EITF75 Systems and Signals

Lecture 2
LTI systems: convolutions, impulse responses (and more)

Fredrik Rusek

EITF75 Systems and Signals

LTI systems

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Linear system

$$x(n) = \alpha x_1(n) + \beta x_2(n)$$
$$\Longleftrightarrow$$
$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

Time invariant system

$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

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LTI systems

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$$\Longleftrightarrow$$
$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

Time invariant system

$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

LTI systems have compact mathematical representation

We next provide two ways the reach the representation

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$$x(n) \rightarrow \boxed{\text{LTI system}} \rightarrow y(n)$$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Method I (not in book, but I find it illuminating)

Input x(n): A sequence of numbers

$$\ldots,\ 0.4,\ -0.3,\ 1.2,\ 0.8,\ \ldots$$

Output y(n): A sequence of numbers

$$\ldots,\ -0.6,\ -0.34,\ 3.8,\ -1.8,\ \ldots$$

In the Linear algebra course, how did we represent a sequence of numbers?

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In the Linear algebra course, how did we represent a sequence of numbers? With a vector

OUT IN

$$\begin{bmatrix} \ldots \\ -0.6 \\ -0.34 \\ 3.8 \\ -1.8 \\ \ldots \end{bmatrix} = \qquad \begin{bmatrix} \ldots \\ 0.4 \\ -0.3 \\ 1.2 \\ 0.8 \\ \ldots \end{bmatrix}$$

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Why is the linear algebra course dealing so much with matrices?

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In the Linear algebra course, how did we represent a sequence of numbers? With a vector

$$\begin{array}{ccc} \text{OUT} & & \text{IN} \\ \begin{bmatrix} \cdots \\ -0.6 \\ -0.34 \\ 3.8 \\ -1.8 \\ \cdots \end{bmatrix} & = A & \begin{bmatrix} \cdots \\ 0.4 \\ -0.3 \\ 1.2 \\ 0.8 \\ \cdots \end{bmatrix} \end{array}$$

Why is the linear algebra course dealing so much with matrices? Because every linear function can be represented by a matrix

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$x(n) \rightarrow$ LTI system $\rightarrow y(n)$

A system is LTI if-and-only if:

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Method I (not in book, but I find it illuminating)

Summary so far: A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

where $\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$ $\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix}$

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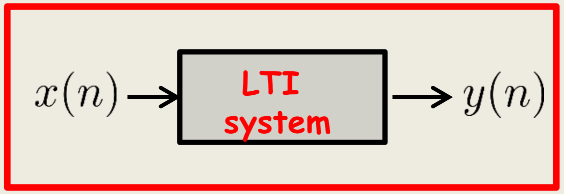

A system is LTI if-and-only if:

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Method I (not in book, but I find it illuminating)

A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

where $\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$ $\mathbf{y} = \begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix}$

But, our system is LTI, not only linear,
so this imposes restrictions on A
i.e., A must have a special structure

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Method I (not in book, but I find it illuminating)

A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \end{bmatrix}$$

Let us understand this special structure

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$x(n) \rightarrow$ LTI system $\rightarrow y(n)$

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Method I (not in book, but I find it illuminating)

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Assume $x(n) = \delta(n)$

Let us understand this special structure

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Method I (not in book, but I find it illuminating)

A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} A_{11} \\ A_{21} \\ A_{31} \end{matrix}} & A_{12} & A_{13} & \cdots \\ & A_{22} & A_{23} & \cdots \\ & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} \boxed{1} \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

Assume $x(n) = \delta(n)$

The output must be the first column of A

Let us understand this special structure

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Method I (not in book, but I find it illuminating)

A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

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Assume $x(n) = \delta(n-1)$

Let us understand this special structure

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$x(n) \rightarrow$ LTI system $\rightarrow y(n)$

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A linear system can be represented as $\mathbf{y} = A\mathbf{x}$

$$\begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ \boxed{1} \\ 0 \\ \vdots \end{bmatrix}$$

Assume $x(n) = \delta(n-1)$

The output must be the second column of A

Let us understand this special structure

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$$x(n) \to \boxed{\text{LTI system}} \to y(n)$$

A system is LTI if-and-only if:

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$$\begin{bmatrix} A_{11} \\ A_{21} \\ A_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

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Now recall that system is time-invariant

Implication?

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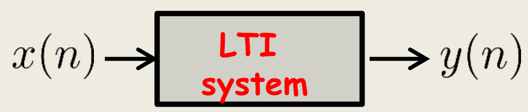

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Implication?

The outputs should be The same, but one step delayed

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$x(n) \rightarrow$ **LTI system** $\rightarrow y(n)$

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$$\boxed{\begin{bmatrix} 0 \\ A_{11} \\ A_{21} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

i.e.

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Now recall that system is time-invariant

Implication?

The outputs should be The same, but one step delayed

$$\boxed{\begin{bmatrix} 0 \\ A_{11} \\ A_{21} \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{12} \\ A_{22} \\ A_{32} \\ \vdots \end{bmatrix}} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & \cdots \\ A_{21} & A_{22} & A_{23} & \cdots \\ A_{31} & A_{32} & A_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}$$

i.e. equal values along all diagonals

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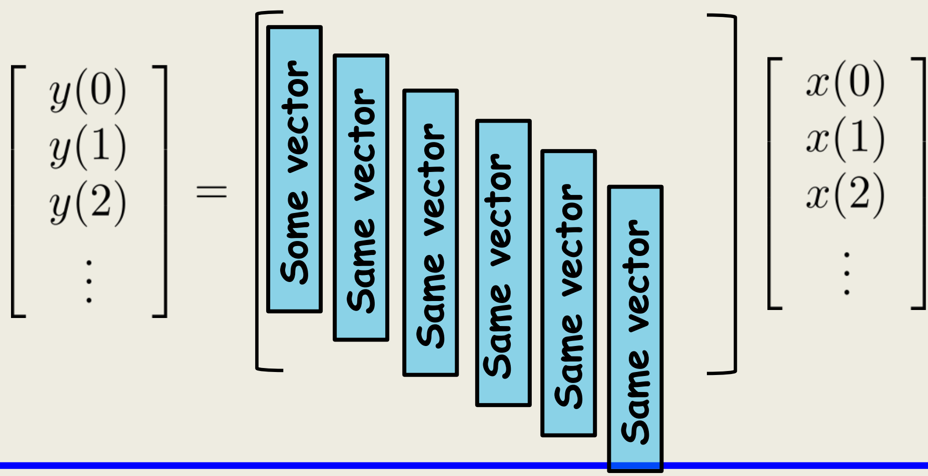
Summary. An LTI system is any discrete-time system that can be described by



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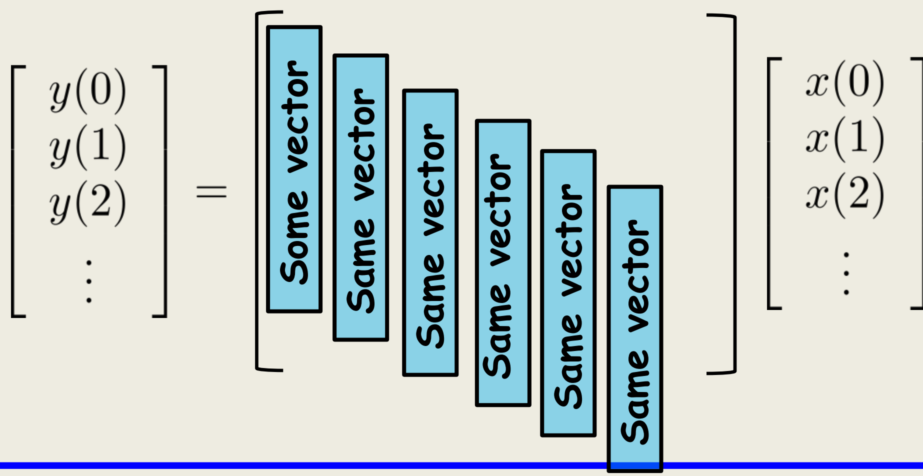
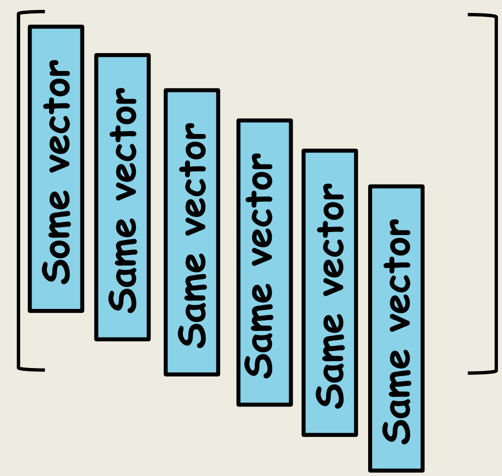


