EITF75 Systems and Signals

Lecture 11
More on DFTs

Fredrik Rusek

EITF75 Systems and Signals

Recap

For a sequence $x(n)$ of arbitrary length, the N-point DFT is defined as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

and the inverse transform (IDFT) as

$$x_{\mathrm{IDFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } n = 0, 1, \ldots, N-1$$

DFTF

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \mathrm{e}^{-i2\pi n f}$$

Formula for DTFT

EITF75 Systems and Signals

Recap

For a sequence $x(n)$ of arbitrary length, the N-point DFT is defined as

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Suitable for computers

- Fast to compute
- Discrete)

DFTF

$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \mathrm{e}^{-i2\pi n f}$$

Suitable for analysis (by human)

EITF75 Systems and Signals

Convolutions

In real life, the input signal is very long, does not start/stop. Previous method (Lecture 10) fails

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Three common methods

Overlap-add   to be described

Overlap-save

Overlap-discard

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add

Visualization of input

.... x(57) x(58) x(59) x(60) x(1835466) ....

Visualization of impulse response

h(0) h(M-1)

EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add

Partion the input in blocks of length L

.... | $x_1(n)$ | $x_2(n)$ | etc | ....

Visualization of impulse response

| h(0) h(M-1) |

EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add

Partion the input in blocks of length L

.... | $x_1(n)$ | $x_2(n)$ | etc | ....

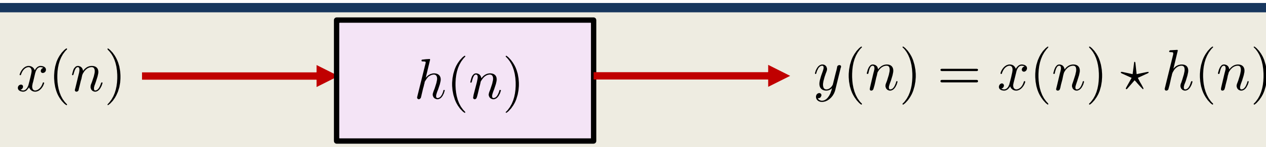
Visualization of impulse response

h(0) h(M-1)

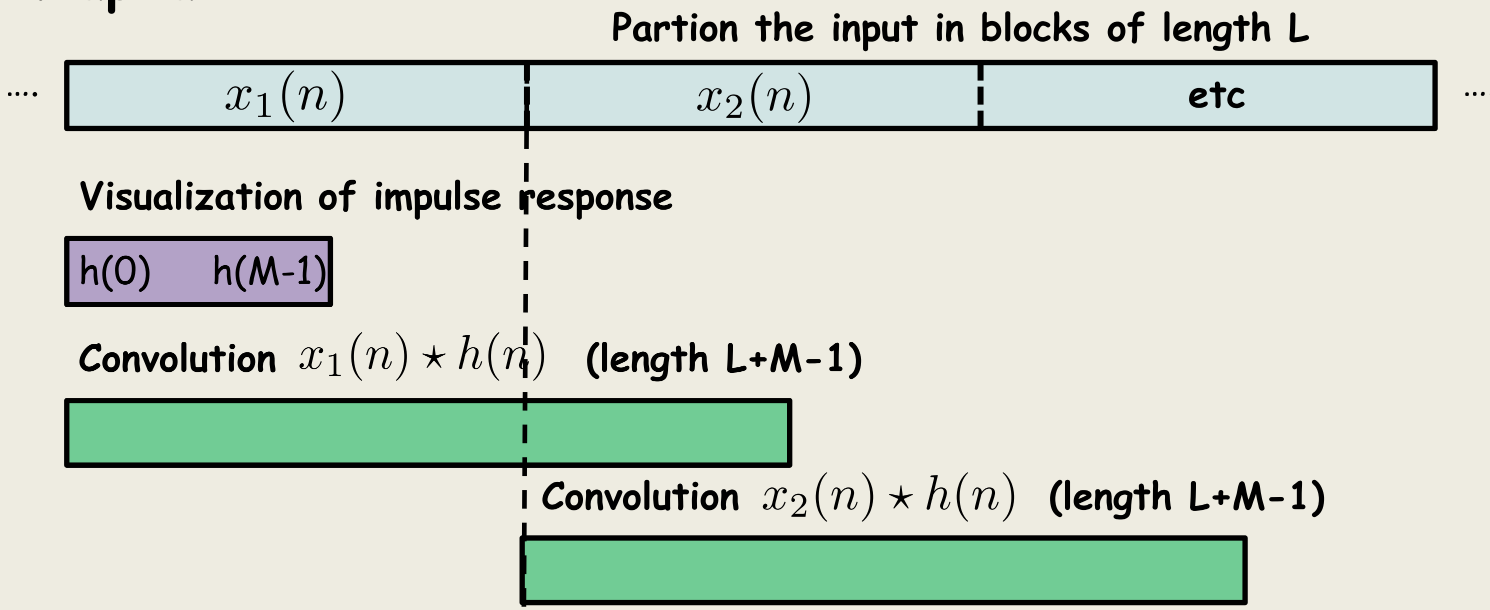
Convolution $x_1(n) \star h(n)$ (length L+M-1)

EITF75 Systems and Signals

Convolutions



Overlap-add

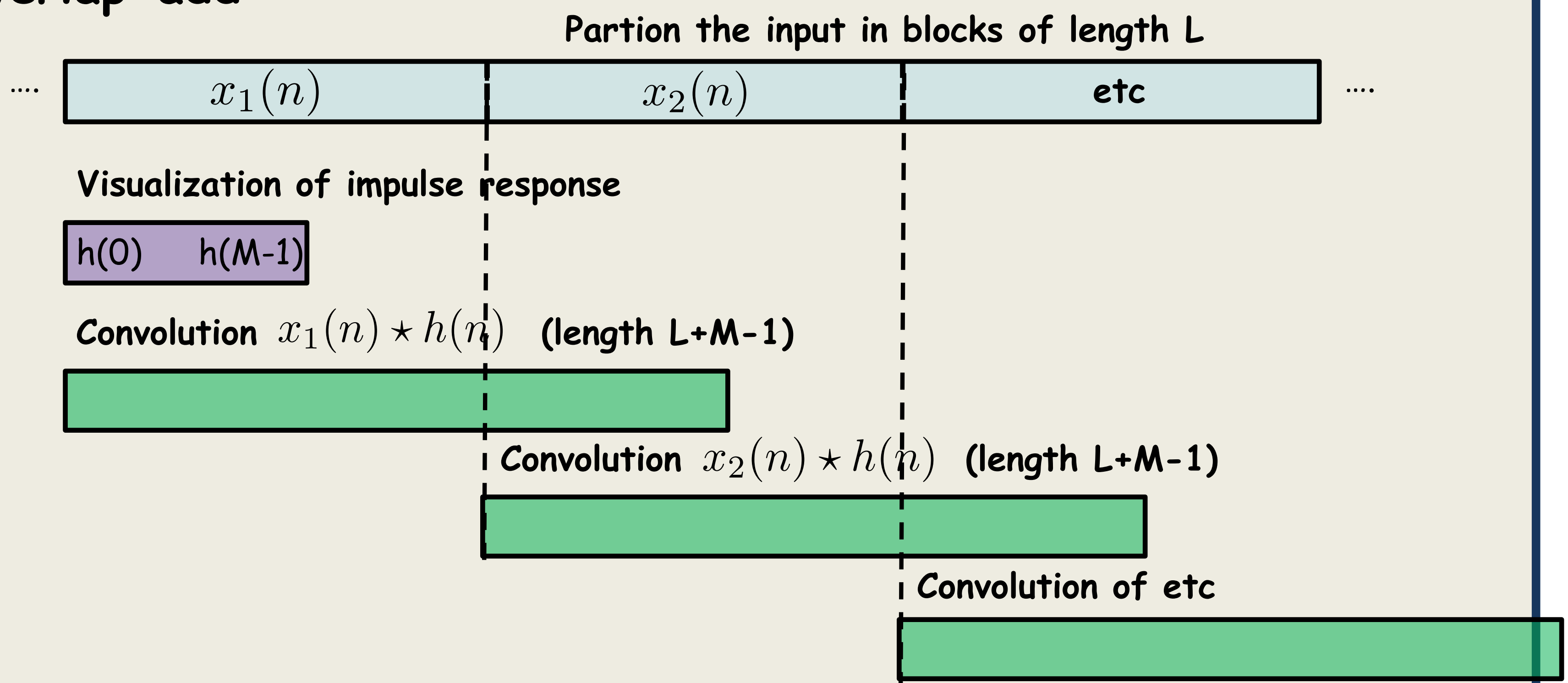


EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add

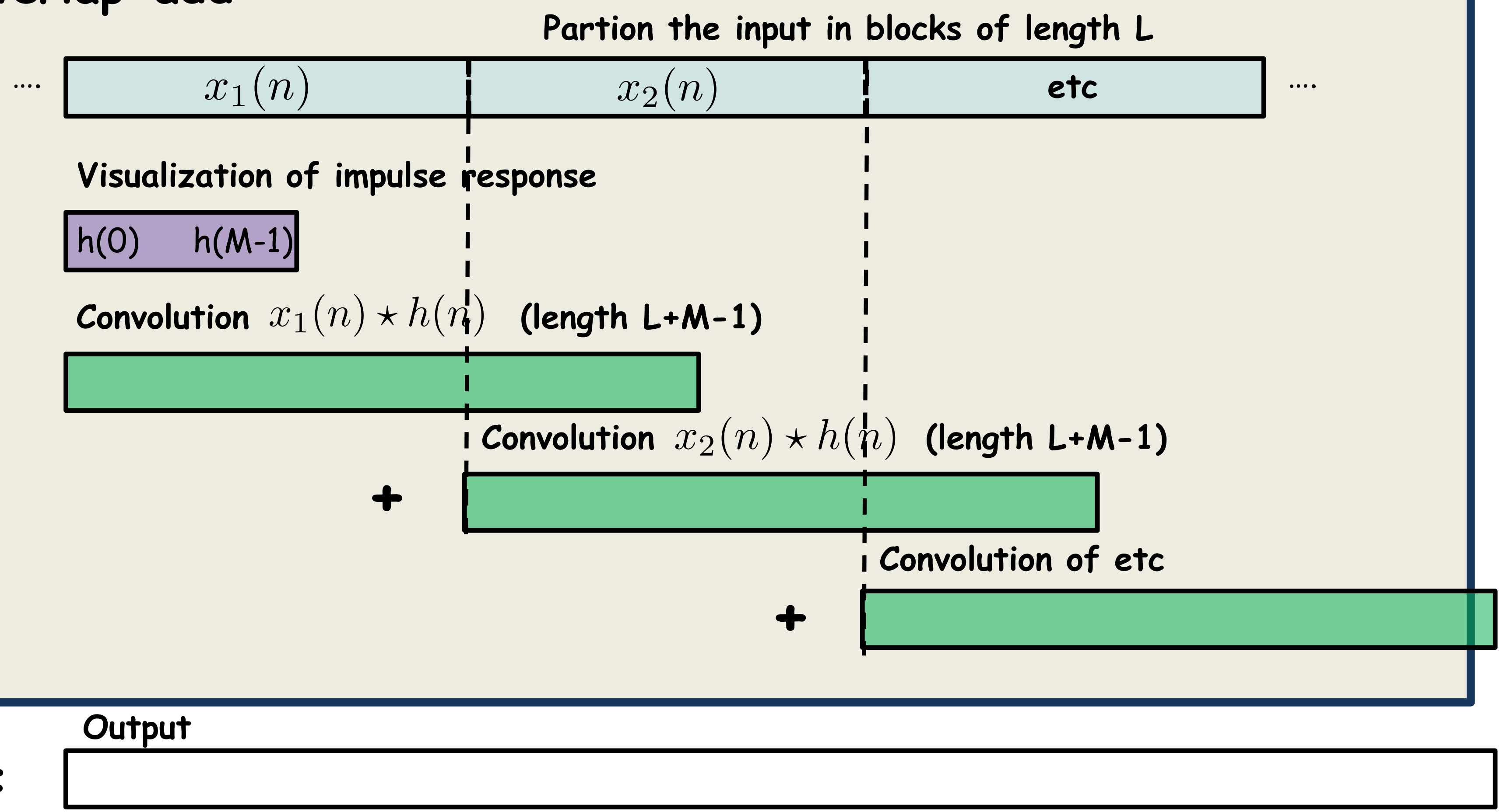


EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add

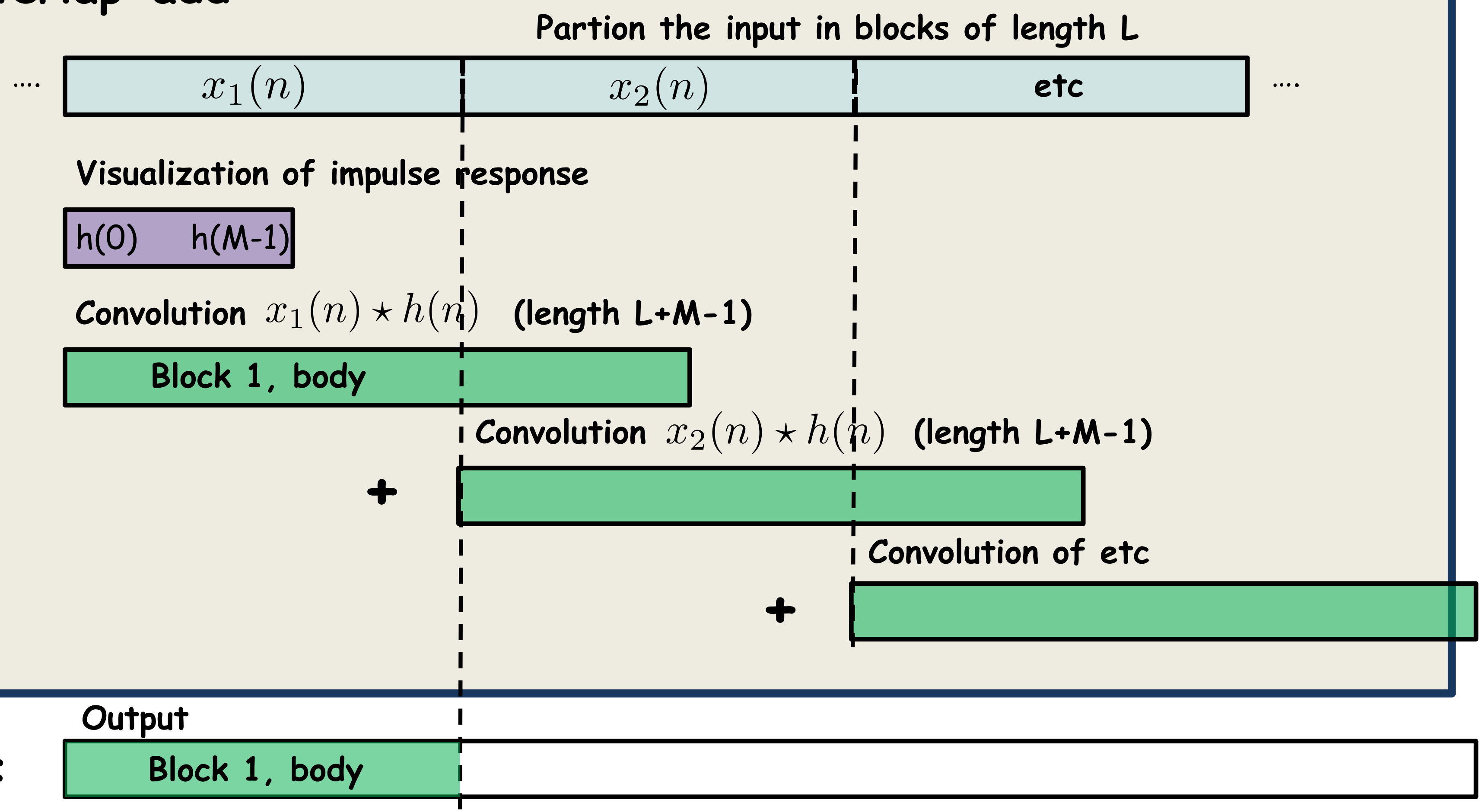


EITF75 Systems and Signals

Convolutions

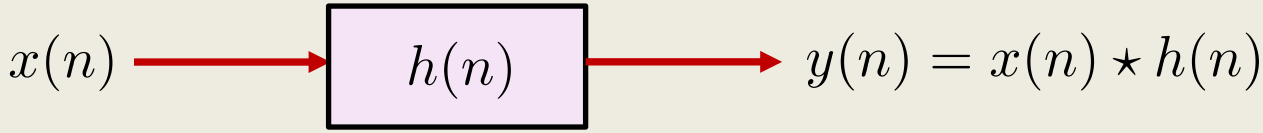
$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add

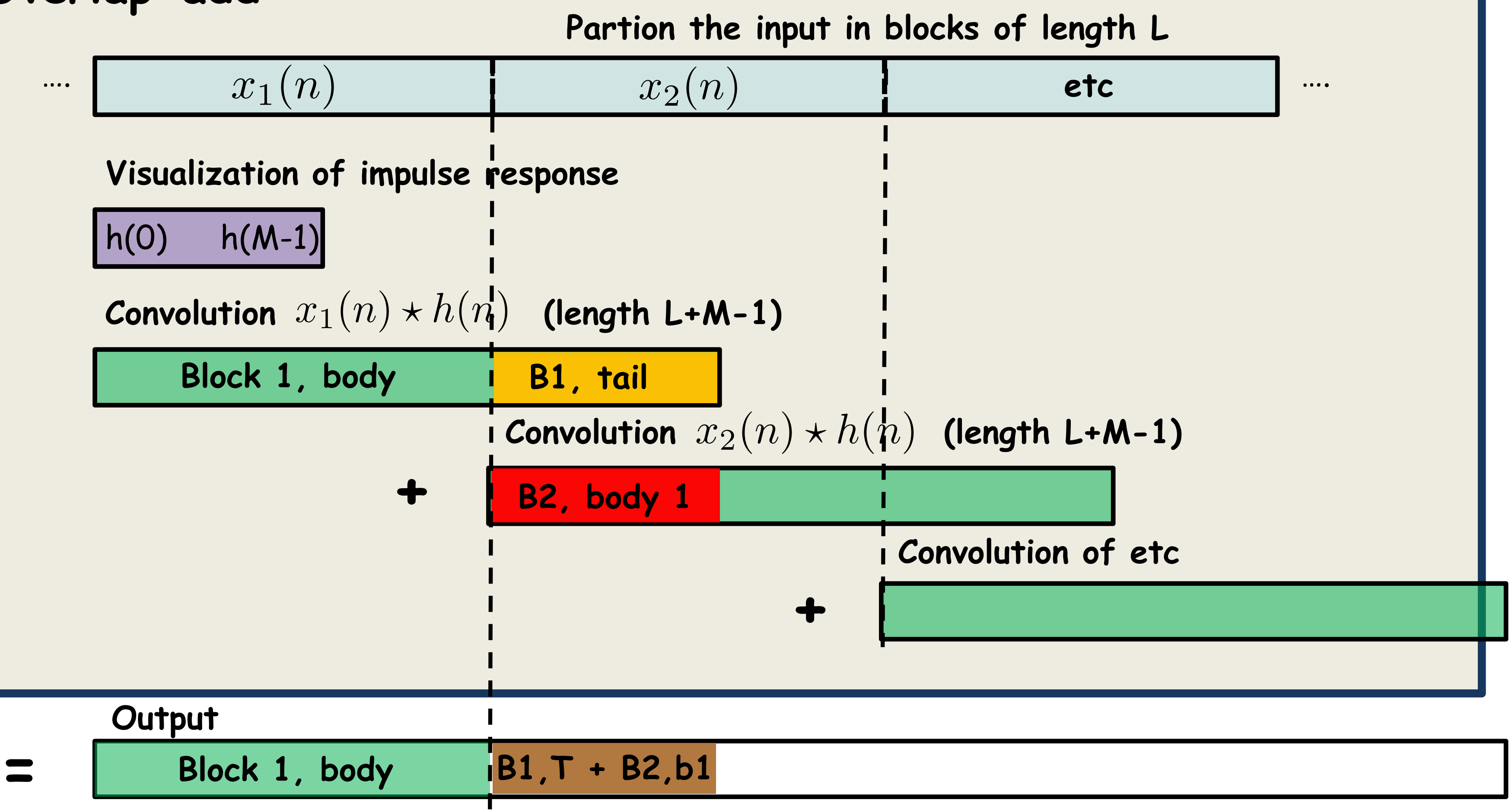


EITF75 Systems and Signals

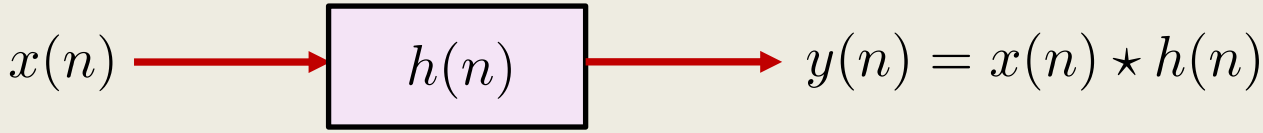
Convolutions



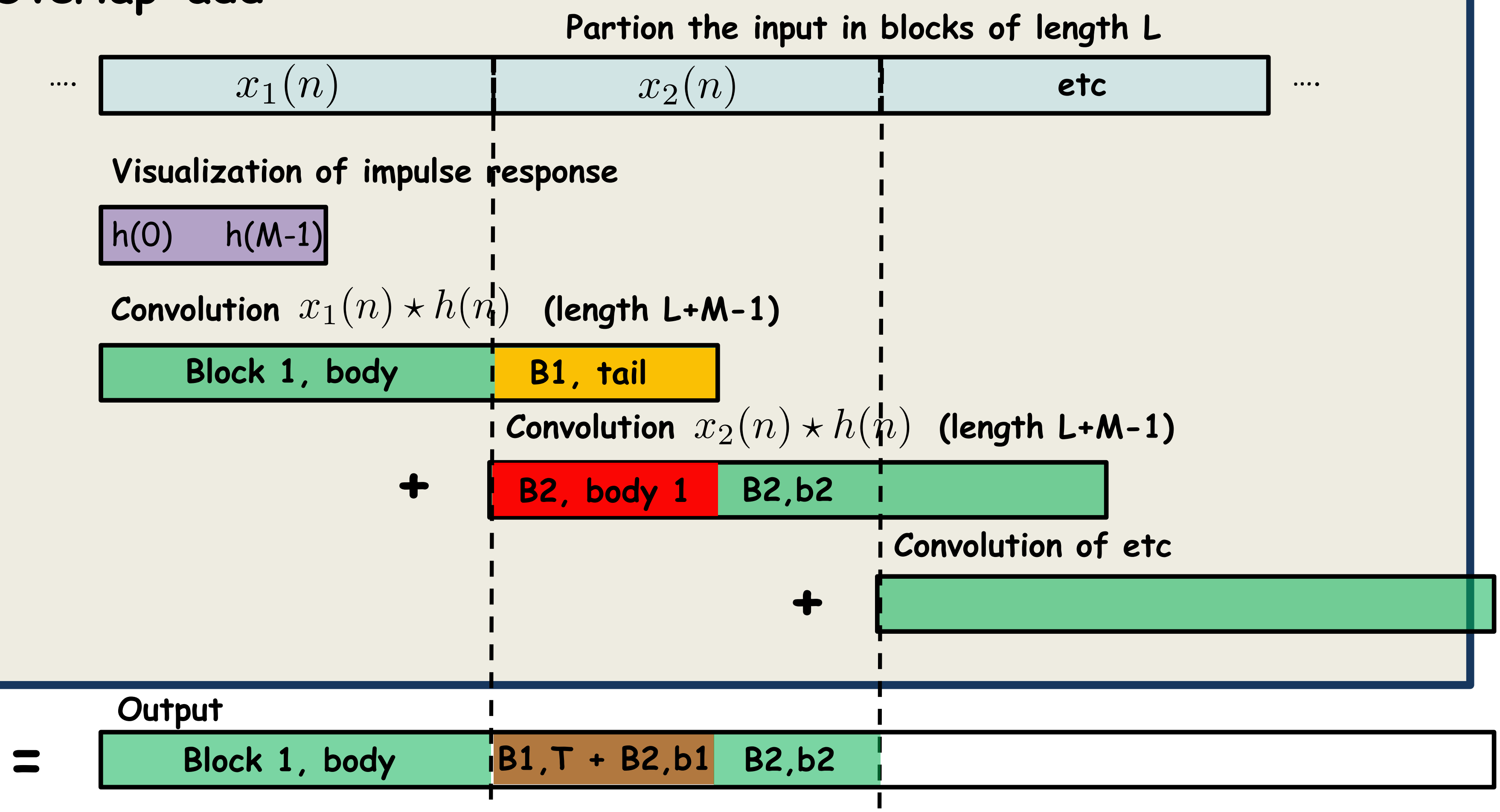
Overlap-add



Convolutions



Overlap-add

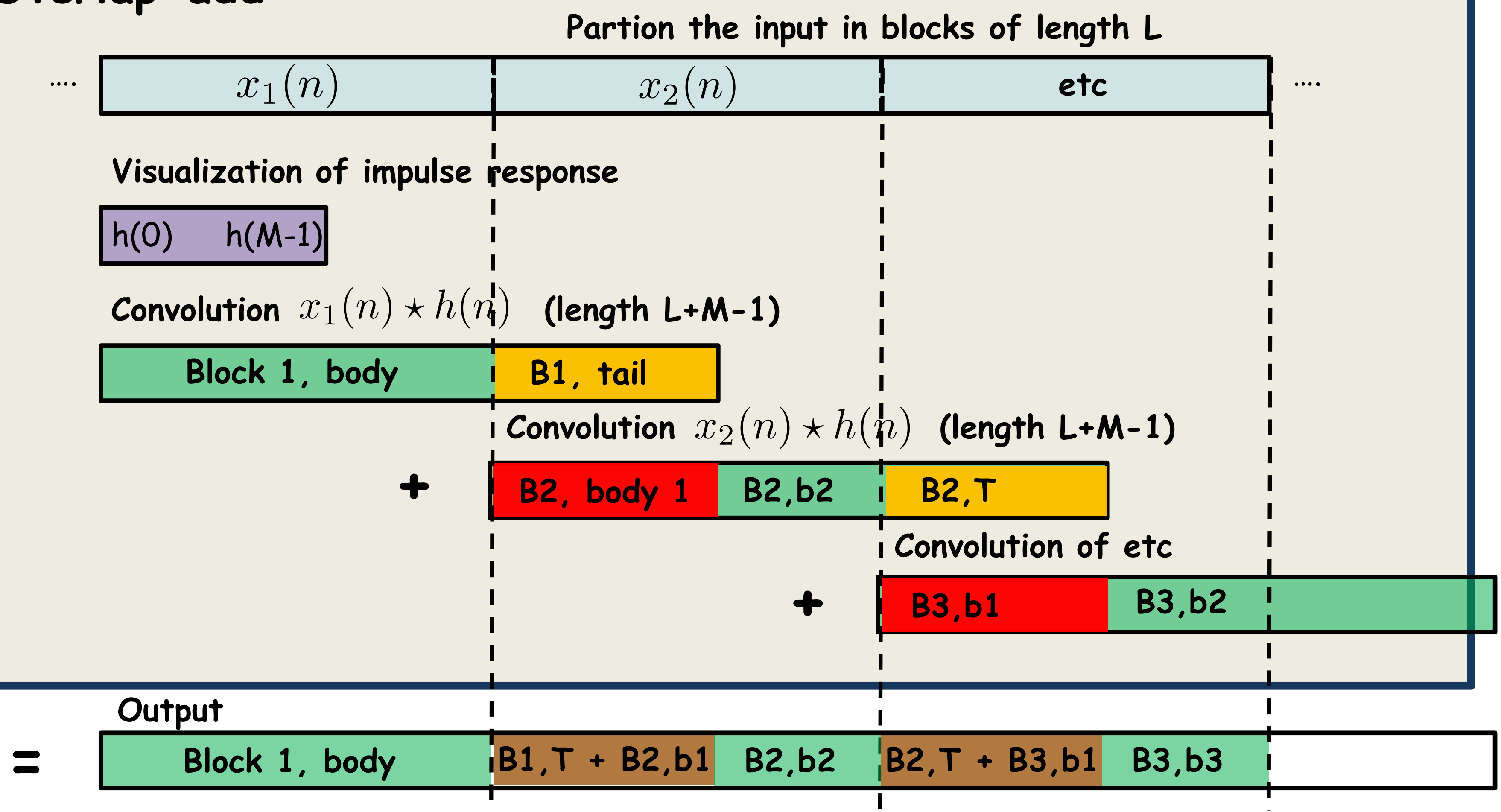


EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow h(n) \rightarrow y(n) = x(n) \star h(n)$

