EITF75 Systems and Signals

Lecture 10
The Discrete Fourier transform
…or…
Honey, I shrank the Discrete-time Fourier Transform

Fredrik Rusek

EITF75 Systems and Signals

Background and motivation for yet another transform

For discrete-time systems, we have treated two transforms:

1. The z-transform
   - Good for solving difference equations
   - Good for algebraic analysis of systems
   - Limited insight from plotting

2. The discrete-time Fourier transform (DTFT)
   - Excellent for understanding characteristics of systems
   - Insights from plotting
   - Relates to reality (bandwidth)

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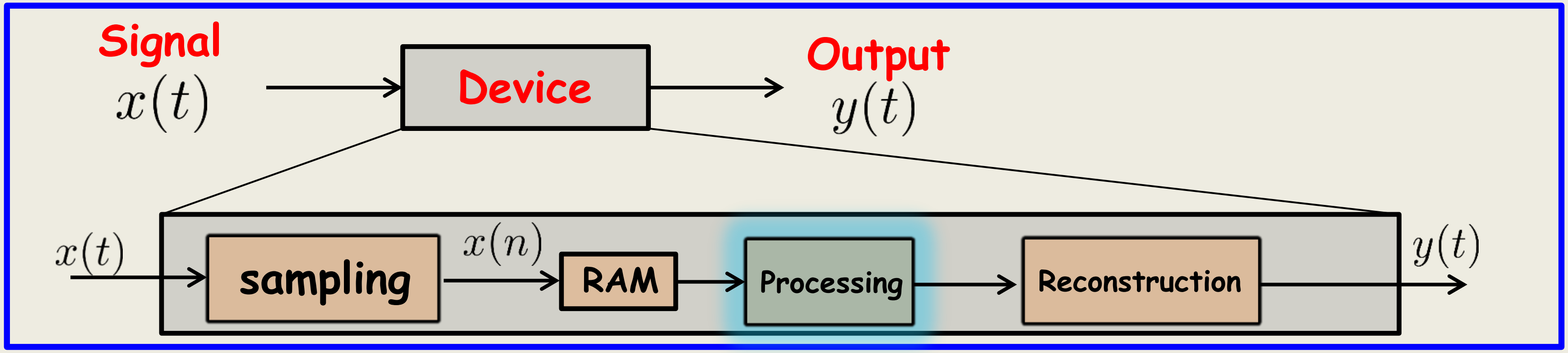

Consider the Processing module. It should process the signal according to the application at hand. (Can be more than a filter, for example an entire C/C++ program)

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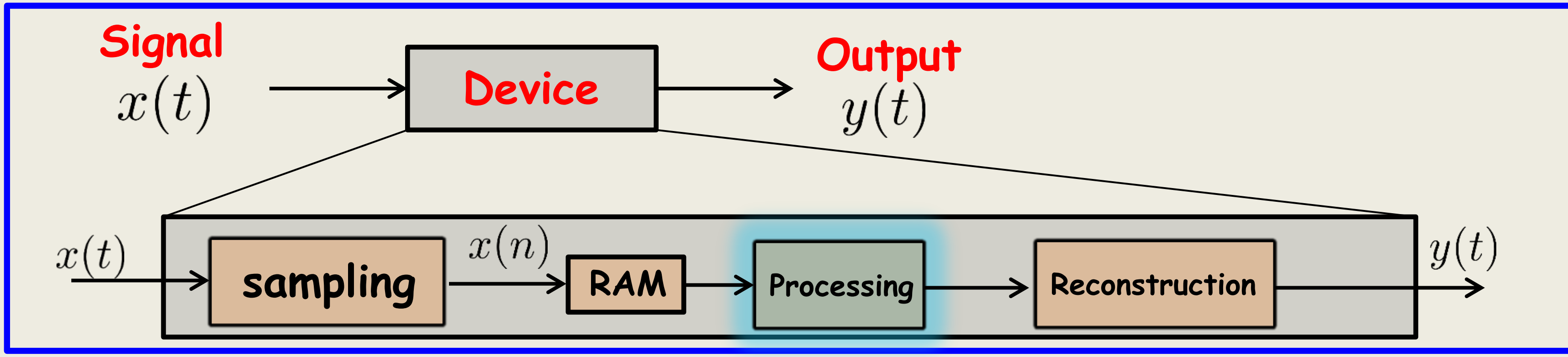

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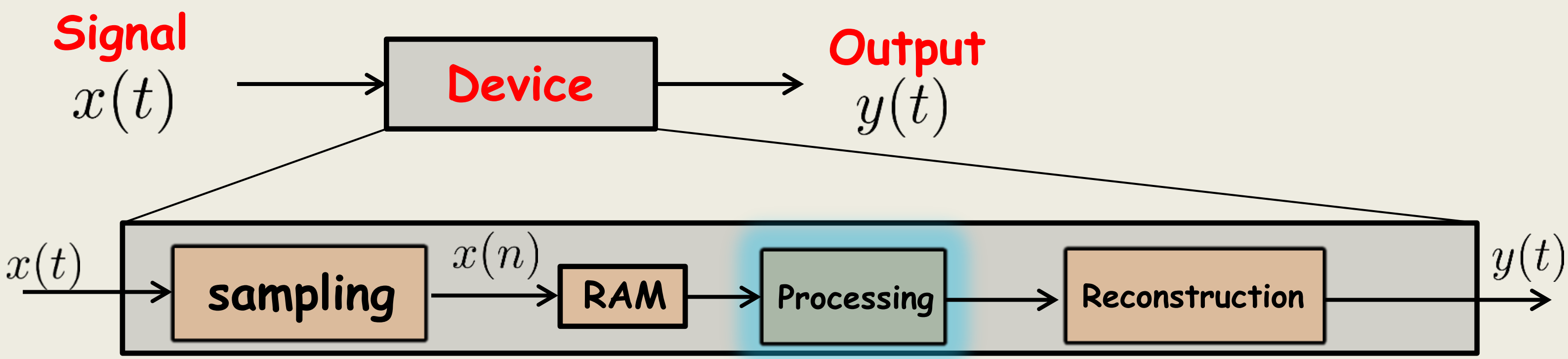

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Hint: Why couldn't a digital processor use x(t) directly?

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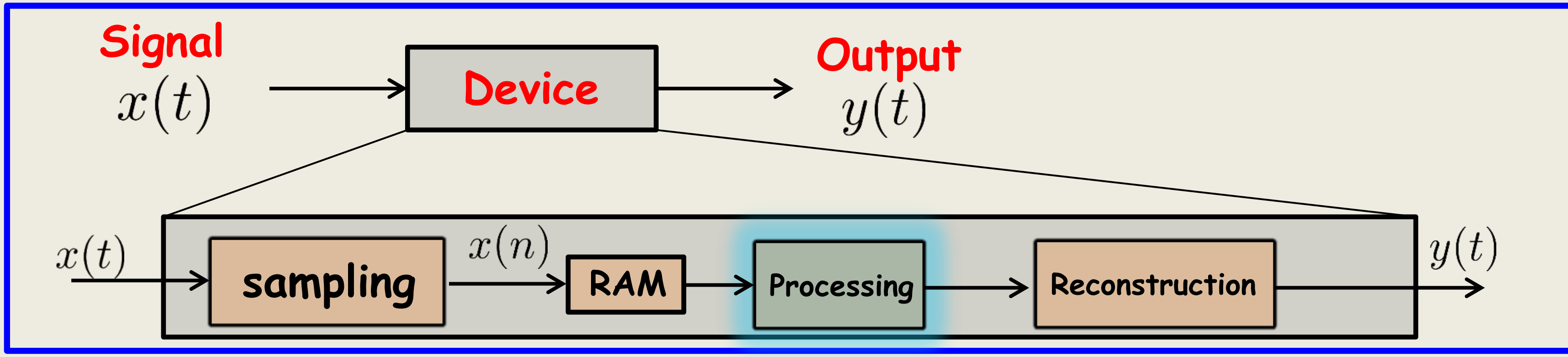

