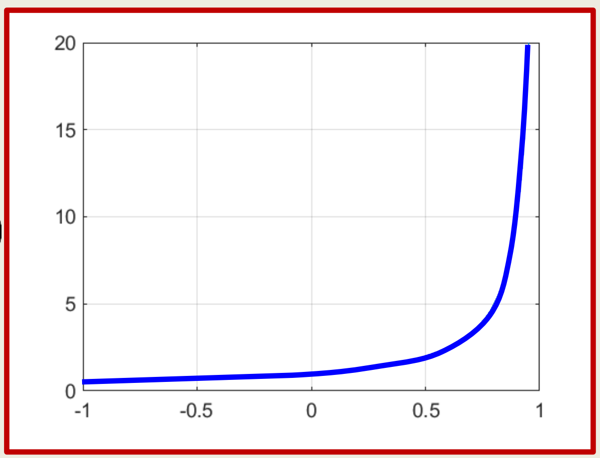
EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$\boxed{S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1}$$

$S(a)$



a

Logics and a Question:

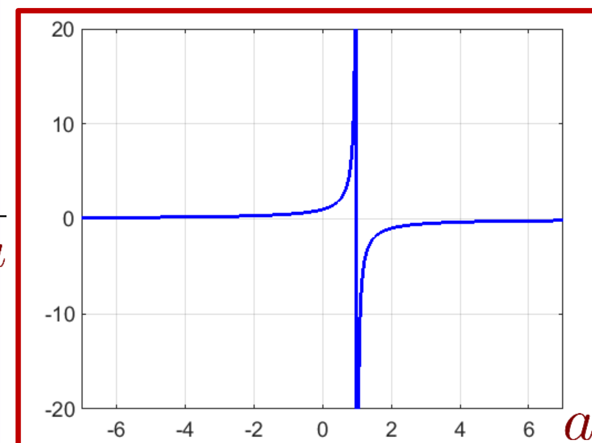
- We know that

$$S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1$$

- However, the function $\dfrac{1}{1-a}$

  can be evaluated for all $a \neq 1$

$$\frac{1}{1-a}$$



a

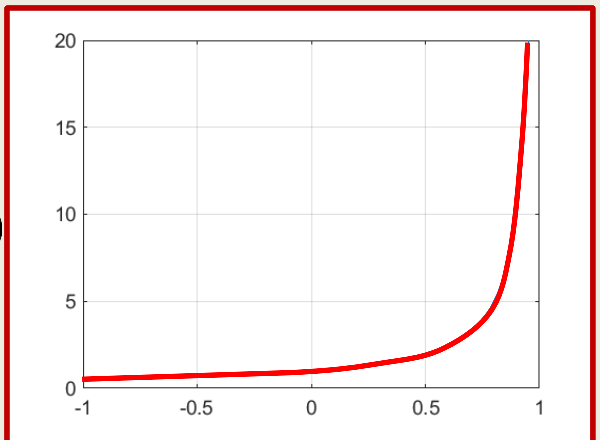
EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$\boxed{S(a) = \sum_{n=0}^{\infty} a^n \;=\; \frac{1}{1-a}, \quad |a| < 1}$$

$S(a)$



a

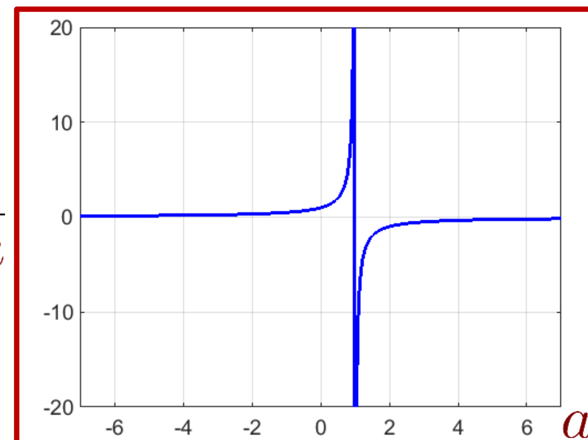
Logics and a Question:

- We know that

$$S(a) = \sum_{n=0}^{\infty} a^n \;=\; \frac{1}{1-a}, \quad |a| < 1$$

- However, the function $\dfrac{1}{1-a}$ can be evaluated for all $a \neq 1$

$\dfrac{1}{1-a}$



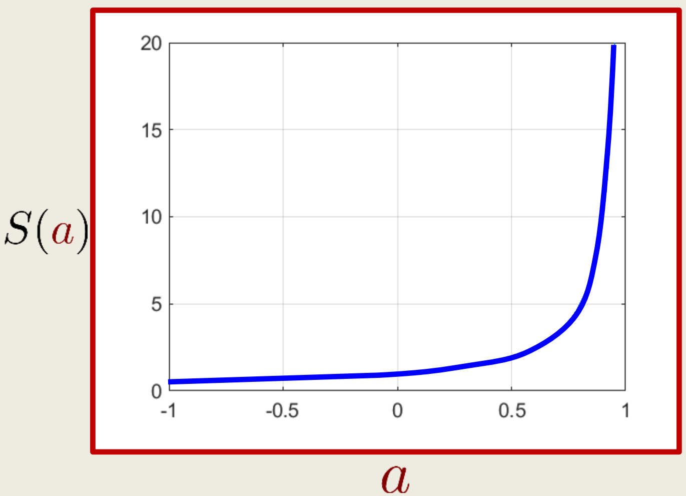
a

EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$\boxed{S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1}$$



Logics and a Question:

- We know that

$$S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1$$

- However, the function $\dfrac{1}{1-a}$

  can be evaluated for all $a \neq 1$

- Is it correct to say that

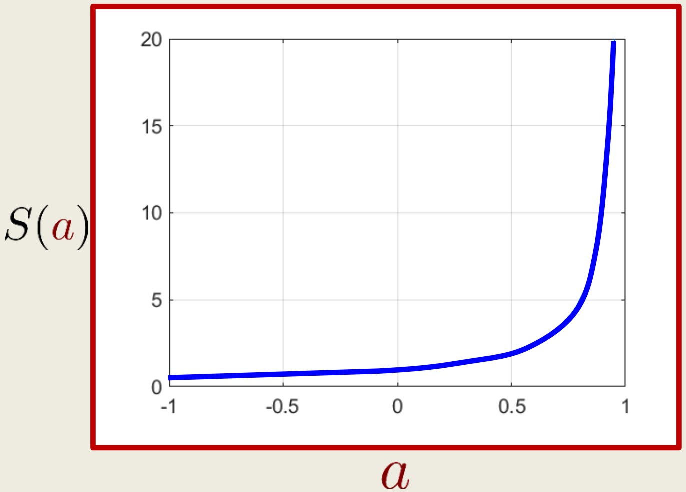
$$S(a) = \frac{1}{1-a}$$

EITF75 Systems and Signals

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$\boxed{S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1}$$



Logics and a Question:

- We know that

$$S(a) = \sum_{n=0}^{\infty} a^n \quad = \frac{1}{1-a}, \quad |a| < 1$$

- However, the function $\dfrac{1}{1-a}$

  can be evaluated for all $a \neq 1$

- Is it correct to say that

$$S(a) = \frac{1}{1-a}$$

NO

EITF75, z-transform

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

EITF75, z-transform

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

What is the z-transform?

- A map from sequences to complex valued functions

What is H(z)?

- A complex function of a complex number

EITF75, z-transform

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

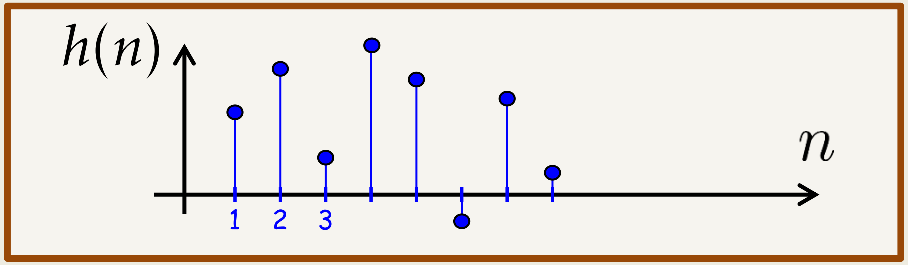
What is the z-transform?

- A map from sequences to complex valued functions

What is H(z)?

- A complex function of a complex number

Illustration



z-transform

EITF75, z-transform

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

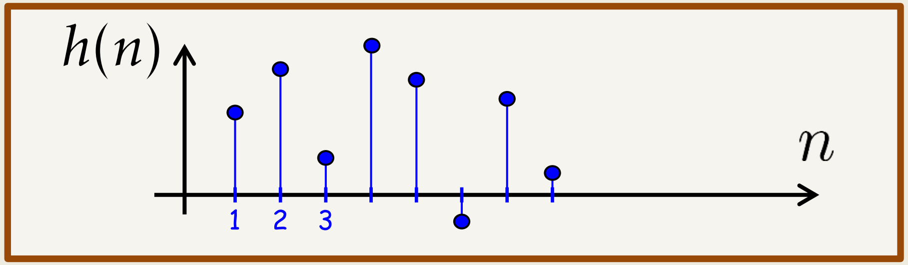
What is the z-transform?

- A map from sequences to complex valued functions

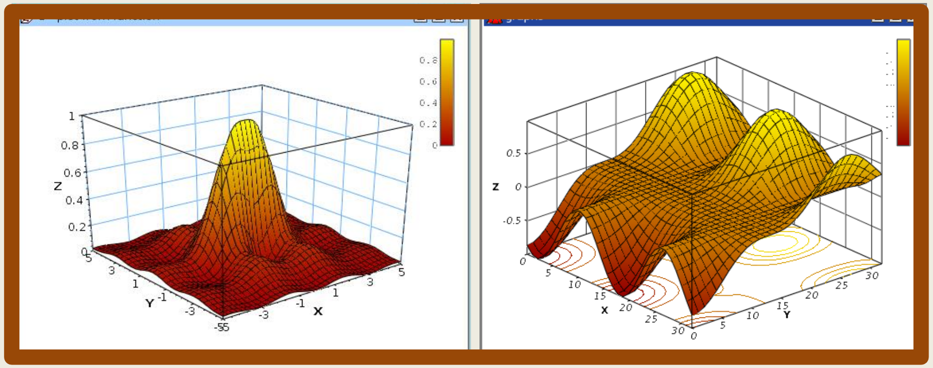
What is H(z)?