Alternative formulation of course-goal:
Understand properties of a matrix of the form



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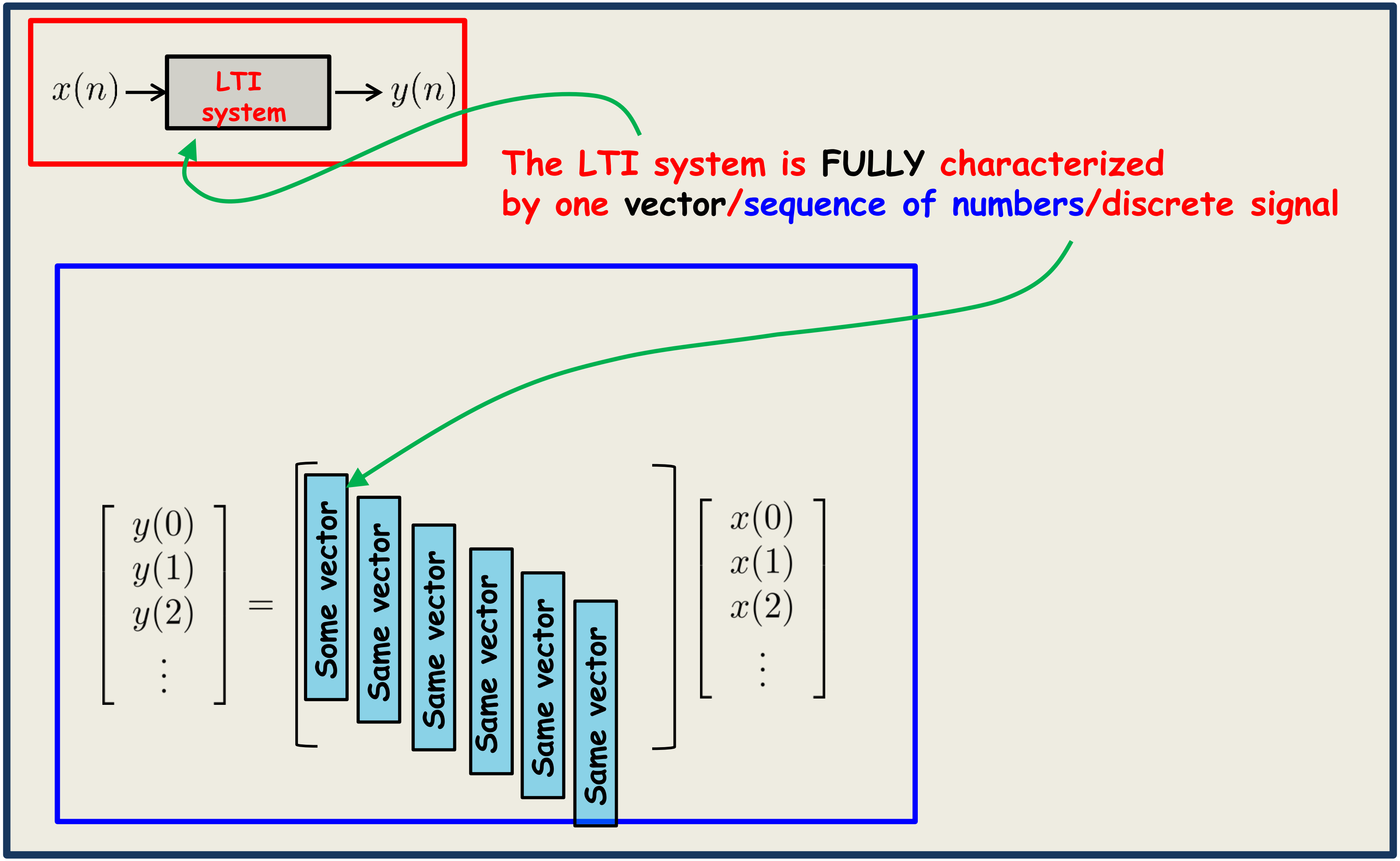

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Method II (to reach the same conclusion: an LTI system can be described by a signal)

Input signal	Output signal	
$\delta(n)$	$h(n)$	Impulse response (definition)

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$x(k)\delta(n-k)$	$x(k)h(n-k)$	Linearity (scaling part)

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$\delta(n-k)$	$h(n-k)$	Time-invariant
$x(k)\delta(n-k)$	$x(k)h(n-k)$	Linearity (scaling part)
$\sum_k x(k)\delta(n-k)$	$\sum_k x(k)h(n-k)$	Linearity (summation part)

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$\delta(n-k)$	$h(n-k)$
$x(k)\delta(n-k)$	$x(k)h(n-k)$
$x(n)=\sum_k x(k)\delta(n-k)$	$\sum_k x(k)h(n-k) = y(n)$

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$x(n)=\sum_k x(k)\delta(n-k)$	$\boxed{\sum_k x(k)h(n-k) = y(n)}$

Final result: The system is described by h(n)

The formula is named convolution. Super-important

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Summary

LTI

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

Input/Output relation (Convolution)

$$y(n) = \sum_{k} x(k)h(n-k)$$

Short-hand notation

$$y(n) = x(n) \star h(n)$$

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Agenda

Today

Get familiar with $y(n) = x(n) \star h(n)$ through some examples

For what $h(n)$ do we have BIBO stability?

See relationship between $h(n)$ and $\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$

Some notes on correlation functions

In the long run

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Some notes on correlation functions

In the long run (Loosely speaking)

Study $\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$ in detail via z-transform, and 2 types of Fourier transforms

The sampling-reconstruction issues

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Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k)h(n-k)$

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Example

Given: Input signal and impulse response

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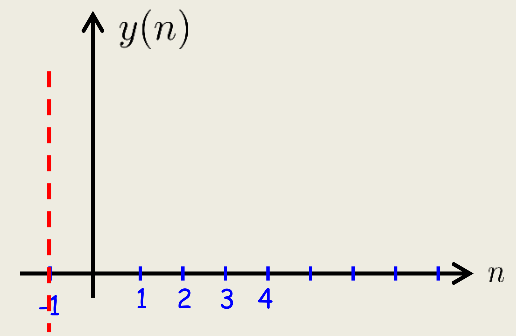
Find: Output signal $y(n) = \sum_{k} x(k)h(n-k)$

Let us start with computation of

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$



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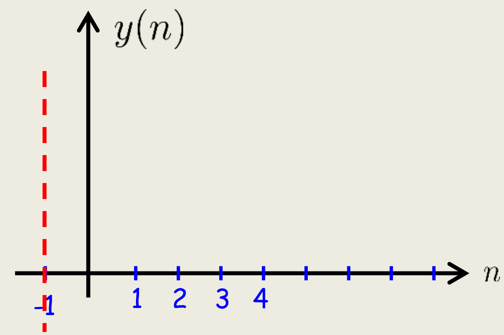