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Because it is continuous, and a processor is digital

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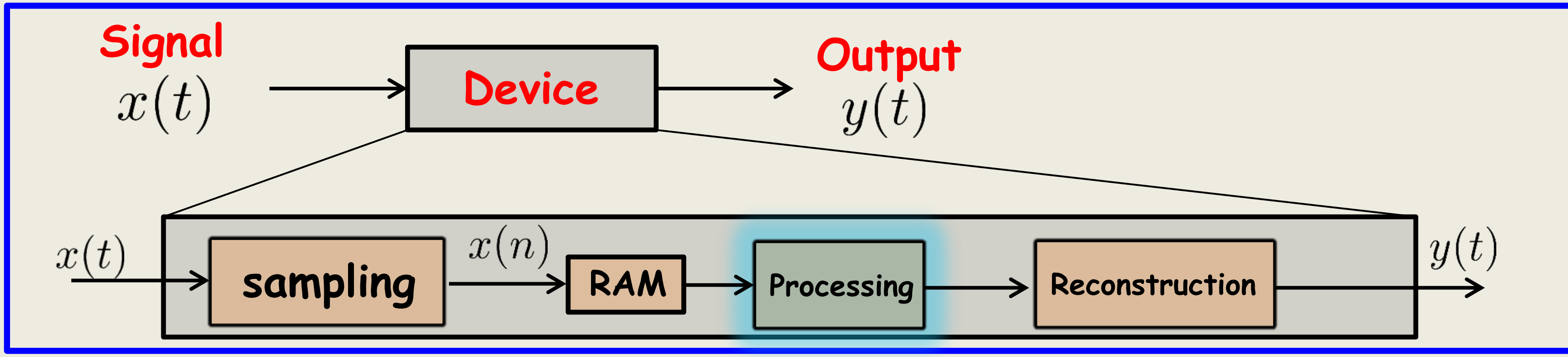

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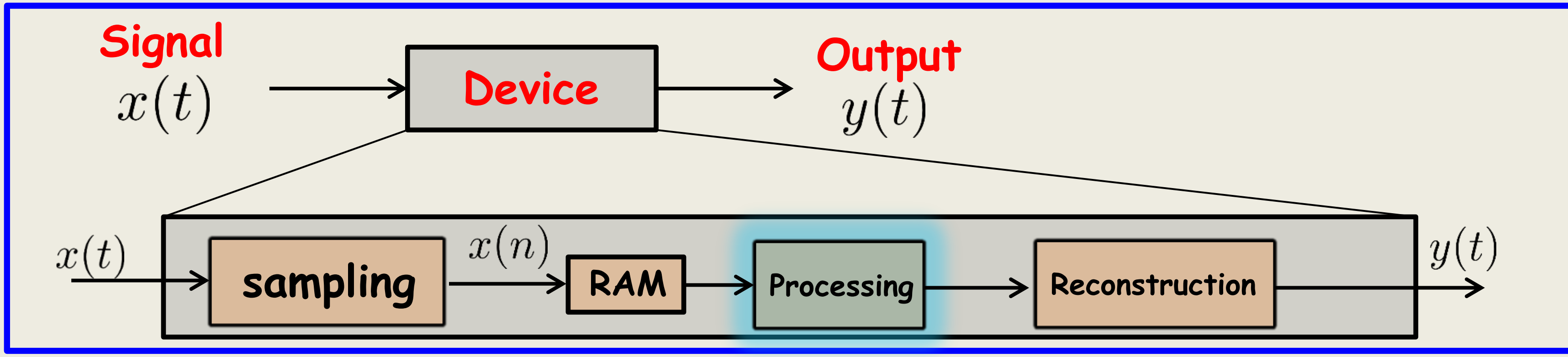

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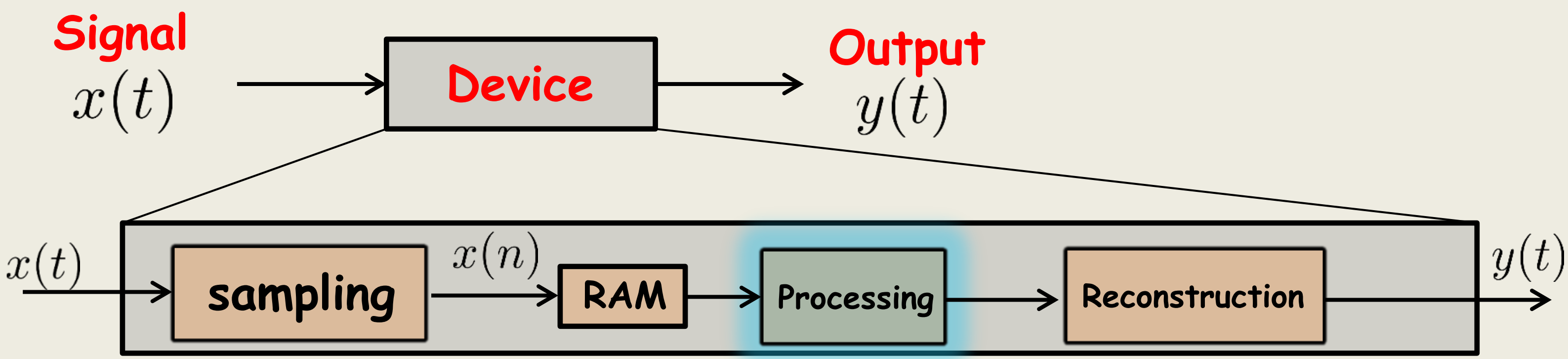

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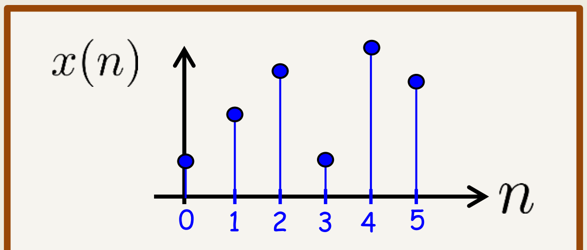
Besides, the DTFT is terribly inefficient



These 6 numbers, are in the frequency domain represened by…

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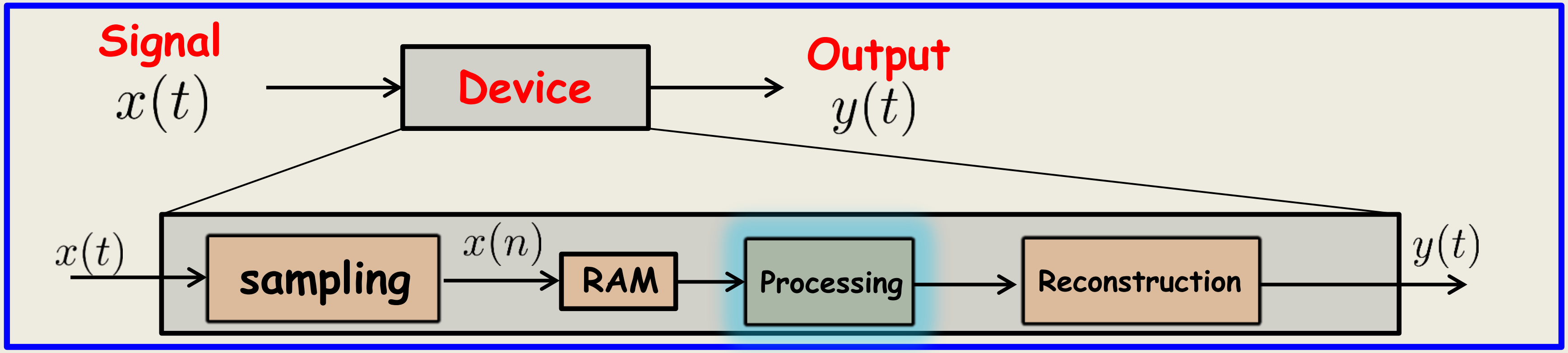
Background and motivation for yet another transform