- A complex function of a complex number

Illustration



z-transform



If we want to plot H(z), we need 2 plots, one for the real part, one for the imaginary

EITF75, z-transform

Definition

Illustration

The z-transform of $h(n)$ is defined as

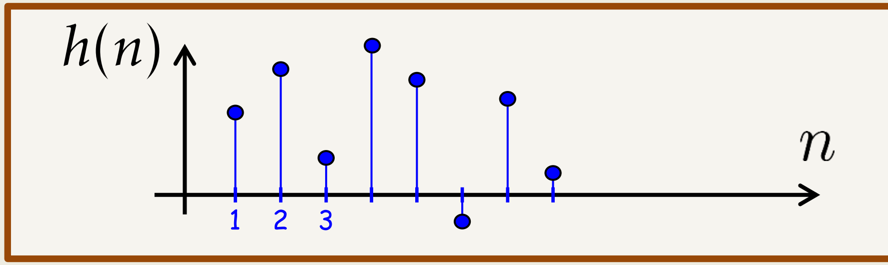
$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

What is the z-transform?

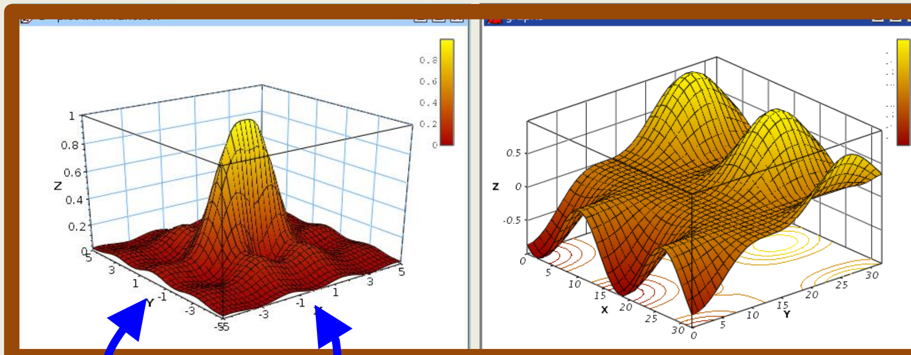
- A map from sequences to complex valued functions

What is H(z)?

- A complex function of a complex number



$\mathrm{Re}\{H(z)\}$ $\mathrm{Im}\{H(z)\}$



$\mathrm{Re}\{z\}$ $\mathrm{Im}\{z\}$

EITF75, z-transform

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

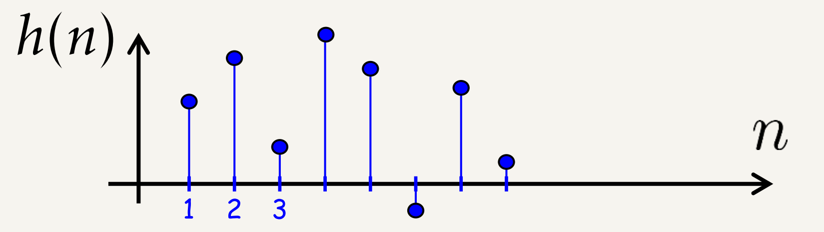
What is the z-transform?

- A map from sequences to complex valued functions

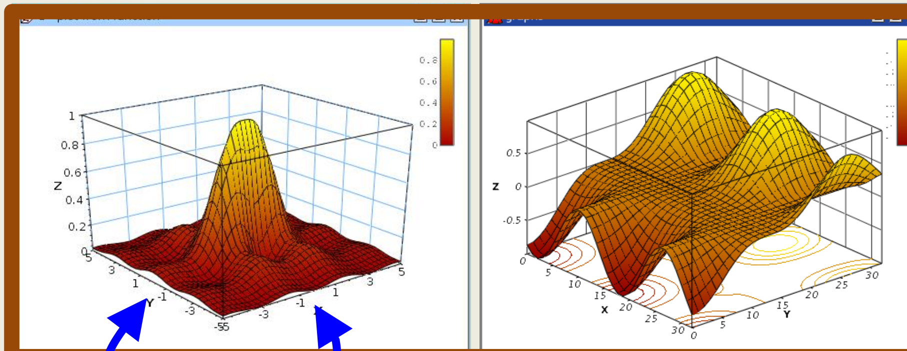
What is H(z)?

- A complex function of a complex number

Illustration



$\mathrm{Re}\{H(z)\}$ $\mathrm{Im}\{H(z)\}$



$\mathrm{Re}\{z\}$ $\mathrm{Im}\{z\}$

Important: h(n) and H(z) contain the same information

EITF75, z-transform

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

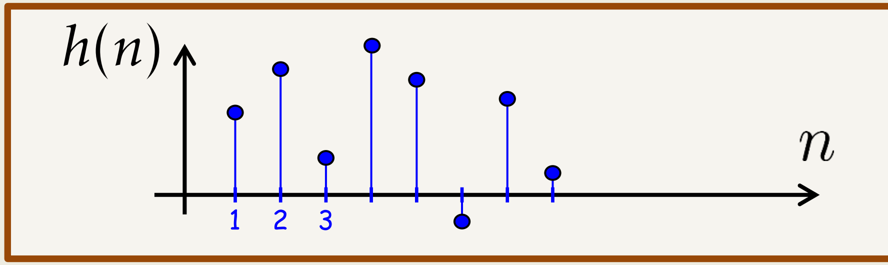
What is the z-transform?

- A map from sequences to complex valued functions

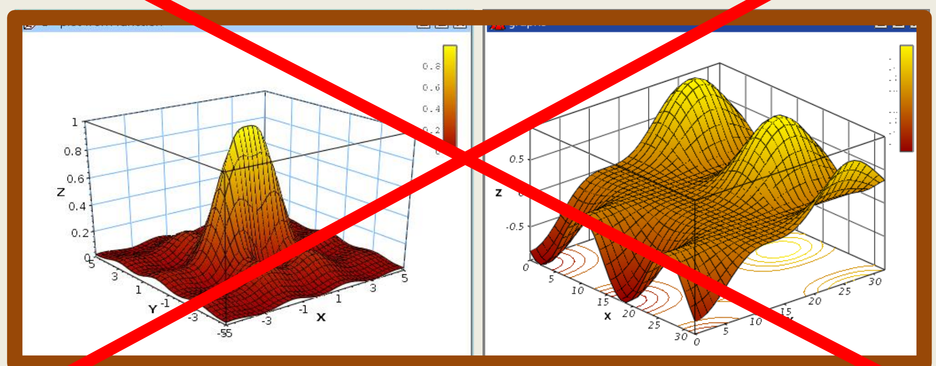
What is H(z)?

- A complex function of a complex number

Illustration



$\mathrm{Re}\{H(z)\}$ $\mathrm{Im}\{H(z)\}$



Luckily, very seldom we need to plot these figures

EITF75, z-transform

Definition

Illustration

The z-transform of $h(n)$ is defined as

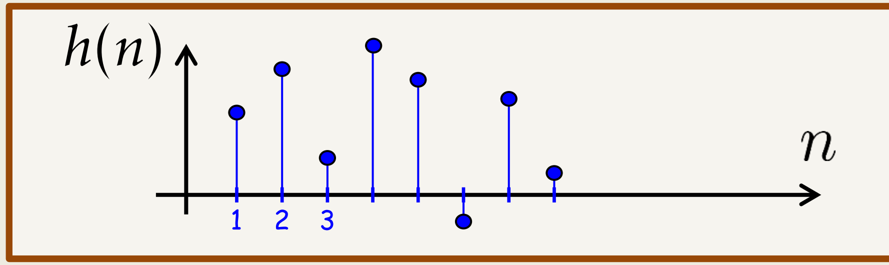
$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

What is the z-transform?

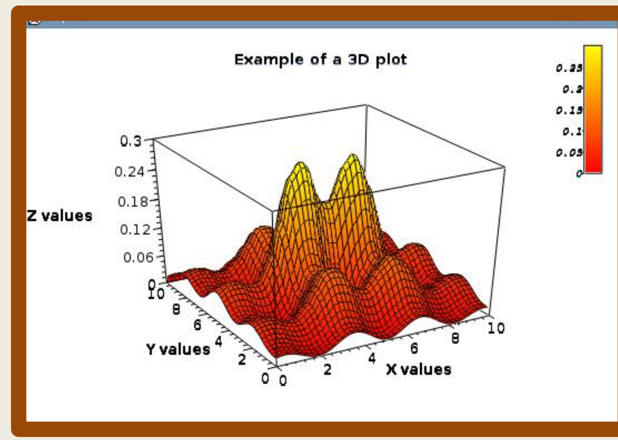
- A map from sequences to complex valued functions

What is H(z)?