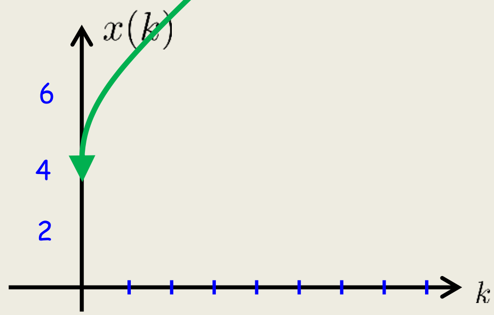

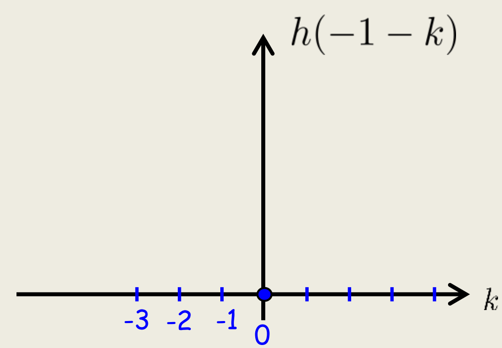

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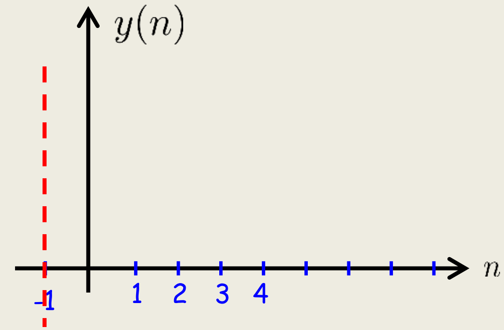

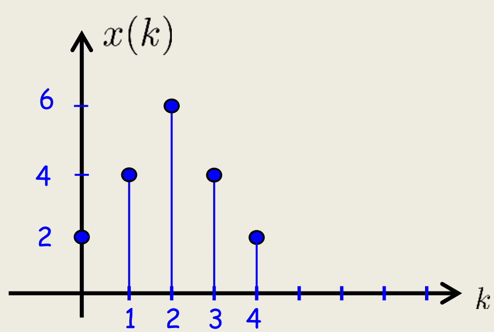

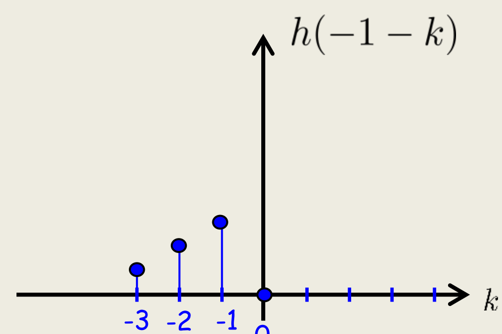

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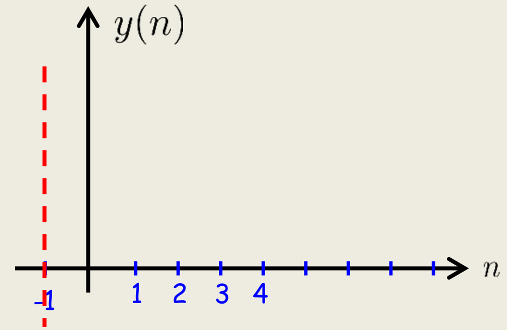
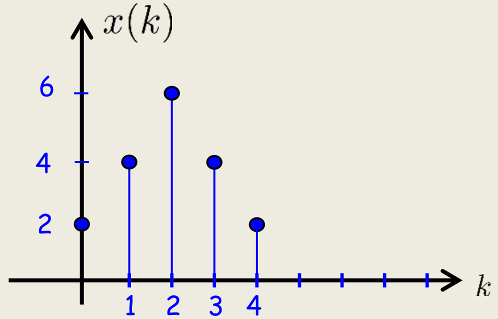
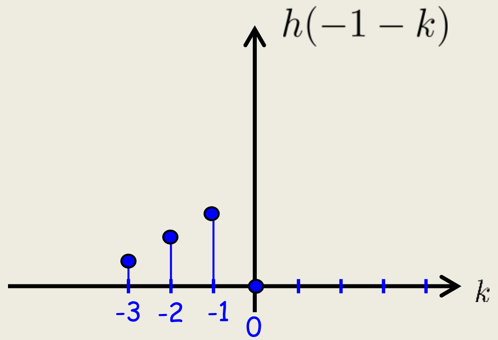






EITF75 Systems and Signals

Example

Given: Input signal and impulse response

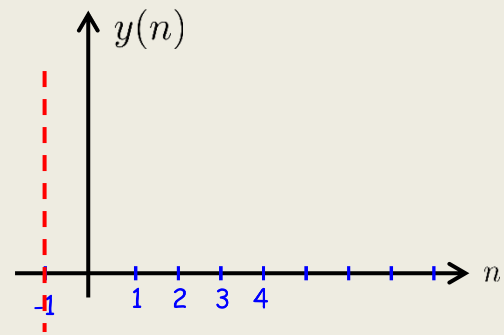
$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

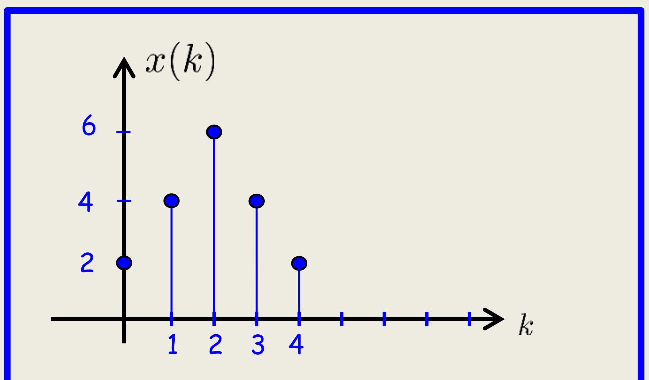
Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

Let us start with computation of

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$



$$\sum_k$$



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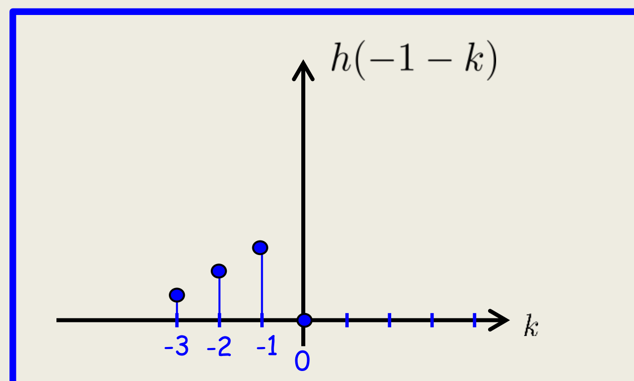


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

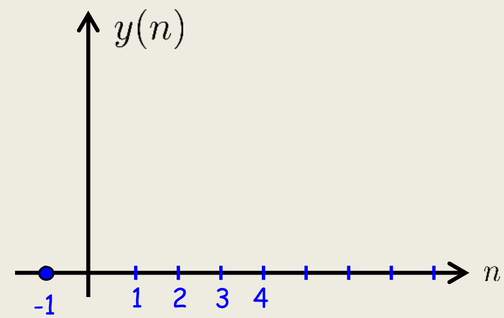
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