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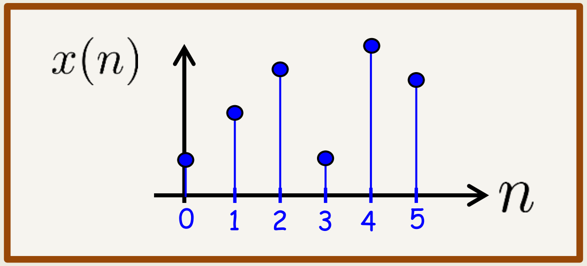

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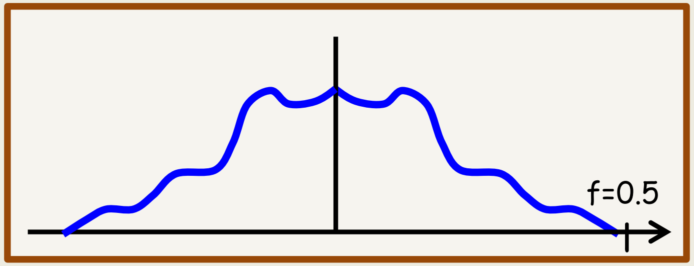

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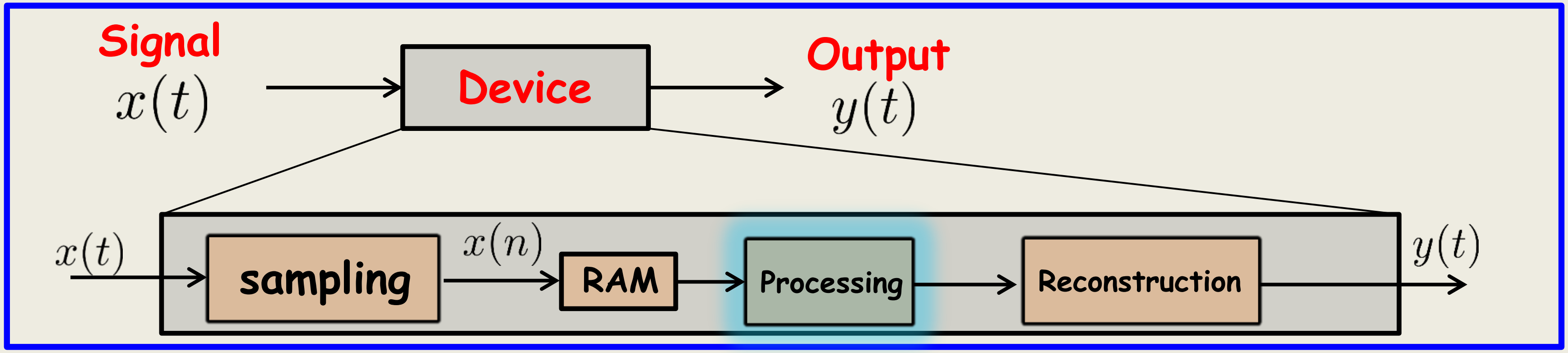

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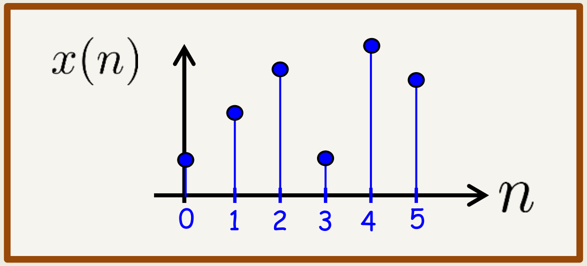

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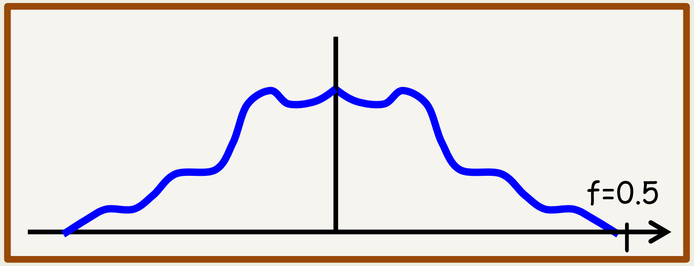
It should be possible to Fourier represent x(n) by 6 numbers as well



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Background and motivation for yet another transform

The discrete Fourier Transform (DFT) in one sentence:
A Fourier version of x(n) with 6 numbers

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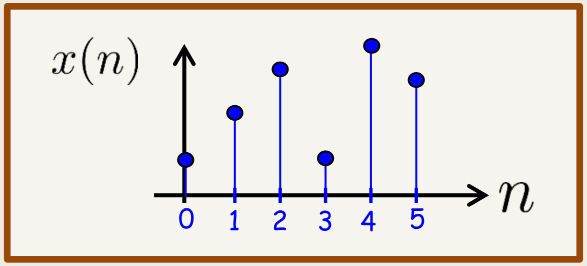

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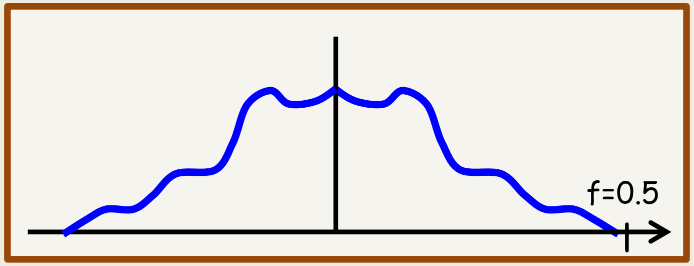

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Honey, I shrank the Discrete-time Fourier Transform

Easy to remember: Less words, less numbers

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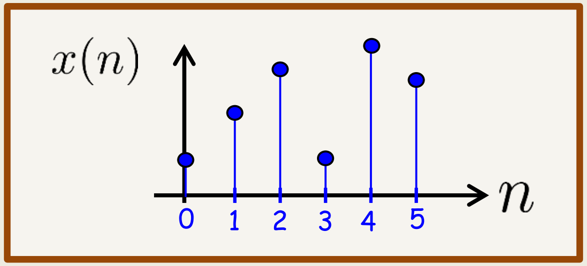

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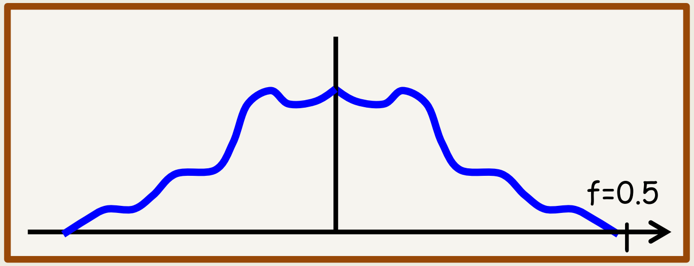

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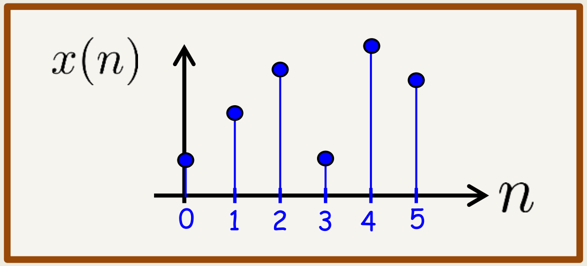