- A complex function of a complex number



$$\left| H(z) \right|$$



Magnitude of H(z) is typically shown

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$

Obvious from definition

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\quad 0\quad \ldots\ \}$	$\Leftrightarrow$	1

Obvious from definition

Important to remember

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1} \quad 0 \quad \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}

Obvious from definition

Important to remember

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \left\{ \underline{1} \quad 0 \quad \ldots \right\}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof:

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{ \underline{1} \quad 0 \quad \ldots \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof: **set** $y(n) = x(n-1)$ Y(n) is x(n) one step delayed

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\quad 0\quad \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof: **set** $y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_{n} y(n) z^{-n}$

Definition of Y(z)

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{ \underline{1} \quad 0 \quad \ldots \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof: **set** $y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_n y(n) z^{-n} = \sum_n x(n-1) z^{-n}$

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\quad 0\quad \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof: set $y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_n y(n)z^{-n} = \sum_n x(n-1)z^{-n}$

$$= z^{-1}\sum_n x(n-1)z^{-(n-1)}$$

Pull out one z^{-1}

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{ \underline{1} \quad 0 \quad \ldots \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof: set $y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_n y(n)z^{-n} = \sum_n x(n-1)z^{-n}$

$$= z^{-1}\sum_n x(n-1)z^{-(n-1)} = z^{-1}\sum_m x(m)z^{-m}$$

Variable change: m=n-1

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{ \underline{1} \quad 0 \quad \ldots \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$

Proof: set $y(n) = x(n-1) \quad \Leftrightarrow \quad Y(z) = \sum_n y(n) z^{-n} = \sum_n x(n-1) z^{-n}$

Definition of X(z)

$$= z^{-1} \sum_n x(n-1) z^{-(n-1)} = z^{-1} \sum_m x(m) z^{-m} = z^{-1} X(z)$$

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\quad 0\quad \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$
$h_1(n) = \{\ \underline{3}\quad 2\quad 1\ \}$	$\Leftrightarrow$	$H_1(z) = 3 + 2z^{-1} + z^{-2}$

Obvious from definition

EITF75, z-transform

Some examples and one property

Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\quad 0\quad \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$
$h_1(n) = \{\ \underline{3}\quad 2\quad 1\ \}$	$\Leftrightarrow$	$H_1(z) = 3 + 2z^{-1} + z^{-2}$
$h_2(n) = \{\ \underline{0}\quad 3\quad 2\quad 1\ \}$	$\Leftrightarrow$	$H_2(z) =$

Can use definition or the delay property

EITF75, z-transform

Some examples and one property

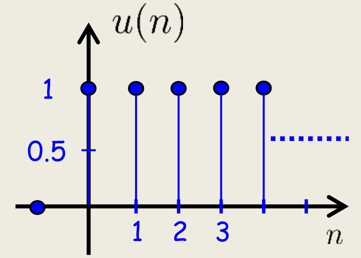
Function	$\Leftrightarrow$	z-transform
$h(n)$	$\Leftrightarrow$	$H(z) = h(0) + h(1)z^{-1} + h(2)z^{-2} + \cdots$
$\delta(n) = \{\ \underline{1}\quad 0\quad \ldots\ \}$	$\Leftrightarrow$	1
$\delta(n-k)$	$\Leftrightarrow$	z^{-k}
$h(n-k)$	$\Leftrightarrow$	$z^{-k}H(z)$
$h_1(n) = \{\ \underline{3}\quad 2\quad 1\ \}$	$\Leftrightarrow$	$H_1(z) = 3 + 2z^{-1} + z^{-2}$
$h_2(n) = \{\ \underline{0}\quad 3\quad 2\quad 1\ \}$	$\Leftrightarrow$	$H_2(z) = 0 + 3z^{-1} + 2z^{-2} + z^{-3} = z^{-1}\left(3 + 2z^{-1} + z^{-2}\right)$

Can use definition or the delay property

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n}$$



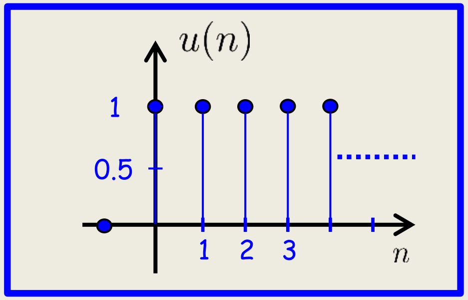
EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n}$$

Geometric series

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}}$$



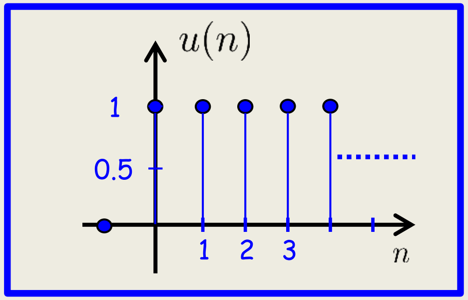
EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-z^{-1}} \qquad \text{if } |z| > 1$$

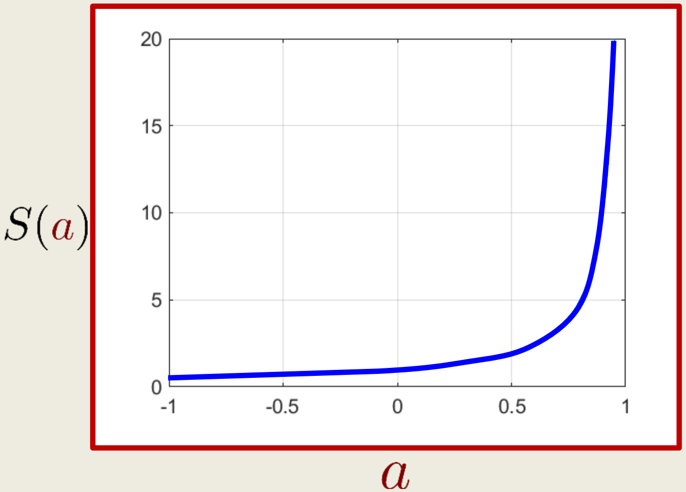


Superimportant not to forget this

Something completely different

Let us define an asymptotic geometric sum, where we show the explicit dependency on a

$$\boxed{S(a) = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1}$$



Logics and a Question:

- We know that

$$S(a) = \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \quad |a| < 1$$

- However, the function $\dfrac{1}{1-a}$ can be evaluated for all $a \neq 1$

- Is it correct to say that

$$S(a) = \frac{1}{1-a}$$

NO

Recall

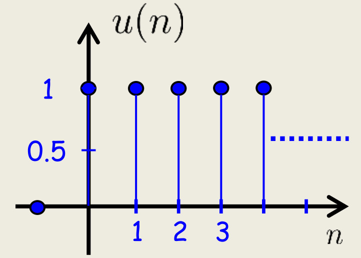
EITF75, z-transform

Some important examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-z^{-1}} \qquad \text{if } |z| > 1 \text{ (ROC)}$$



Summary. We **cannot** say that

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1-z^{-1}}$$

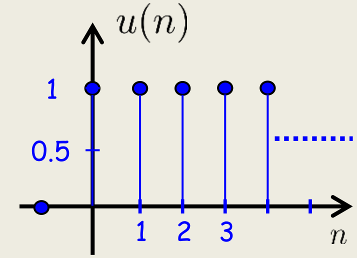
EITF75, z-transform

Some important examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-z^{-1}} \qquad \text{if } |z| > 1 \text{ (ROC)}$$



Summary. We **cannot** say that

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1-z^{-1}}$$

Because, e.g., $H(1/2) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{-n} = \infty$

but $\frac{1}{1-(1/2)^{-1}} = -1$

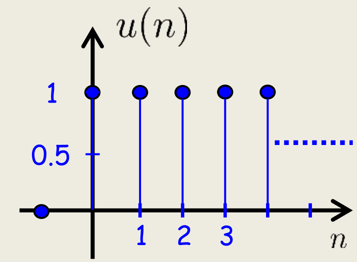
EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} z^{-n}$$

$$= \frac{1-(z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-z^{-1}} \qquad \text{if } |z|>1 \text{ (ROC)}$$



Summary. We **can** say that

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1-z^{-1}}$$

$$\text{ROC} \quad |z|>1$$

Because, e.g., $H(2) = \sum_{n=0}^{\infty} 2^{-n} = 2$

and $\frac{1}{1-2^{-1}} = 2$

CORRECTION

Definition

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$

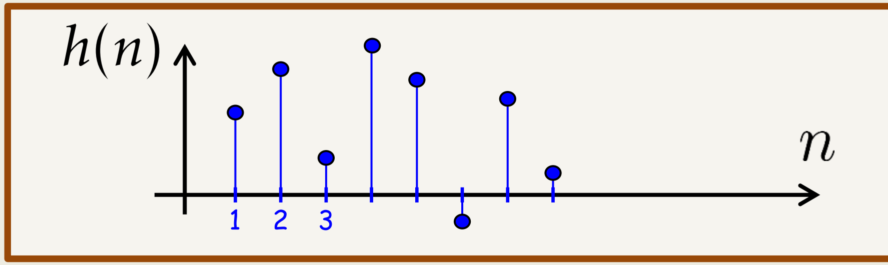
What is the z-transform?

- A map from sequences to complex valued functions

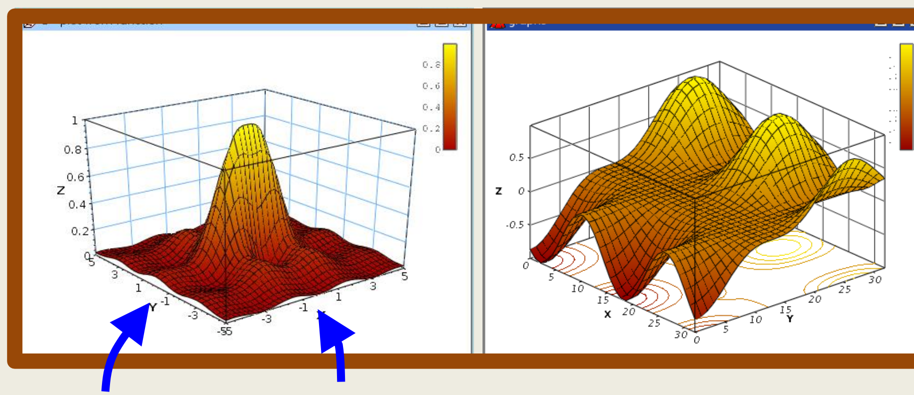
What is H(z)?