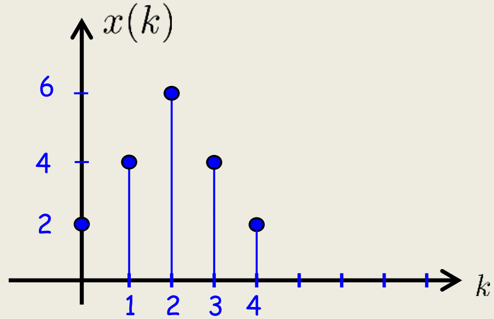
$$y(n) = \{0\ \}$$

Let us start with computation of

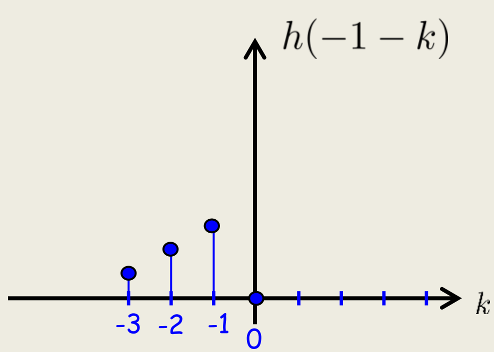
$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$$



$$\sum_k$$



$$\bullet$$



$$= 0$$

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

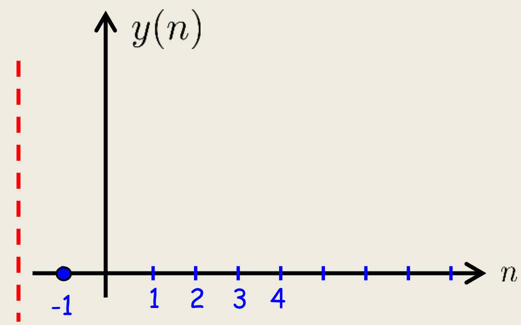
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

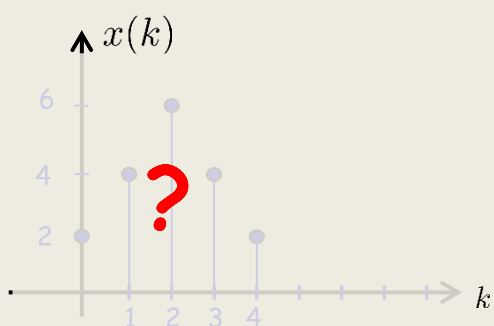
$$y(n) = \{0\ \}$$

computation of

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$



$$\sum_k$$



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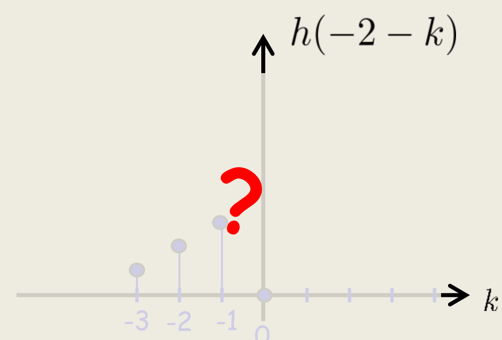


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

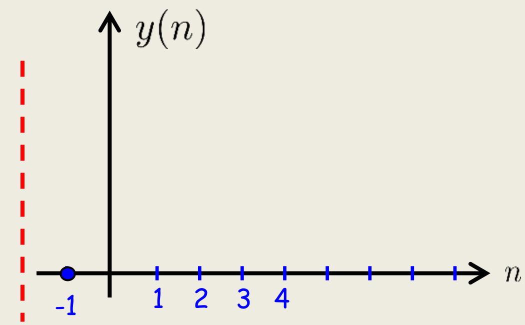
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$y(n) = \{0\ \}$$

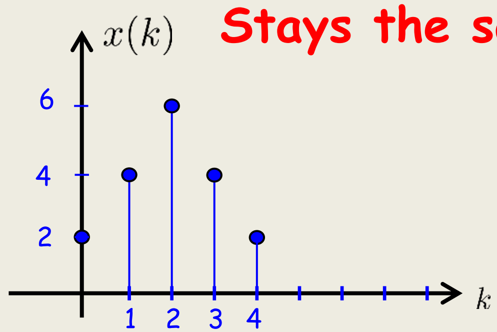
computation of

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$



$\sum_k$

Stays the same



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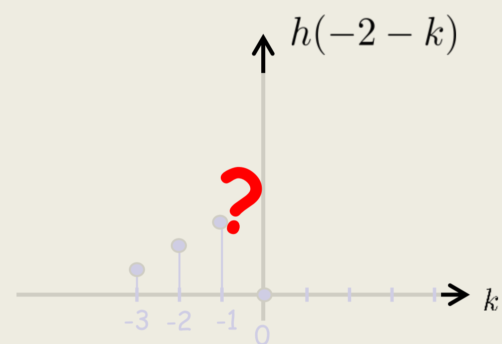


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

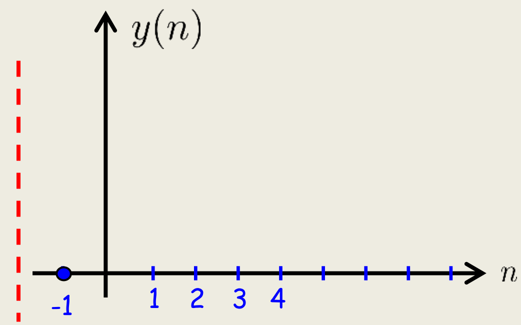
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_{k} x(k) \cdot h(n-k)$

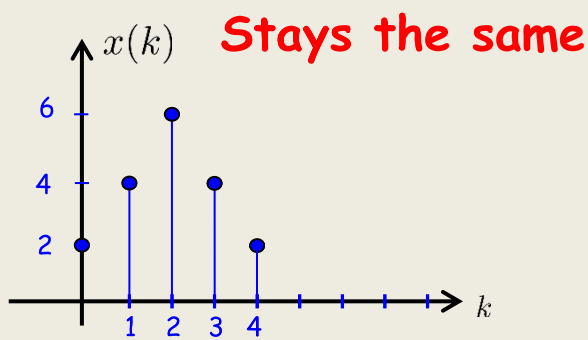
$$y(n) = \{0\ \}$$

computation of

$$y(\color{red}{-2}) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$



$$\sum_{k}$$



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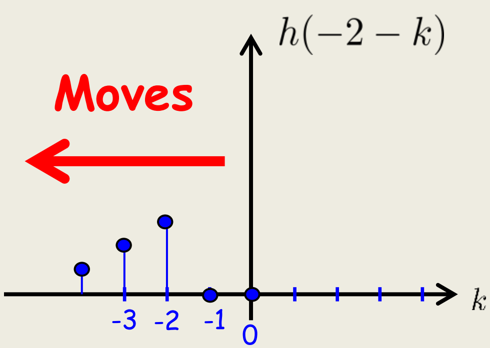


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

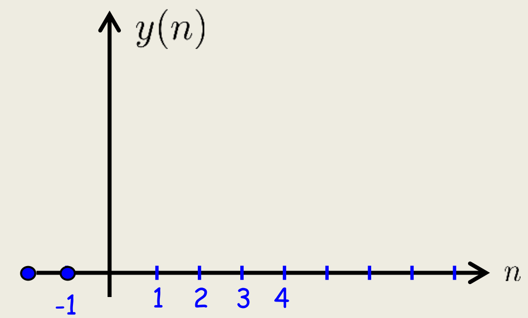
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

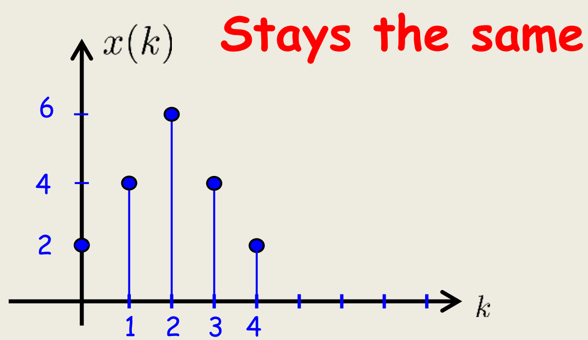
$$y(n) = \{0\ \}$$

computation of

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$



$\sum_k$



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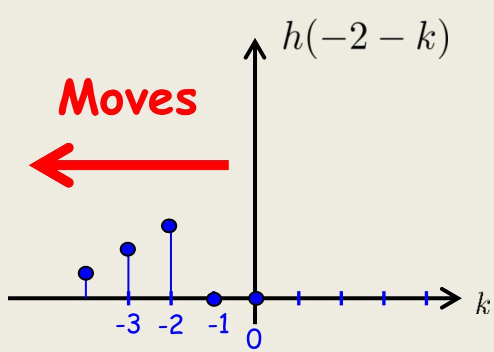