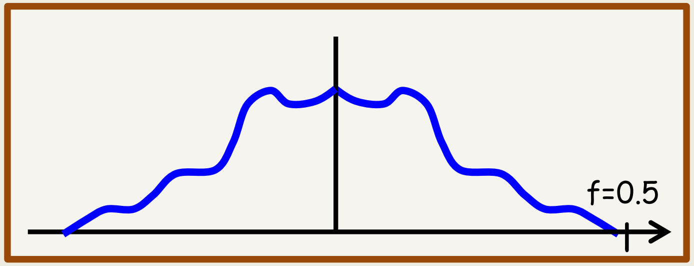

The DTFT is periodic

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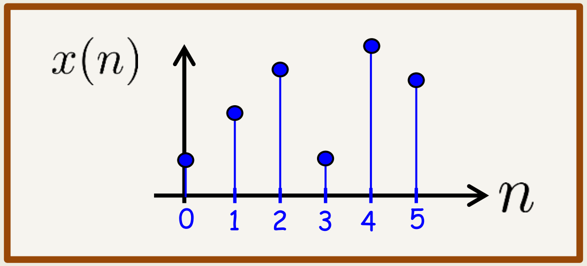

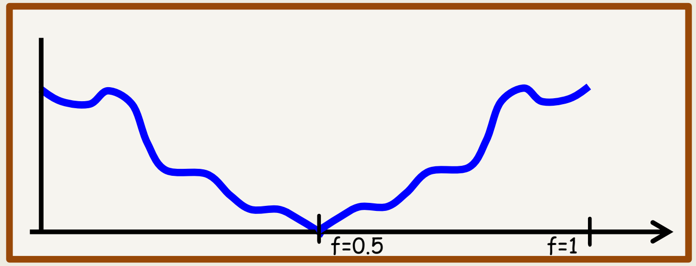

We can represent it like this

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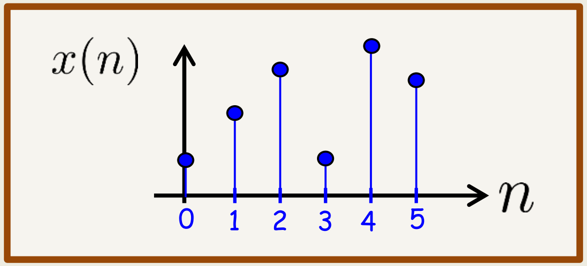
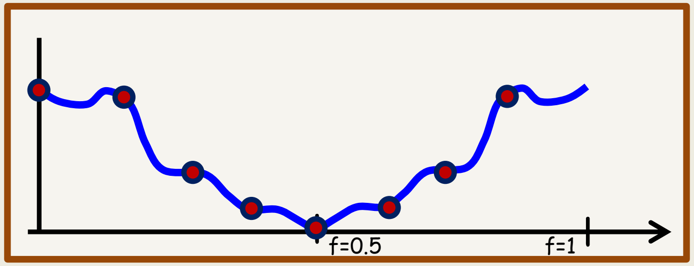


We don't assume N taps in x(n)



Assume that we take N samples of the DTFT

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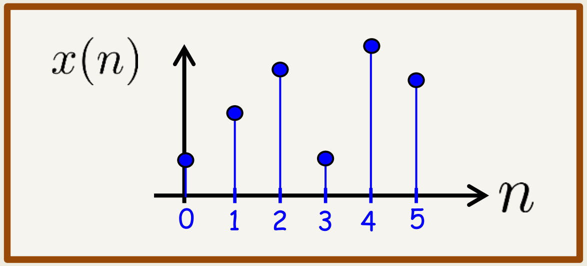

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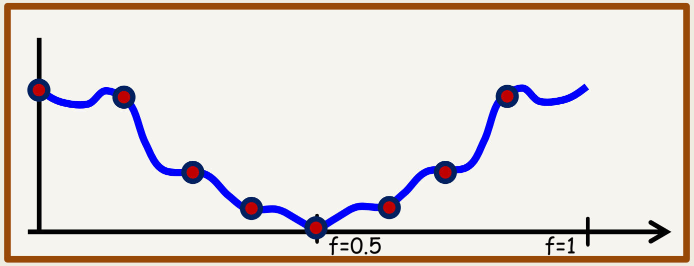

Assume that we take N samples of the DTFT

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-i2\pi nf}$$

Formula for DTFT

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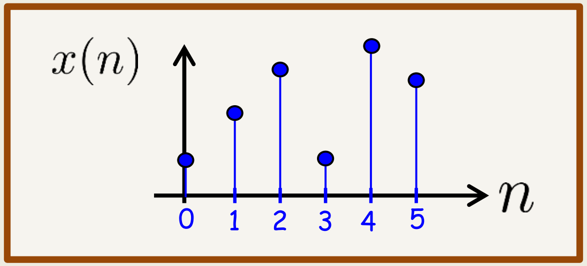

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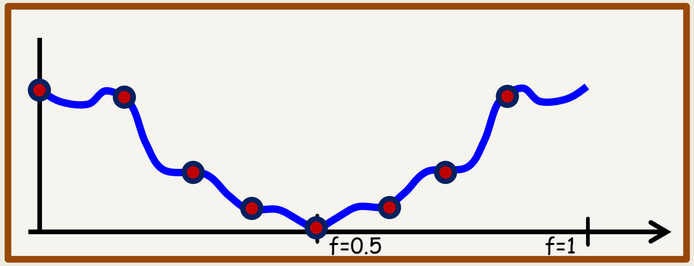

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Formula for DTFT

Samples located at $f = \dfrac{k}{N}$

$$k = 0,\ 1,\ \ldots,\ N-1$$

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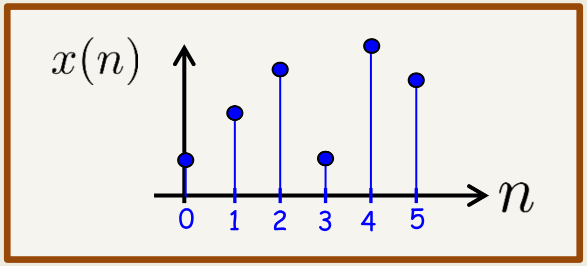

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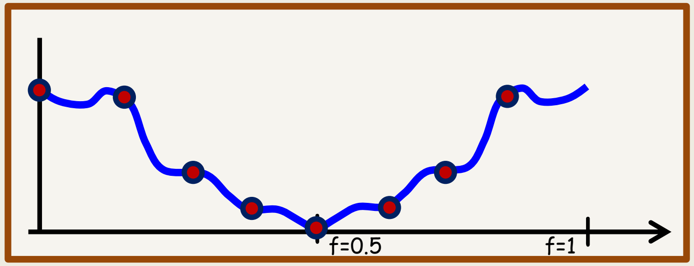

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$$X(k) = \sum_{m=-\infty}^{\infty} x(m) \mathrm{e}^{-i2\pi m k/N}$$

Formula for samples

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Warning: This is not yet the DFT. It is only samples of the DTFT

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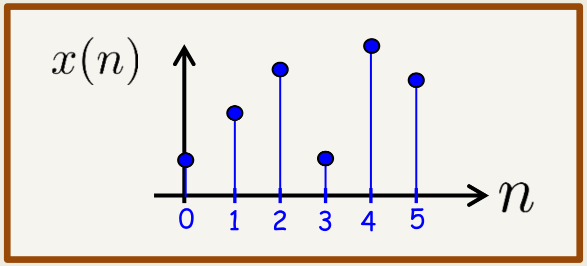

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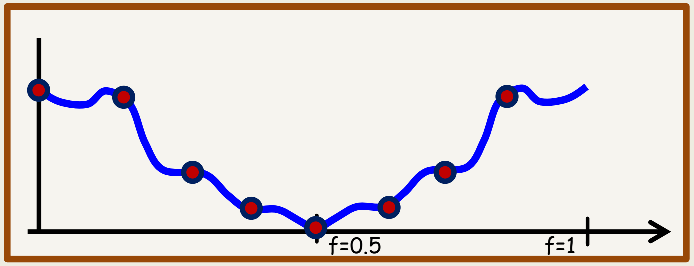