Overlap-add

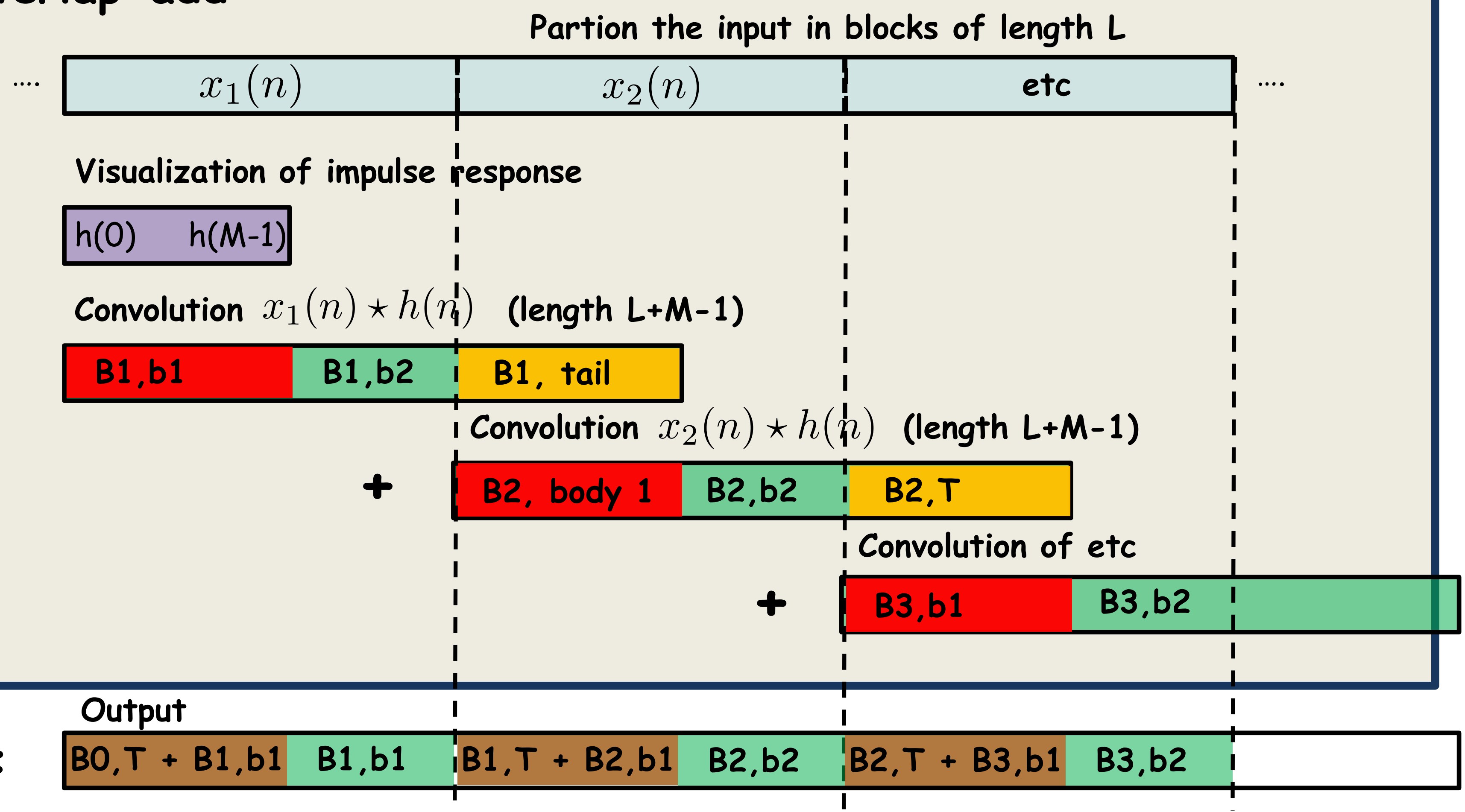


EITF75 Systems and Signals

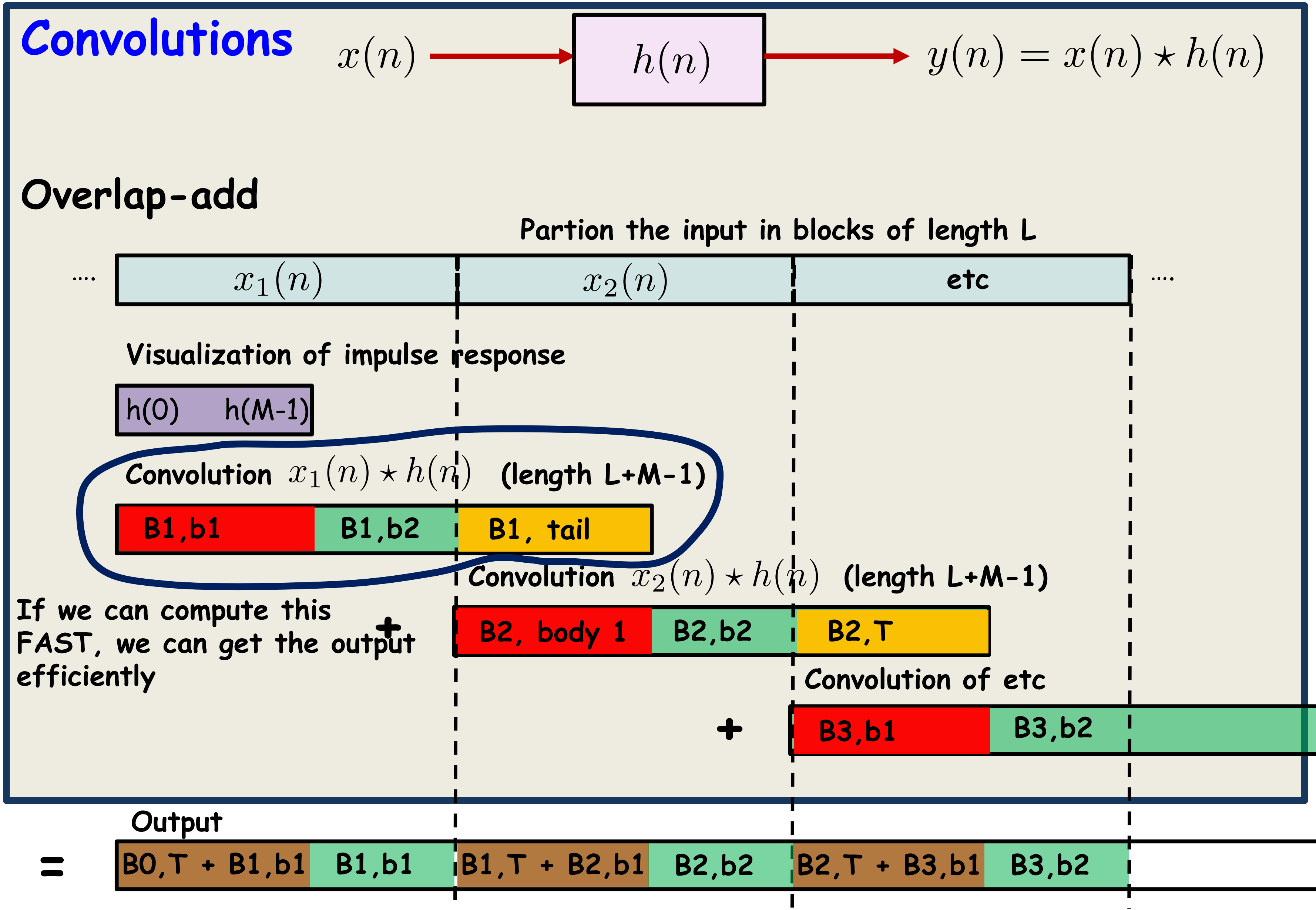
Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add



EITF75 Systems and Signals



EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add

Partion the input in blocks of length L

.... | $x_1(n)$ | $x_2(n)$ | etc | ....

Visualization of impulse response

h(0) h(M-1)

Convolution $x_1(n) \star h(n)$ (length L+M-1)

Zero-pad Block 1 to length N=L+M-1

$x_1(n)$ 0 0 0 0 ... 0

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add

Partion the input in blocks of length L

....	$x_1(n)$	$x_2(n)$	etc	....

Visualization of impulse response

h(0)	h(M-1)

Convolution $x_1(n) \star h(n)$ (length L+M-1)

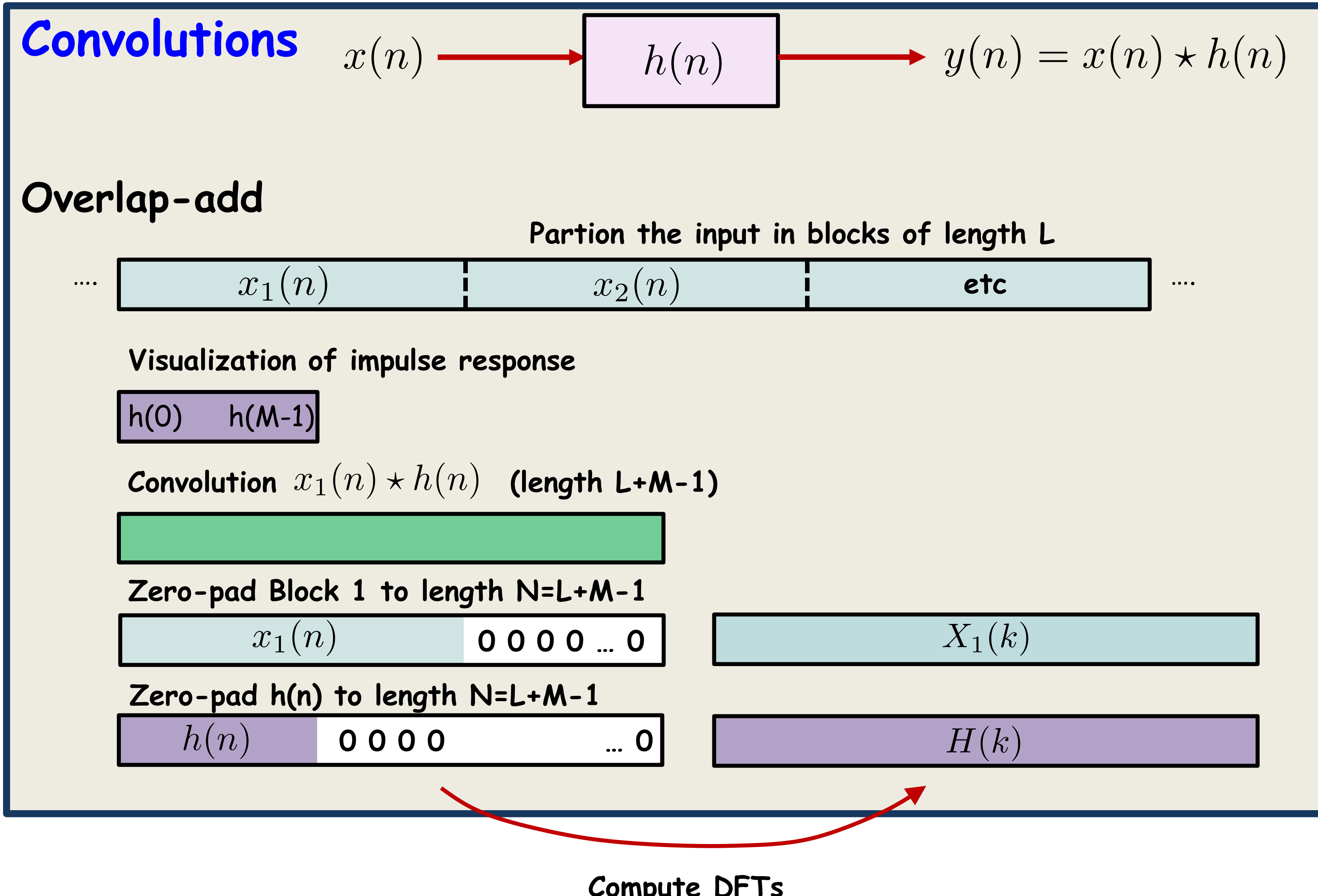
Zero-pad Block 1 to length N=L+M-1

$x_1(n)$	0 0 0 0 … 0

Zero-pad h(n) to length N=L+M-1

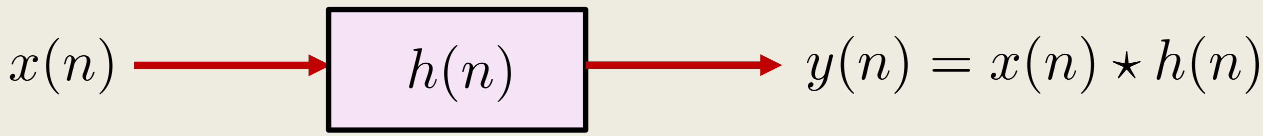
$h(n)$	0 0 0 0 … 0

EITF75 Systems and Signals

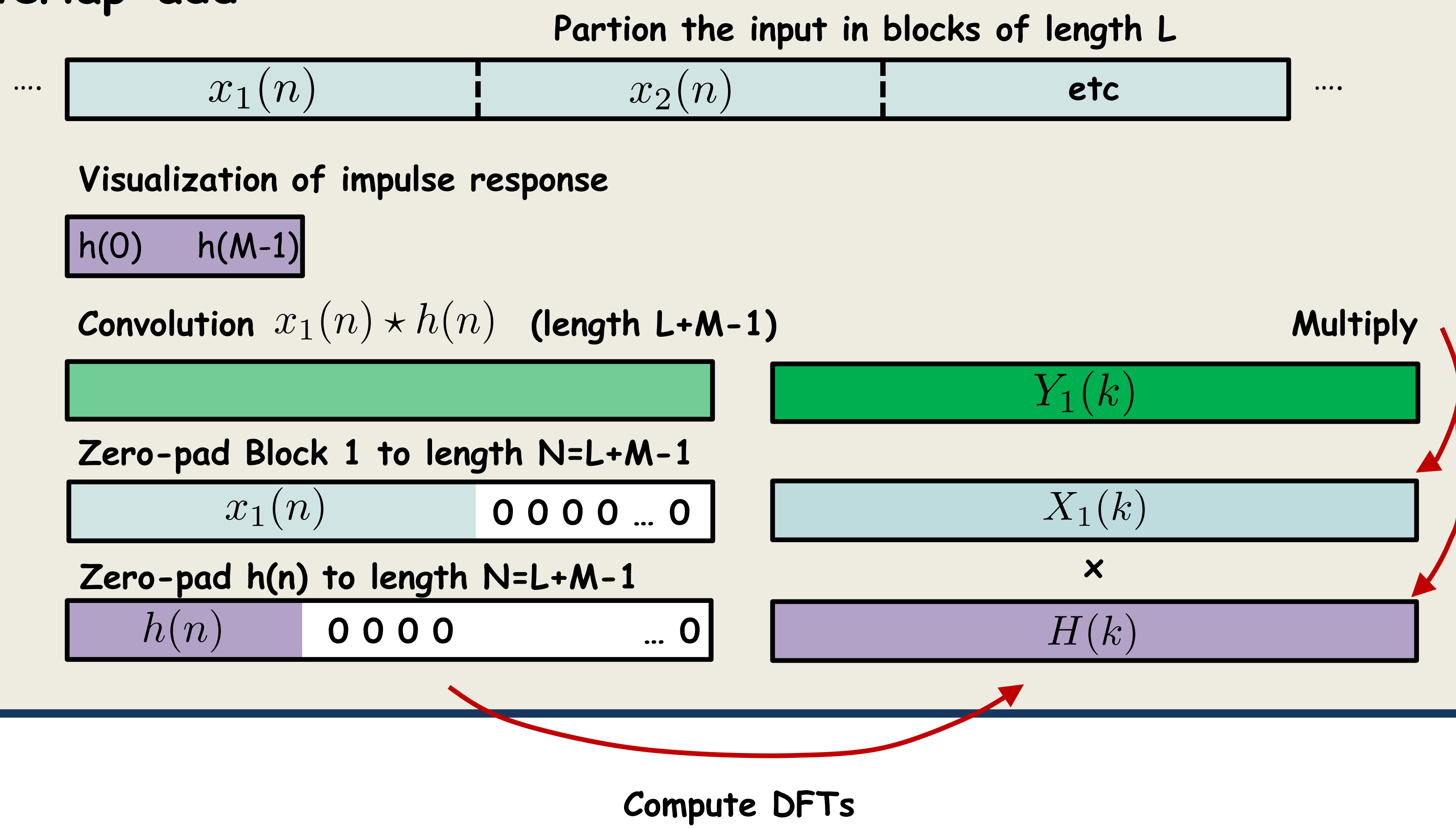


EITF75 Systems and Signals

Convolutions

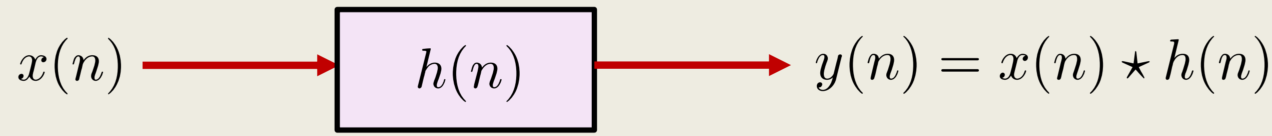


Overlap-add

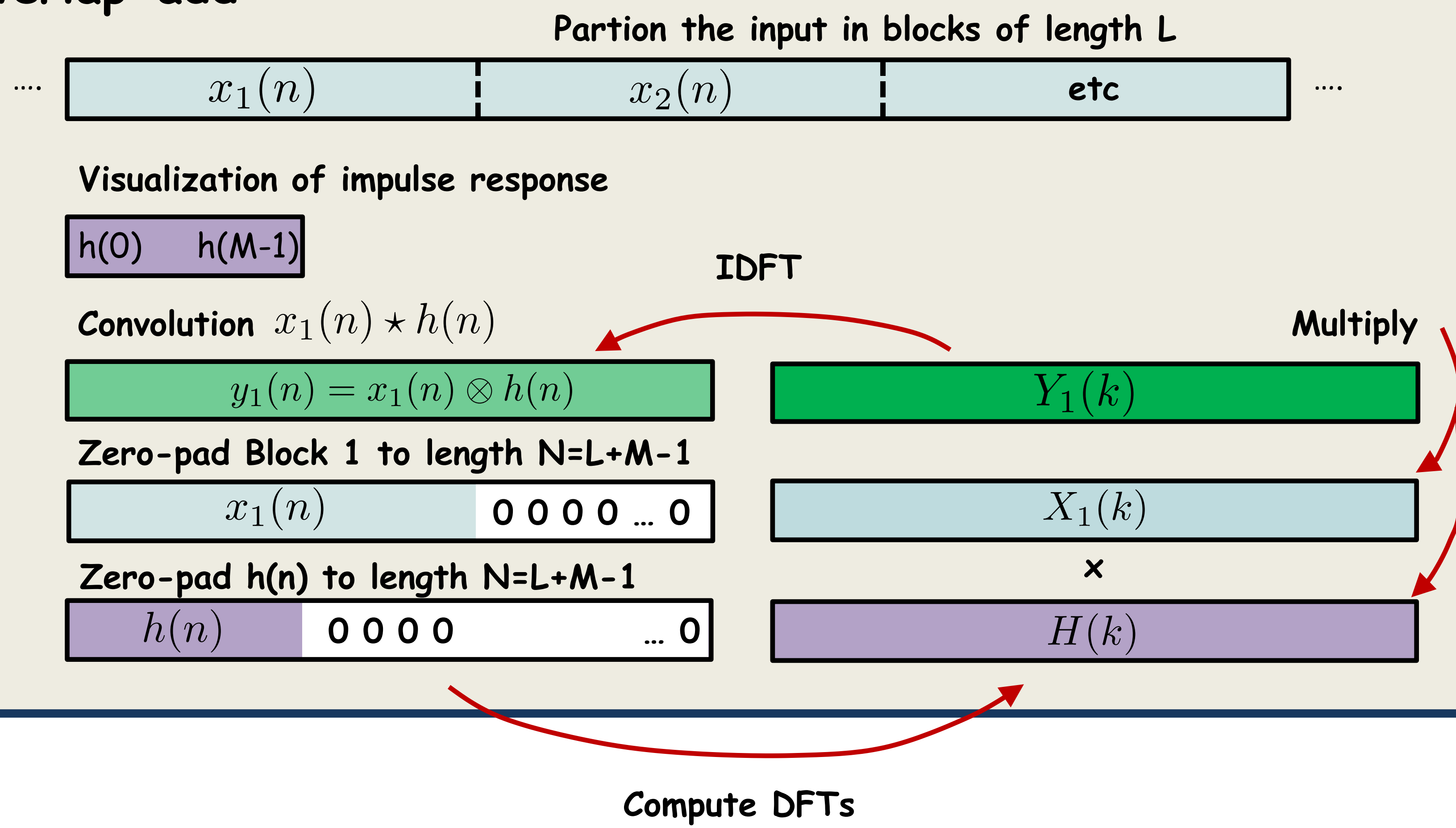


EITF75 Systems and Signals

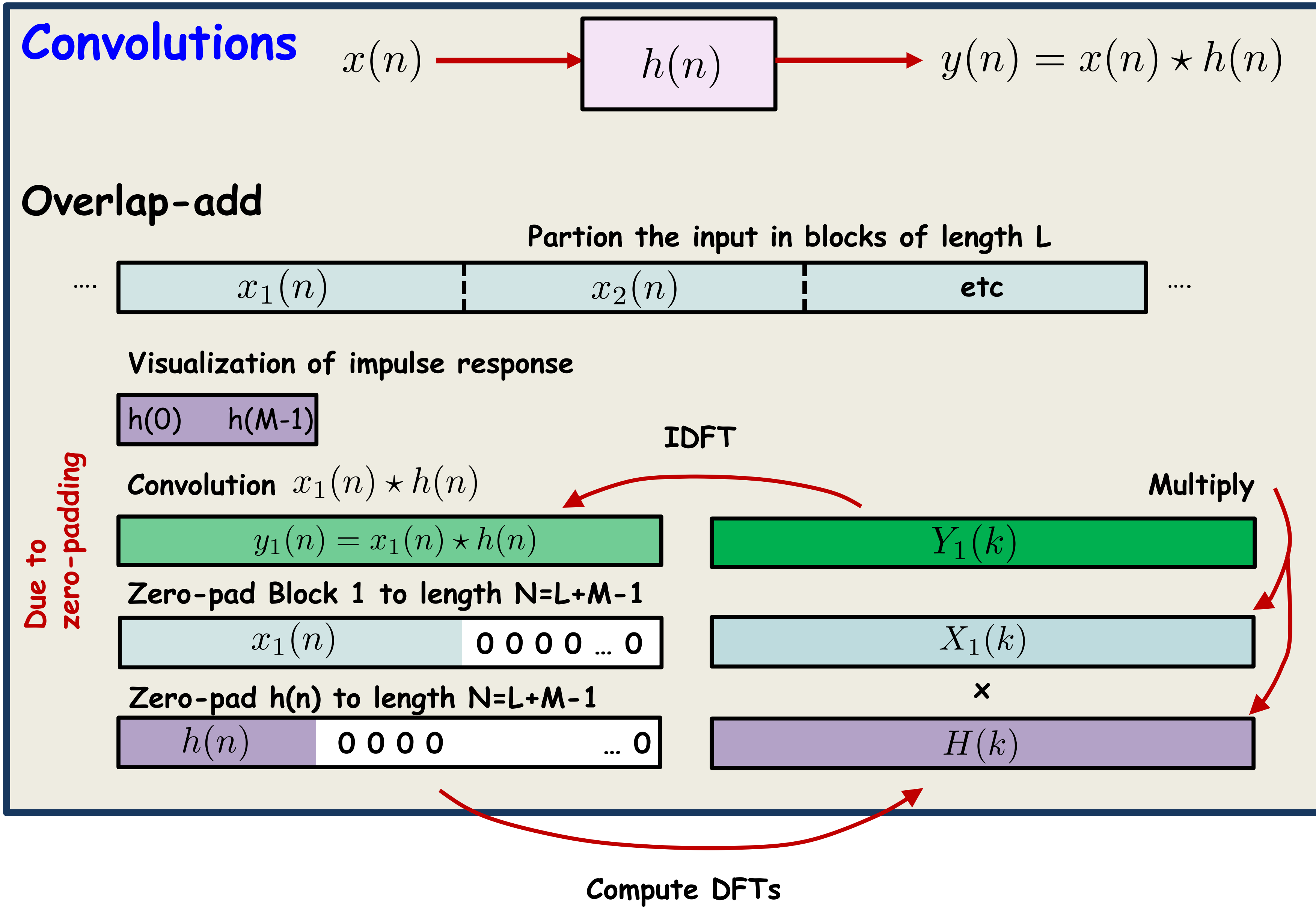
Convolutions



Overlap-add



EITF75 Systems and Signals

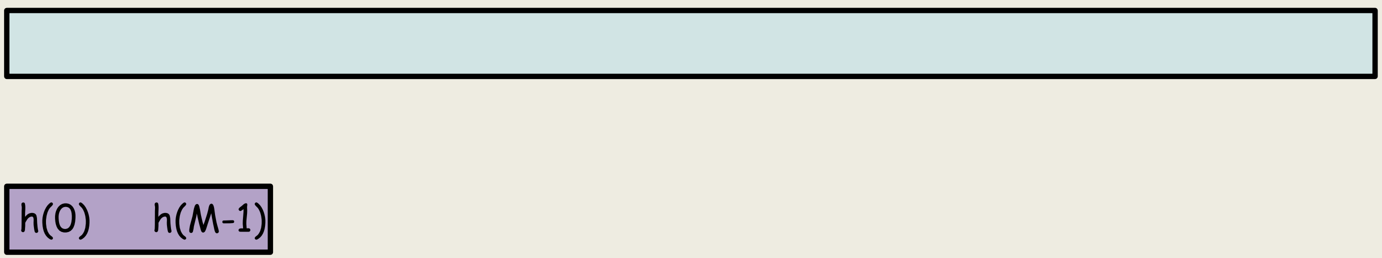


EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

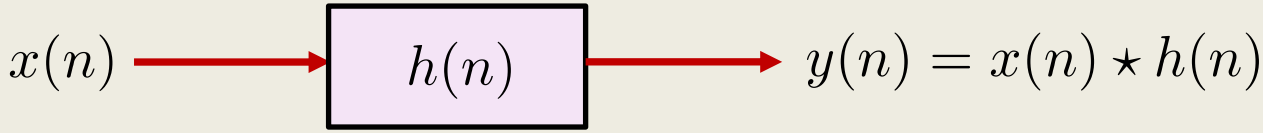
Overlap-add. **SUMMARY**



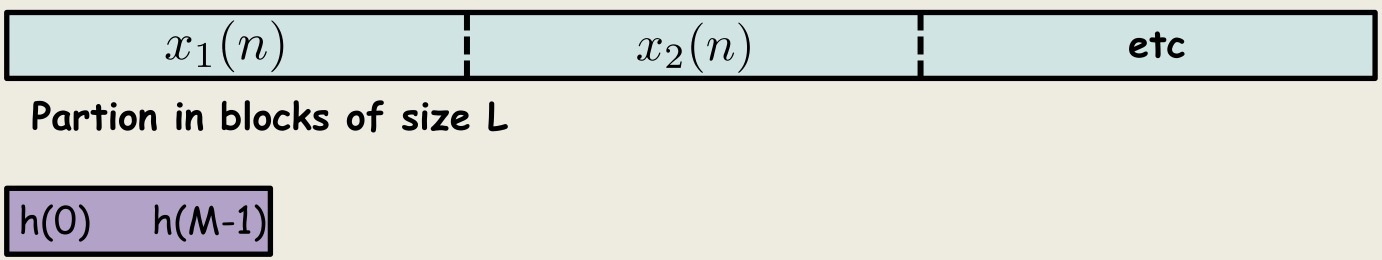
Compute DFTs

EITF75 Systems and Signals

Convolutions



Overlap-add. SUMMARY STEP 1



EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$

Overlap-add. **SUMMARY** **STEP 2**

$x_1(n)$	$x_2(n)$	etc

Partion in blocks of size L

h(0) h(M-1)