$= 0$

EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

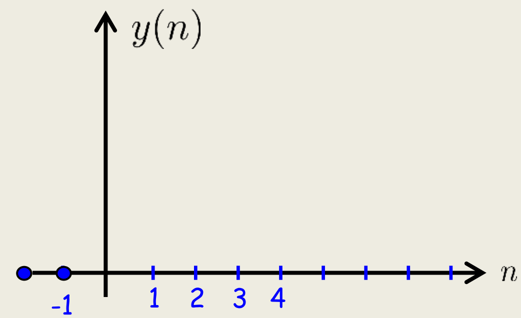
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

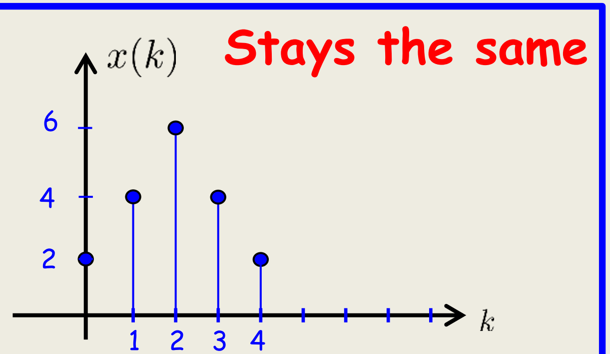
$$y(n) = \{0\ \}$$

computation of

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)h(-2-k)$$



$$\sum_k$$



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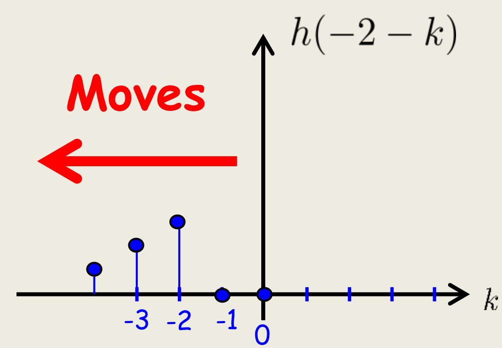


$$= 0$$

"Home work 1"
Verify that causal x(n) and causal h(n) yields causal y(n)

"Home work 2"
If x(n) starts at -3, and h(n) at -4. When does y(n) start?

"Home work 3"
etc

EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

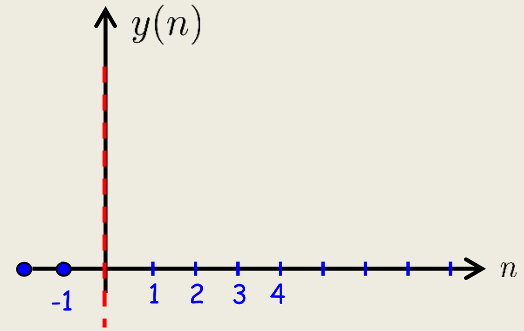
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

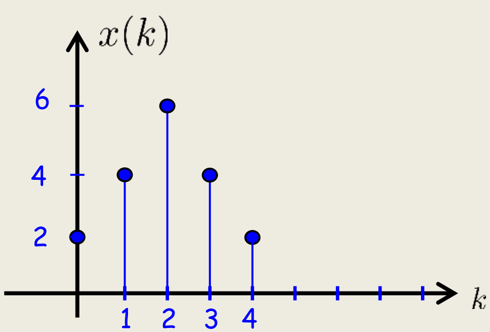
$$y(n) = \{0\ \}$$

computation of

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$



$$\sum_k$$



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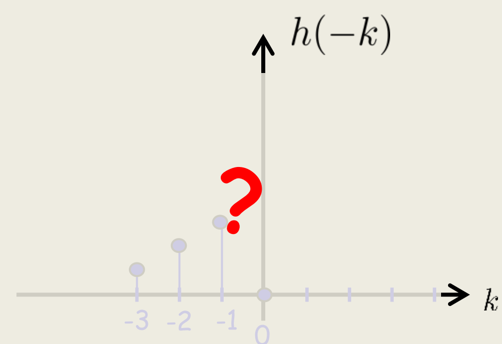


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

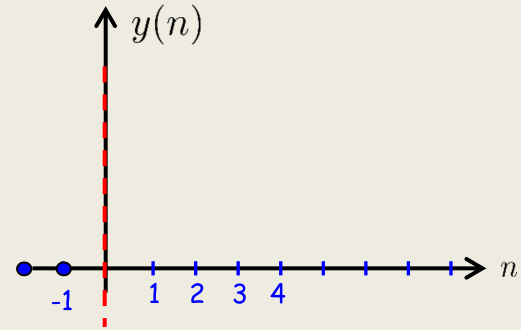
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

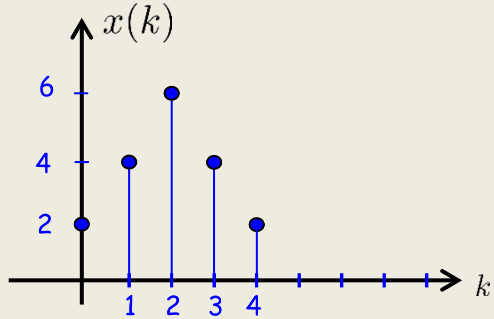
$$y(n) = \{0\ \}$$

computation of

$$y(\color{red}{0}\color{black}) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$



$$\sum_k$$



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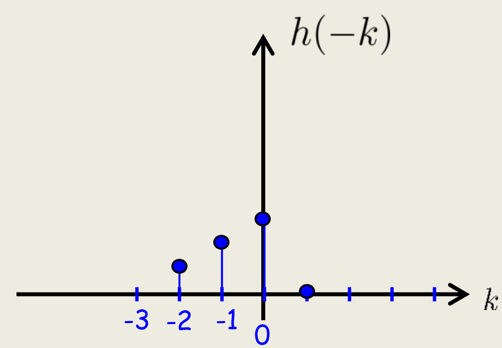


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

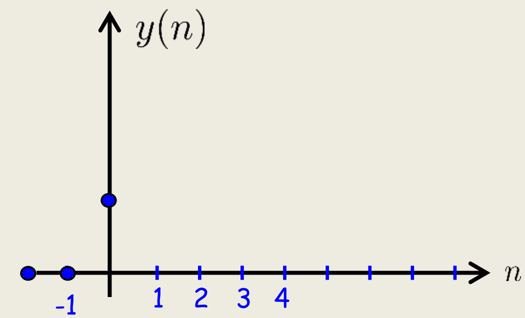
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

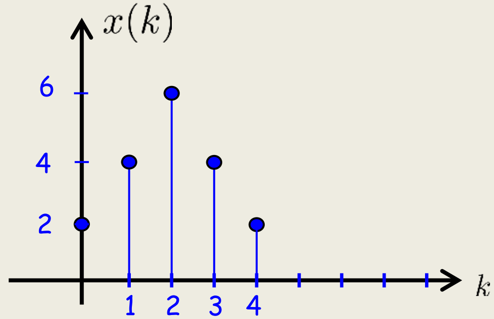
$$y(n) = \{0\ \ \underline{6}\}$$

computation of

$$y(\mathbf{0}) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$



$\sum_k$



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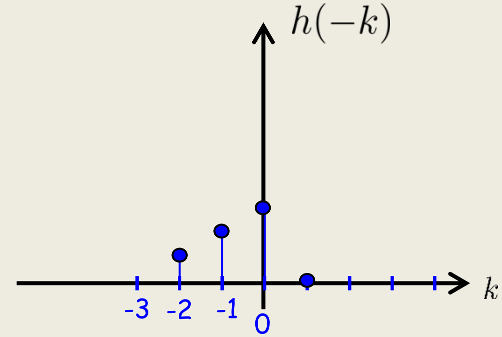