- A complex function of a complex number

Illustration



$\mathrm{Re}\{H(z)\}$ $\mathrm{Im}\{H(z)\}$



$\mathrm{Re}\{z\}$ $\mathrm{Im}\{z\}$

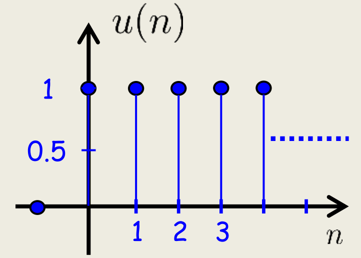
Important: h(n) and H(z)+ROC contain The same information

EITF75, z-transform

Some *important* examples

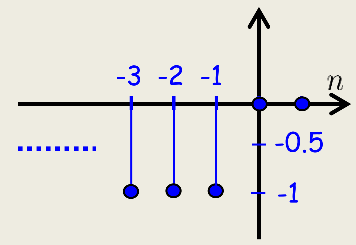
$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\text{ROC} \quad |z| > 1$$



$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n}$$

$$h(n) = -u(-n-1)$$



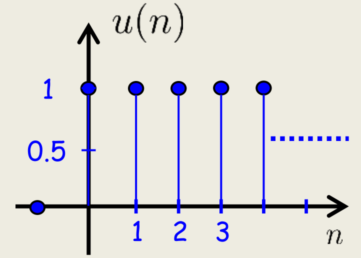
Anti-causal step

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

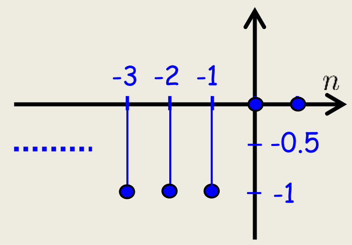
$$\text{ROC} \quad |z| > 1$$



$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} z^{-n}$$

$$h(n) = -u(-n-1)$$



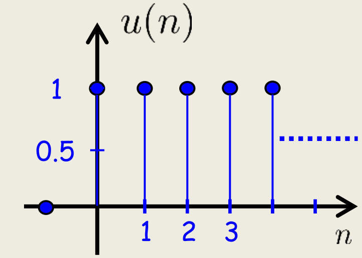
Anti-causal step

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1-z^{-1}}$$

$$\text{ROC} \quad |z| > 1$$

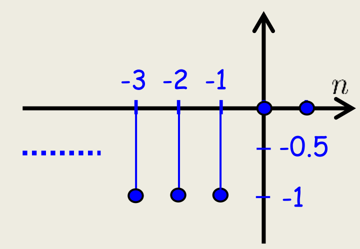


$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=1}^{\infty} z^{n}$$

Variable change

$$h(n) = -u(-n-1)$$



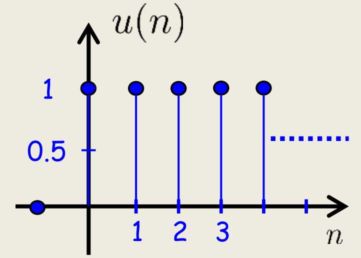
Anti-causal step

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

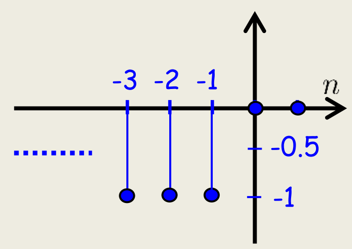
$$\text{ROC} \quad |z| > 1$$



$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n} \\ &= -\sum_{n=-\infty}^{-1} z^{-n} \quad = \quad -\sum_{n=0}^{\infty} z^{n} + 1 \end{aligned}$$

Trick to get a standard geometric series

$$h(n) = -u(-n-1)$$



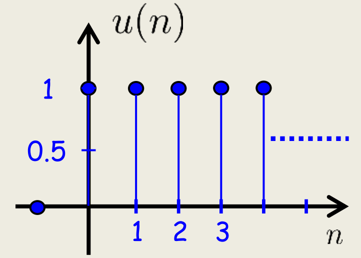
Anti-causal step

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\text{ROC} \quad |z| > 1$$



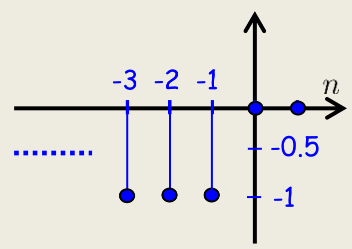
$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1$$

$$= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z}$$

Let us first ignore the ROC

$$h(n) = -u(-n-1)$$



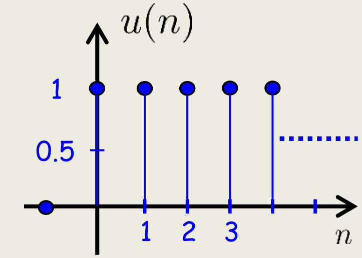
Anti-causal step

EITF75, z-transform

Some *important* examples

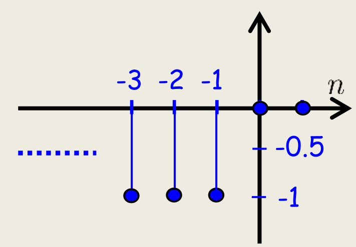
$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\text{ROC} \quad |z| > 1$$



$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n}$$
$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1$$
$$= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z}$$
$$= \frac{1}{1-z^{-1}}$$

$$h(n) = -u(-n-1)$$



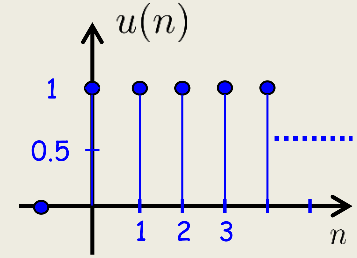
Anti-causal step

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1-z^{-1}}$$

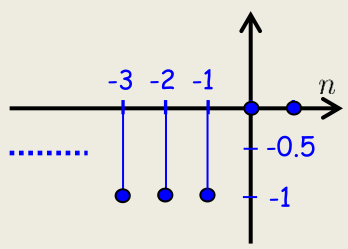
$$\text{ROC} \quad |z| > 1$$



$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n} \\ &= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1 \\ &= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z} \\ &= \frac{1}{1-z^{-1}} \end{aligned}$$

Equal transforms if ROC left unspecified

$$h(n) = -u(-n-1)$$



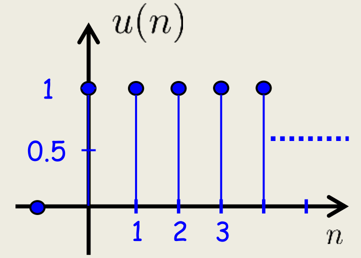
Anti-causal step

EITF75, z-transform

Some important examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

ROC $|z| > 1$



$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1)z^{-n}$$

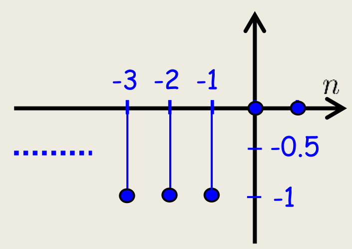
Let's specify the ROC

$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1$$

$$= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z}$$

$$= \frac{1}{1-z^{-1}}$$

$$h(n) = -u(-n-1)$$



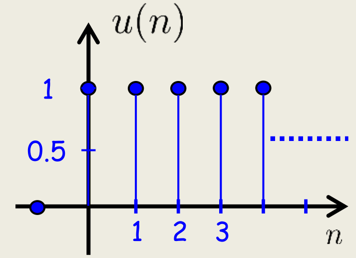
Anti-causal step

EITF75, z-transform

Some *important* examples

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

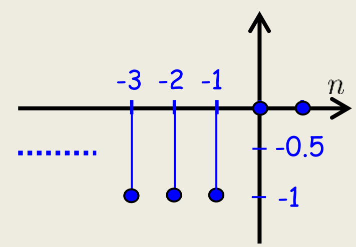
$$\text{ROC} \quad |z| > 1$$



$$
\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n} \\
&= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1 \\
&= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z} \\
&= \frac{1}{1-z^{-1}} \qquad \text{ROC} \quad |z| < 1
\end{aligned}
$$

Let's specify the ROC

$$h(n) = -u(-n-1)$$



Anti-causal step

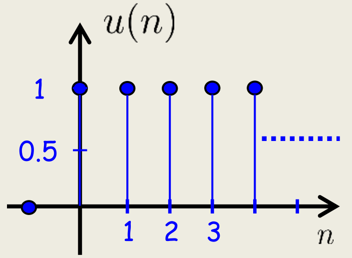
EITF75, z-transform

Some *important* examples

Causal step

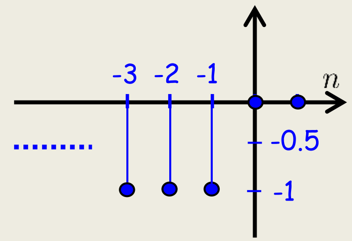
$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\mathrm{ROC} \quad |z| > 1$$



$$h(n) = -u(-n-1) \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\mathrm{ROC} \quad |z| < 1$$



Anti-causal step

Summary. Different sequences have the same H(z) but different ROCs

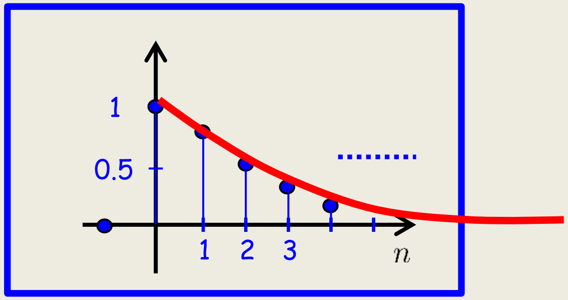
We must specify ROC, otherwise ambiguous

EITF75, z-transform

More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

Exponentially damped step

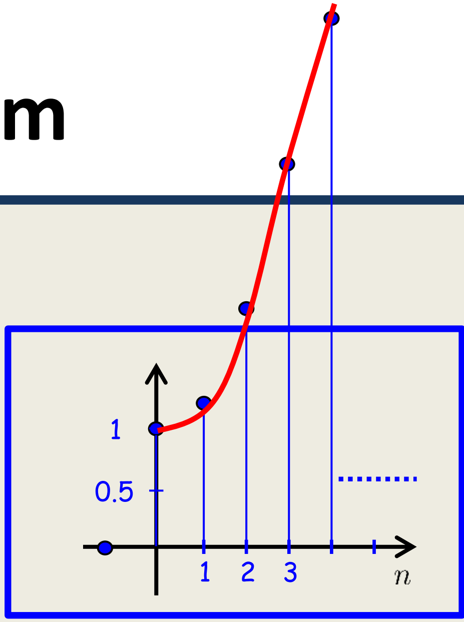


EITF75, z-transform

More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

To think about: Will it be possible to compute the z-transform for a>1, i.e., a signal that grows to infinity?