Zero-pad to length N=L+M-1

$h(n)$	0 0 0 0 … 0

$H(k)$

Compute DFT (pre-processing)

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. SUMMARY STEP 2

$x_1(n)$	$x_2(n)$	etc

Partion in blocks of size L

h(n) no longer needed

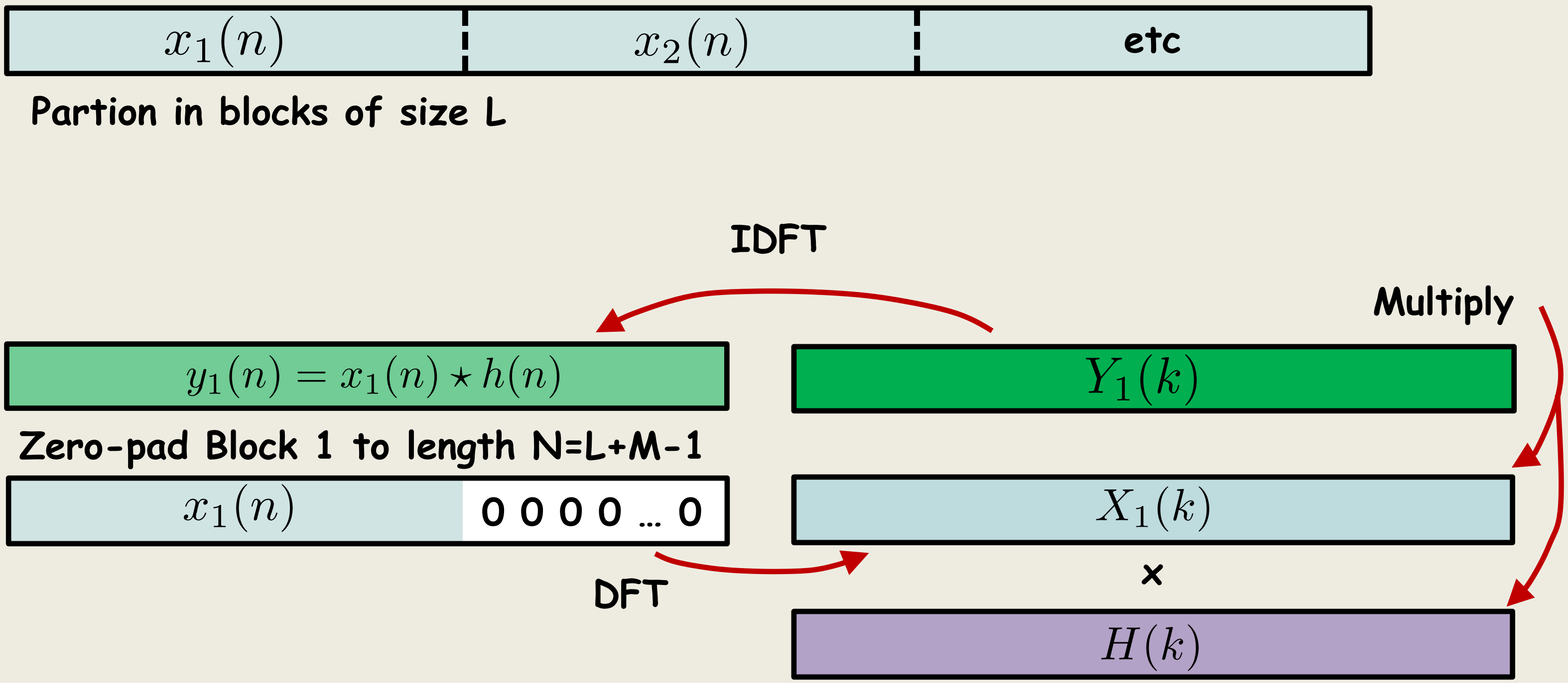
$H(k)$

EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add. SUMMARY STEP 3 For block 1



EITF75 Systems and Signals

Convolutions

$x(n)$ → $h(n)$ → $y(n) = x(n) \star h(n)$

Overlap-add. **SUMMARY** **STEP 4** For block 1

$x_2(n)$	etc

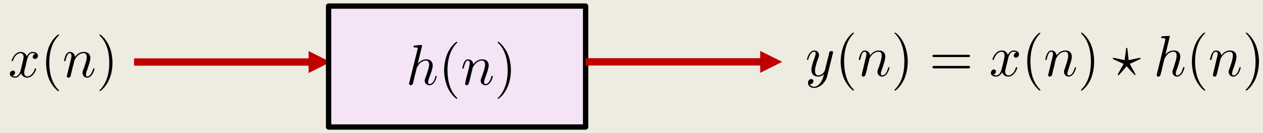
Partion in blocks of size L

$y_1(n) = x_1(n) \star h(n)$

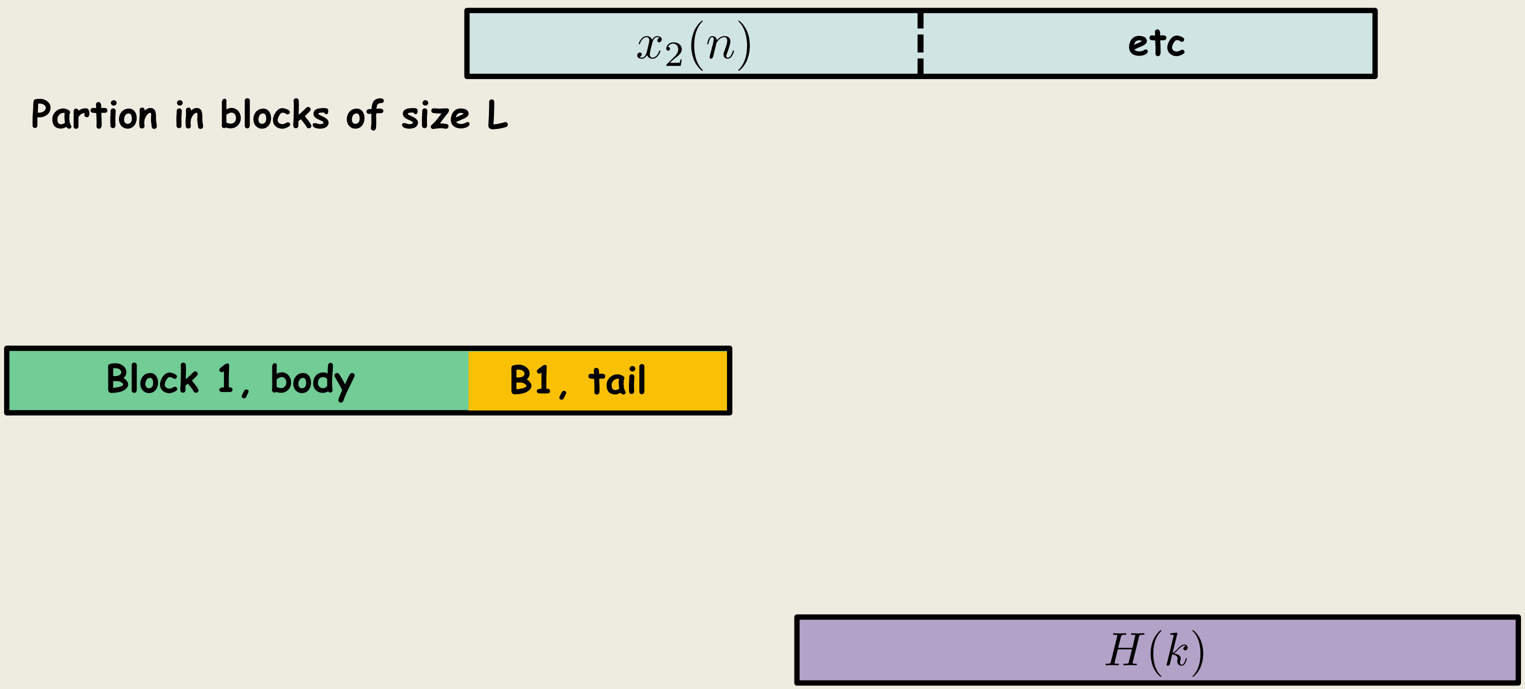
$H(k)$

EITF75 Systems and Signals

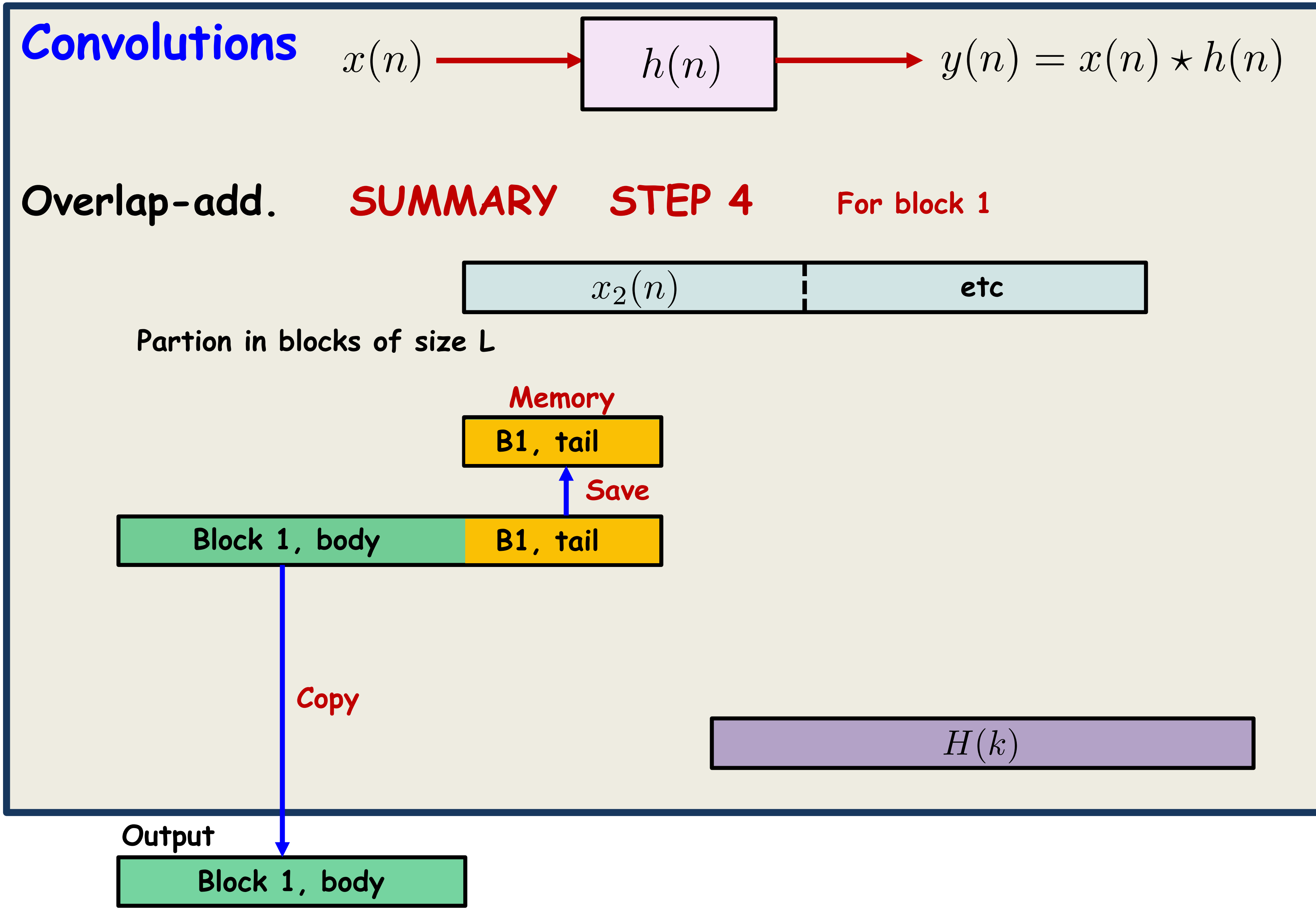
Convolutions



Overlap-add. SUMMARY STEP 4 For block 1



EITF75 Systems and Signals



EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. SUMMARY STEP 4 For block 1

$x_2(n)$	etc

Partion in blocks of size L

Memory

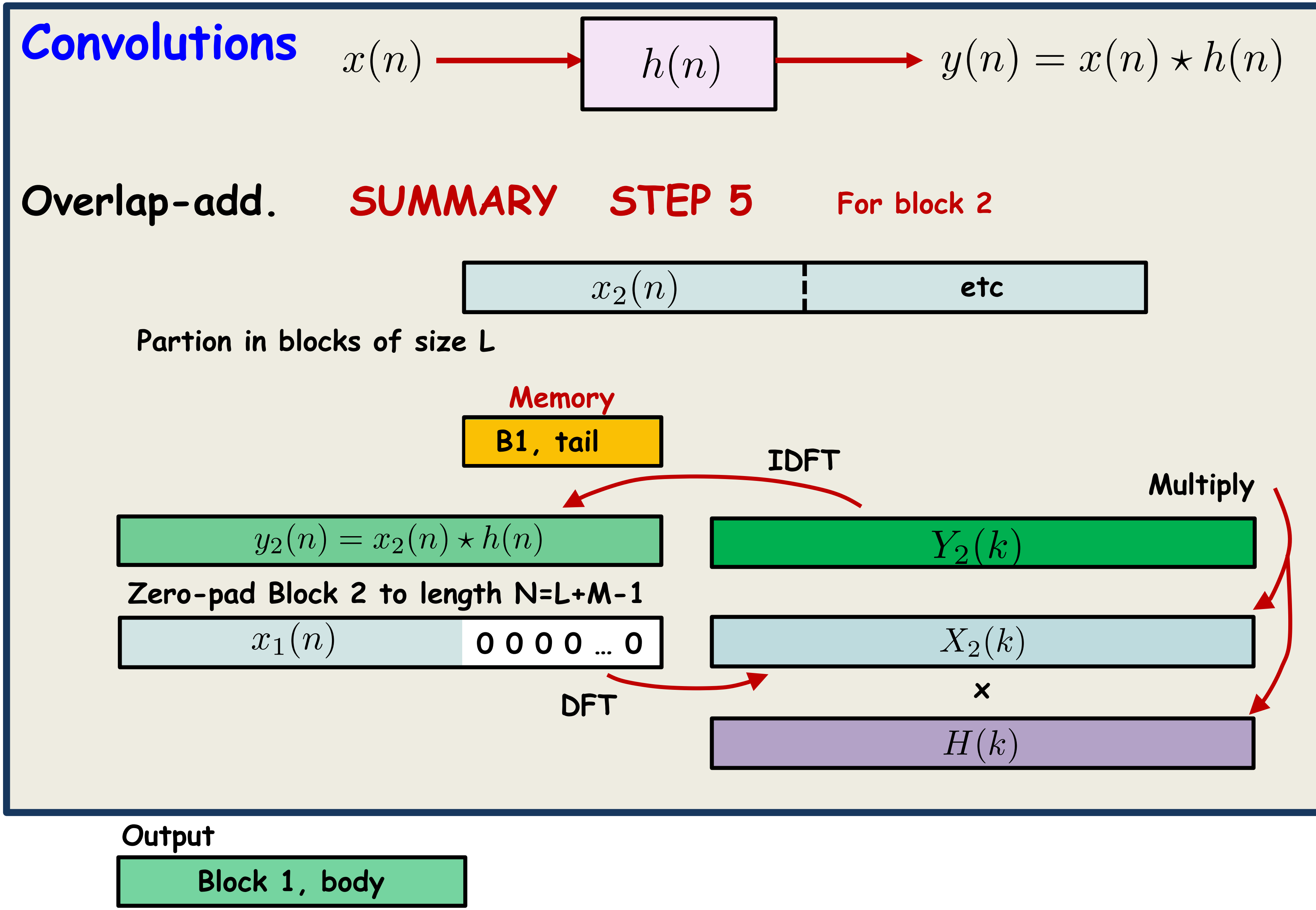
B1, tail

$H(k)$

Output

Block 1, body

EITF75 Systems and Signals



EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$

Overlap-add. SUMMARY STEP 5 For block 2

etc

Partion in blocks of size L

Memory

B1, tail

$y_2(n) = x_2(n) \star h(n)$

$H(k)$

Output

Block 1, body

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. SUMMARY STEP 5 For block 2

etc

Partion in blocks of size L

Memory

B1, tail

$y_2(n) = x_2(n) \star h(n)$

$H(k)$

Output

Block 1, body

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **SUMMARY** **STEP 6** For block 2

etc

Partion in blocks of size L

Memory

B1, tail

B2, body 1	B2,b2	B2,T

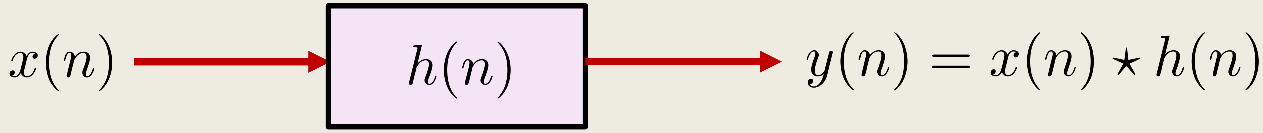
$H(k)$

Output

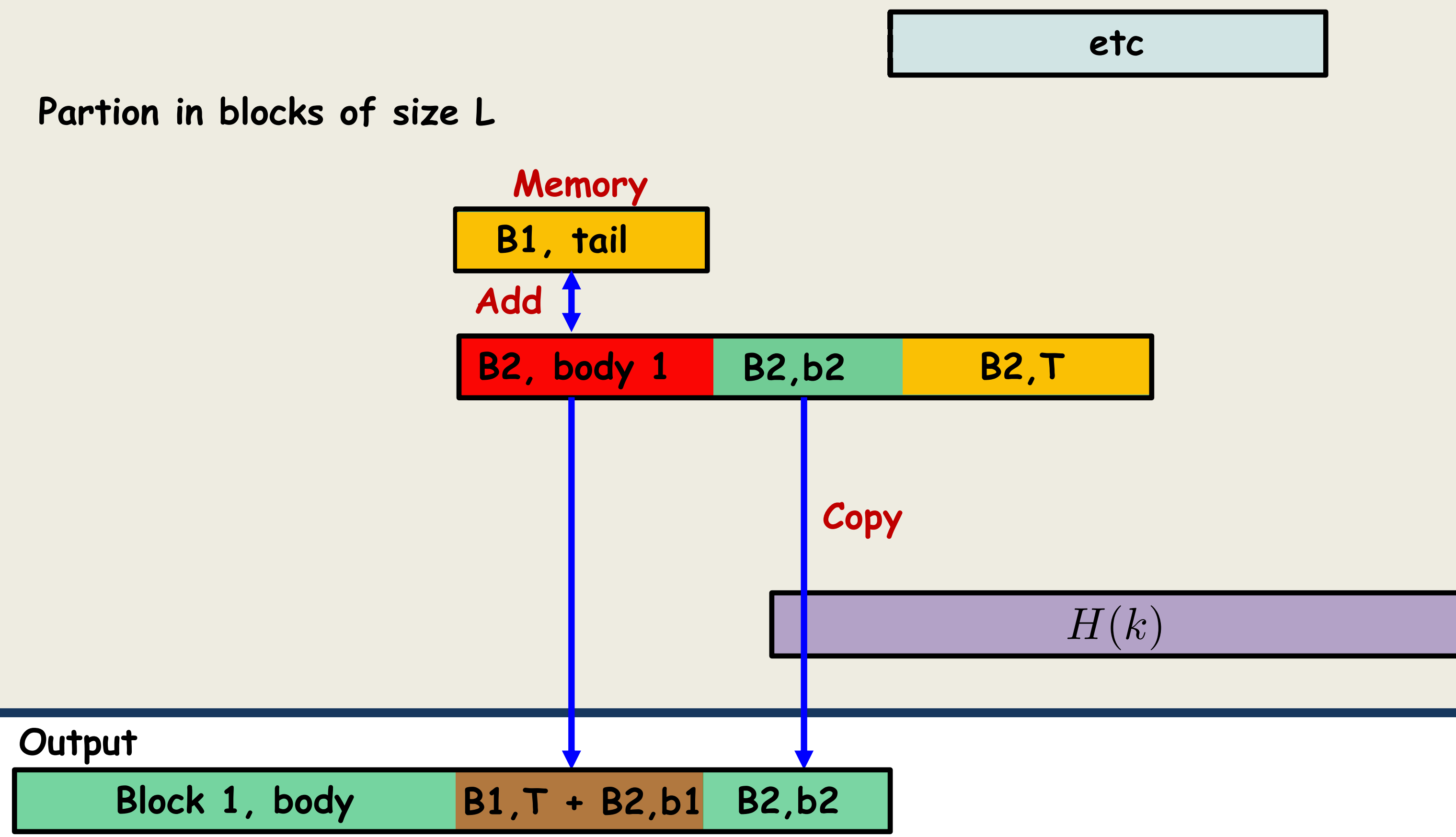
Block 1, body

EITF75 Systems and Signals

Convolutions



Overlap-add. SUMMARY STEP 6 For block 2



EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$

Overlap-add. SUMMARY STEP 6 For block 2

etc

Partion in blocks of size L

Memory

B2,T

Save

B2,T

$H(k)$

Output

Block 1, body	B1,T + B2,b1	B2,b2

EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$

Overlap-add. SUMMARY STEP 6 For block 2

etc

Partion in blocks of size L

Memory

B2,T

$H(k)$

Output

Block 1, body	B1,T + B2,b1	B2,b2

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **SUMMARY** **Repeat 5&6** For blocks >2

etc

Partion in blocks of size L

Memory

B2,T

$H(k)$

Output

Block 1, body	B1,T + B2,b1	B2,b2

EITF75 Systems and Signals

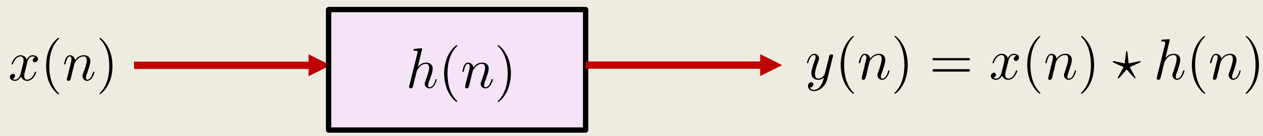
Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **Computational Complexity**

EITF75 Systems and Signals

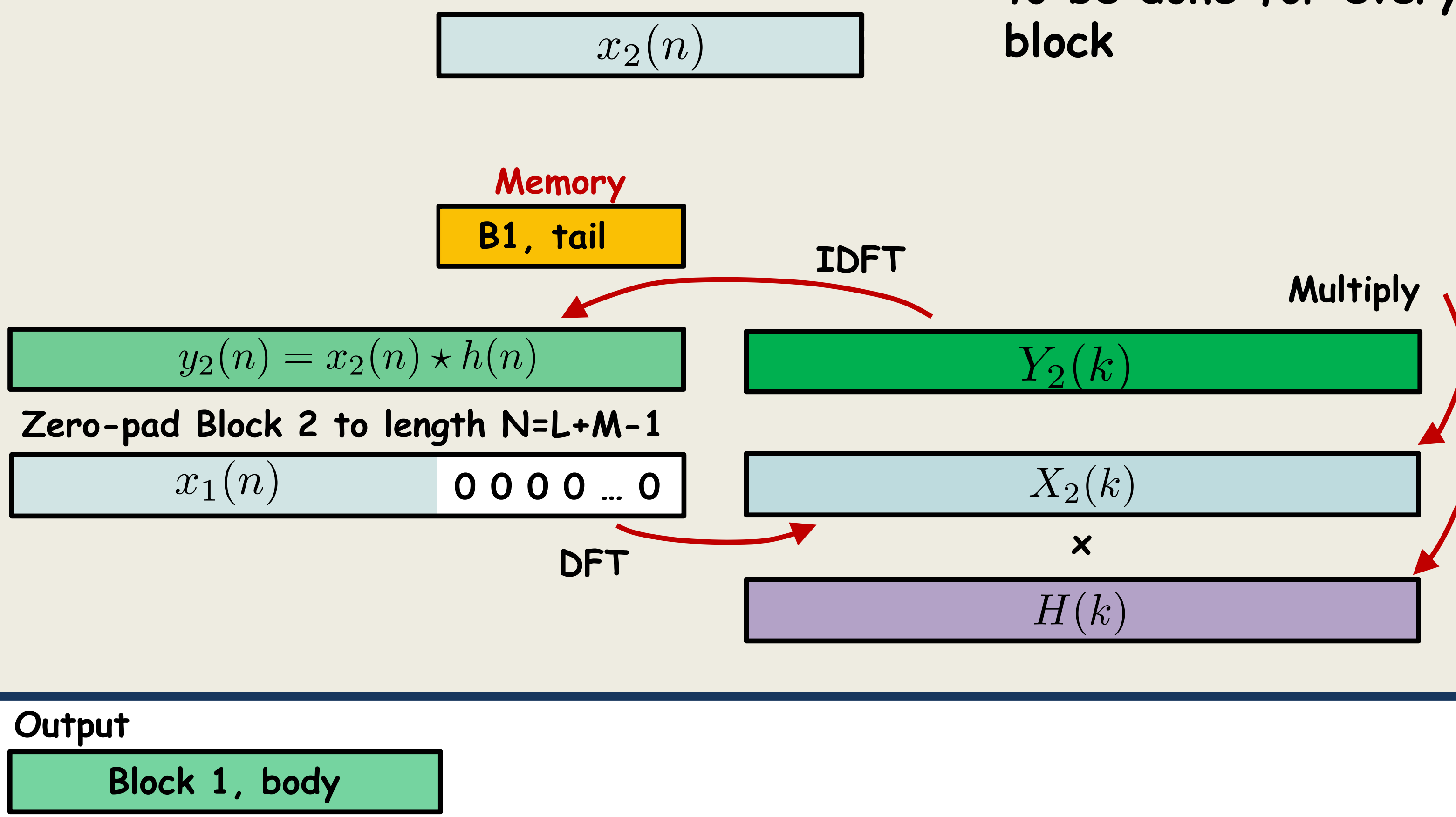
Convolutions



Overlap-add.