$= 2 \cdot 3 = 6$

EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

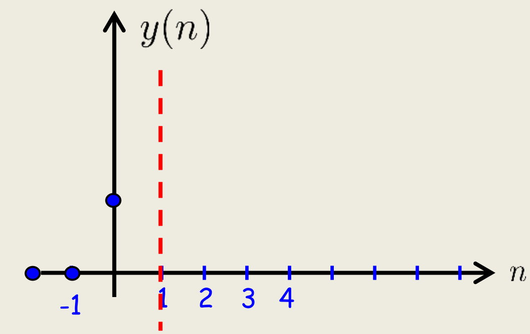
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

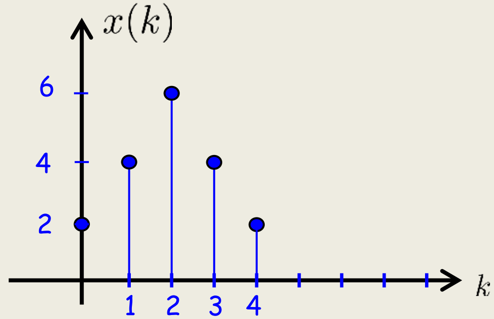
$$y(n) = \{0\ \ \underline{6}\}$$

computation of

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$



$\sum_k$



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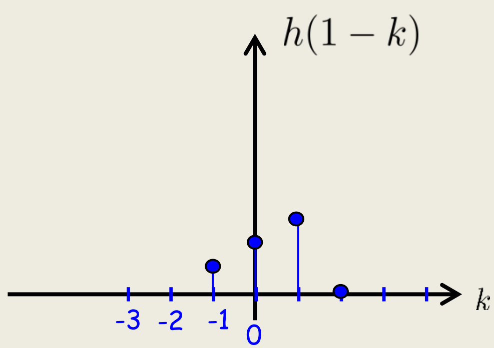


EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

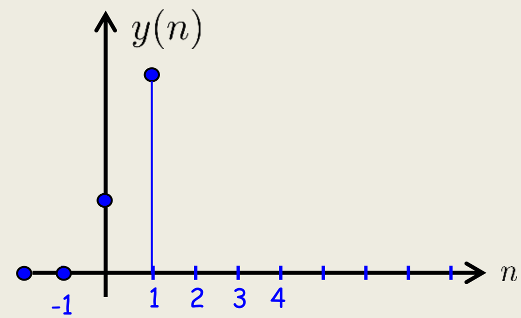
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

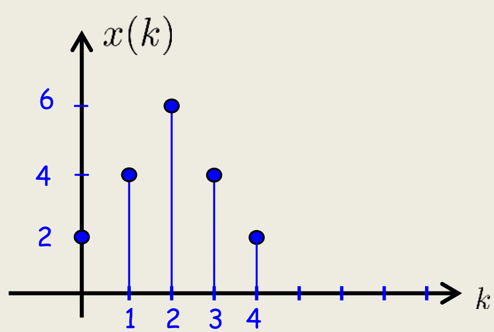
$$y(n) = \{0\ \ \underline{6}\ \ 16\}$$

computation of

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$



$\sum_k$



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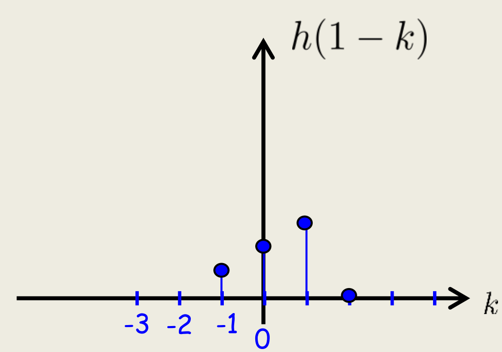


$$= 2 \cdot 2 + 4 \cdot 3 = 16$$

EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

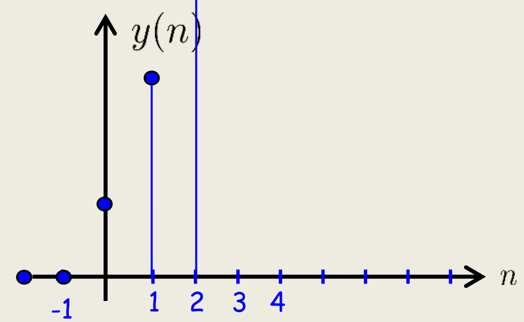
$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

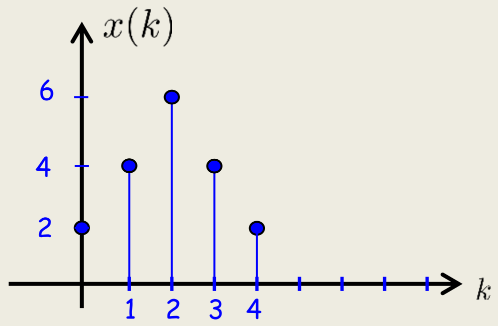
$$y(n) = \{0\ \ \underline{6}\ \ 16\ \ 28\}$$

computation of

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$



$\sum_k$



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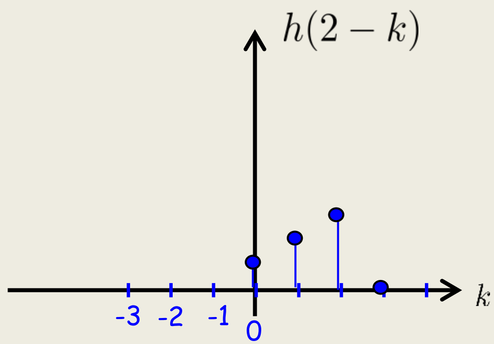

$$= 2 \cdot 1 + 4 \cdot 2 + 6 \cdot 3 = 28$$

EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$y(n) = \{0\ \ \underline{6}\ \ 16\ \ 28\ \ 28\ \ 20\ \ 8\ \ 2\}$$

Repeating gives

EITF75 Systems and Signals

Example

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$y(n) = \{\underline{6}\ \ 16\ \ 28\ \ 28\ \ 20\ \ 8\ \ 2\}$$

EITF75 Systems and Signals

Example

Three more methods

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

EITF75 Systems and Signals

Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

EITF75 Systems and Signals

Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$n = 0$

$h(0-k)$	1	2	$\underline{3}$					
$x(k)$			$\underline{2}$	4	6	4	2	
$h(0-k)x(k)$			$\underline{6}$					$\sum = \underline{6} = y(0)$

EITF75 Systems and Signals

Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$n = 1$

$h(1-k)$	1	2	$\underline{3}$				
$x(k)$		$\underline{2}$	4	6	4	2	
$h(1-k)x(k)$		4	12				$\sum = 16 = y(1)$

EITF75 Systems and Signals

Example

Three more methods
Method 1

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Find: Output signal $y(n) = \sum_k x(k) \cdot h(n-k)$

$$= 3x(n) + 2x(n-1) + x(n-2)$$

$n = 2$

$h(2-k)$	1	2	$\underline{3}$			
$x(k)$	$\underline{2}$	4	6	4	2	
$h(2-k)x(k)$	2	8	18			$\sum = 28 = y(2)$

EITF75 Systems and Signals

Example

Three more methods
Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Put numbers in a table and multiply

	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$	12	18	12	6
2	4	8	12	8	4
1	2	4	6	4	2

Example

Three more methods

Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

Sum the diagonals

	$\underline{2}$	4	6	4	2	
	$\underline{6}$	16	28	28	20	
$\underline{3}$	$\underline{6}$	12	18	12	6	8
2	4	8	12	8	4	2
1	2	4	6	4	2	

EITF75 Systems and Signals

Example

Three more methods
Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

$$y(n) = \{\underline{6}\ \ 16\ \ 28\ \ 28\ \ 20\ \ 8\ \ 2\}$$

	$\underline{2}$	4	6	4	2
$\underline{3}$	$\underline{6}$	12	18	12	6
2	4	8	12	8	4
1	2	4	6	4	2