Exponentially amplified step

EITF75, z-transform

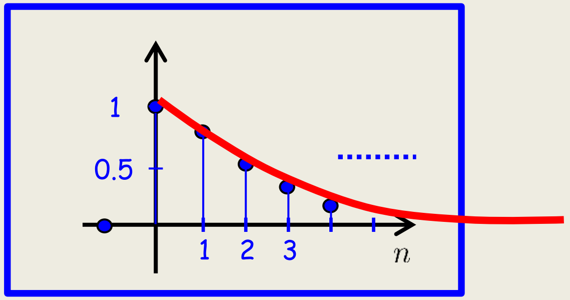
More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

Not much to comment upon…

Exponentially damped step



EITF75, z-transform

More important examples

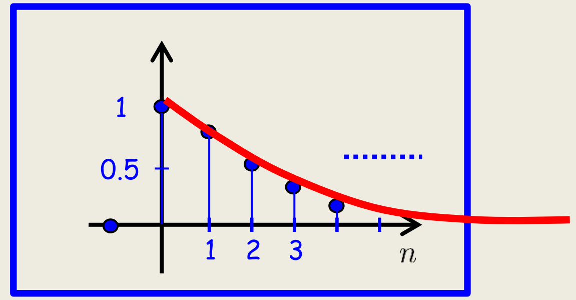
$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

Geometric series

$$= \frac{1 - (a \cdot z^{-1})^{\infty+1}}{1 - z^{-1}}$$

Exponentially damped step



EITF75, z-transform

More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

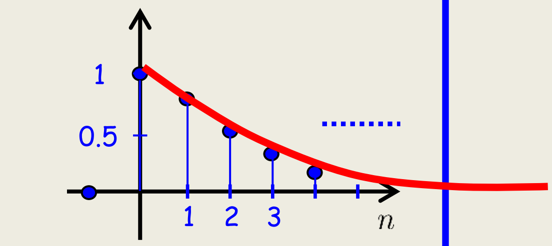
$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

Geometric series

$$= \frac{1 - \left(a \cdot z^{-1}\right)^{\infty+1}}{1 - z^{-1}}$$

Convergent when this is <1

Exponentially damped step



EITF75, z-transform

More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

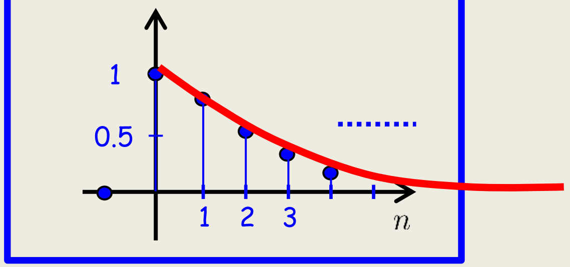
Geometric series

$$= \frac{1-(a \cdot z^{-1})^{\infty+1}}{1-z^{-1}}$$

ROC $|z| > |a|$

$$= \frac{1}{1-a \cdot z^{-1}}$$

Exponentially damped step



EITF75, z-transform

More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

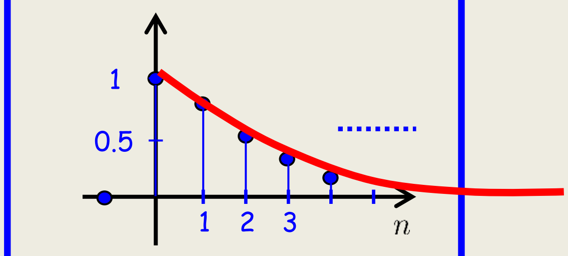
Geometric series

$$= \frac{1-(a \cdot z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-a \cdot z^{-1}}$$

ROC $\quad |z| > |a|$

Exponentially damped step



Summary. The z-transform can be computed even if the sequnce is Infinite/has infinite energy.

EITF75, z-transform

More important examples

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

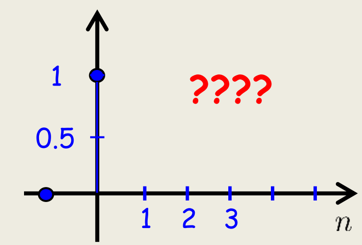
$$= \sum_{n=0}^{\infty} \left(a \cdot z^{-1}\right)^n$$

Geometric series

$$= \frac{1-(a \cdot z^{-1})^{\infty+1}}{1-z^{-1}}$$

$$= \frac{1}{1-a \cdot z^{-1}}$$

ROC $|z| < |a|$



Homework: Find h(n) having the same transform, but "opposite" ROC

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots + x(0) + \ldots + x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

EITF75, z-transform

Some general rules about the ROC

$$X(z) = x(0) + \ldots + x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be causal

EITF75, z-transform

Some general rules about the ROC

$$X(z) = x(0) + \ldots \boxed{+x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots}$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

EITF75, z-transform

Some general rules about the ROC

$$X(z) = x(0) + \ldots \boxed{+ x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots}$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

Thus, z, cannot be too small.

How small? Depends on x(n)

EITF75, z-transform

Some general rules about the ROC

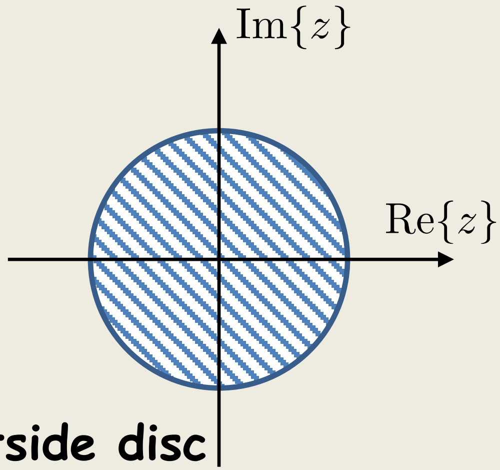
$$X(z) = x(0) + \ldots \boxed{+ x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots}$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

Thus, z, cannot be too small.



Hence, the ROC says that
"z should be larger than something"

ROC is outside disc

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots + x(0) \quad .$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be anti-causal

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \boxed{\ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots} + x(0)$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be anti-causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \boxed{\ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots} + x(0)$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be anti-causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

Thus, z, cannot be too large.

EITF75, z-transform

Some general rules about the ROC

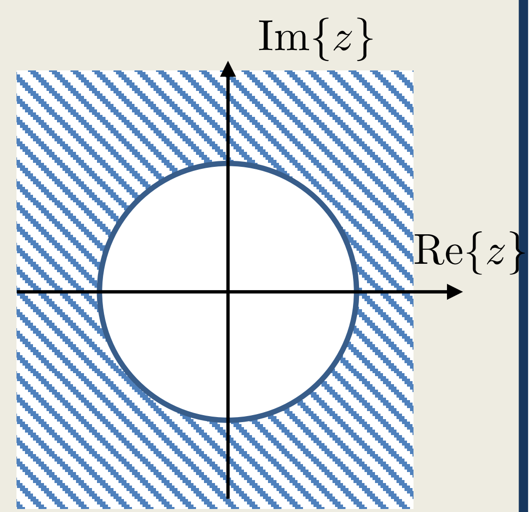
$$X(z) = \boxed{\ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots} + x(0)$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be anti-causal

If X(z) exists (meaning that it is not infinity) then this cannot be too large

Thus, z, cannot be too large.



Hence, the ROC says that
"z should be smaller than something"

ROC is inside disc

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \boxed{\ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots} + x(0) + \boxed{\ldots + x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots}$$

Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be neither

If X(z) exists (meaning that it is not infinity) then these cannot be large

Thus, z, cannot be too large or too small

EITF75, z-transform

Some general rules about the ROC

$$X(z) = \boxed{\ldots + x(-1001)z^{1001} + x(-1000)z^{1000} + \ldots} + x(0) + \boxed{\ldots + x(1000)z^{-1000} + x(1001)z^{-1001} + \ldots}$$

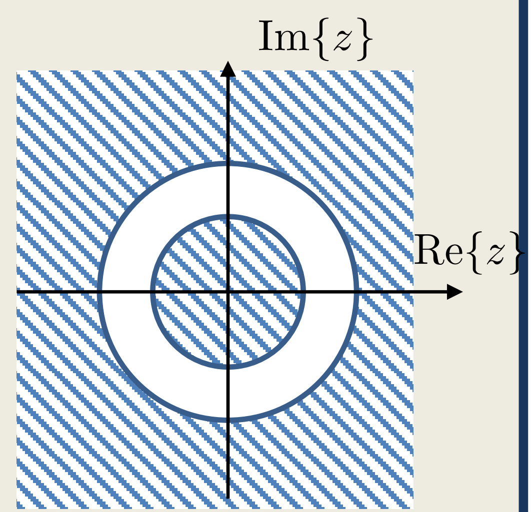
Here is a formula for X(z) emphasizing x(n) far away from n=0