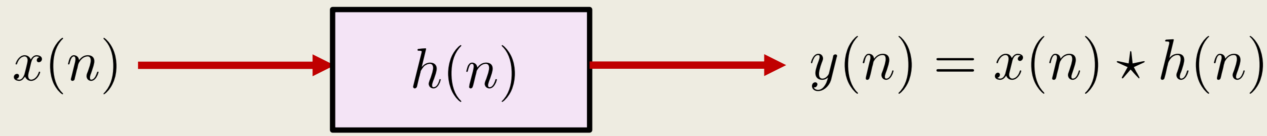
Computational Complexity

What is below needs to be done for every block



EITF75 Systems and Signals

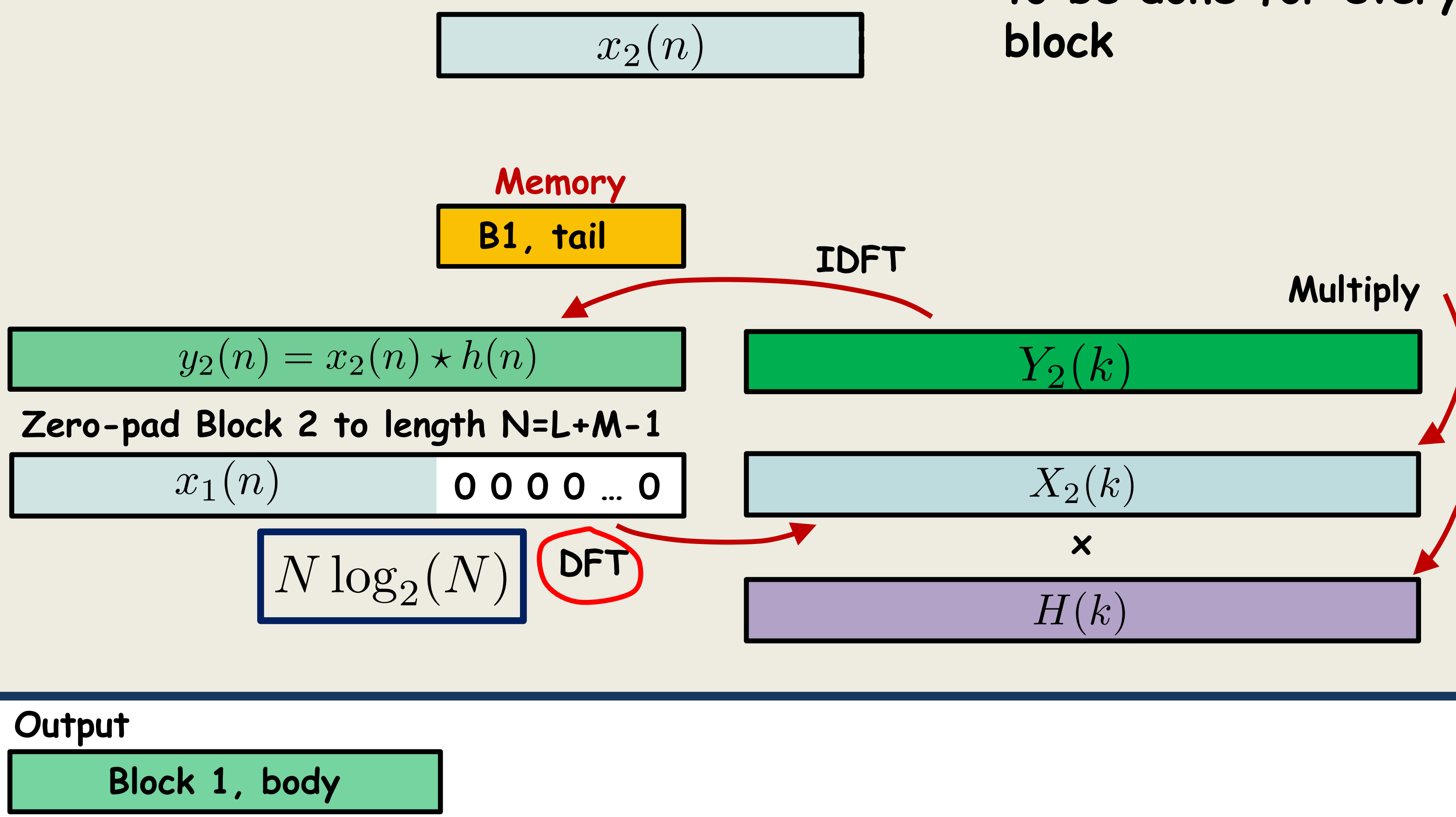
Convolutions



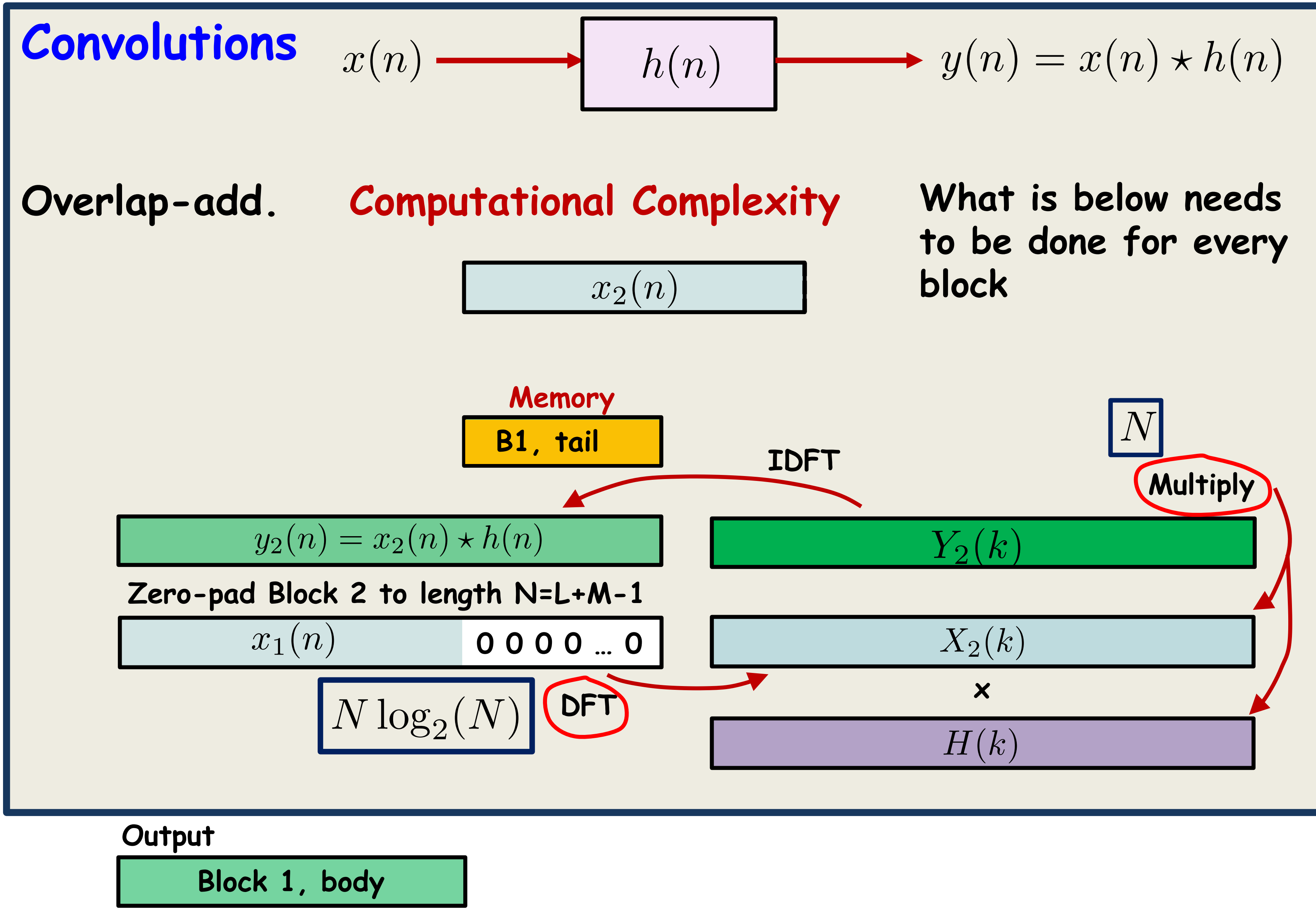
Overlap-add.

Computational Complexity

What is below needs to be done for every block



EITF75 Systems and Signals



EITF75 Systems and Signals

Convolutions

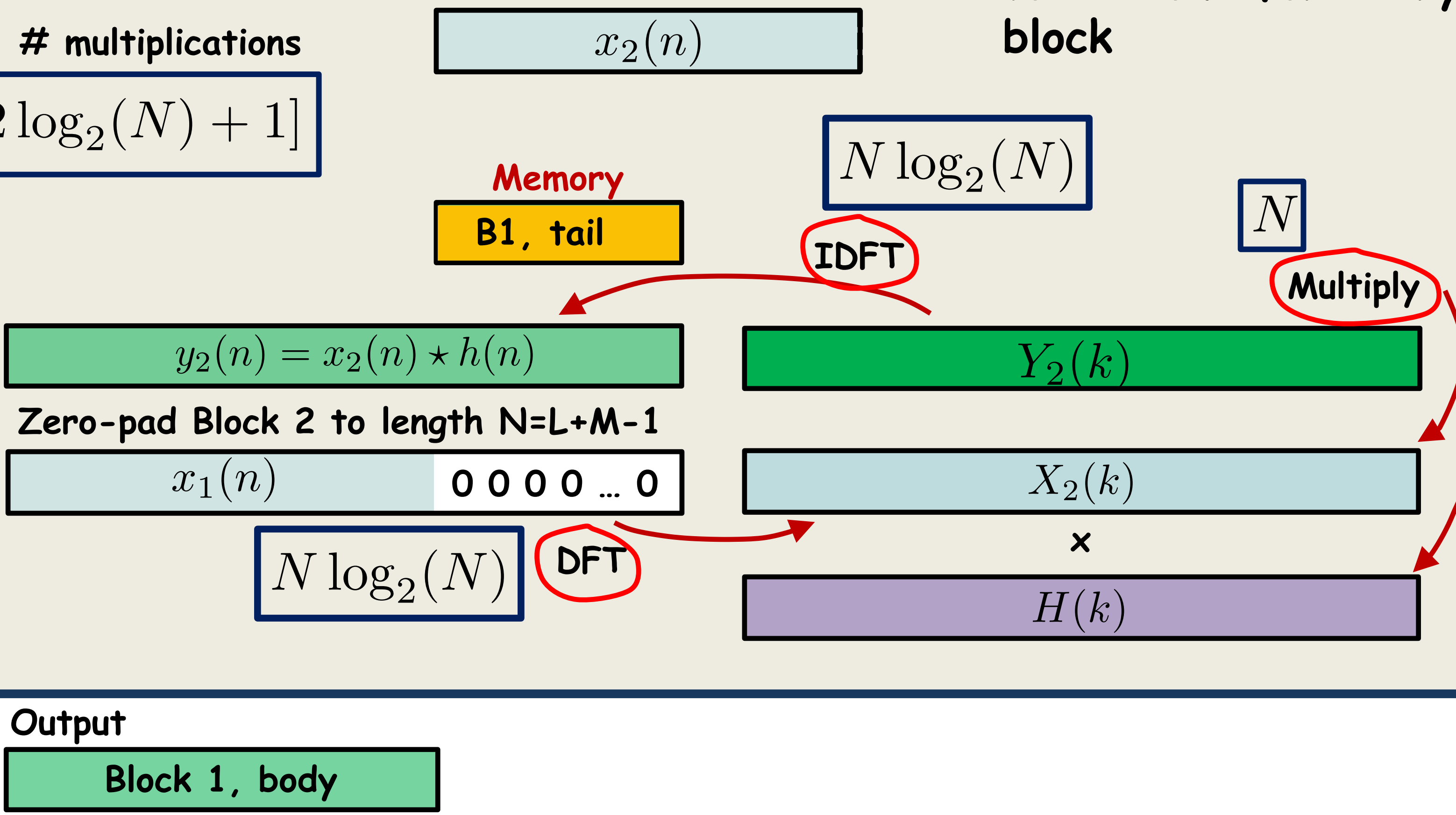
$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **Computational Complexity**

What is below needs to be done for every block

Total # multiplications

$$\boxed{N\left[2\log_2(N) + 1\right]}$$



EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **Computational Complexity**

Total # multiplications

$$\boxed{N\left[2\log_2(N)+1\right]}$$

For these multiplications, we get L = N-M outputs

#multiplications/output

$$\boxed{\frac{N}{N-M}\left[2\log_2(N)+1\right]}$$

$$\boxed{\frac{N}{N-M} \approx 1,\ L \gg M}$$

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **Computational Complexity**

Total # multiplications

$$\boxed{N\left[2\log_2(N)+1\right]}$$

For these multiplications, we get L = N-M outputs

#multiplications/output

$$\boxed{\frac{N}{N-M}\left[2\log_2(N)+1\right]}$$

Easy to solve numerically (no closed form)

$$\boxed{\frac{\partial}{\partial N}\frac{N}{N-M}\left[2\log_2(N)+1\right]=0}$$

$$\boxed{\frac{N}{N-M}\approx 1,\ L \gg M}$$

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-add. **Computational Complexity**

Total # multiplications

$$\boxed{N\,[2\log_2(N) + 1]}$$

For these multiplications, we get L = N-M outputs

#multiplications/output

$$\boxed{\frac{N}{N-M}\,[2\log_2(N) + 1]}$$

Easy to solve numerically (no closed form)

$$\boxed{\frac{\partial}{\partial N}\frac{N}{N-M}\,[2\log_2(N) + 1] = 0}$$

$$\boxed{\frac{N}{N-M} \approx 1,\ L \gg M}$$

Convolutions

In real life, the input signal is very long, does not start/stop. Previous method fails

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Three common methods

Overlap-add to be described

Overlap-save to be briefly described

Overlap-discard not to be described

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-save

Partion the input in blocks of length L

....	$x_1(n)$	$x_2(n)$	etc	....

Visualization of impulse response

h(0)	h(M-1)

Zero-pad Block 1 to length N=L+M-1

0 0 0 0 … 0	$x_1(n)$

EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$

Overlap-save

Partion the input in blocks of length L

.... $x_1(n)$ | $x_2(n)$ | etc ....

Visualization of impulse response

h(0) h(M-1)

Zero-pad Block 1 to length N=L+M-1

0 0 0 0 … 0 $x_1(n)$

Circular Convolution of zero-padded signal and h(n)

EITF75 Systems and Signals

Convolutions

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$

Overlap-save

Partion the input in blocks of length L

....	$x_1(n)$	$x_2(n)$	etc	....

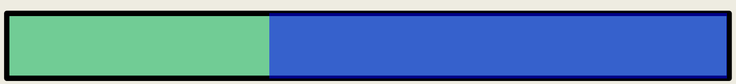
Visualization of impulse response

h(0) h(M-1)

Zero-pad Block 1 to length N=L+M-1

0 0 0 0 … 0	$x_1(n)$

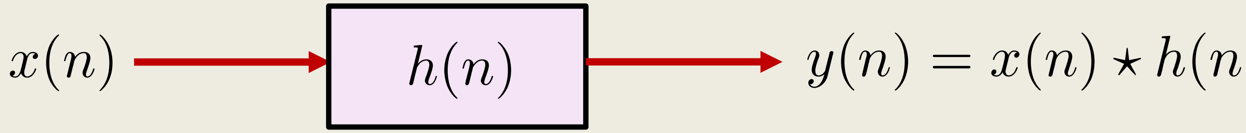
Circular Convolution of zero-padded signal and h(n)



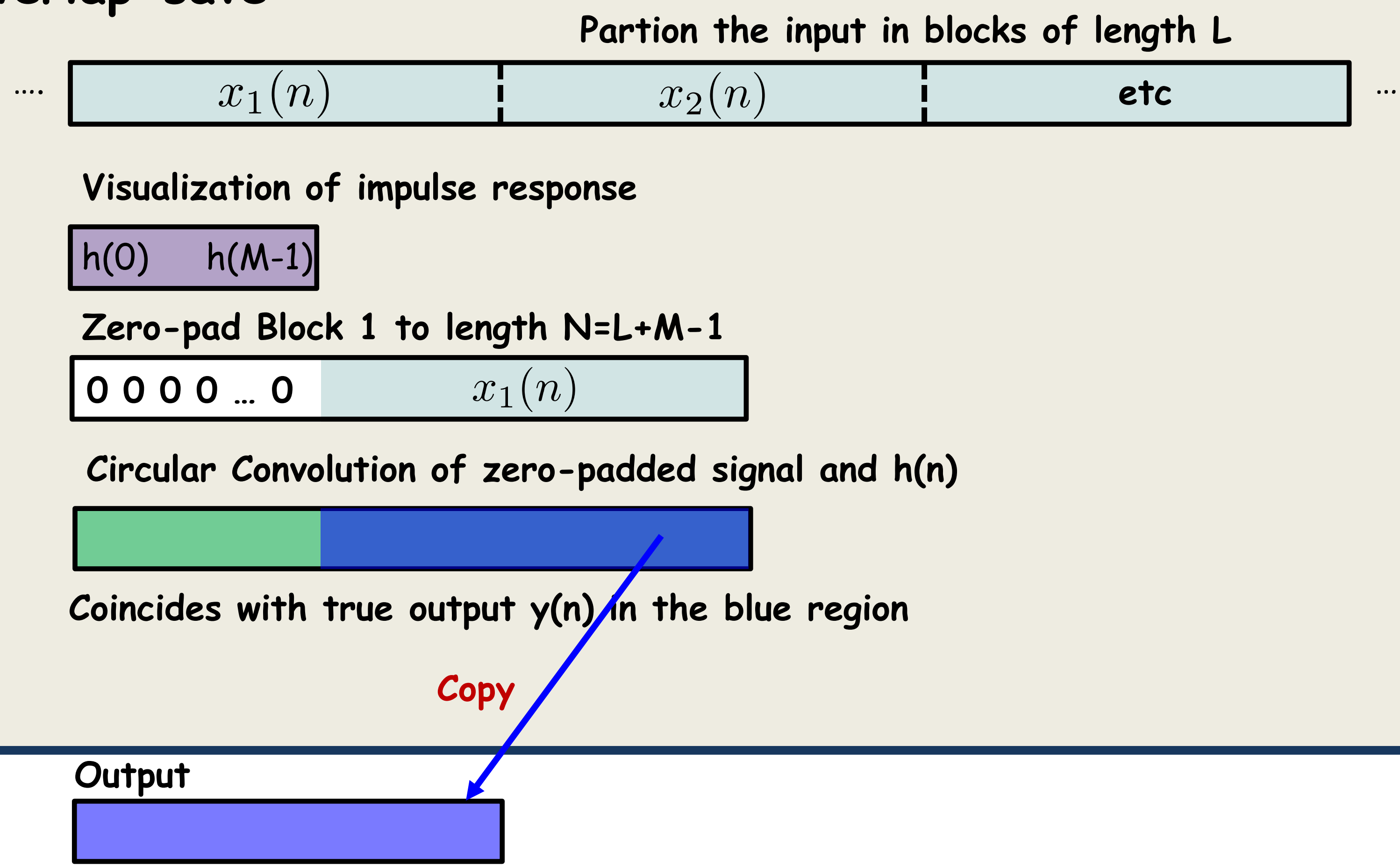
Coincides with true output y(n) in the blue region

EITF75 Systems and Signals

Convolutions



Overlap-save



EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Overlap-save

Partion the input in blocks of length L

.... | $x_1(n)$ | $x_2(n)$ | etc | ....

Visualization of impulse response

h(0) h(M-1)

Block 2: Borrow a bit from block 1

$x_2(n)$

Output

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

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Circular Convolution of above signal and h(n)

Output

EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

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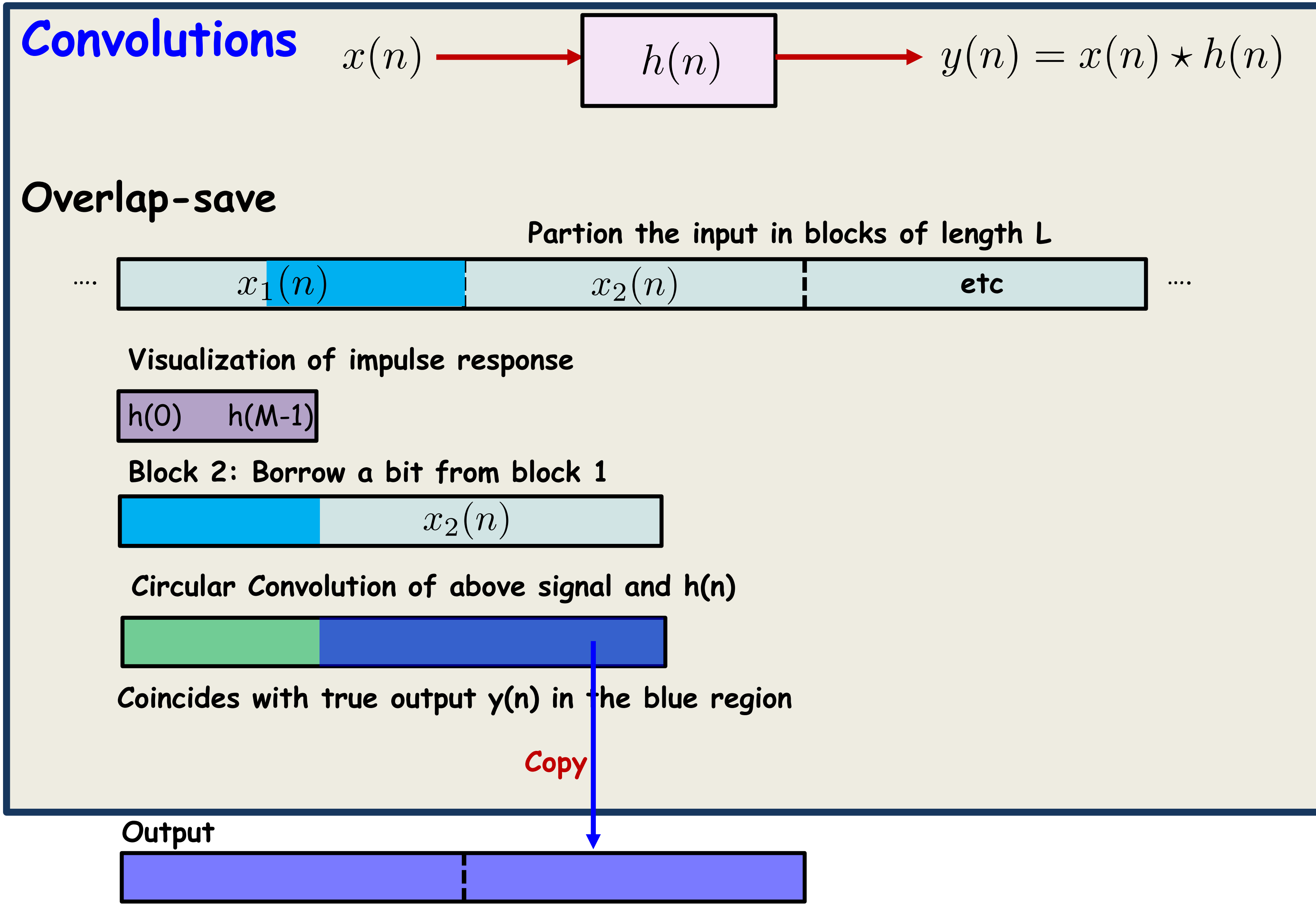
EITF75 Systems and Signals



EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Assume an IIR filter

Implications for overlap-add/save ?