$\underline{6}$ 16 28 28 20 8 2

Result

EITF75 Systems and Signals

Example

Three more methods
Method 2

Given: Input signal and impulse response

$$x(n) = \{\underline{2}\ 4\ 6\ 4\ 2\}$$

$$h(n) = \{\underline{3}\ 2\ 1\}$$

$$y(n) = \{\underline{6}\ \ 16\ \ 28\ \ 28\ \ 20\ \ 8\ \ 2\}$$

	$\underline{2}$	4	6	4	2	
	$\underline{6}$	16	28	28	20	**Result**
$\underline{3}$	$\underline{6}$	12	18	12	6	
2	4	8	12	8	4	8
1	2	4	6	4	2	2

EITF75 Systems and Signals

Make sure that you understand why a convolution of a **length K signal** with a **length L signal** has **length K+L-1**

EITF75 Systems and Signals

Example

Three more methods
Method 3: Analytical solution

$$x(n) = u(n) \qquad h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$$

EITF75 Systems and Signals

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EITF75 Systems and Signals

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Minor trick. Not really needed, but slightly simpler.
Try without the trick at home

EITF75 Systems and Signals

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EITF75 Systems and Signals

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EITF75 Systems and Signals

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EITF75 Systems and Signals

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EITF75 Systems and Signals

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$$x(n) = u(n) \qquad h(n) = \left(\frac{1}{2}\right)^n \cdot u(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \ldots = 2 - \left(\frac{1}{2}\right)^n \qquad n \geq 0$$

EITF75 Systems and Signals

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$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k = \ldots = 2 - \left(\frac{1}{2}\right)^n \qquad n \geq 0$$

Thus $y(n) = \left[2 - \left(\frac{1}{2}\right)^n\right] \cdot u(n)$

EITF75 Systems and Signals

Standard Properties

Commutativity

$$\boxed{x_1(n) \star x_2(n) = x_2(n) \star x_1(n)}$$

Associativity

$$\boxed{x_1(n) \star [x_2(n) \star x_3(n)] = [x_1(n) \star x_2(n)] \star x_3(n)}$$

Distributivity

$$\boxed{x_1(n) \star [x_2(n) + x_3(n)] = x_1(n) \star x_2(n) + x_1(n) \star x_3(n)}$$

EITF75 Systems and Signals

$$\boxed{x_1(n) \star x_2(n) = x_2(n) \star x_1(n)}$$ Commutativity

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Some consequences

$$\boxed{x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)} \iff \boxed{h(n) \rightarrow \boxed{x(n)} \rightarrow y(n)}$$

EITF75 Systems and Signals

$$\boxed{x_1(n) \star x_2(n) = x_2(n) \star x_1(n)}$$

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Distributivity

Some consequences

$$x(n) \to \boxed{h_1(n)} \to \boxed{h_2(n)} \to y(n) \quad \Longleftrightarrow \quad x(n) \to \boxed{h(n) = h_1(n) \star h_2(n)} \to y(n)$$

$$\Longleftrightarrow$$

$$x(n) \to \boxed{h_2(n)} \to \boxed{h_1(n)} \to y(n)$$

EITF75 Systems and Signals

$$x_1(n) \star x_2(n) = x_2(n) \star x_1(n)$$

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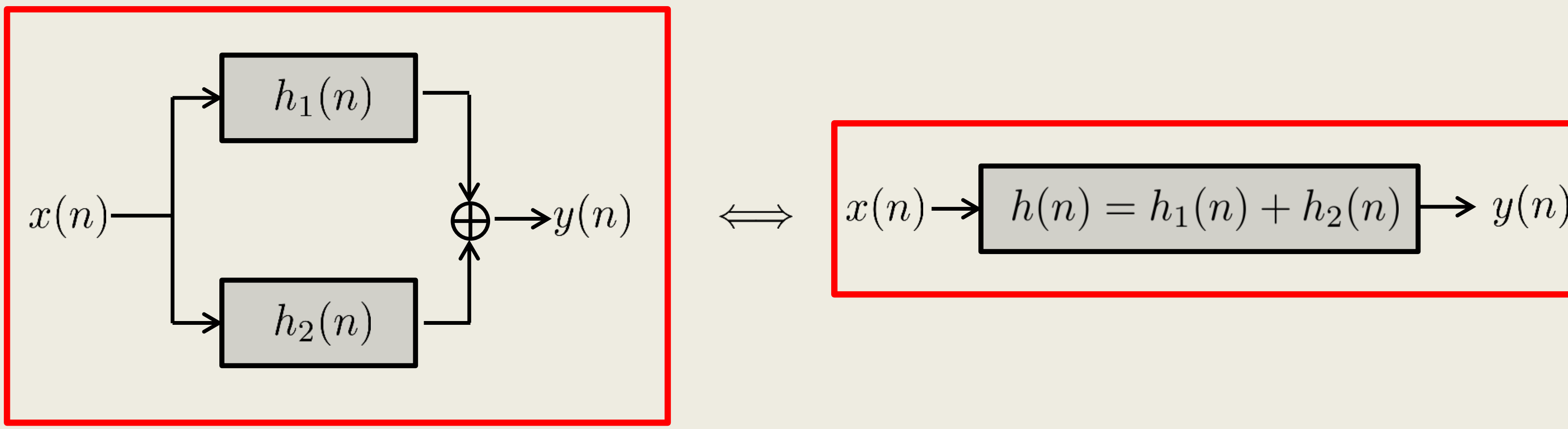

EITF75 Systems and Signals

BIBO stability

A system is BIBO stable if $|x(n)| < M_x \implies |y(n)| < M_y < \infty$

EITF75 Systems and Signals

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For an LTI system

$$|y(n)| = \left|\sum_k x(k)h(n-k)\right|$$

EITF75 Systems and Signals

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$$|y(n)| = \left| \sum_k x(k)h(n-k) \right|$$

$$\leq \sum_k |x(k)||h(n-k)| \qquad |a+b| \leq |a| + |b|$$

EITF75 Systems and Signals

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$$
\begin{aligned}
|y(n)| &= \left| \sum_k x(k)h(n-k) \right| \\
&\leq \sum_k |x(k)||h(n-k)| \qquad |a+b| \leq |a| + |b| \\
&\leq \sum_k M_x |h(n-k)|
\end{aligned}
$$

EITF75 Systems and Signals

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$$\leq \sum_k |x(k)||h(n-k)| \qquad |a+b| \leq |a| + |b|$$

$$\leq \sum_k M_x |h(n-k)| = \sum_k M_x |h(k)|$$

An LTI system is stable if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

EITF75 Systems and Signals

Relation to difference equations

We have seen that an LTI system is fully described by an impulse response $h(n)$

EITF75 Systems and Signals

Relation to difference equations

We have seen that an LTI system is fully described by an impulse response h(n)

We have also mentioned that difference equations are important for LTI systems

$$\sum_{k} a(k)y(n-k) = \sum_{\ell} b(\ell)x(n-\ell)$$

Relation to difference equations

We have seen that an LTI system is fully described by an impulse response h(n)

We have also mentioned that difference equations are important for LTI systems

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

This means that every impulse response h(n) is equivalent to a difference equation

We now investigate this

EITF75 Systems and Signals

Relation to difference equations

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Suppose $a(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$ $\qquad b(\ell) = 0,\ \ell > L, \ell < 0$

EITF75 Systems and Signals

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This is a convolution, with a finite length impulse response b(n)

EITF75 Systems and Signals

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The class of systems described by difference equations encompasses LTI systems with finite length impulse responses

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$$\begin{aligned} y(2) &= -a_1 y(1) + b_0 x(2) \\ &= (-a_1)^3 y(-1) + b_0 x(2) + (-a_1) b_0 x(1) + (-a_1)^2 b_0 x(0) \end{aligned}$$

EITF75 Systems and Signals

Relation to difference equations

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

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Pattern recognition, suitably done at home, gives

$$y(n) = \sum_{k=0}^{n} (-a_1)^k b_0 x(n-k) + (-a_1)^{n+1} y(-1)$$

EITF75 Systems and Signals

Relation to difference equations

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Convolution

EITF75 Systems and Signals

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$$\begin{aligned} y(n) &= \sum_{k=0}^{n} (-a_1)^k b_0 x(n-k) + (-a_1)^{n+1} y(-1) \\ &= \sum_{k=0}^{n} h(k) x(n-k) + (-a_1)^{n+1} y(-1) \end{aligned}$$

$$h(k) = (-a_1)^k b_0 u(k)$$

Infinite Impulse response (IIR)

EITF75 Systems and Signals

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Not Linear system
Not time-invariant

EITF75 Systems and Signals

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Note: Only makes sense to assume

$$|a_1| < 1$$

So, transient will fade out and "after a while it is LTI"

EITF75 Systems and Signals

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EITF75 Systems and Signals

Relation to difference equations

$$\sum_k a(k)y(n-k) = \sum_\ell b(\ell)x(n-\ell)$$

Every impulse response corresponds to one difference equation.