Assume x(n) to be neither

If X(z) exists (meaning that it is not infinity) then these cannot be large

Thus, z, cannot be too large or too small

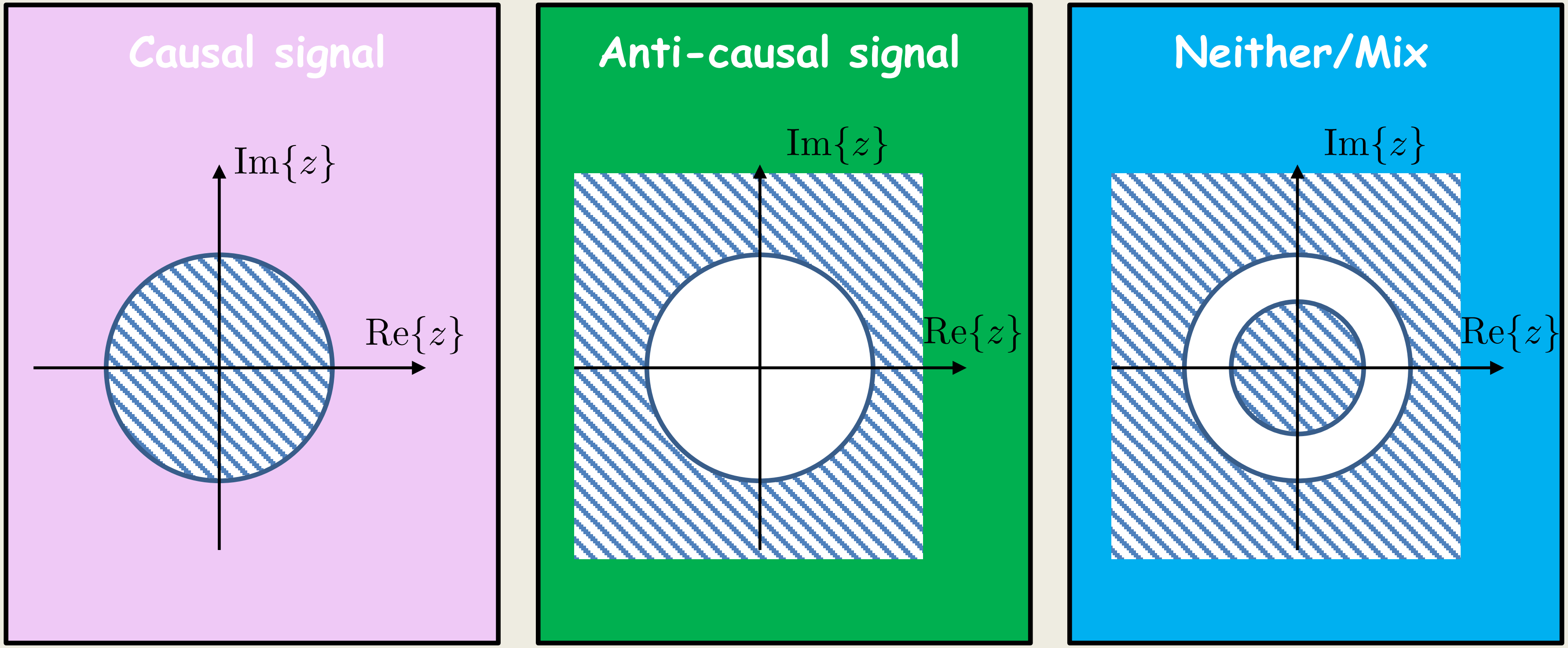
Hence, the ROC says that
"z should be smaller than something,
but larger than something else"



ROC is the white area

EITF75, z-transform

Summary: What is the general shape of the ROC?



EITF75, z-transform

Homework

Given:

$$x(n) = \left(\frac{1}{2}\right)^{|n|} \qquad \text{for all } n$$

Find: The z-transform $X(z)$ of $x(n)$.

To try later: The z-transform obtained is also the transform for a causal sequence $x_c(n)$ (if we ignore ROCs).
Find that sequence, compare, and think about how important the ROC is.

EITF75, z-transform

Convention

If we are given an $X(z)$, and assume that the signal $x(n)$ is causal, then we can be a bit sloppy with the ROC

This is what we do in this (most) of this course

In other words. There are many $x(n)$ for the same $X(z)$, and the ROC specifies the particular one. However, there is only one that is causal.

EITF75, z-transform

Illustration

Sequence	Transform	ROC
$x_1(n)$		
$x_2(n)$		
$x_3(n)$		
$x_4(n)$		
$x_5(n)$		
$x_6(n)$		
$x_7(n)$		

Assume a bunch of *different* sequences

EITF75, z-transform

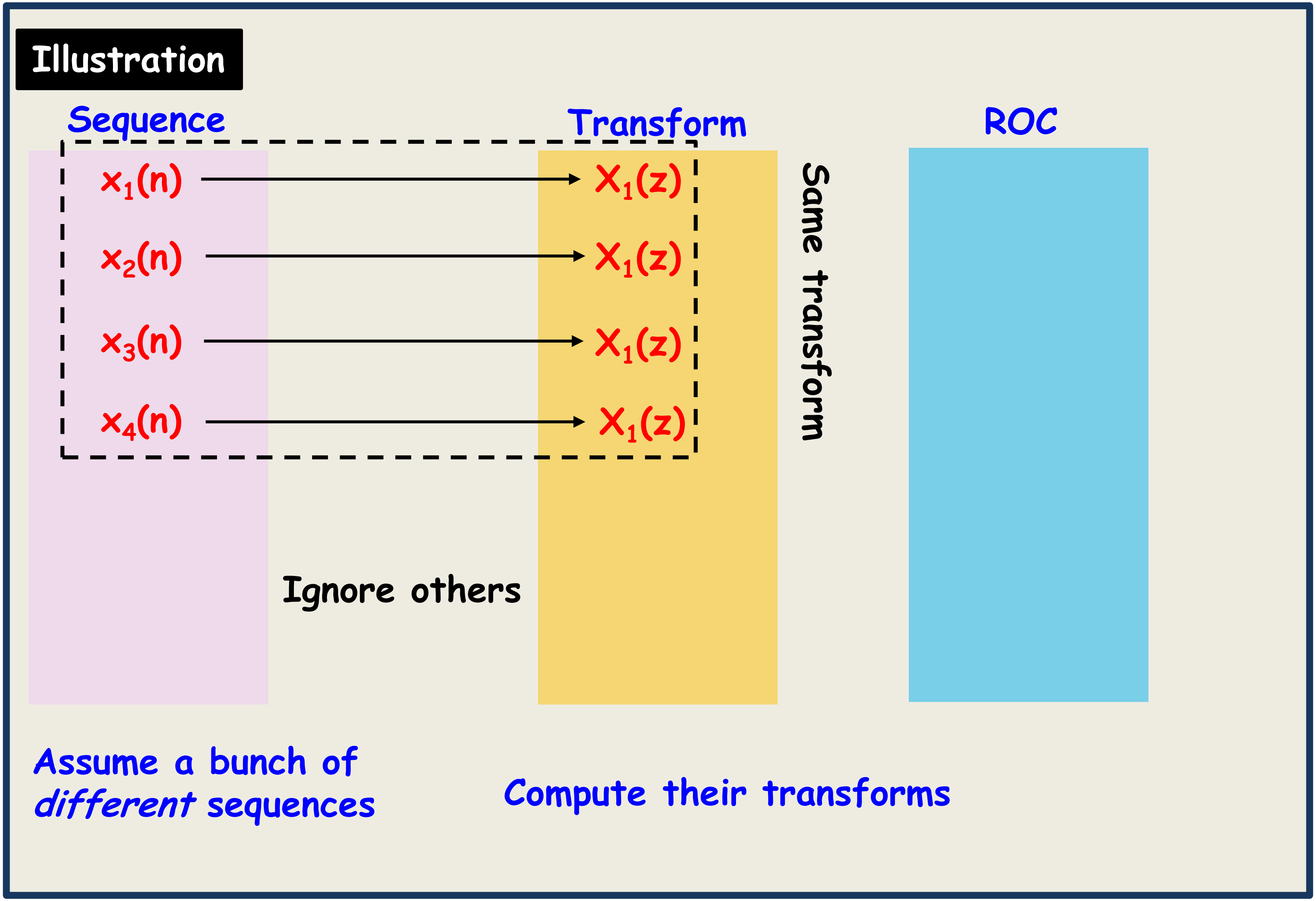
Illustration

Sequence	Transform	ROC
$x_1(n)$ →	$X_1(z)$	
$x_2(n)$ →	$X_1(z)$	
$x_3(n)$ →	$X_1(z)$	
$x_4(n)$ →	$X_1(z)$	
$x_5(n)$ →	$X_2(z)$	
$x_6(n)$ →	$X_2(z)$	
$x_7(n)$ →	$X_2(z)$	