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Assume an IIR filter

Implications for overlap-add/save ?   Does not work, since L>M for overlap to work

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Assume an IIR filter

Implications for overlap-add/save ? Does not work, since L>M for overlap to work

However, an IIR filter is implemented via a difference equation

$$y(n) = \sum_{\ell=1}^{L} a_\ell y(n-\ell) + \sum_{k=0}^{K} b_k x(n-k)$$

L+K+1 multiplications to get 1 output

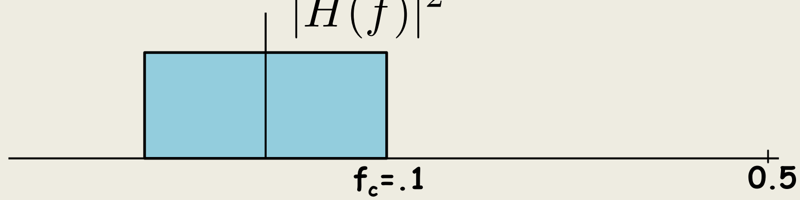
EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

Target



EITF75 Systems and Signals

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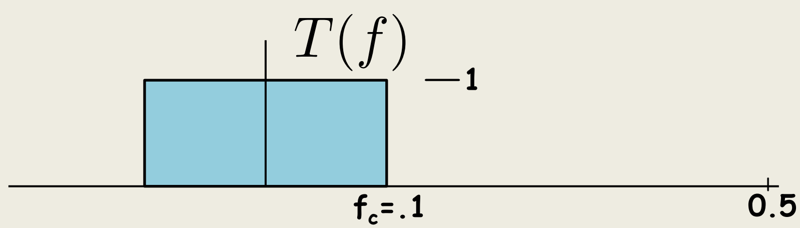

<u>FIR:</u> $$H(f) = \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$$

EITF75 Systems and Signals

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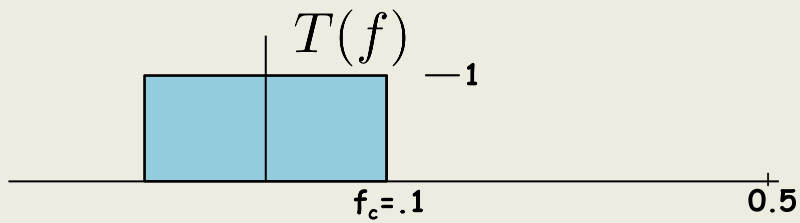


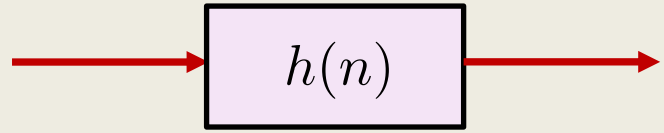
FIR: $$H(f) = \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$$

Error $$E(f) = \sqrt{T(f)} - \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$$

EITF75 Systems and Signals

Convolutions

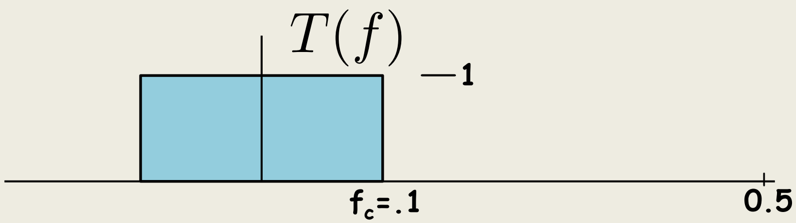
$x(n)$



$y(n) = x(n) \star h(n)$

Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

Target



FIR: $H(f) = \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$ **Error** $E(f) = \sqrt{T(f)} - \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$

To be minimized: $\delta = \int_{-0.5}^{0.5} |E(f)|^2 \mathrm{d}f$

EITF75 Systems and Signals

Convolutions

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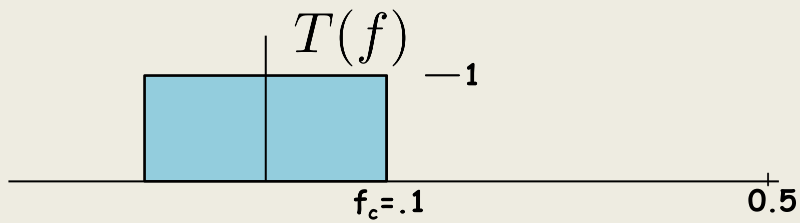

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To be minimized: $$\delta = \int_{-0.5}^{0.5} |E(f)|^2 \mathrm{d}f = \int_{-0.5}^{0.5} |H(f)|^2 \mathrm{d}f + \int_{-0.5}^{0.5} T(f)\mathrm{d}f - 2\mathrm{R}\left\{\int_{-0.5}^{0.5} \sum_{n=0}^{N-1} b_n T(f) \mathrm{e}^{-i2\pi f n} \mathrm{d}f\right\}$$

EITF75 Systems and Signals

Convolutions

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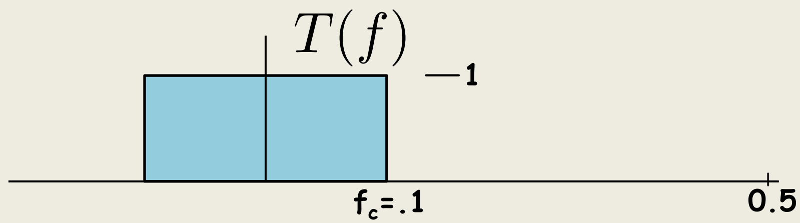

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Error $$E(f) = \sqrt{T(f)} - \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$$

Parseval

To be minimized: $$\delta = \int_{-0.5}^{0.5} |E(f)|^2 \mathrm{d}f = \sum_{n=0}^{N-1} |b_n|^2 + \int_{-0.5}^{0.5} T(f)\mathrm{d}f - 2\mathrm{R}\left\{ \int_{-0.5}^{0.5} \sum_{n=0}^{N-1} b_n T(f) \mathrm{e}^{-i2\pi f n} \mathrm{d}f \right\}$$

EIT75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

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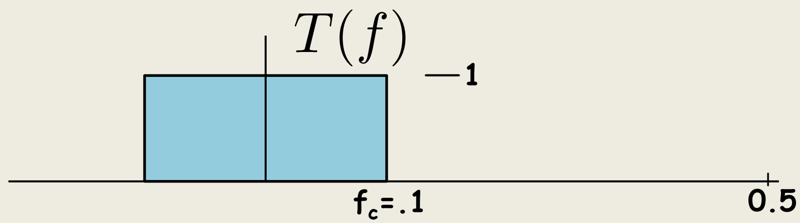

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Parseval

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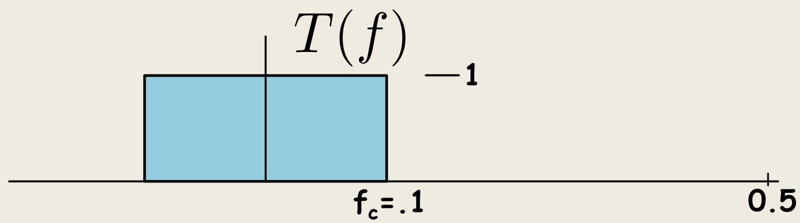
EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\left\{ \begin{array}{l} \textbf{FIR} \\ \textbf{IIR} \end{array} \right.$

Target



FIR: $H(f) = \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$ **Error** $E(f) = T(f) - \sum_{n=0}^{N-1} b_n \mathrm{e}^{-i2\pi f n}$

To be minimized: $\delta = \int_{-0.5}^{0.5} |E(f)|^2 \mathrm{d}f$

$$= \sum_{n=0}^{N-1} |b_n|^2 + 0.2$$

$$-2 \int_{-0.5}^{0.5} \sum_{n=0}^{N-1} b_n T(f) \mathrm{e}^{-i2\pi f n} \mathrm{d}f$$

From symmetry: real

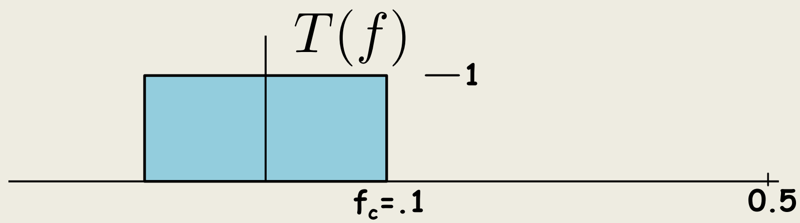
EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\left\{ \begin{array}{l} \textbf{FIR} \\ \textbf{IIR} \end{array} \right.$

Target



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To be minimized: $$\delta = \int_{-0.5}^{0.5} |E(f)|^2 \mathrm{d}f = \sum_{n=0}^{N-1} |b_n|^2 + 0.2 - 2\sum_{n=0}^{N-1} b_n \int_{-0.1}^{0.1} T(f)\mathrm{e}^{-i2\pi f n}\mathrm{d}f$$

Change order

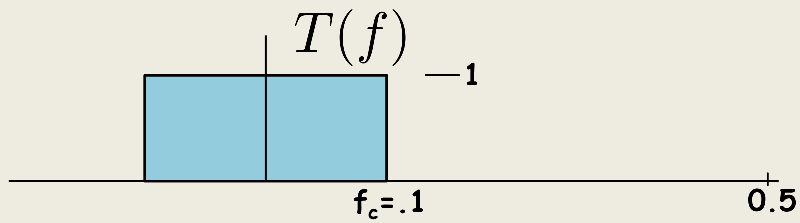
EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

Target



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$$T_n = \int_{-0.1}^{0.1} T(f)\mathrm{e}^{-i2\pi f n}\mathrm{d}f = \cdots = \frac{\sin(\pi n/5)}{\pi n}$$

Integral independent of b_n

EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

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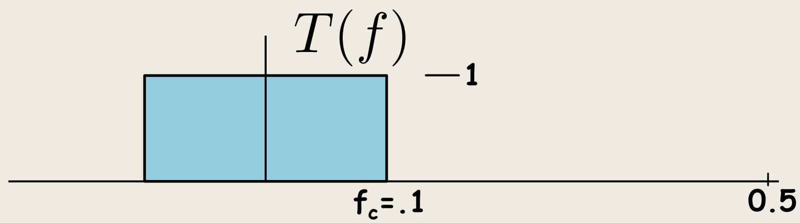

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$$\min_{\{b_n\}_{n=0}^{N-1}} \left[\sum_{n=0}^{N-1} |b_n|^2 - 2 \sum_{n=0}^{N-1} b_n T_n + 0.2 \right]$$

$$\sum_{n=0}^{N-1} |b_n|^2 \quad + \quad 0.2$$

$$-2 \sum_{n=0}^{N-1} b_n T_n$$

Optimization problem

EITF75 Systems and Signals

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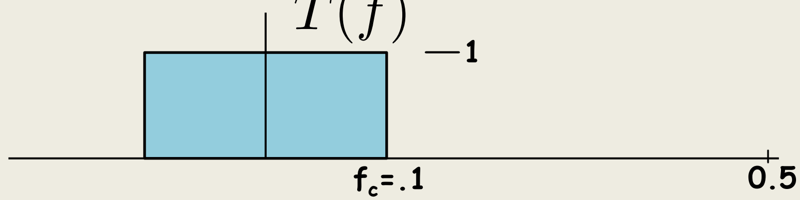
Target



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$$\min_{\{b_n\}_{n=0}^{N-1}} \left[\sum_{n=0}^{N-1} |b_n|^2 - 2 \sum_{n=0}^{N-1} b_n T_n + 0.2 \right]$$

$$\boxed{\frac{\partial}{\partial b_n} \cdots = 0 \rightarrow b_n = T_n}$$

Optimization problem

$$T_n = \int_{-0.1}^{0.1} T(f) \mathrm{e}^{-i2\pi f n} \mathrm{d}f = \cdots = \frac{\sin(\pi n/5)}{\pi n}$$

EITF75 Systems and Signals

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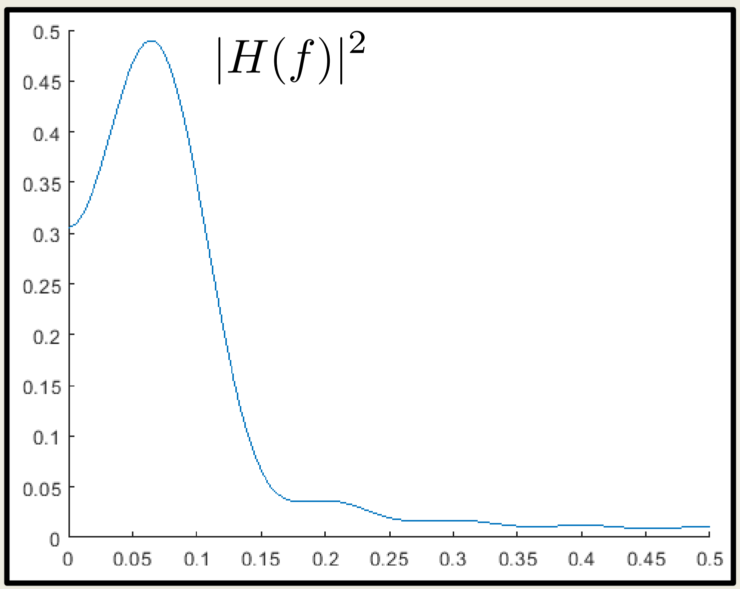
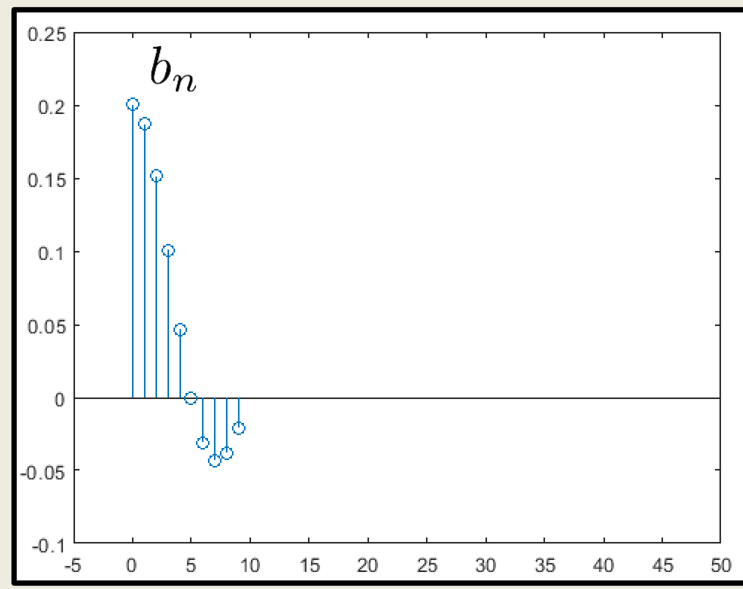
$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\begin{cases} \textbf{FIR} \\ \textbf{IIR} \end{cases}$

FIR, N=10





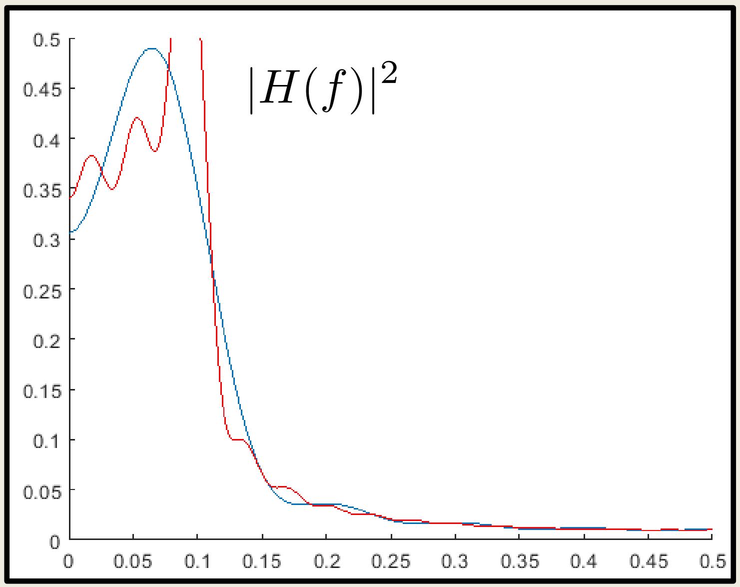
EITF75 Systems and Signals

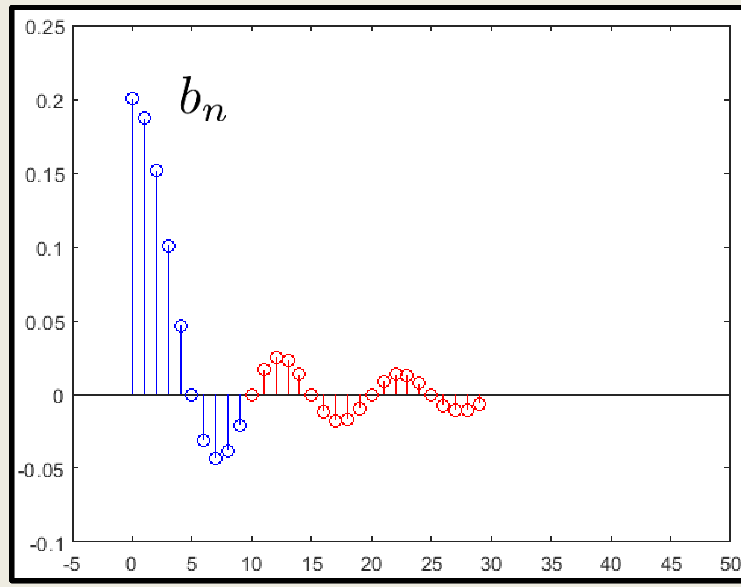
Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

FIR, N=10, N=30





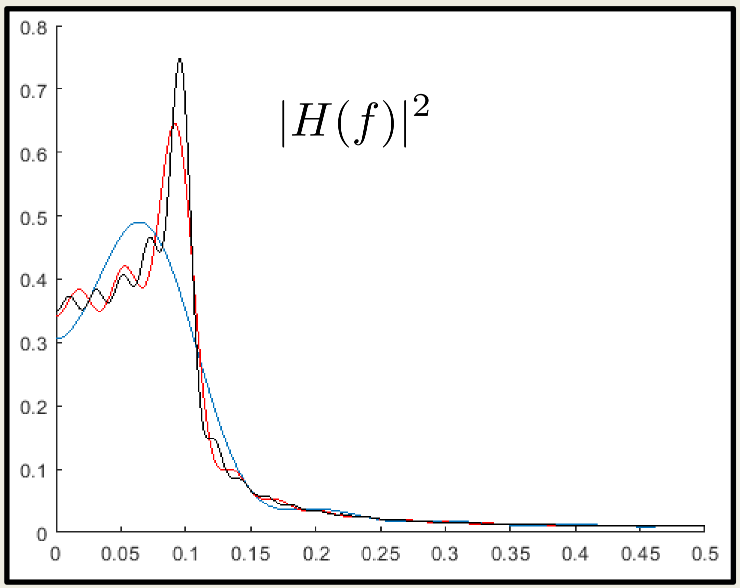
EITF75 Systems and Signals

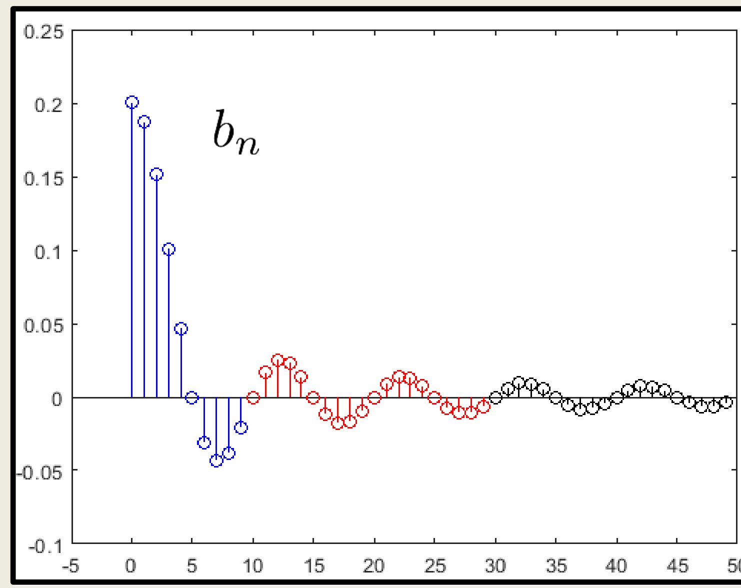
Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

Case study: Low pass filter implemented as $\left\{ \begin{array}{l} \textbf{FIR} \\ \textbf{IIR} \end{array} \right.$

FIR, N=10, N=30, N=50





EITF75 Systems and Signals

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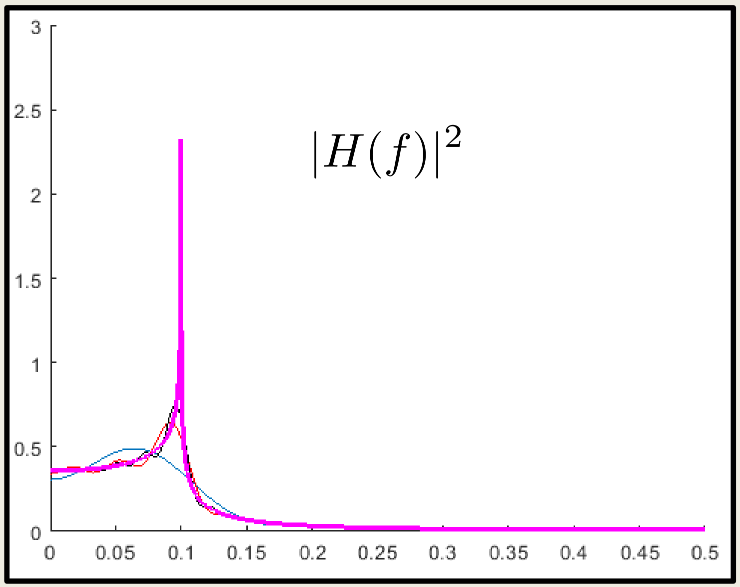
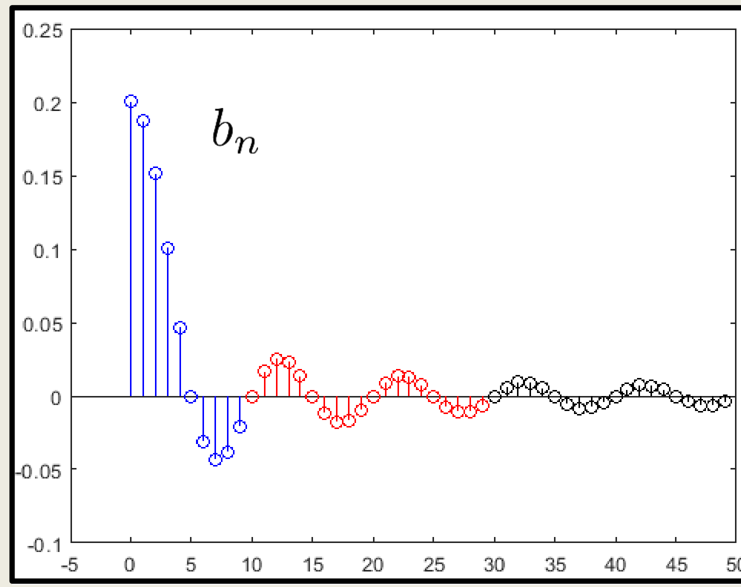
FIR, N=10, N=30, N=50, N=1000





EITF75 Systems and Signals

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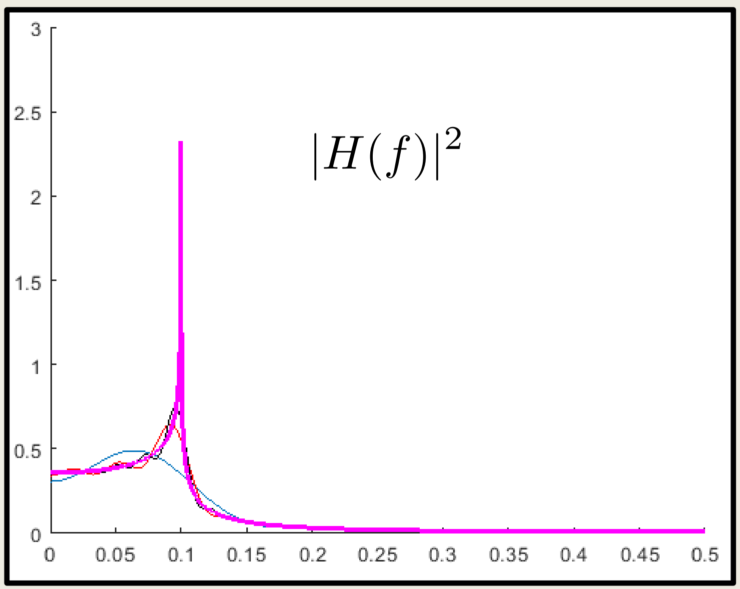

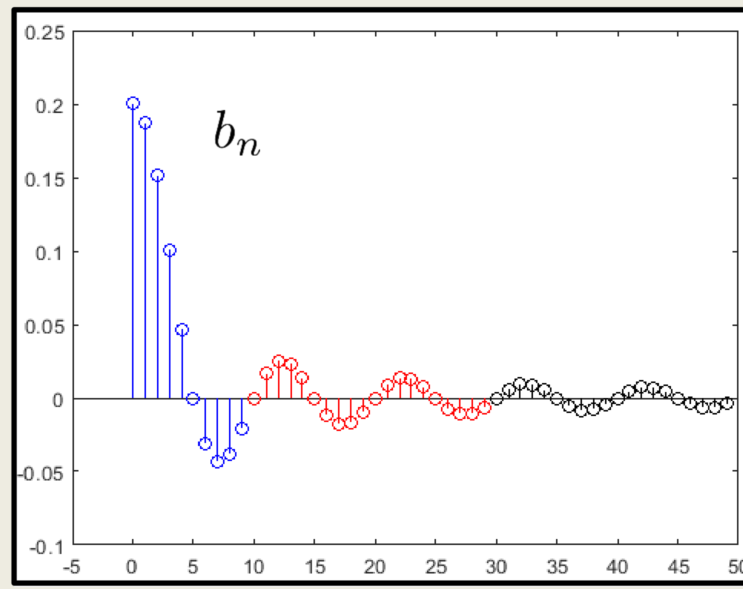

EITF75 Systems and Signals

Convolutions

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Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

IIR, Butterworth filter, P poles, cutoff f_c=0.1

```
>> [B,A]=butter(4,2*.1)
B =
    0.0048    0.0193    0.0289    0.0193    0.0048
A =
    1.0000   -2.3695    2.3140   -1.0547    0.1874
```

Numerator polynomial

Denominator polynomial

EITF75 Systems and Signals

Convolutions

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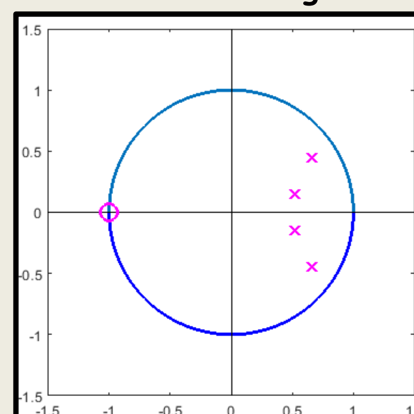
Numerator polynomial

Denominator polynomial

Pole-zero diagram



EITF75 Systems and Signals

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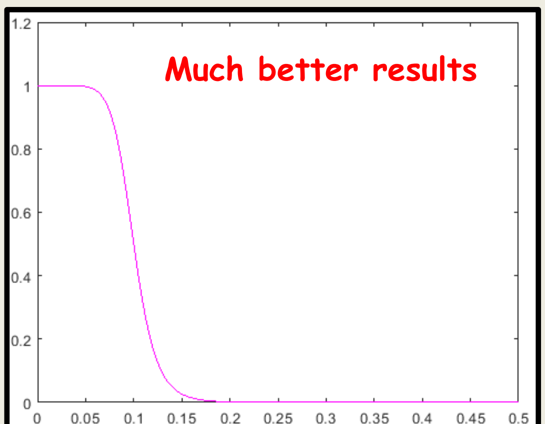
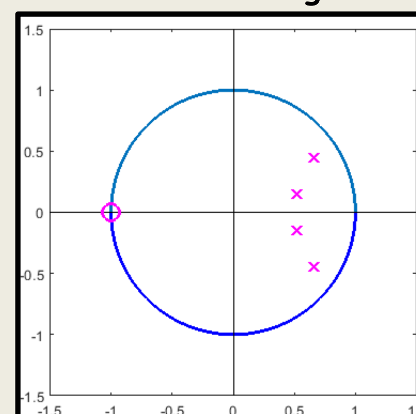
Numerator polynomial

Denominator polynomial



Pole-zero diagram



EITF75 Systems and Signals

Convolutions

$$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) \star h(n)$$

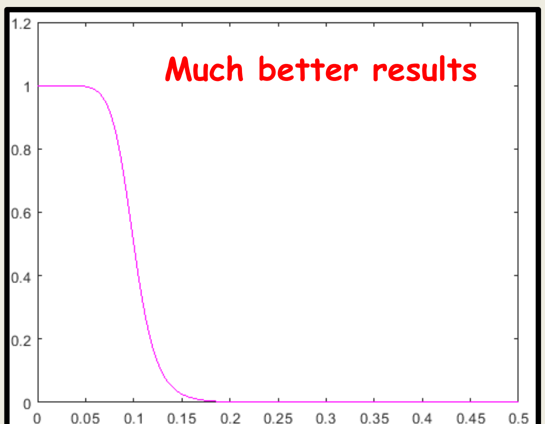
Case study: Low pass filter implemented as $\begin{cases} \text{FIR} \\ \text{IIR} \end{cases}$