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If samples are representing X(f), then we should be able to get x(n) back from them

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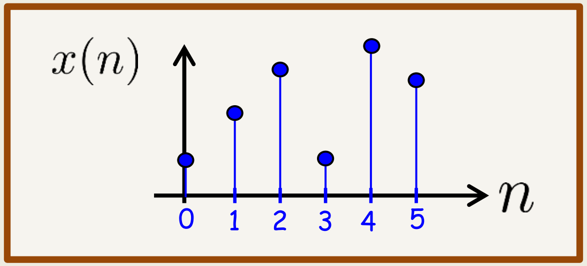

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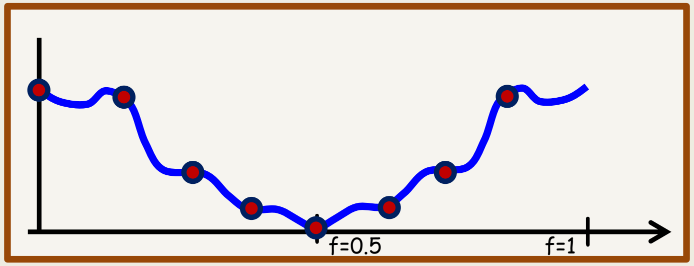

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Let us try to invert using this formula

$$x(n) \stackrel{?}{=} \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k}$$

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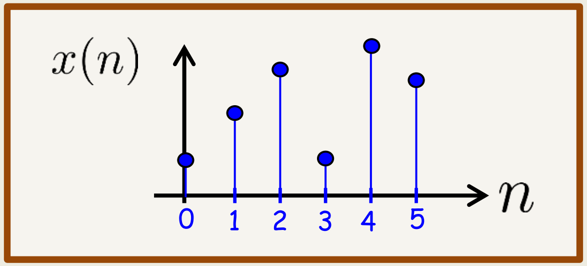

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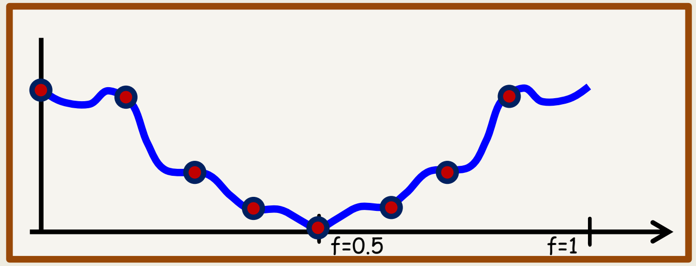

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Plug in

$$X(k) = \sum_{m=-\infty}^{\infty} x(m) e^{-i2\pi mk/N}$$

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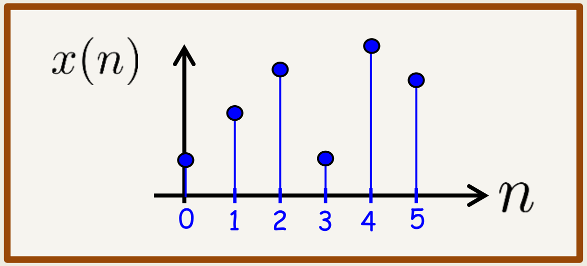

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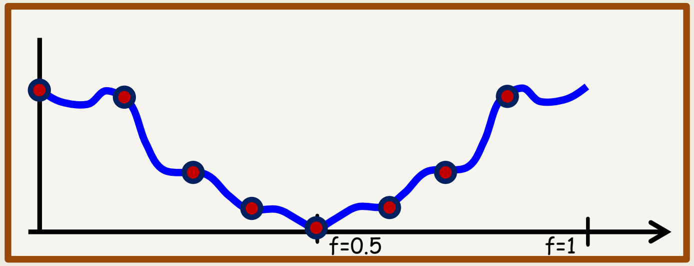

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$$X(k) = \sum_{m=-\infty}^{\infty} x(m)\mathrm{e}^{-i2\pi mk/N}$$

Standard approach when facing two sums ?

If samples are representing X(f), then we should be able to get x(n) back from them

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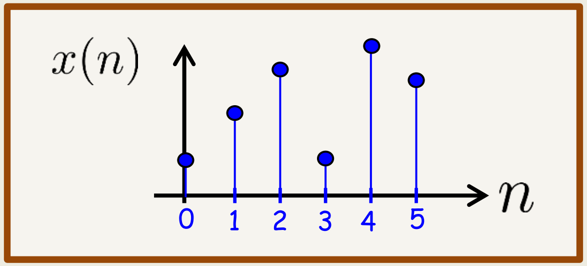

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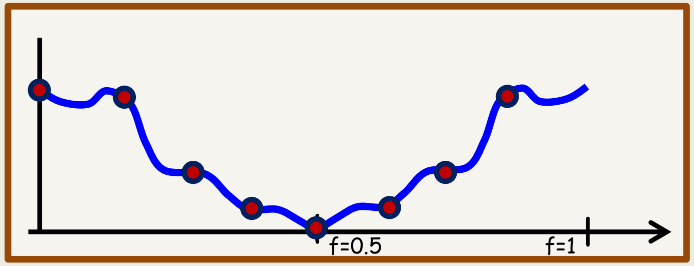

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$$X(k) = \sum_{m=-\infty}^{\infty} x(m) e^{-i2\pi mk/N}$$

$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot \sum_{k=0}^{N-1} e^{j2\pi \cdot \frac{n-m}{N} \cdot k}$$

Switch order

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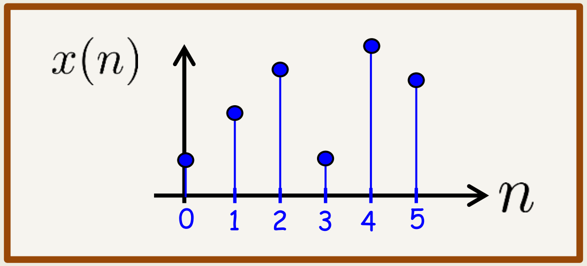

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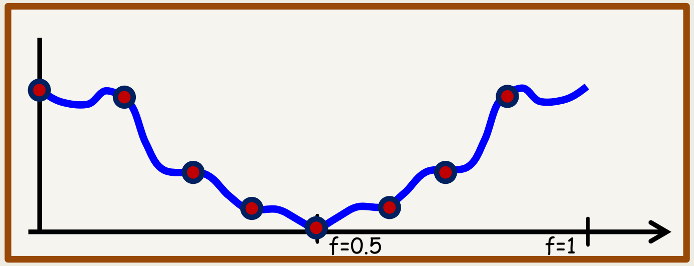

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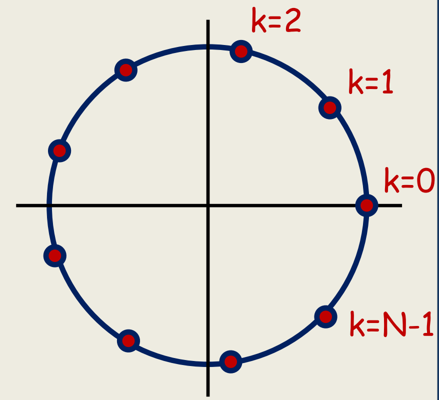

$$n \neq m$$

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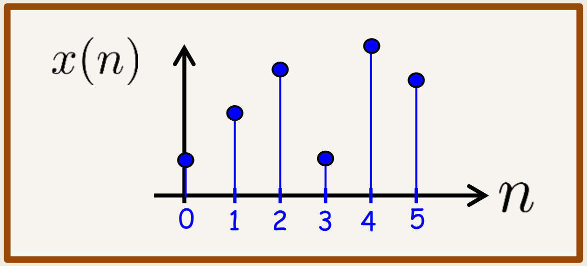