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EITF75 Systems and Signals

Brief info on correlation

Not focal point of course, but highly important in signal processing

Correlation measures similarity between two signals

EITF75 Systems and Signals

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Correlation measures similarity between two signals

Auto correlation	Cross correlation
$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(n) \star x(-n)$	$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(n) \star x(-n)$
Measures similarity between time shifted versions of the same signal	Measures similarity between time shifted versions of different signals

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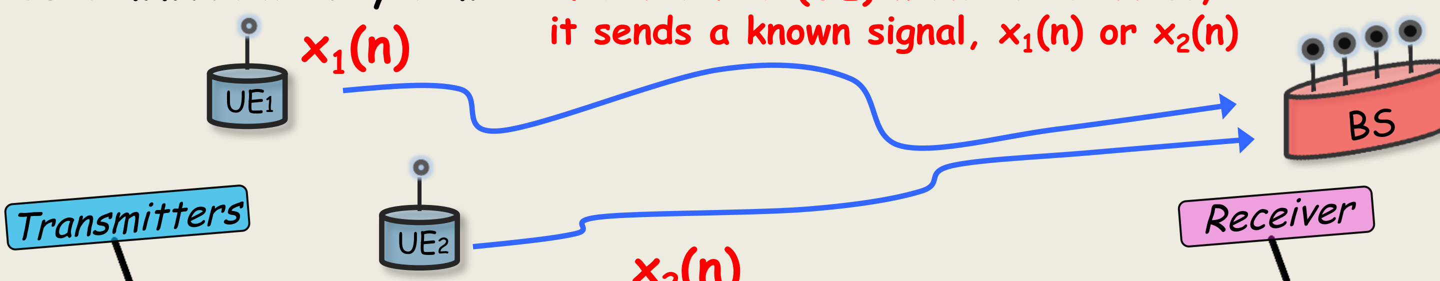
Cross correlation

$$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(n) \star x(-n)$$

Measures similarity between time shifted versions of different signals

Example: 5G communication system

When a user (UE) wants to connect, it sends a known signal, $x_1(n)$ or $x_2(n)$



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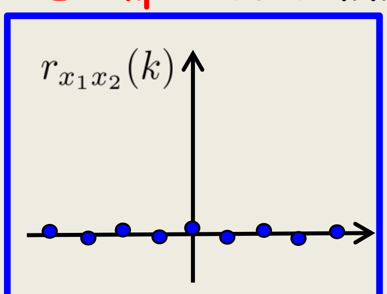
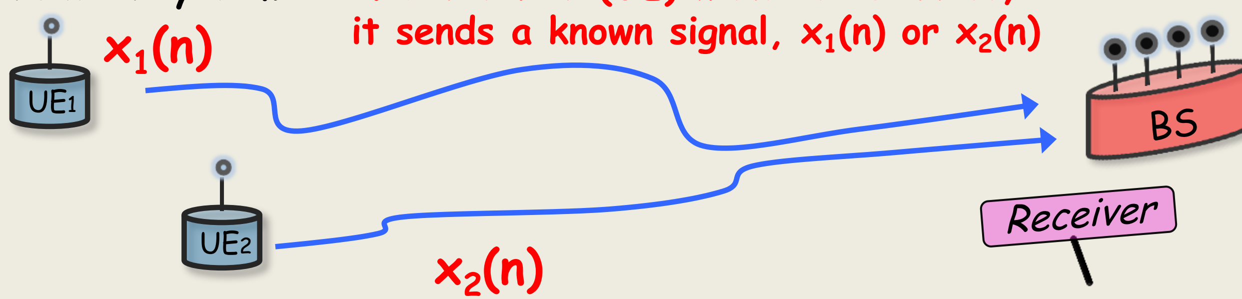
Measures similarity between time shifted versions of different signals

Example: 5G communication system



When a user (UE) wants to connect, it sends a known signal, $x_1(n)$ or $x_2(n)$



Cross correlation between $x_1(n)$ and $x_2(n)$ should be small (to know who is connecting)

EITF75 Systems and Signals

Brief info on correlation

Not focal point of course, but highly important in signal processing

Correlation measures similarity between two signals

Auto correlation

$$r_{xx}(k) = \sum_{n=-\infty}^{\infty} x(n)x(n-k) = x(n) \star x(-n)$$

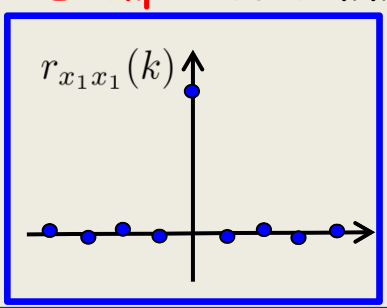
Measures similarity between time shifted versions of the same signal

Cross correlation

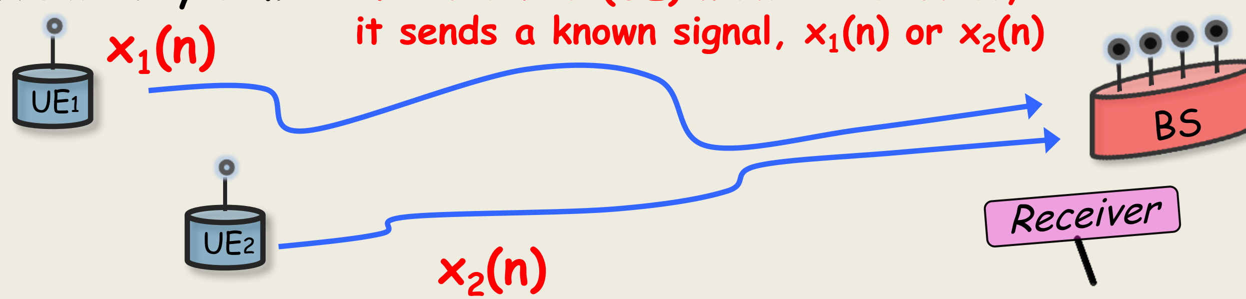
$$r_{yx}(k) = \sum_{n=-\infty}^{\infty} y(n)x(n-k) = y(n) \star x(-n)$$

Measures similarity between time shifted versions of different signals

Example: 5G communication system



When a user (UE) wants to connect, it sends a known signal, $x_1(n)$ or $x_2(n)$



Auto correlation of $x_1(n)$ (and $x_2(n)$) should be delta (to know when a user is connecting)

Brief info on correlation

Cross correlation for input and output signals

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$$

$$\begin{aligned} r_{yx}(k) &= y(k) \star x(-k) \\ &= x(k) \star h(k) \star x(-k) \\ &= h(k) \star x(k) \star x(-k) \\ &= h(k) \star r_{xx}(k) \end{aligned}$$

$$\begin{aligned} r_{yy}(k) &= y(k) \star y(-k) \\ &= x(k) \star h(k) \star x(-k) \star h(-k) \\ &= r_{hh}(k) \star r_{xx}(k) \end{aligned}$$