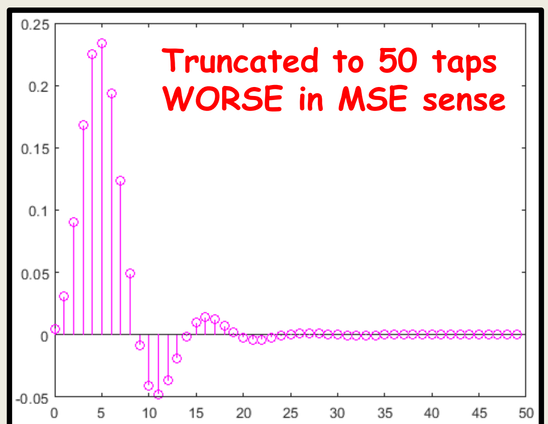
IIR, Butterworth filter, P poles, cutoff f_c=0.1

```
>> [B,A]=butter(4,2*.1)
B =
     0.0048     0.0193     0.0289     0.0193     0.0048
A =
     1.0000    -2.3695     2.3140    -1.0547     0.1874
```

Numerator polynomial

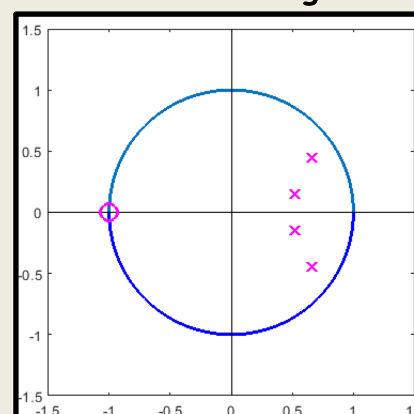
Denominator polynomial






```
>> h=filter(B,A,[1 zeros(1,1000)]);
```

Pole-zero diagram



EITF75 Systems and Signals

Convolutions

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n) = x(n) \star h(n)$$

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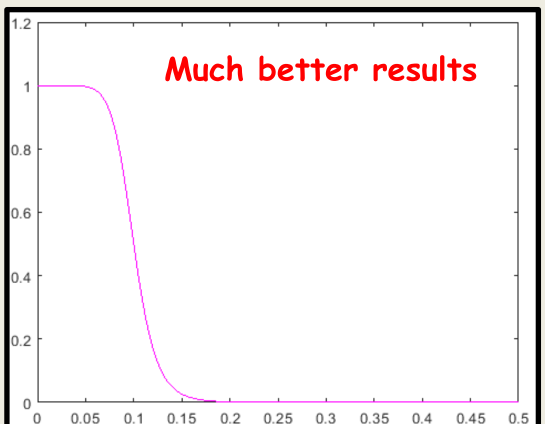
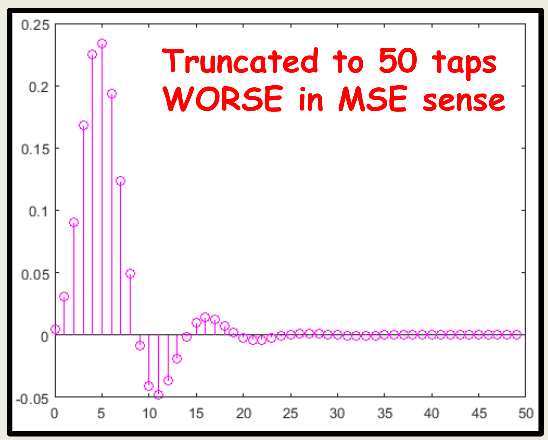
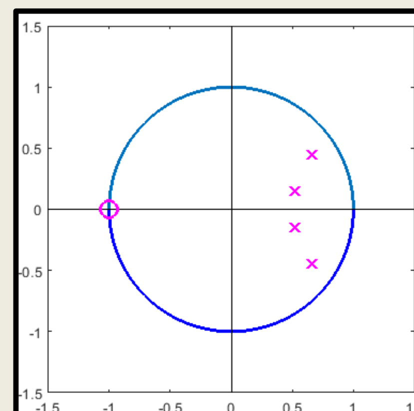
IIR, Butterworth filter, P poles, cutoff f_c=0.1

```
>> [B,A]=butter(8,2*.1)
B =
    0.0000    0.0002    0.0007    0.0013    0.0017    0.0013    0.0007    0.0002    0.0000
A =
    1.0000   -4.7845   10.4450  -13.4577   11.1293   -6.0253    2.0793   -0.4172    0.0372
```







Pole-zero diagram

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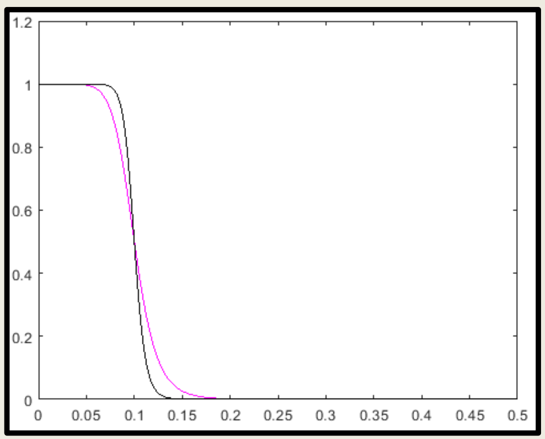

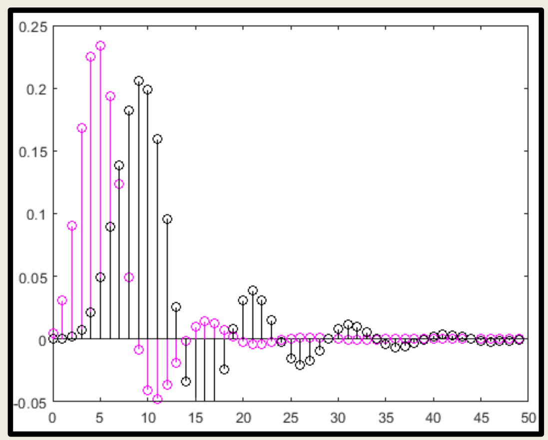

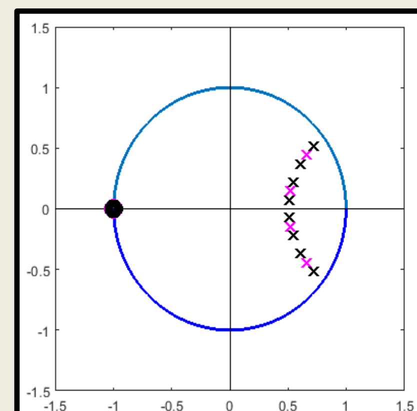

Pole-zero diagram

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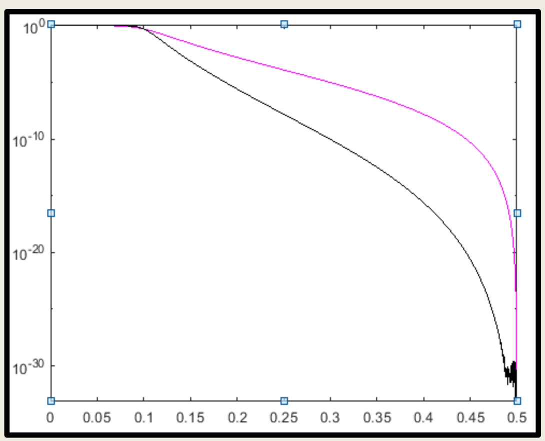
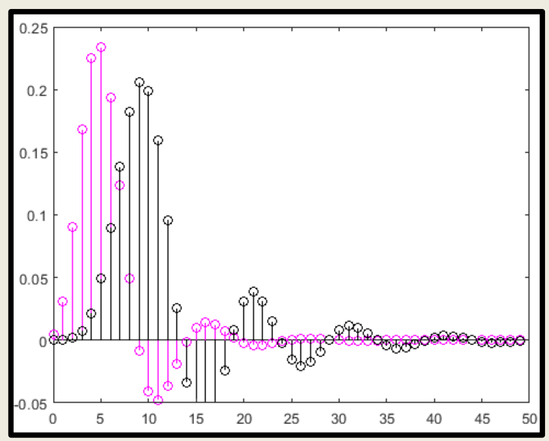
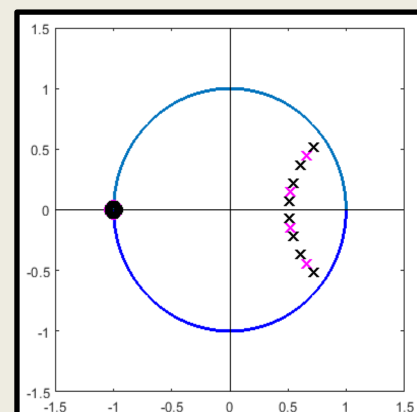


Log scale





Pole-zero diagram

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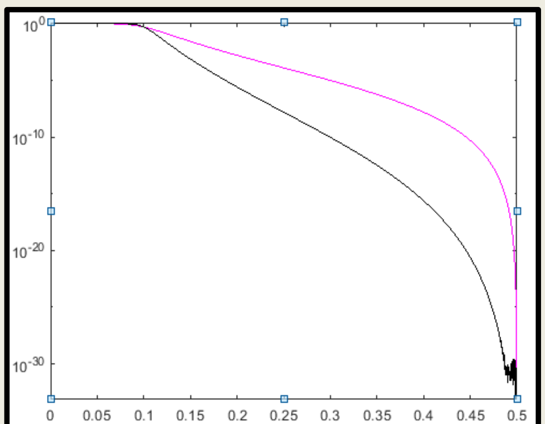

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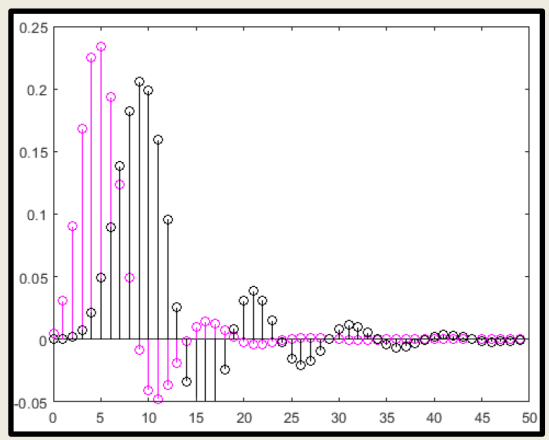

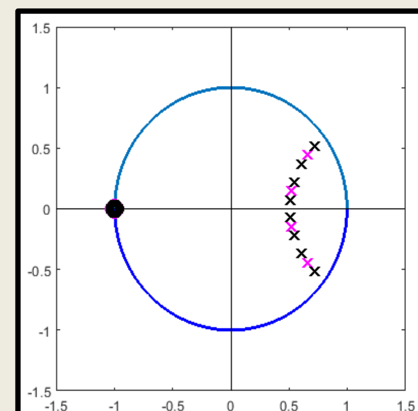

Pole-zero diagram

Lesson learded: IIR filters superior. Simple implementation, good results

EITF75 Systems and Signals

DFT of sinusoids

$$x(n) = \cos\left(2\pi \frac{k_0}{N} n\right)$$

Find DFT

Note, for k_0 an integer, an integer number of periods

EITF75 Systems and Signals

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$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n}$$

By definition

EITF75 Systems and Signals

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$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi\cdot\frac{k}{N}\cdot n}$$

$$= \sum_{n=0}^{N-1} \frac{1}{2}\cdot\left[e^{j2\pi\cdot\frac{k_0}{N}\cdot n} + e^{-j2\pi\cdot\frac{k_0}{N}\cdot n}\right]\cdot e^{-j2\pi\cdot\frac{k}{N}\cdot n}$$

Euler

EITF75 Systems and Signals

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$$= \sum_{n=0}^{N-1} \frac{1}{2} \cdot e^{-j2\pi \cdot \frac{k-k_0}{N} \cdot n} + \sum_{n=0}^{N-1} \frac{1}{2} \cdot e^{-j2\pi \cdot \frac{k+k_0}{N} \cdot n}$$

Preparation for geometric series

EITF75 Systems and Signals

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$$= \sum_{n=0}^{N-1} \frac{1}{2} \cdot e^{-j2\pi \cdot \frac{k-k_0}{N} \cdot n} + \sum_{n=0}^{N-1} \frac{1}{2} \cdot e^{-j2\pi \cdot \frac{k+k_0}{N} \cdot n}$$

$$= \frac{N}{2} \cdot \left[\delta(k - k_0 \mod N) + \delta(k + k_0 \mod N) \right]$$

We have seen this before (lecture 10)

EITF75 Systems and Signals

DFT of sinusoids

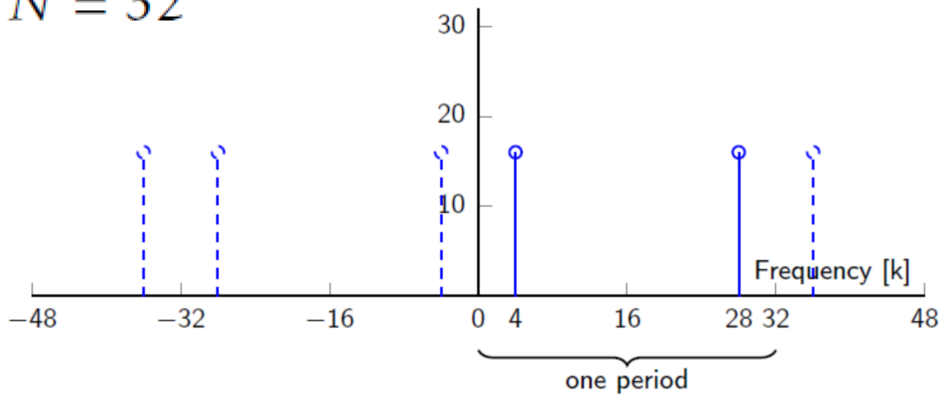
$$x(n) = \cos\left(2\pi \frac{k_0}{N} n\right)$$

Find DFT

Note, for k_0 an integer, an integer number of periods

$k_0 = 4$

$N = 32$



$$X(k) = \frac{N}{2} \cdot [\delta(k - k_0 \mod N) + \delta(k + k_0 \mod N)]$$

EITF75 Systems and Signals

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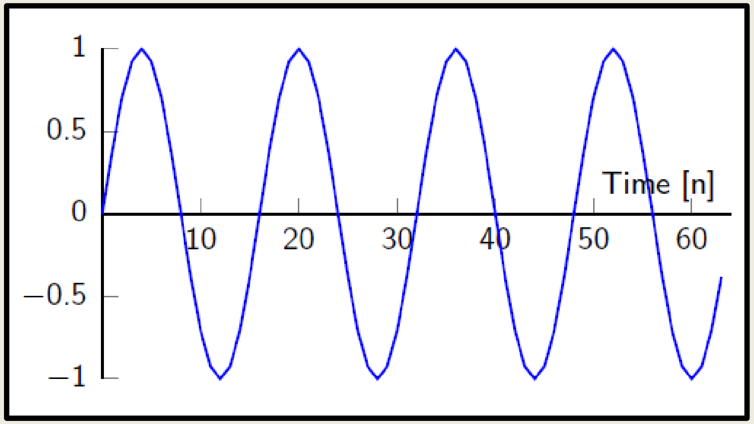

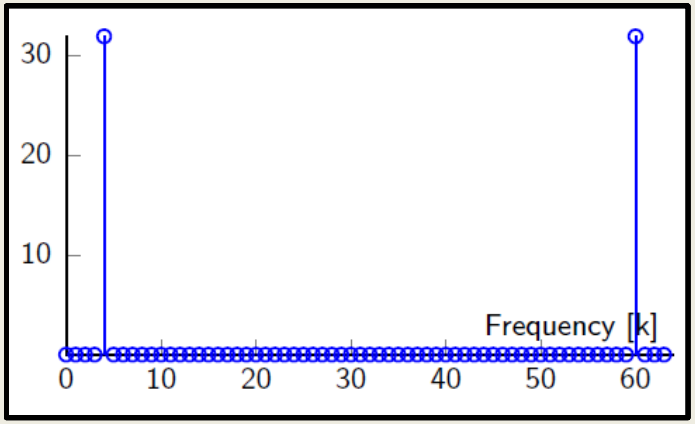

EITF75 Systems and Signals

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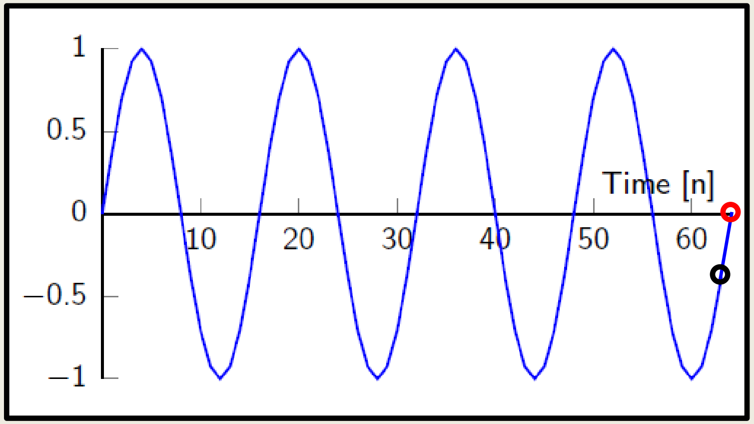

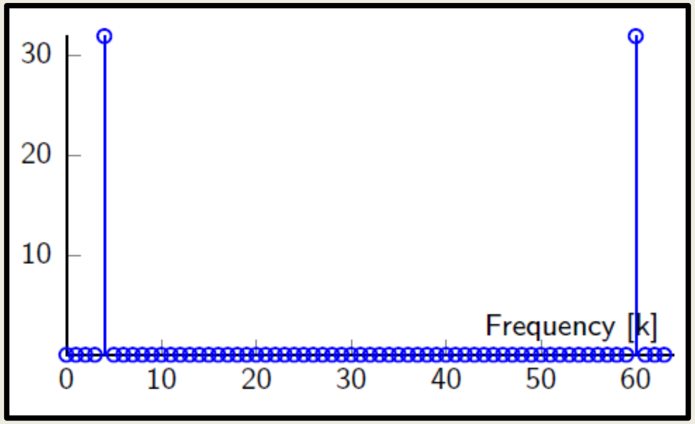

This IS an even number of periods

EITF75 Systems and Signals

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Find DFT

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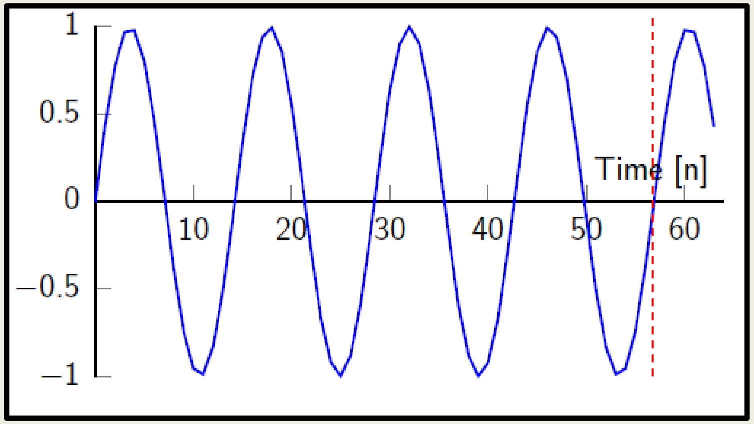


??

What would happen if k_0 not an integer

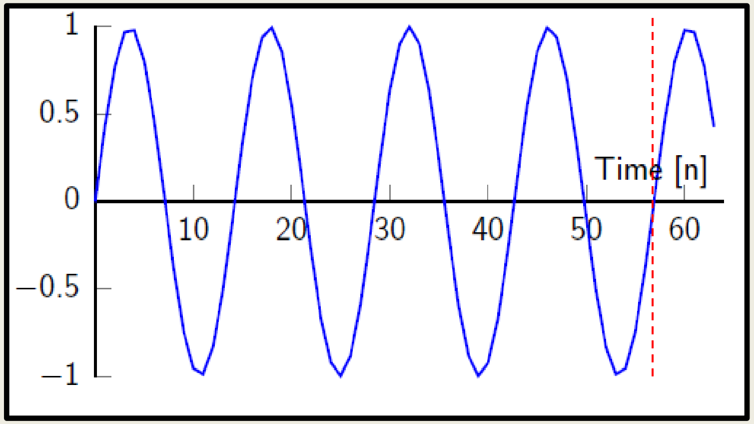
EITF75 Systems and Signals

DFT of sinusoids

$$x(n) = \cos\left(2\pi \frac{k_0}{N} n\right)$$

Find DFT

Note, for k_0 an integer, an integer number of periods



Essentially, we get a delta at k_0, but this is not an integer, so not computed by the DFT

What would happen if k_0 not an integer

EITF75 Systems and Signals

DFT of sinusoids

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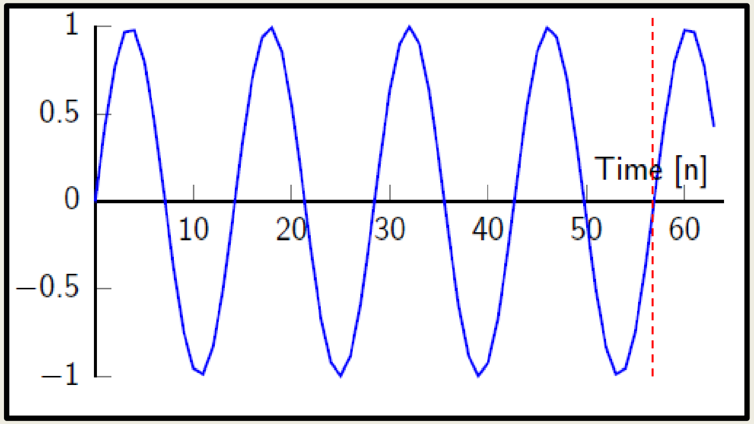


For the above case, $4<k_0<5$

Essentially, we get a delta at k_0, but this is not an integer, so not computed by the DFT

Spectral leakage: We get something around k_0

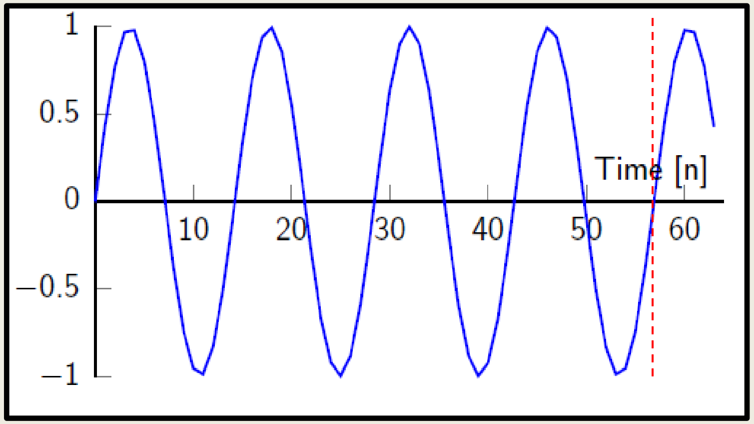
EITF75 Systems and Signals

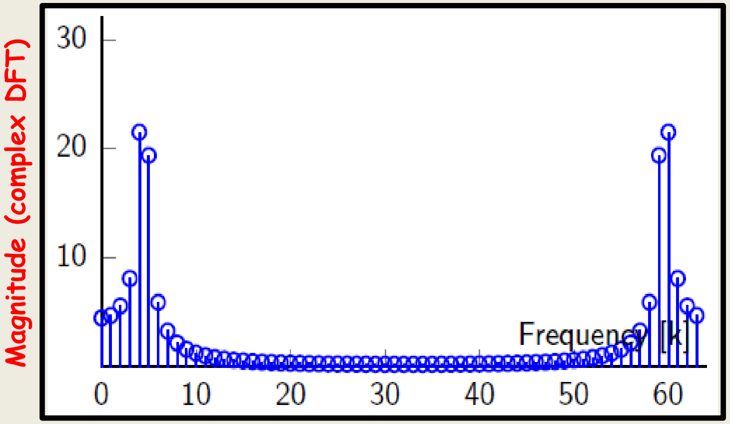
DFT of sinusoids

$$x(n) = \cos\left(2\pi \frac{k_0}{N} n\right)$$

Find DFT

Note, for k_0 an integer, an integer number of periods





For the above case, $4<k_0<5$

Observe something around 4-5

EITF75 Systems and Signals

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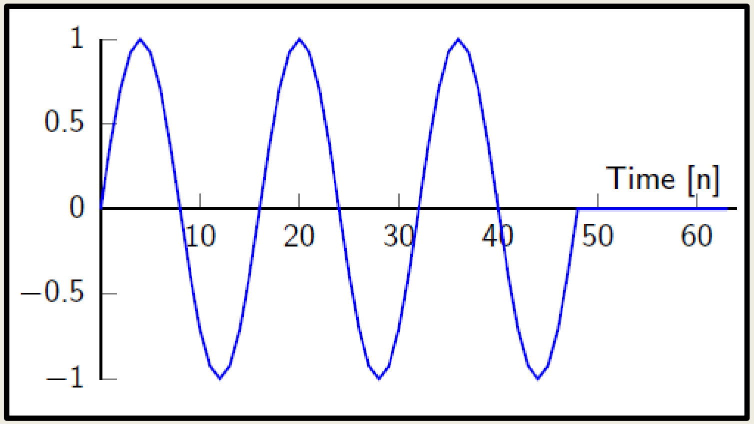

??