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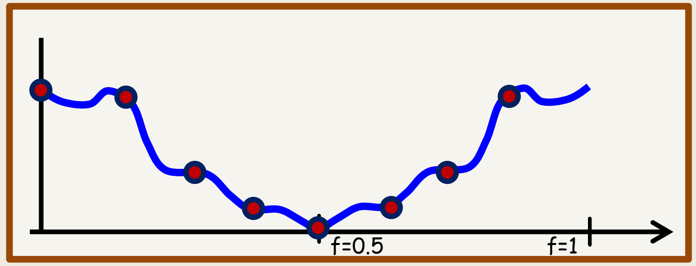

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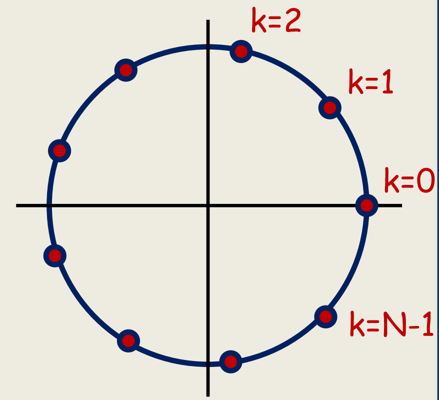

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If samples are representing X(f), then we should be able to get x(n) back from them

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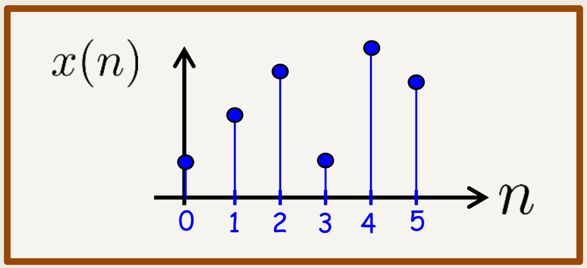
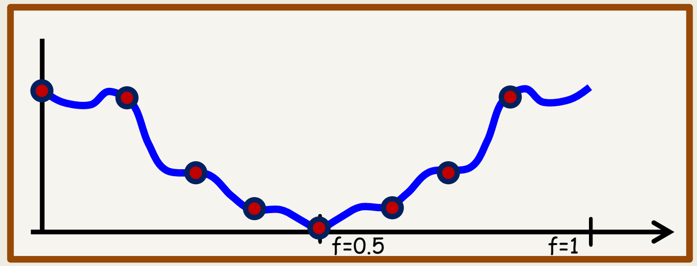
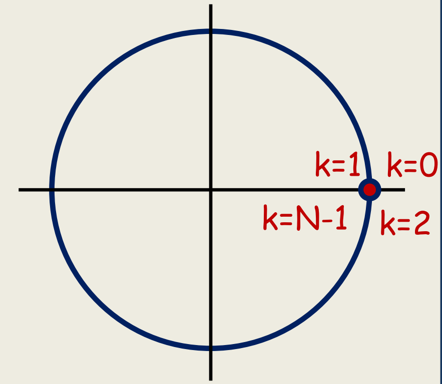


We don't assume N taps in x(n)



Assume that we take N samples of the DTFT



$$n = m$$

Let us try to invert using this formula

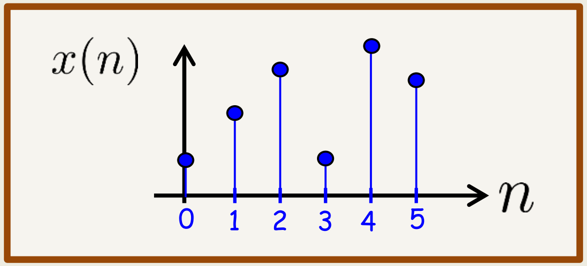
$$x(n) \overset{?}{=} \frac{1}{N}\cdot\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k} = \frac{1}{N}\cdot\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x(m)\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{m}{N}\cdot k}\cdot\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k}$$

$$X(k) = \sum_{m=-\infty}^{\infty} x(m)\mathrm{e}^{-i2\pi mk/N}$$

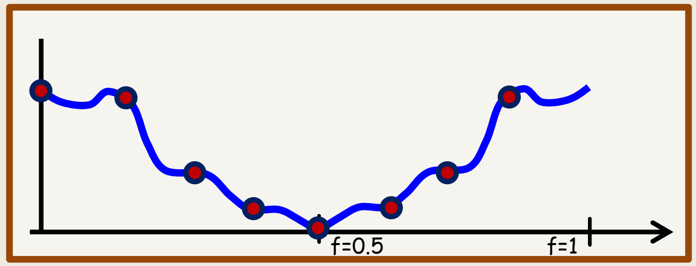
$$= \frac{1}{N}\cdot\sum_{m=-\infty}^{\infty} x(m)\underbrace{\sum_{k=0}^{N-1}\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n-m}{N}\cdot k}}_{= N}$$

If samples are representing X(f), then we should be able to get x(n) back from them

EITF75 Systems and Signals



We don't assume N taps in x(n)



Assume that we take N samples of the DTFT

Let us try to invert using this formula

$$x(n) \overset{?}{=} \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k} = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} x(m) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{m}{N} \cdot k} \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k}$$

$$X(k) = \sum_{m=-\infty}^{\infty} x(m) \mathrm{e}^{-i2\pi mk/N}$$

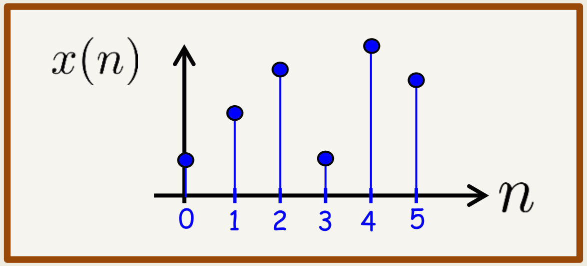
$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot \sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n-m}{N} \cdot k}$$

$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot N \cdot \delta(n - m \mod N)$$

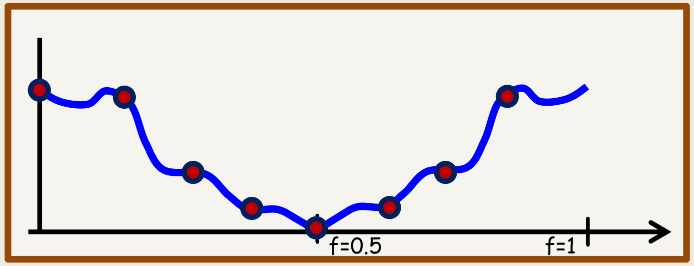
If samples are representing X(f), then we should be able to get x(n) back from them

Compact notation.