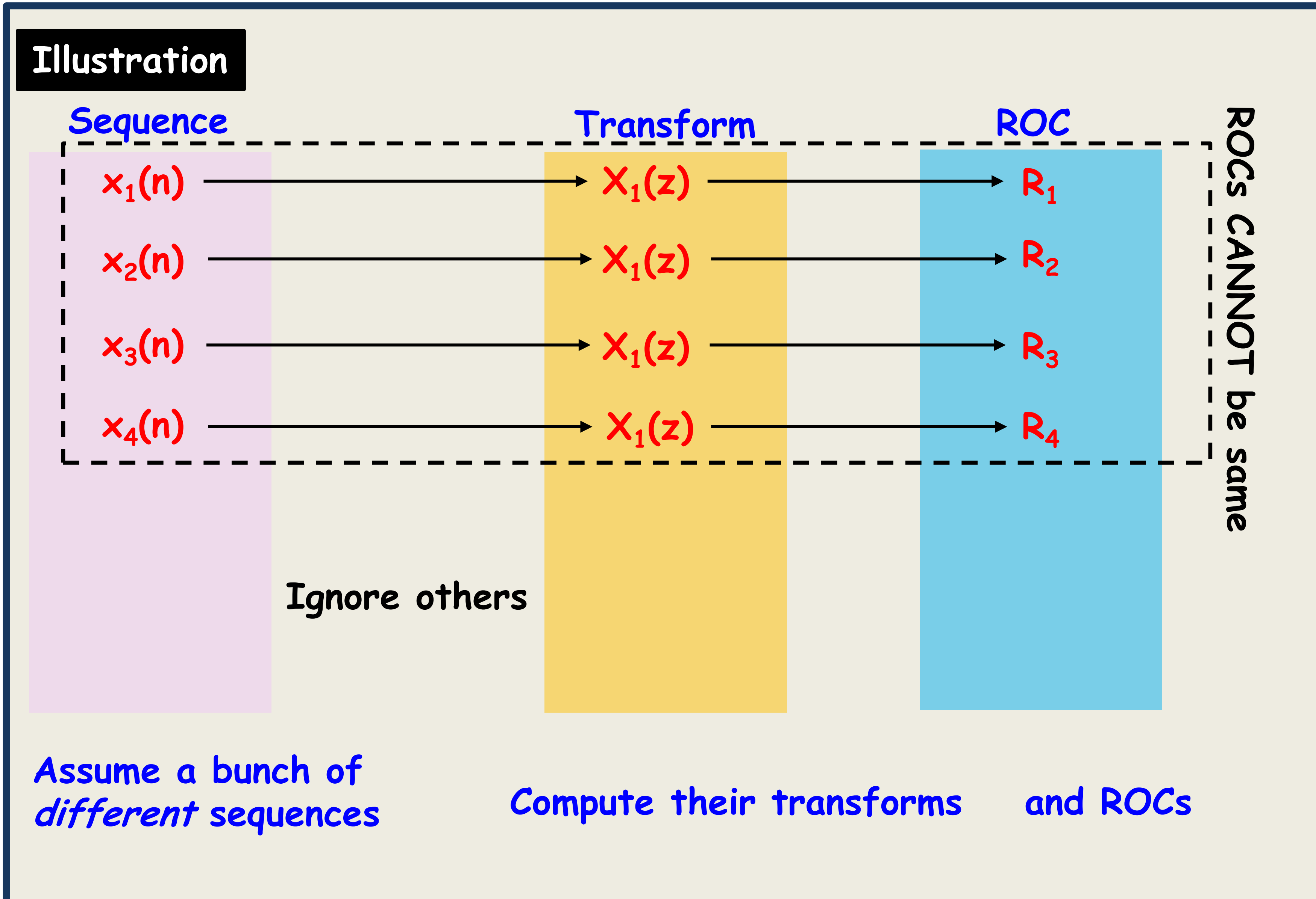
Assume a bunch of *different* sequences

Compute their transforms

EITF75, z-transform

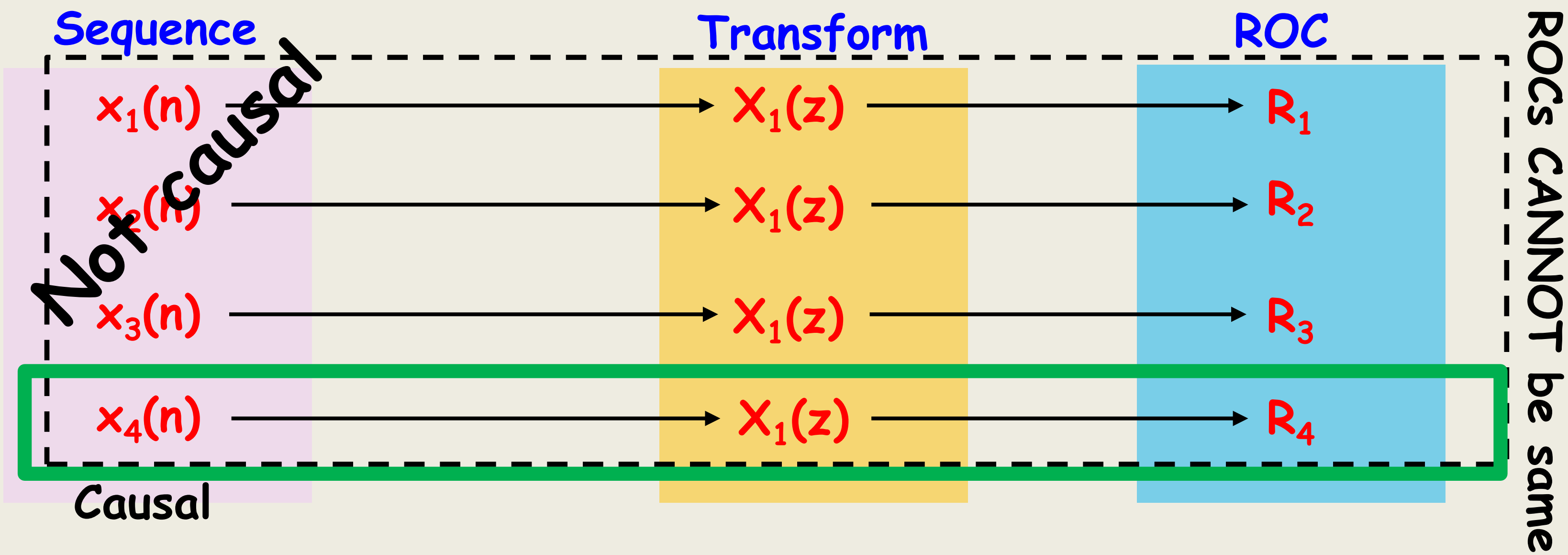


EITF75, z-transform



EITF75, z-transform

Illustration



Exactly one of the ROCs corresponds to a causal signal

EITF75, z-transform

Illustration

Sequence	Transform	ROC
$x_1(n)$ →	$X_1(z)$ →	R_1
$x_2(n)$ →	$X_1(z)$ →	R_2
$x_3(n)$ →	$X_1(z)$ →	R_3
$x_4(n)$ →	$X_1(z)$ →	R_4

Not causal

Causal

ROCs CANNOT be same

Exactly one of the ROCs corresponds to a causal signal

So, if we know $X_1(z)$ and that we work with causal $x(n)$, we can establish $x_4(n)$ without knowing the ROC

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n}$$

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n}$$

Only way forward: Plug in formula for convolution

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n} = \sum_n \boxed{\sum_k h(k)x(n-k)} z^{-n}$$

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n} = \sum_n \sum_k h(k)x(n-k)z^{-n}$$

$$= \sum_n \sum_k h(k)x(n-k)z^{-(n-k)} z^{-k}$$

Trick: multipliction with $z^k z^{-k}$

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n} = \sum_n \sum_k h(k)x(n-k)z^{-n}$$

$$= \sum_n \sum_k h(k)x(n-k)z^{-(n-k)}z^{-k}$$

$$= \sum_k h(k)z^{-k} \cdot \sum_n x(n-k)z^{-(n-k)}$$

Some terms not dependent on n

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n} = \sum_n \sum_k h(k) x(n-k) z^{-n}$$

$$= \sum_n \sum_k h(k) x(n-k) z^{-(n-k)} z^{-k}$$

$$= \sum_k h(k) z^{-k} \cdot \sum_n x(n-k) z^{-(n-k)}$$

Do a variable change: n-k = m

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n)z^{-n} = \sum_n \sum_k h(k)x(n-k)z^{-n}$$

$$= \sum_n \sum_k h(k)x(n-k)z^{-(n-k)}z^{-k}$$

$$= \sum_k h(k)z^{-k} \cdot \sum_m x(m)z^{-m}$$

EITF75, z-transform

Super important property: Convolution becomes multiplication

$$y(n) = h(n) * x(n) \quad \Leftrightarrow \quad Y(z) = H(z)X(z)$$

Proof

$$Y(z) = \sum_n y(n) z^{-n} = \sum_n \sum_k h(k)x(n-k)z^{-n}$$

$$= \sum_n \sum_k h(k)x(n-k)z^{-(n-k)}z^{-k}$$

$$= \sum_k h(k)z^{-k} \cdot \sum_m x(m)z^{-m}$$

$$= H(z)X(z)$$

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month,
get 5% interest

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month,
get 5% interest

Strategy:

- Both signals are causal -> ignore ROCs

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

Strategy:

- Both signals are causal -> ignore ROCs
- Try to obtain Y(z)

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

Strategy:

- Both signals are causal -> ignore ROCs
- Try to obtain Y(z)
- Invert it to get y(n)

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

We proved the delay-property earlier

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}}$$

We computed this earlier today

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z)$$

Solve for Y(z)

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month,
get 5% interest

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z)$$

Interlude: The solution is a convolution between x(n) and the inverse of the green part

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z) \quad = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

We are now done, and only need to invert

EITF75, z-transform

Interlude: How to invert Y(z)

We have not dealt with this at all, so we are not supposed to know at this stage

EITF75, z-transform

Interlude: How to invert Y(z)

We have not dealt with this at all, so we are not supposed to know at this stage

But we know one thing

$$\boxed{h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - a \cdot z^{-1}}}$$

EITF75, z-transform

Interlude: How to invert Y(z)

We have not dealt with this at all, so we are not supposed to know at this stage

But we know one thing

$$\boxed{h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - a \cdot z^{-1}}}$$

So, if we encounter $\frac{1}{1 - a \cdot z^{-1}}$ we can invert that one: $a^n \cdot u(n)$

EITF75, z-transform

Interlude: How to invert Y(z)

So, if we encounter $\frac{1}{1-a\cdot z^{-1}}$ we can invert that one: $a^n\cdot u(n)$

But now we had $\frac{1}{1-1.05\cdot z^{-1}}\cdot\frac{100}{1-z^{-1}}$

EITF75, z-transform

Interlude: How to invert Y(z)

So, if we encounter $\dfrac{1}{1-a\cdot z^{-1}}$ we can invert that one: $a^n \cdot u(n)$

But now we had $\dfrac{1}{1-1.05\cdot z^{-1}} \cdot \dfrac{100}{1-z^{-1}}$

Can we somehow turn $\dfrac{1}{1-1.05\cdot z^{-1}} \cdot \dfrac{100}{1-z^{-1}}$ into $\dfrac{1}{1-a\cdot z^{-1}}$?

EITF75, z-transform

Interlude: How to invert Y(z)

So, if we encounter $\dfrac{1}{1-a\cdot z^{-1}}$ we can invert that one: $a^n \cdot u(n)$

But now we had $\dfrac{1}{1-1.05\cdot z^{-1}} \cdot \dfrac{100}{1-z^{-1}}$

Can we somehow turn $\dfrac{1}{1-1.05\cdot z^{-1}} \cdot \dfrac{100}{1-z^{-1}}$ into $\dfrac{1}{1-a\cdot z^{-1}}$?