DFT of integer k_0, but with zero-padding

EITF75 Systems and Signals

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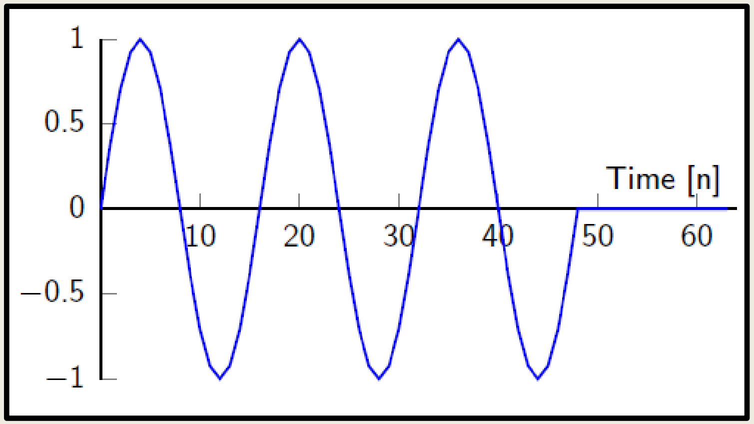

DFT of integer k_0, but with zero-padding

We cannot get a delta at k=4, since that is what we obtain without zero-padding

Convolution theorem duality [edit]

It can also be shown that:

$$\mathcal{F}\{\mathbf{x} \cdot \mathbf{y}\}_k \triangleq \sum_{n=0}^{N-1} x_n \cdot y_n \cdot e^{-i\frac{2\pi}{N}kn}$$

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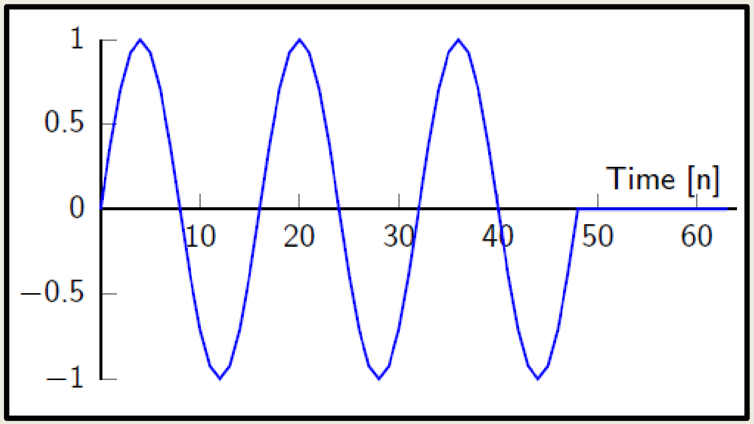

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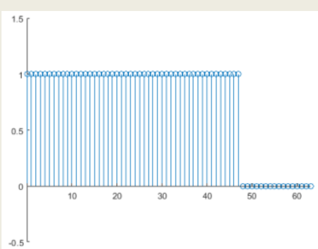

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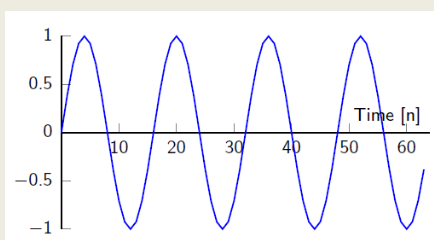

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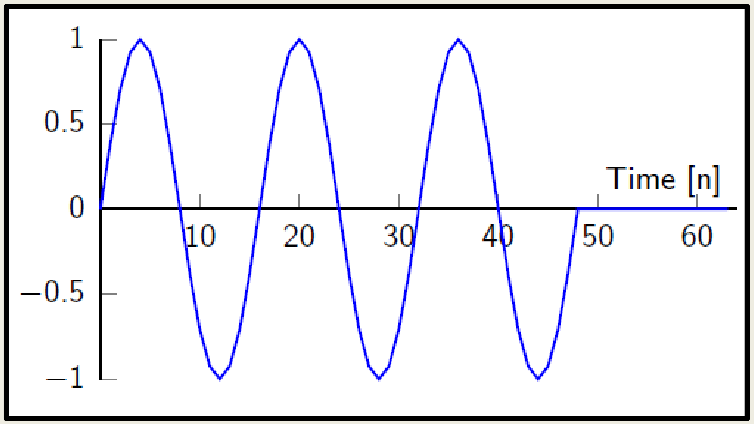
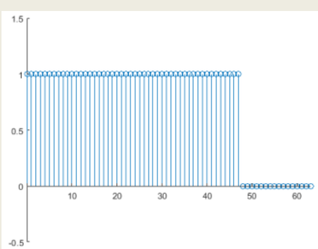


DFT of zero-padded signal will be circular convolution of DFTs of the two bottom pictures

=



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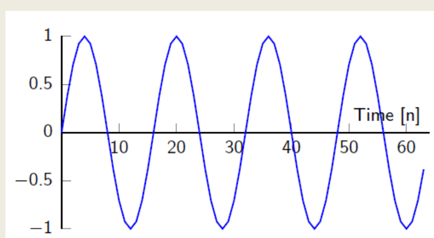

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EITF75 Systems and Signals

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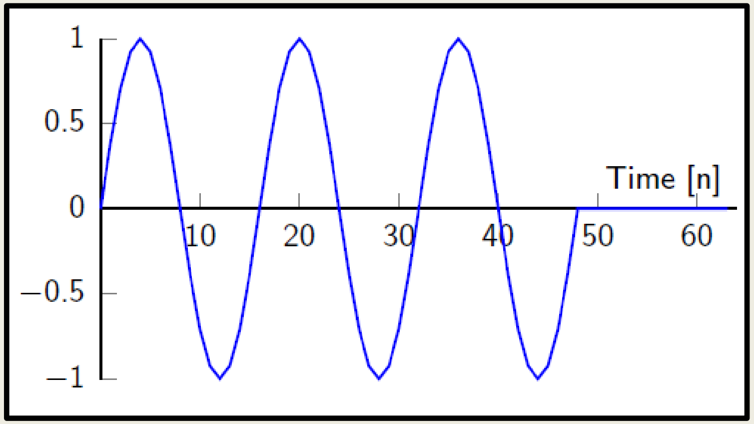

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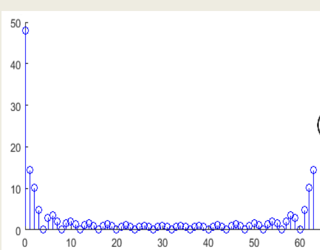
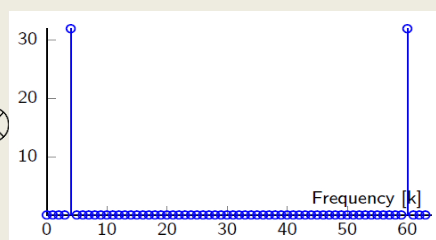
Magnitude (complex DFT)



$\otimes$



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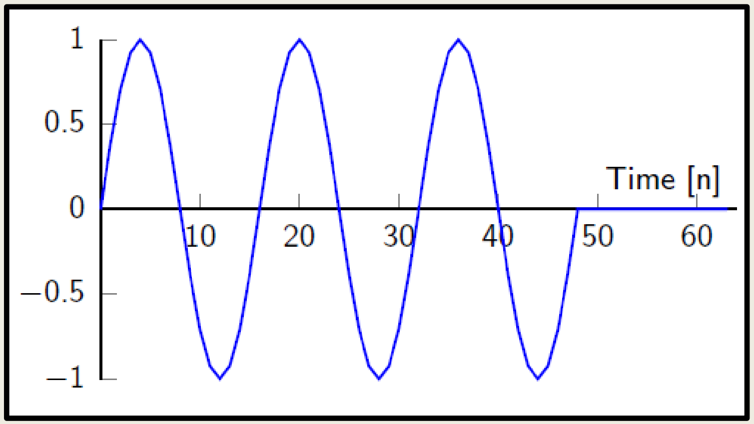

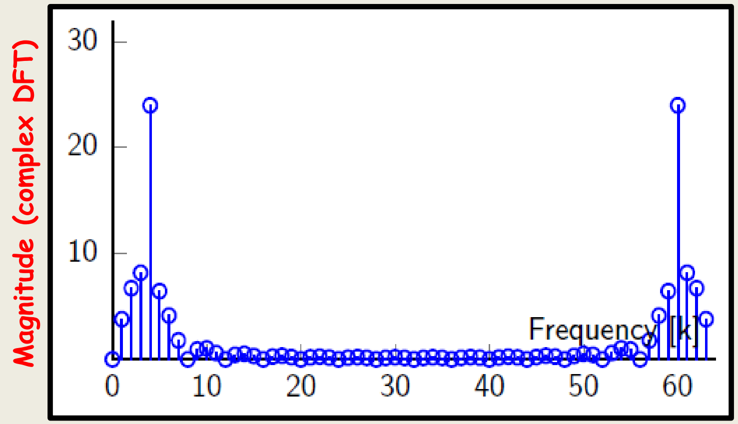

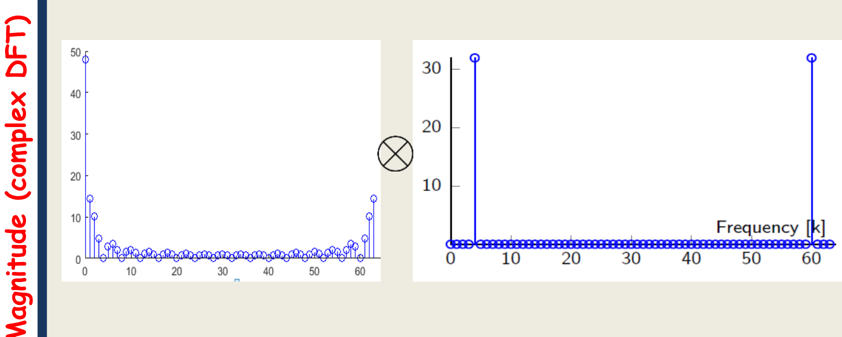

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Wikipedia
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EITF75 Systems and Signals

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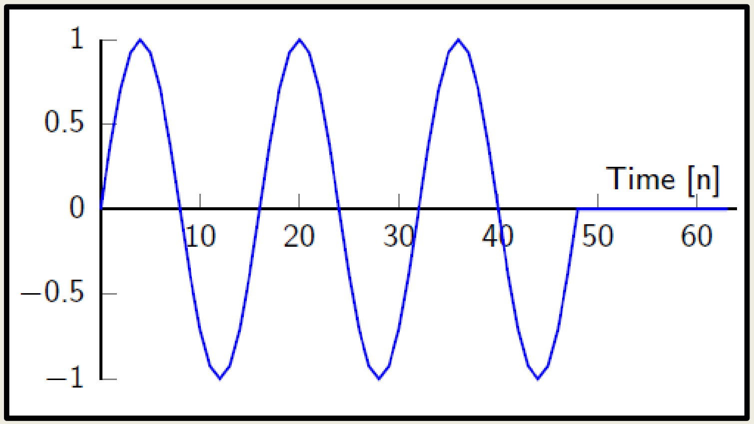

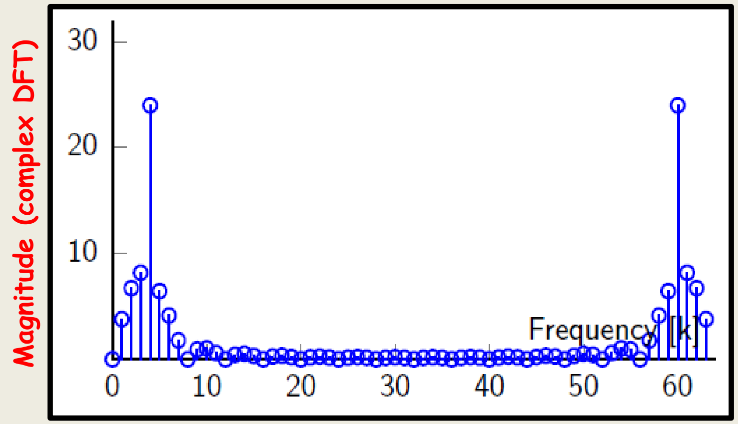

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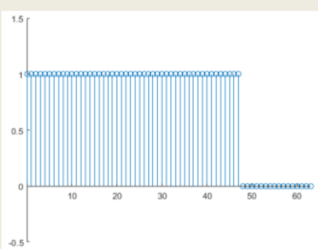

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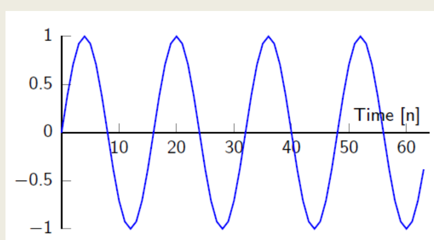

Convolution theorem duality [edit]

It can also be shown that:

$$\mathcal{F}\{\mathbf{x}\cdot\mathbf{y}\}_k \triangleq \sum_{n=0}^{N-1} x_n \cdot y_n \cdot e^{-i\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}(\mathbf{X} * \mathbf{Y_N})_k,$$ which is the circular convolution of $\mathbf{X}$ and $\mathbf{Y}$.

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Can we see the peak at n=4 sharper if we window with something else ?

EITF75 Systems and Signals

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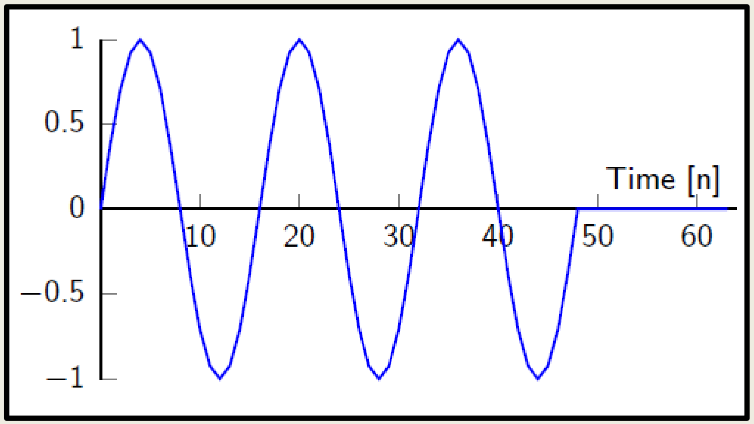
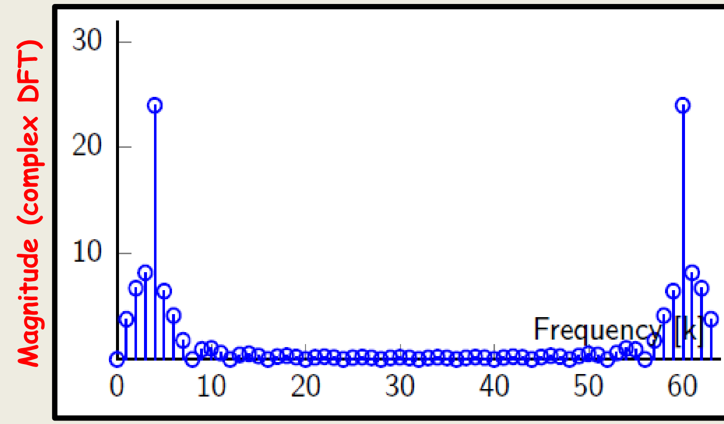
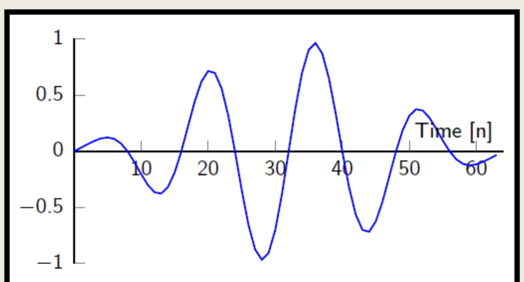
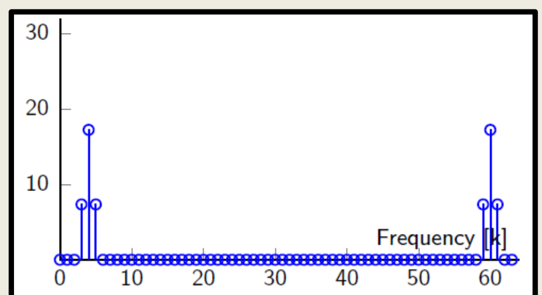






DFT $\longleftrightarrow$



Hamming window $w_{\text{hamming}}(n) = 0.54 + 0.46\cos\left(2\pi \cdot \frac{1}{N-1} \cdot \left(n - \frac{N-1}{2}\right)\right)$

EITF75 Systems and Signals

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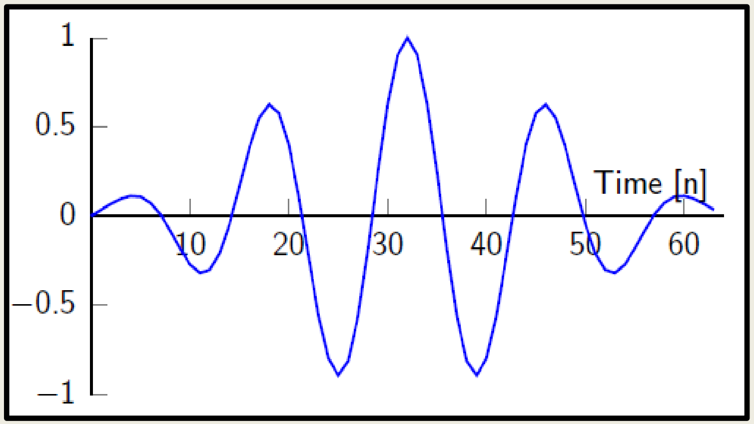

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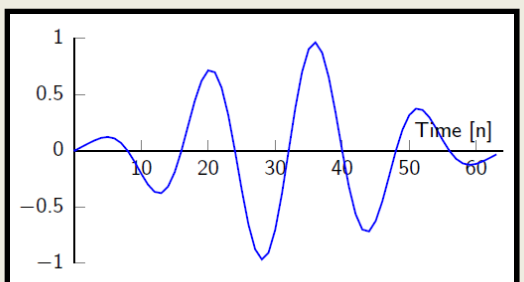

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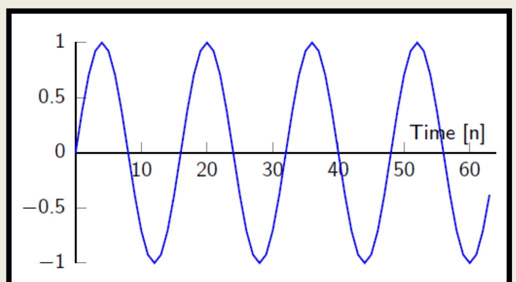

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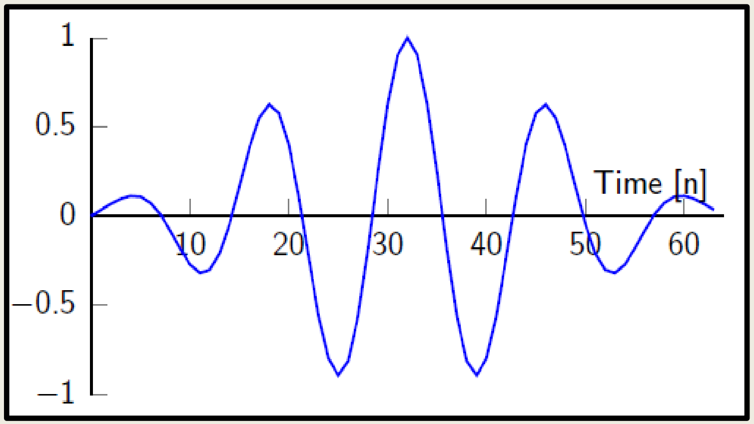

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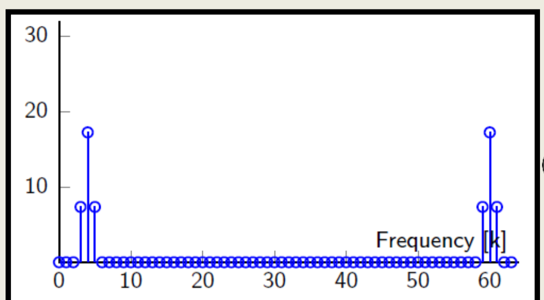
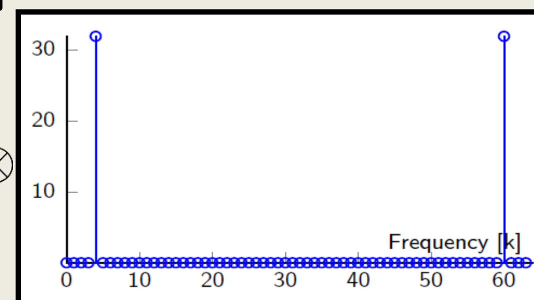


$\otimes$



EITF75 Systems and Signals

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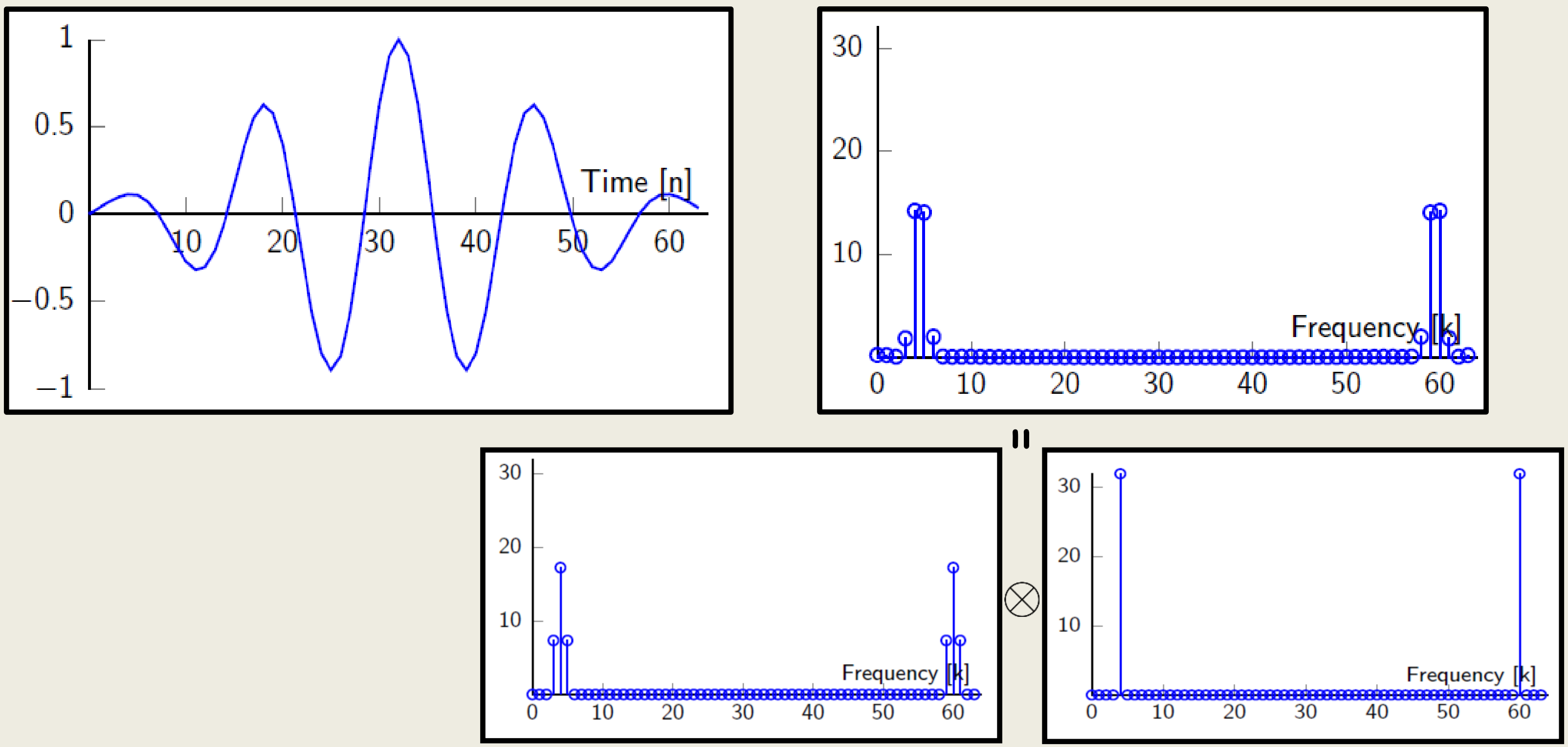
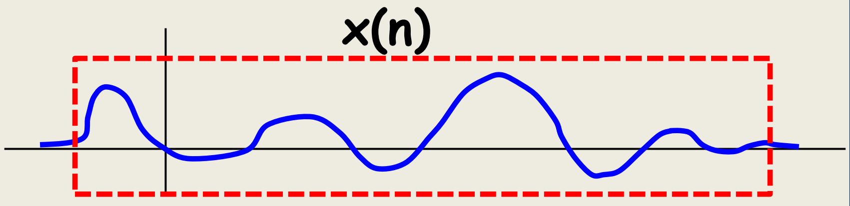


EITF75 Systems and Signals

Summary of windowing:

1. If we encounter an unknown signal x(n)



EITF75 Systems and Signals

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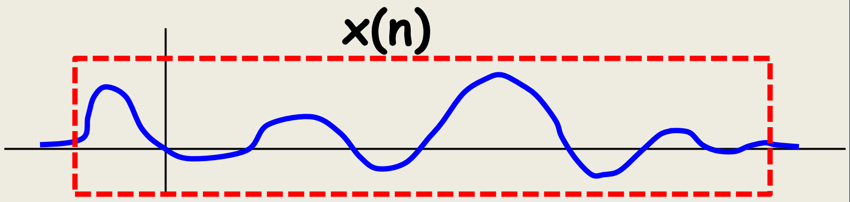

EITF75 Systems and Signals

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EITF75 Systems and Signals

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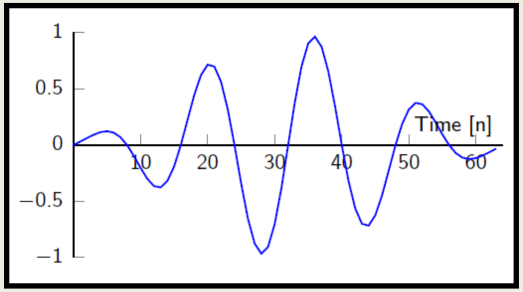
4. By windowing with a Hamming window



EITF75 Systems and Signals

Application of DFT in 5G: Orthogonal Frequency division multiplexing

Data signal $x(n) = \ldots +1\ -1\ -1\ +1\ +1\ -1\ \ldots$ (very long)

Channel $h(n) = h(0)\ h(1)\ h(2) \qquad h(M-1)$ (quite long, say 50 taps)

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EITF75 Systems and Signals

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Remedy: use overlap-add

EITF75 Systems and Signals

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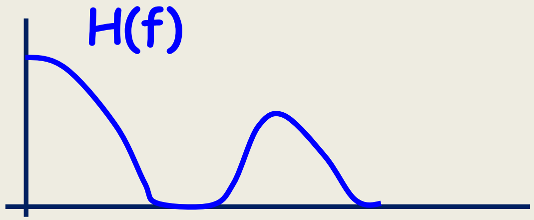

EITF75 Systems and Signals

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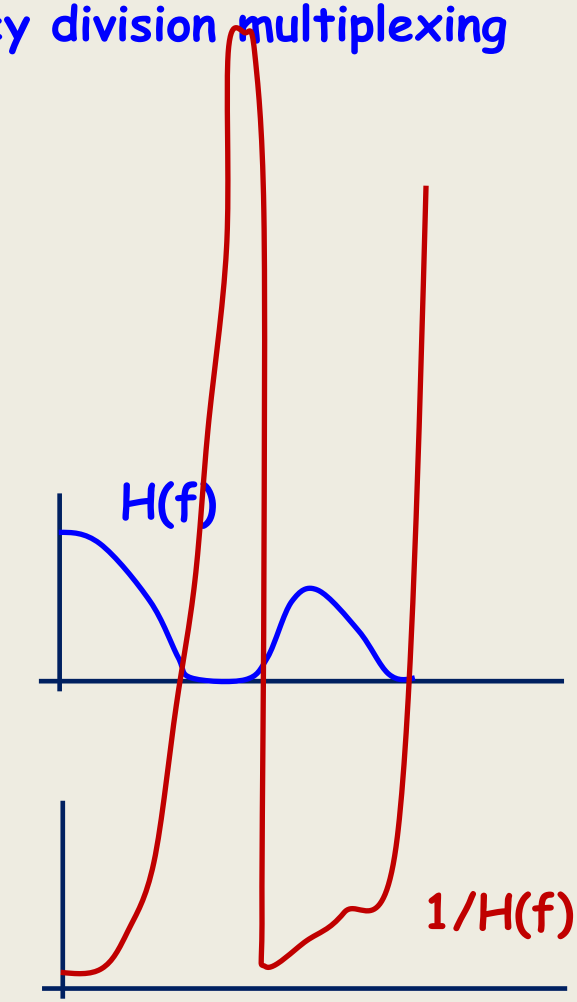