EITF75 Systems and Signals



We don't assume N taps in x(n)



Assume that we take N samples of the DTFT

In general $x(n) \neq \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k} = \frac{1}{N} \cdot \sum_{k=0}^{N-1} \sum_{m=-\infty}^{\infty} x(m) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{m}{N} \cdot k} \cdot \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k}$

$$X(k) = \sum_{m=-\infty}^{\infty} x(m) \mathrm{e}^{-i2\pi mk/N}$$

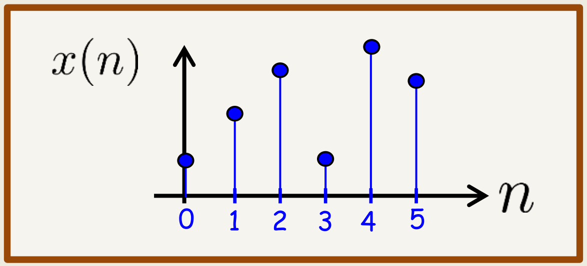
$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot \sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n-m}{N} \cdot k}$$

If samples are representing X(f), then we should be able to get x(n) back from them

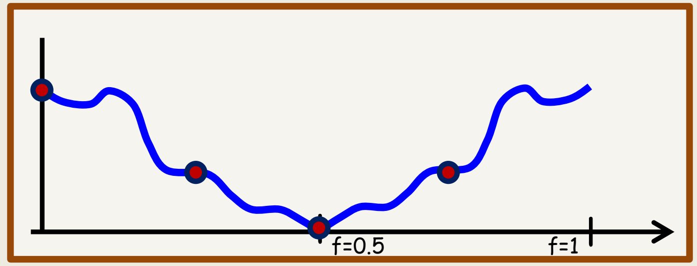
$$= \frac{1}{N} \cdot \sum_{m=-\infty}^{\infty} x(m) \cdot N \cdot \delta(n - m \mod N)$$

$$= \sum_{m=-\infty}^{\infty} x(n - mN)$$

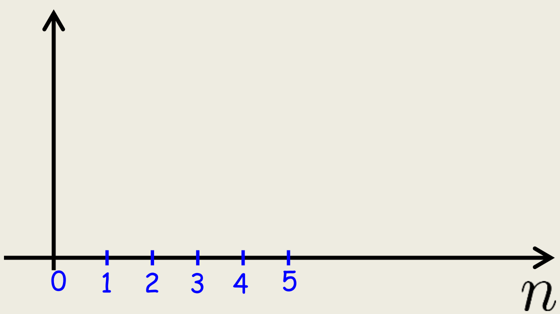
EITF75 Systems and Signals



There are 6 taps here



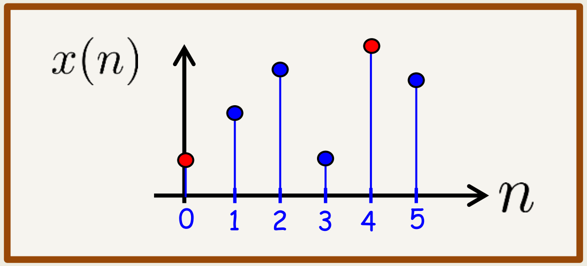
Assume that we take N=4 samples of the DTFT



Let us plot this

$$\frac{1}{N}\cdot\sum_{k=0}^{N-1} X(k)e^{j2\pi\cdot\frac{n}{N}\cdot k} = \sum_{m=-\infty}^{\infty} x(n-mN)$$

EITF75 Systems and Signals



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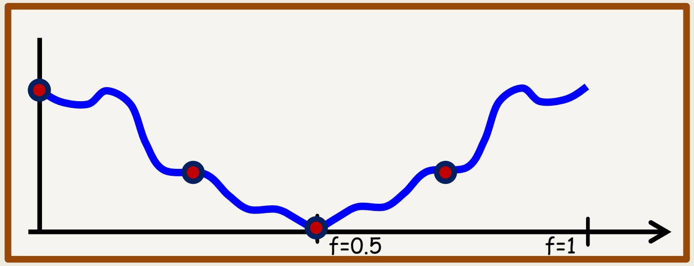
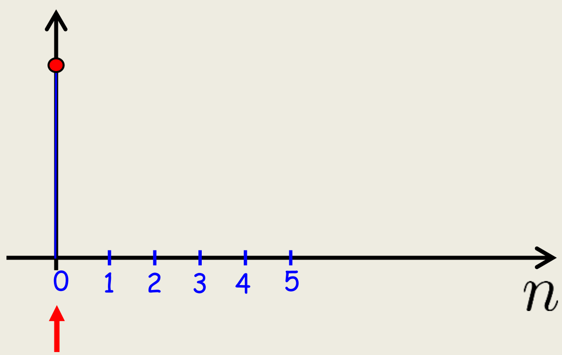
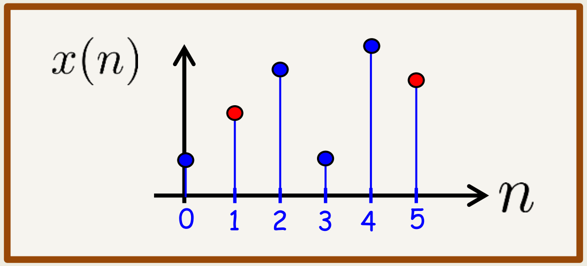


Assume that we take N=4 samples of the DTFT



$$\frac{1}{N}\cdot\sum_{k=0}^{N-1}X(k)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k} = \sum_{m=-\infty}^{\infty}x(n-mN) \qquad x(0)+x(4)$$

EITF75 Systems and Signals



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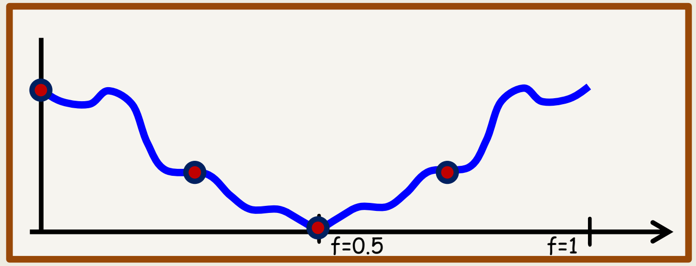
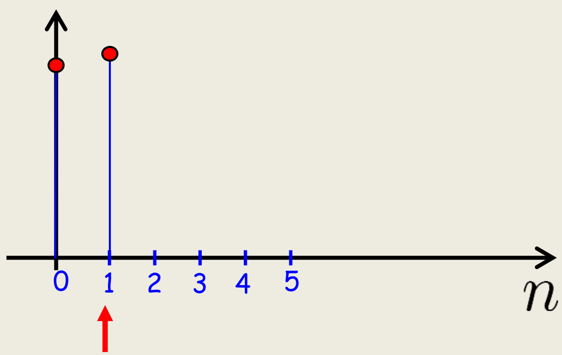
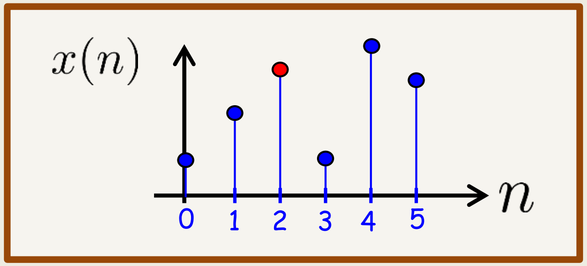


Assume that we take N=4 samples of the DTFT

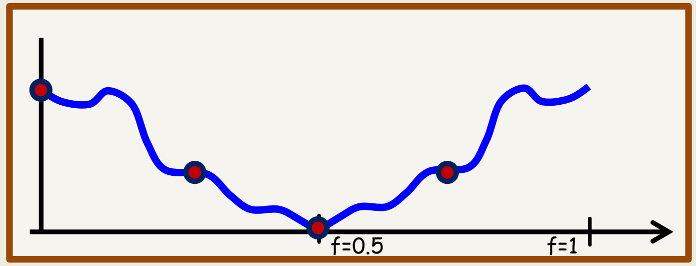


$$\frac{1}{N}\cdot\sum_{k=0}^{N-1}X(k)e^{j2\pi\cdot\frac{n}{N}\cdot k} = \sum_{m=-\infty}^{\infty}x(n-mN) \qquad x(1)+x(5)$$