Yes, by PARTIAL FRACTION EXPANSION

$$\boxed{\frac{1}{1-1.05\cdot z^{-1}} \cdot \frac{100}{1-z^{-1}} = \frac{A}{1-1.05\cdot z^{-1}} + \frac{B}{1-z^{-1}}}$$

EITF75, z-transform

Interlude: How to invert Y(z)

Four cases exist of PFE

1. Roots in denominator are real and distinct
2. Roots in denominator are distinct complex conjugated pairs
3. Roots are real, but with multiplicity
4. Roots are complex conjugated pairs with multiplicity

Yes, by PARTIAL FRACTION EXPANSION

$$\frac{1}{1-1.05\cdot z^{-1}}\cdot\frac{100}{1-z^{-1}}=\frac{A}{1-1.05\cdot z^{-1}}+\frac{B}{1-z^{-1}}$$

EITF75, z-transform

Interlude: How to invert Y(z)

Four cases exist of PFE

1. Roots in denominator are real and distinct
2. Roots in denominator are distinct complex conjugated pairs
3. ~~Roots are real, but with multiplicity~~
4. ~~Roots are complex conjugated pairs with multiplicity~~

WE ONLY DO 1-2. Assumed that you can do these PFEs

Yes, by PARTIAL FRACTION EXPANSION

$$\frac{1}{1-1.05\cdot z^{-1}}\cdot\frac{100}{1-z^{-1}}=\frac{A}{1-1.05\cdot z^{-1}}+\frac{B}{1-z^{-1}}$$

EITF75, z-transform

Interlude: How to invert Y(z)

$$h(n) = a^n \cdot u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - a \cdot z^{-1}}$$

Now easy to invert

$$A \cdot 1.05^n \cdot u(n) + B \cdot u(n)$$

$$\frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}} = \frac{A}{1 - 1.05 \cdot z^{-1}} + \frac{B}{1 - z^{-1}}$$

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month, get 5% interest

$$Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$$

$$X(z) = 100 \cdot \frac{1}{1 - z^{-1}}$$

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z) \quad = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

$$= 100 \cdot \left(\frac{21}{1 - 1.05 \cdot z^{-1}} - \frac{20}{1 - z^{-1}} \right)$$

Partial fraction expansion

EITF75, z-transform

Application of z-transforms: Solving difference equations

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 0$$

Save 100/month,
get 5% interest

$$y(n) = 100 \cdot (21 \cdot 1.05^n - 20)\, u(n)$$

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z) \quad = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

$$= 100 \cdot \left(\frac{21}{1 - 1.05 \cdot z^{-1}} - \frac{20}{1 - z^{-1}} \right)$$

Partial fraction
expansion

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to $z^{-k}\,Y(z)$

Step 2:

Change x(n-k) to $z^{-k}\,X(z)$

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_{k} a(k)y(n-k) = \sum_{\ell} b(\ell)x(n-\ell)$$

Step 1:

Change $y(n-k)$ to $z^{-k}Y(z)$

Step 2:

Change $x(n-k)$ to $z^{-k}X(z)$

Step 3:

Express $Y(z)$ as $H(z)X(z)$

Step 4:

Find the roots of the denominator of $H(z)$

Step 5:

Perform PFE (you may need your calculus book)

Step 6:

Invert each term of the FPE

For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

Summary: How to solve a difference equation in 6 simple steps

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Step 1:

Change y(n-k) to $z^{-k}\,Y(z)$

Step 2:

Change x(n-k) to $z^{-k}\,X(z)$

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:
Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example: $y(n) - 1.27y(n-1) + 0.81y(n-2) = x(n-1) - x(n-2)$

All signals causal

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:
Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example: $Y(z) - 1.27z^{-1}Y(z) + 0.81z^{-2}Y(z) = z^{-1}X(z) - z^{-2}X(z)$

All signals causal

Step 1:

Change y(n-k) to $z^{-k}Y(z)$

Step 2:

Change x(n-k) to $z^{-k}X(z)$

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:

Invert each term of the FPE

For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

$$Y(z)=\frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}\cdot X(z)$$

All signals causal

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:

Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

All signals causal

$$z_{1,2} = 0.9\mathrm{e}^{i\pm\pi/4}$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

Step 4:

Find the roots of the denominator of H(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 6:

Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

All signals causal

$$z_{1,2} = 0.9\mathrm{e}^{i \pm \pi/4}$$

Type II. PFE already done, 1 complex conjugated pair

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 4:

Find the roots of the denominator of H(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 5:

Perform PFE (you may need your calculus book)

Step 3:

Express Y(z) as H(z)X(z)

Step 6:
Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

One further example:

All signals causal

Step 1:

Change $y(n-k)$ to $z^{-k}\,Y(z)$

Step 2:

Change $x(n-k)$ to $z^{-k}\,X(z)$

Step 3:

Express $Y(z)$ as $H(z)X(z)$

Next lecture…

Step 4:

Find the roots of the denominator of $H(z)$

Step 5:

Perform PFE (you may need your calculus book)

Step 6:
Invert each term of the FPE
For FPE-type II, we haven't seen the formula yet

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

Old solution for y(-1)=0

$$y(n) = 100 \cdot (21 \cdot 1.05^n - 20)\, u(n)$$

Where does it go wrong?

$$Y(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot X(z) = \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

$$= 100 \cdot \left(\frac{21}{1 - 1.05 \cdot z^{-1}} - \frac{20}{1 - z^{-1}} \right)$$

Partial fraction expansion

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month, get 5% interest

Start with 1000

Old solution for y(-1)=0

Where does it go wrong?

Already here

Not true: $y(-1) = 1.05 \cdot y(-2) + x(-1)$

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

Old solution for y(-1)=0

Where does it go wrong?

Already here

Not true: $y(-1) = 1.05 \cdot y(-2) + x(-1)$

Therefore not true: $Y(z) = 1.05 \cdot z^{-1} Y(z) + X(z)$

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

Hard to carry on – we need a new method altogether:

The one-sided z-transform

The transform of choice for problems not at rest/have initial conditions

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

Definition

$$Y^+(z) = \sum_{n=0}^{\infty} y(n) z^{-n}$$

Only change is lower limit of summation

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

Definition $$Y^+(z) = \sum_{n=0}^{\infty} y(n) z^{-n}$$

Only change is lower limit of summation

$$x(n) \quad \Leftrightarrow \quad X^+(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n \right]$$

Properties of z-transform remain, except time-delay

EITF75, z-transform

Explanation

sequence

One-sided transform

$$x(n) = \{x(-1),\, x(0),\, x(1),\, \ldots\}$$

n=0

$$x(n) \quad \Leftrightarrow \quad X^{+}(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^{+}(z) + \sum_{n=1}^{k} x(-n) z^{n} \right]$$

Properties of z-transform remain, except time-delay

EITF75, z-transform

Explanation

sequence

$$x(n) = \{x(-1), \boxed{x(0),\ x(1),\ \ldots}\}$$

Only this part matters for one-sided transform

One-sided transform

$$X^+(z)$$

Definition $Y^+(z) = \sum_{n=0}^{\infty} y(n) z^{-n}$

$$x(n) \quad \Leftrightarrow \quad X^+(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n \right]$$

Properties of z-transform remain, except time-delay

EITF75, z-transform

Explanation

sequence

$$x(n) = \{x(-1), \boxed{x(0),\ x(1),\ \ldots}\}$$

n=0

One-sided transform

$$X^{+}(z)$$

$$x(n-1) = \{0,\ x(-1),\ x(0),\ x(1),\ \ldots\}$$

n=0

$$x(n) \quad \Leftrightarrow \quad X^{+}(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^{+}(z) + \sum_{n=1}^{k} x(-n) z^{n} \right]$$

Properties of z-transform remain, except time-delay

EITF75, z-transform

Explanation

sequence

$$x(n) = \{x(-1), \boxed{x(0),\ x(1),\ \ldots}\}$$

n=0

One-sided transform

$$X^+(z)$$

$$x(n-1) = \{0, \boxed{x(-1)} \boxed{x(0),\ x(1),\ \ldots}\}$$

new

$$z^{-1}X^+(z)$$

$$x(n) \quad \Leftrightarrow \quad X^+(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n\right]$$

Properties of z-transform remain, except time-delay

EITF75, z-transform

Explanation

sequence

$$x(n) = \{x(-1), \boxed{x(0),\ x(1),\ \ldots}\}$$

n=0

One-sided transform

$$X^+(z)$$

$$x(n-1) = \{0, \boxed{x(-1)} \boxed{x(0),\ x(1),\ \ldots}\}$$

new

$$z^{-1}X^+(z) + x(-1)$$

$$x(n) \quad \Leftrightarrow \quad X^+(z)$$

$$x(n-k) \quad \Leftrightarrow \quad z^{-k} \cdot \left[X^+(z) + \sum_{n=1}^{k} x(-n) z^n \right]$$

Properties of z-transform remain, except time-delay

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

So,

$$Y^+(z) = 1.05 \cdot z^{-1} Y^+(z) + 1.05 \cdot y(-1) + X^+(z)$$

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

So,

$$Y^+(z) = 1.05 \cdot z^{-1} Y^+(z) + 1.05 \cdot y(-1) + X^+(z)$$

$$Y^+(z) = \frac{1.05}{1 - 1.05 \cdot z^{-1}} y(-1) + \frac{1}{1 - 1.05 \cdot z^{-1}} X^+(z)$$

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

So,

$$Y^+(z) = 1.05 \cdot z^{-1} Y^+(z) + 1.05 \cdot y(-1) + X^+(z)$$

$$Y^+(z) = \frac{1.05}{1 - 1.05 \cdot z^{-1}} y(-1) + \frac{1}{1 - 1.05 \cdot z^{-1}} X^+(z)$$

$$Y^+(z) = \frac{1050}{1 - 1.05 \cdot z^{-1}} + \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

$$Y^+(z) = 1.05 \cdot z^{-1} Y^+(z) + 1.05 \cdot y(-1) + X^+(z)$$

$$Y^+(z) = \frac{1.05}{1 - 1.05 \cdot z^{-1}} y(-1) + \frac{1}{1 - 1.05 \cdot z^{-1}} X^+(z)$$

$$Y^+(z) = \frac{1050}{1 - 1.05 \cdot z^{-1}} + \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

Only change
From before

EIT F75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month, get 5% interest
Start with 1000

$$y(n) = 100 \cdot (21 \cdot 1.05^n - 20)\, u(n)$$

Solution for $y(-1) = 0$

$$Y^+(z) = \frac{1050}{1 - 1.05 \cdot z^{-1}} + \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

Only change From before

EITF75, z-transform

Solving difference equations with initial conditions

$$y(n) = 1.05 \cdot y(n-1) + x(n)$$

$$x(n) = 100 \cdot u(n) \quad y(-1) = 1000$$

Save 100/month,
get 5% interest
Start with 1000

$$y(n) = 100 \cdot (21 \cdot 1.05^n - 20)\, u(n)$$
$$+1050 \cdot 1.05^n \cdot u(n)$$
$$y(-1) = 1000$$

Solution for

$$y(-1) = 1000$$

$$Y^+(z) = \frac{1050}{1 - 1.05 \cdot z^{-1}} + \frac{1}{1 - 1.05 \cdot z^{-1}} \cdot \frac{100}{1 - z^{-1}}$$

Only change
From before