EITF75 Systems and Signals

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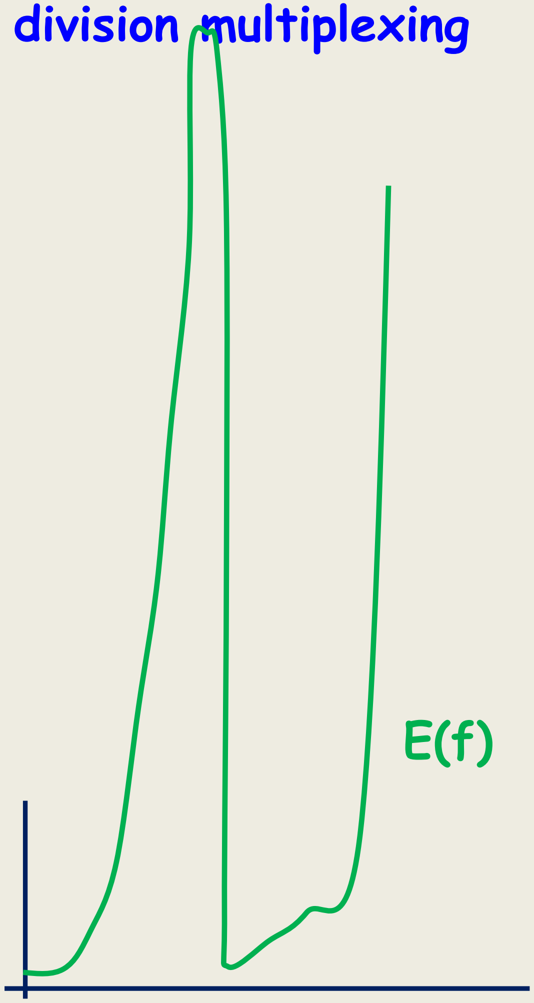

EITF75 Systems and Signals

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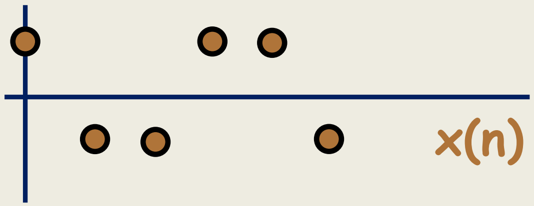

EITF75 Systems and Signals

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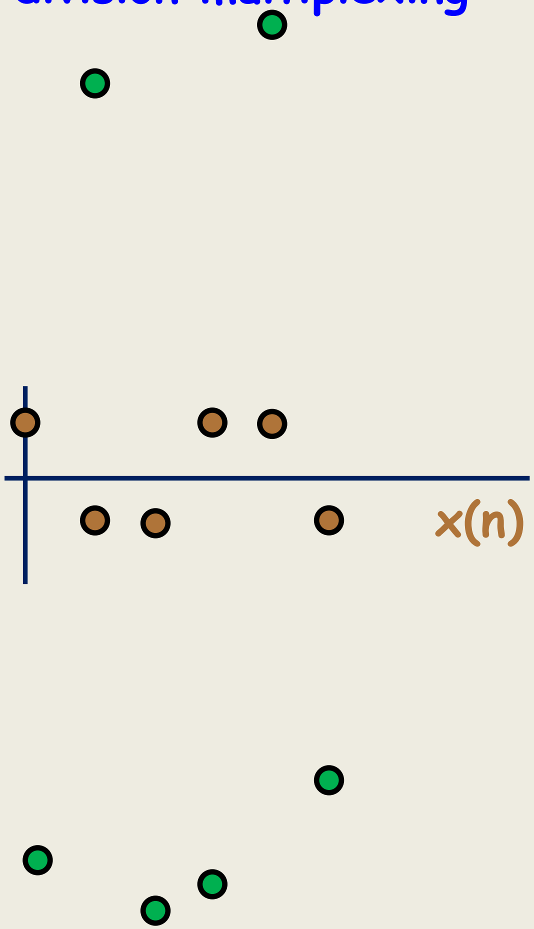

EITF75 Systems and Signals

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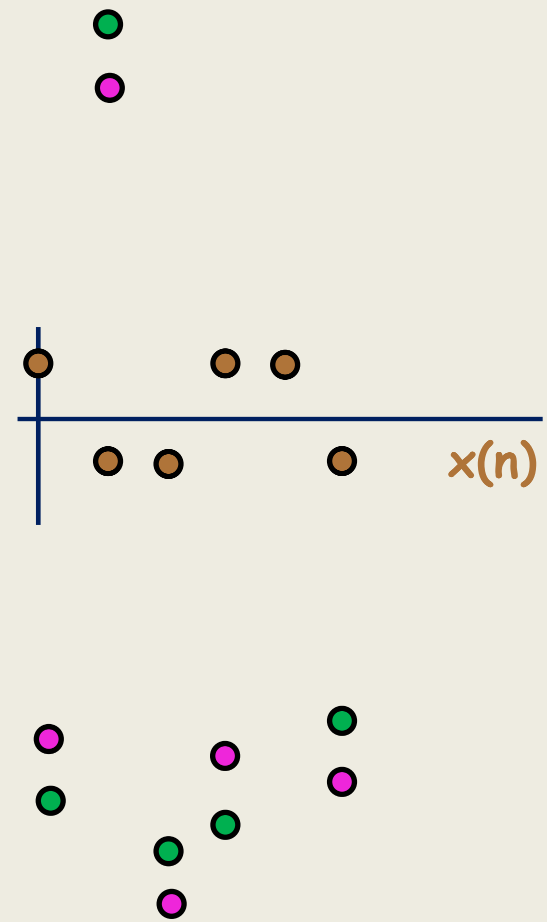
random $\longrightarrow$ $z(n) = x(n) + e(n)$

$\hat{x}(n) = \text{Threshold}[\, z(n) \,]$



EITF75 Systems and Signals

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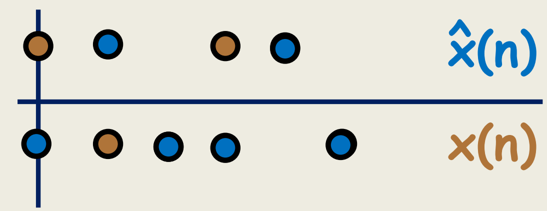

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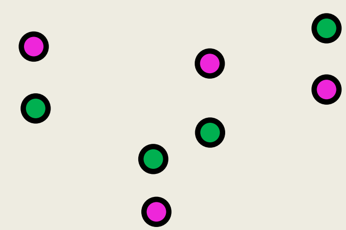

random ⟶ $z(n) = x(n) + e(n)$

50% error ⟶ $\hat{x}(n) = \text{Threshold}[\ z(n)\ ]$

EITF75 Systems and Signals

Application of DFT in 5G: Orthogonal Frequency division multiplexing

Orthogonal Frequency division multiplexing solution

Data signal $x(n)$ = … +1 -1 -1 +1 +1 -1 …

$x_1(n)$	$x_2(n)$	$x_3(n)$

Block the data signal. Block size L

EITF75 Systems and Signals

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Apply IDFT to each block

$a_1(n)$	$a_2(n)$	$a_3(n)$

$a_1(n) = \text{IDFT}(x_1(n))$

$a_2(n) = \text{IDFT}(x_2(n))$

$a_3(n) = \text{IDFT}(x_3(n))$

EITF75 Systems and Signals

Application of DFT in 5G: Orthogonal Frequency division multiplexing

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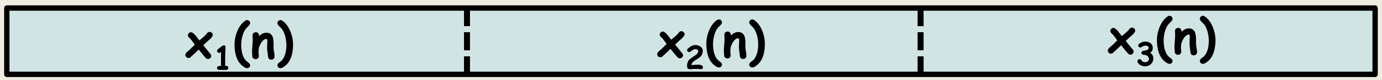
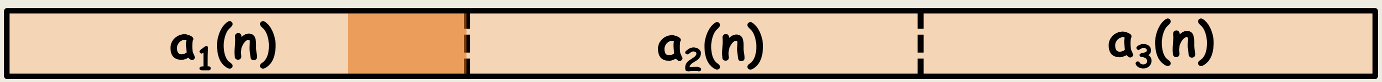


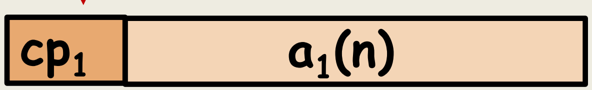
Block the data signal. Block size L

Apply IDFT to each block



$a_k(n) = \text{IDFT}(x_k(n))$

Copy tail and append at front (cyclic prefix)



EITF75 Systems and Signals

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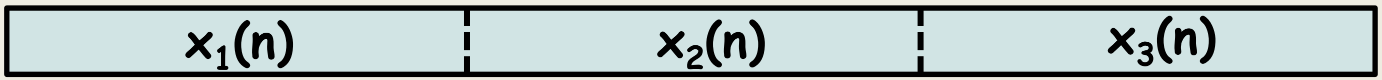

Block the data signal. Block size L

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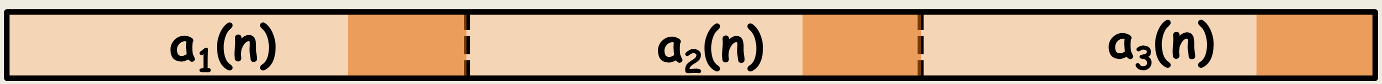
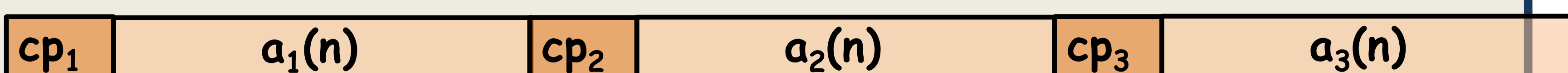


$a_k(n)$ = IDFT($x_k(n)$)

Repeat for all blocks



EITF75 Systems and Signals

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Apply IDFT to each block

$a_1(n)$	$a_2(n)$	$a_3(n)$

$a_k(n)$ = IDFT($x_k(n)$)

Transmit over channel $h(n)$

cp_1	$a_1(n)$	cp_2	$a_2(n)$	cp_3	$a_3(n)$

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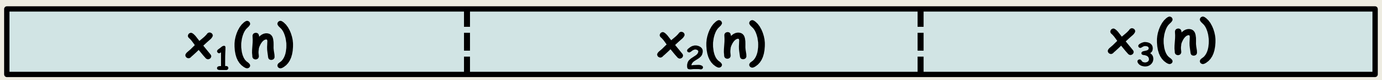
cp_1	$a_1(n)$	cp_2	$a_2(n)$	cp_3	$a_3(n)$
ycp_1	$y_1(n)$	ycp_2	$y_2(n)$	ycp_3	$y_3(n)$

EITF75 Systems and Signals

Application of DFT in 5G: Orthogonal Frequency division multiplexing

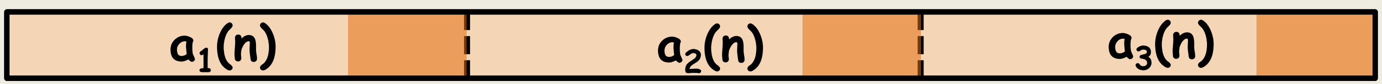
Orthogonal Frequency division multiplexing solution

Data signal $x(n)$ = … +1 -1 -1 +1 +1 -1 …



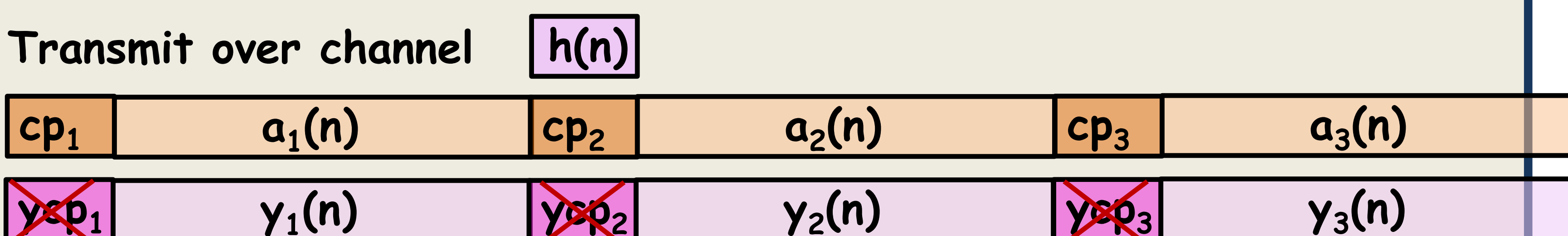
Block the data signal. Block size L

Apply IDFT to each block



$a_k(n) = \mathrm{IDFT}(x_k(n))$

Transmit over channel



Throw away

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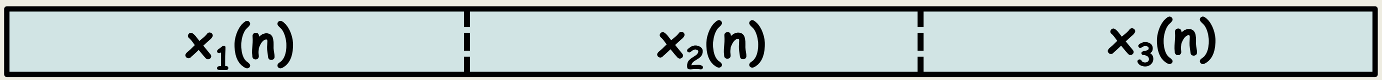

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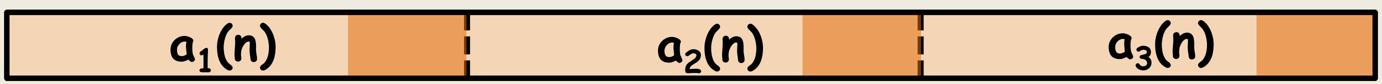
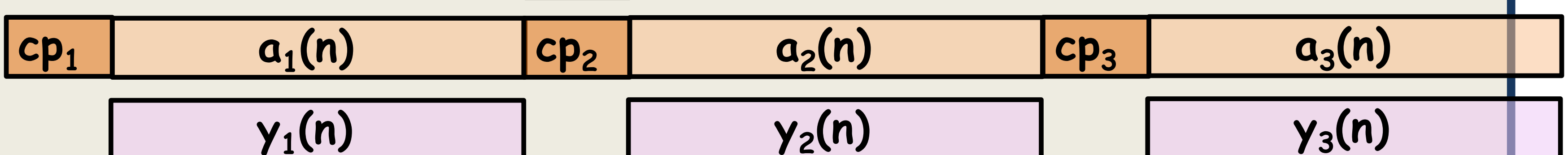


$a_k(n)$ = IDFT($x_k(n)$)

Transmit over channel $h(n)$



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Data signal $x(n) = \ldots +1 \ -1 \ -1 \ +1 \ +1 \ -1 \ \ldots$

$x_1(n)$ | $x_2(n)$ | $x_3(n)$

Block the data signal. Block size L

Apply IDFT to each block

$a_1(n)$ | $a_2(n)$ | $a_3(n)$

$a_k(n) = \mathrm{IDFT}(x_k(n))$

Transmit over channel $h(n)$

cp_1 | $a_1(n)$ | cp_2 | $a_2(n)$ | cp_3 | $a_3(n)$

$y_1(n)$ | $y_2(n)$ | $y_3(n)$

Math formula for $y_1(n)$?

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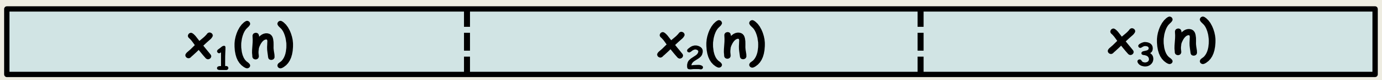

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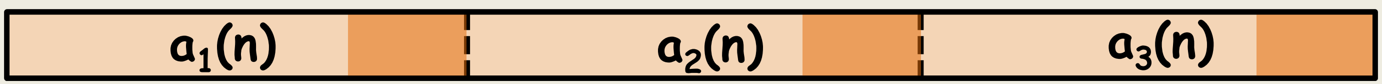

$a_k(n) = \mathrm{IDFT}(x_k(n))$

Transmit over channel $h(n)$

Math formula for $y_1(n)$? $y_1(n) = a_1(n) \otimes h(n) + w_1(n)$

Due to CP

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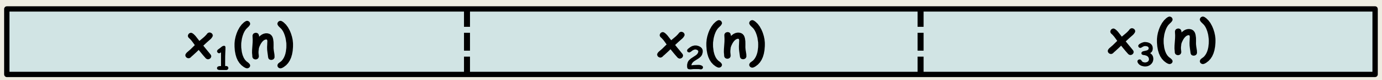
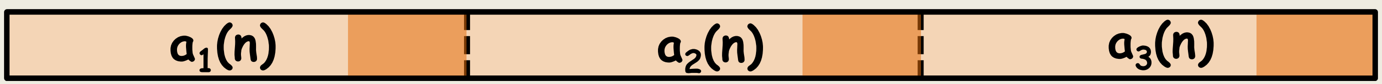


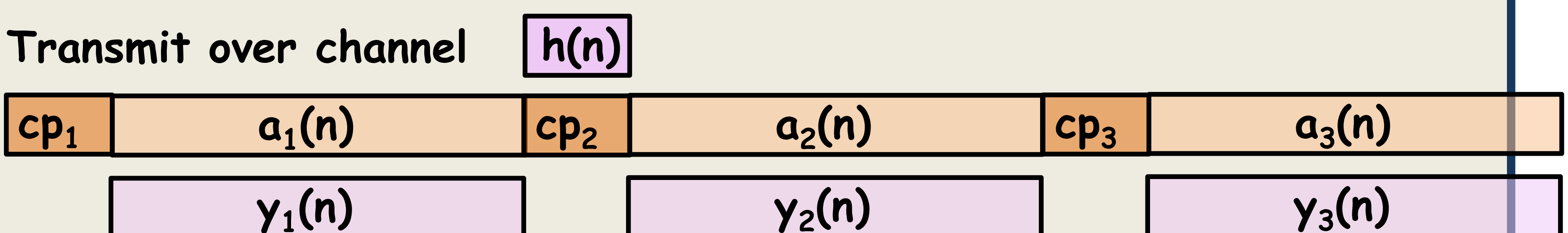
Block the data signal. Block size L

Apply IDFT to each block



$a_k(n)$ = IDFT($x_k(n)$)

Transmit over channel



Math formula for $y_1(n)$? $\quad y_1(n) = a_1(n) \otimes h(n) + w_1(n)$

Apply DFT $\quad Y_1(k) = x_1(k)\, H(k) + W_1(k)$

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Data signal $x(n)$ = … +1 -1 -1 +1 +1 -1 …

$x_1(n)$ | $x_2(n)$ | $x_3(n)$

Block the data signal. Block size L

Apply IDFT to each block

$a_1(n)$ | $a_2(n)$ | $a_3(n)$

$a_k(n) = \mathrm{IDFT}(x_k(n))$

Transmit over channel $h(n)$

cp_1 | $a_1(n)$ | cp_2 | $a_2(n)$ | cp_3 | $a_3(n)$

$y_1(n)$ | $y_2(n)$ | $y_3(n)$

Math formula for $y_1(n)$? $y_1(n) = a_1(n) \otimes h(n) + w_1(n)$

Apply DFT $Z_1(k) = x_1(k) + W_1(k)/H(k)$

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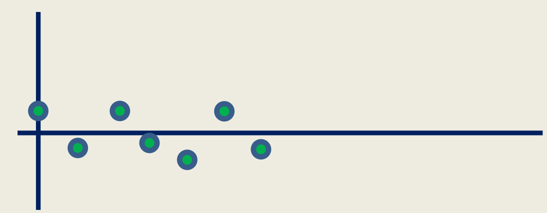
Data signal $x(n)$ = … +1 -1 -1 +1 +1 -1 …

$x_1(n)$

$W_1(k)$



Transmit over channel $h(n)$

cp_1	$a_1(n)$	cp_2	$a_2(n)$	cp_3	$a_3(n)$
	$y_1(n)$		$y_2(n)$		$y_3(n)$

Math formula for $y_1(n)$? $\quad y_1(n) = a_1(n) \otimes h(n) + w_1(n)$

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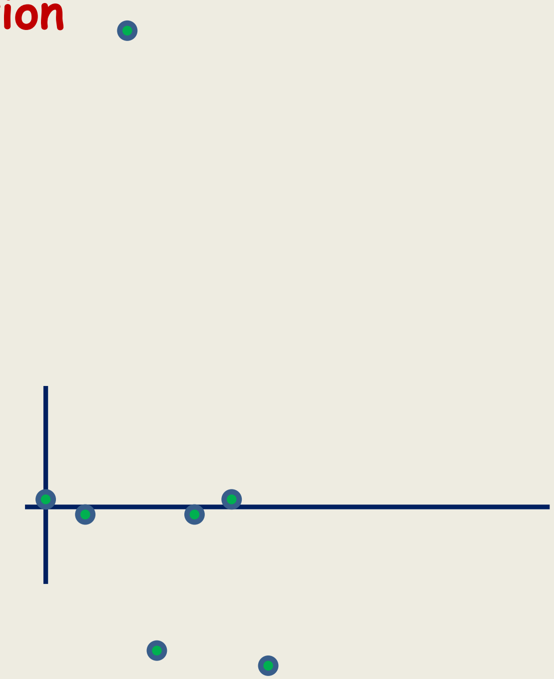
$x_1(n)$

$W_1(k)/H(k)$



Transmit over channel $h(n)$

cp_1	$a_1(n)$	cp_2	$a_2(n)$	cp_3	$a_3(n)$
	$y_1(n)$		$y_2(n)$		$y_3(n)$

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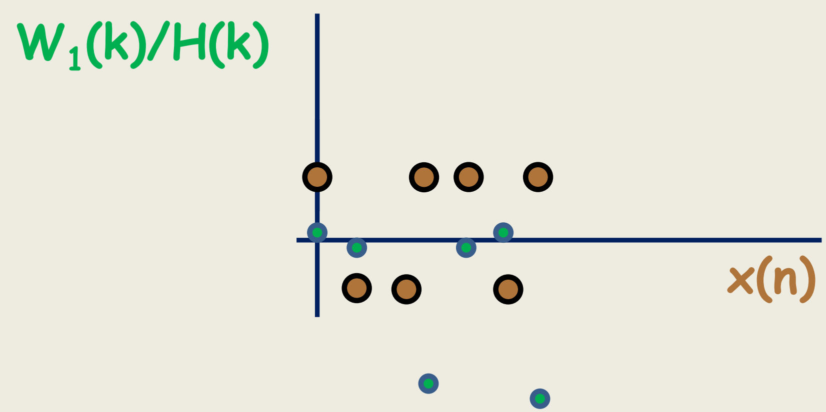
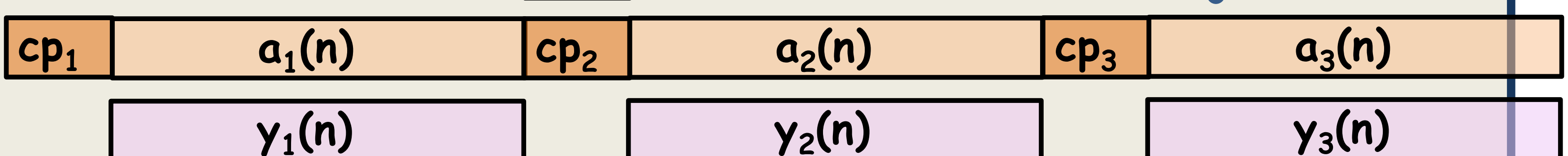
$x_1(n)$



Transmit over channel $h(n)$



Math formula for $y_1(n)$? $y_1(n) = a_1(n) \otimes h(n) + w_1(n)$

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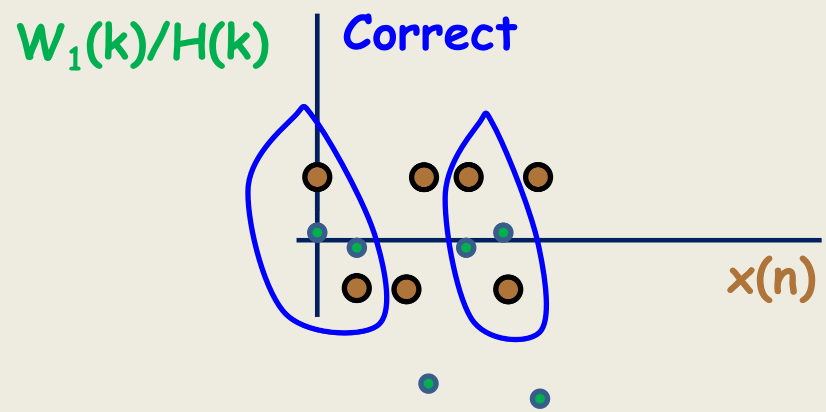
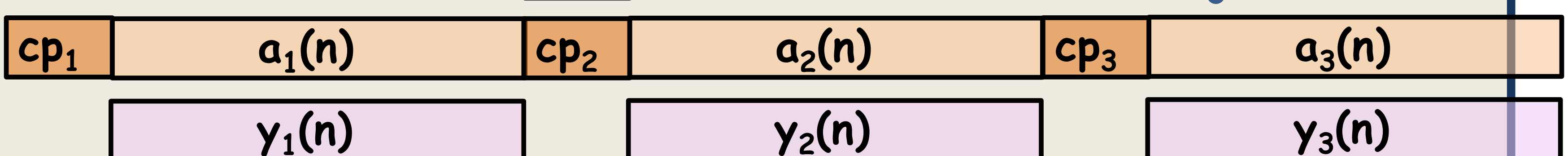
$x_1(n)$



Transmit over channel $h(n)$



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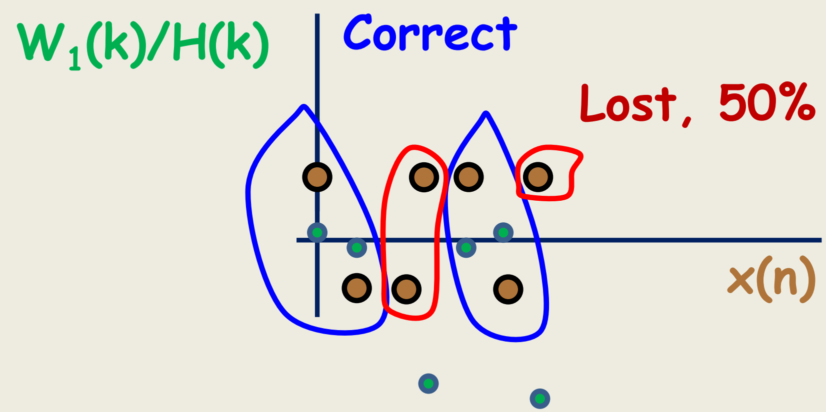
$x_1(n)$



Transmit over channel $h(n)$

cp_1	$a_1(n)$	cp_2	$a_2(n)$	cp_3	$a_3(n)$
	$y_1(n)$		$y_2(n)$		$y_3(n)$

Math formula for $y_1(n)$? $y_1(n) = a_1(n) \otimes h(n) + w_1(n)$

Apply DFT $Z_1(k) = x_1(k) + W_1(k)/H(k)$

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Summary:

By inserting the CP (of length M = channel duration) and using DFT/IDFT, we can recover the data where the channel is good

The transmitter can avoid sending data at bad channel frequencies

FFT can be used for efficient implementation. 1024 - 4096 in 5G

CP is pure rate and power loss, but needed