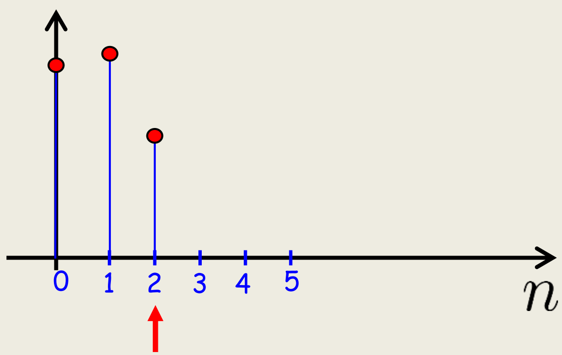
EITF75 Systems and Signals



There are 6 taps here



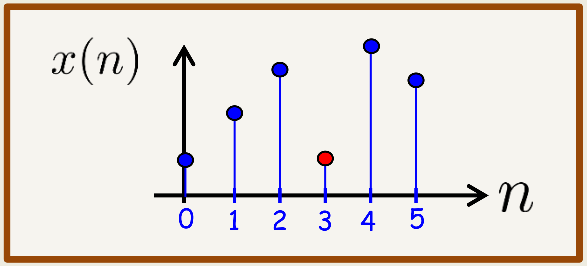
Assume that we take N=4 samples of the DTFT



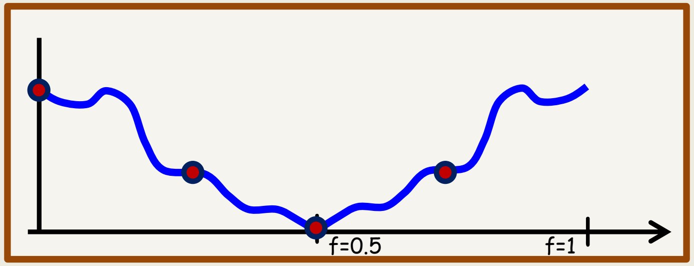
$$\frac{1}{N}\cdot\sum_{k=0}^{N-1}X(k)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k} = \sum_{m=-\infty}^{\infty}x(n-mN)$$

$$\cancel{x(-2)} + x(2) + \cancel{x(6)}$$

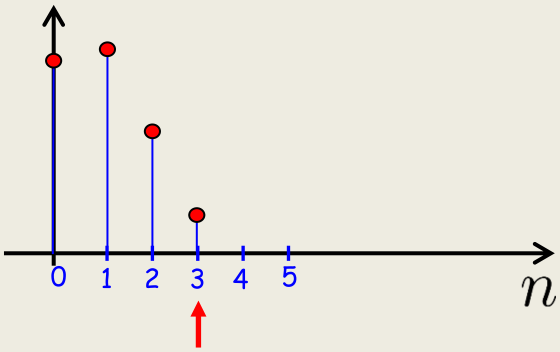
EITF75 Systems and Signals



There are 6 taps here



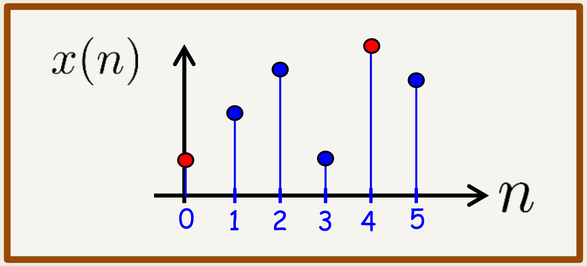
Assume that we take N=4 samples of the DTFT



$$\frac{1}{N}\cdot\sum_{k=0}^{N-1}X(k)e^{j2\pi\cdot\frac{n}{N}\cdot k}=\sum_{m=-\infty}^{\infty}x(n-mN)$$

$$x(3)$$

EITF75 Systems and Signals



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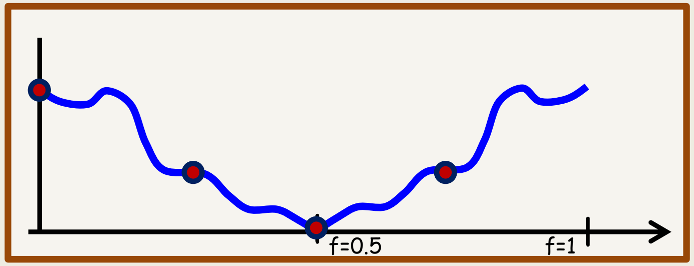
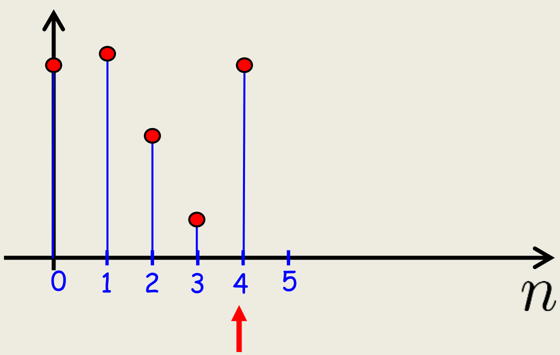
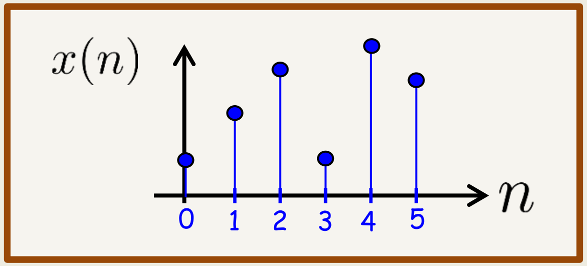


Assume that we take N=4 samples of the DTFT



$$\frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j2\pi \cdot \frac{n}{N} \cdot k} = \sum_{m=-\infty}^{\infty} x(n - mN) \qquad x(0) + x(4)$$

EITF75 Systems and Signals



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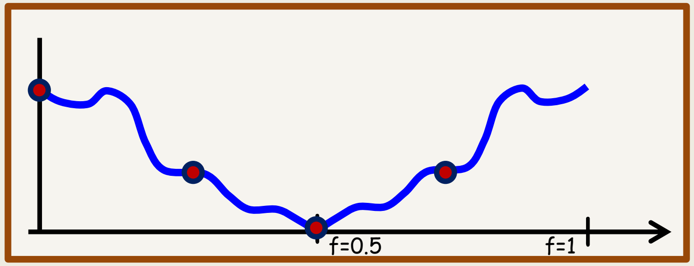
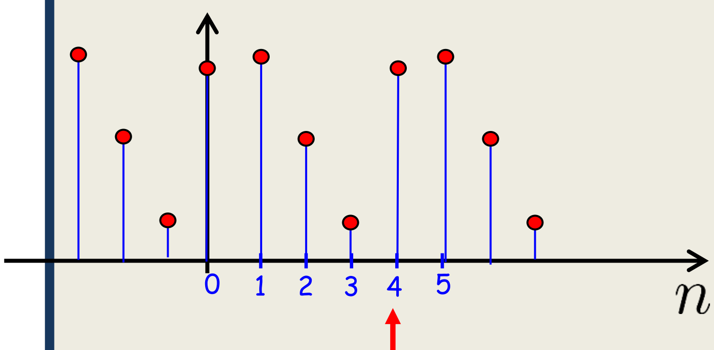


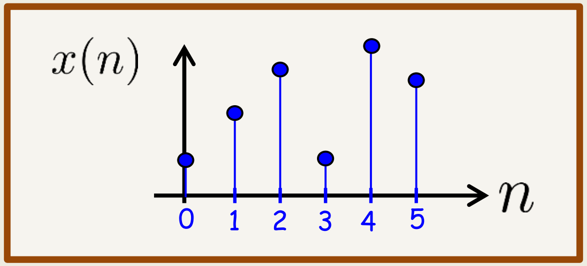
Assume that we take N=4 samples of the DTFT



Periodic sequnce

$$\frac{1}{N}\cdot\sum_{k=0}^{N-1} X(k)e^{j2\pi\cdot\frac{n}{N}\cdot k} = \sum_{m=-\infty}^{\infty} x(n-mN)$$

EITF75 Systems and Signals



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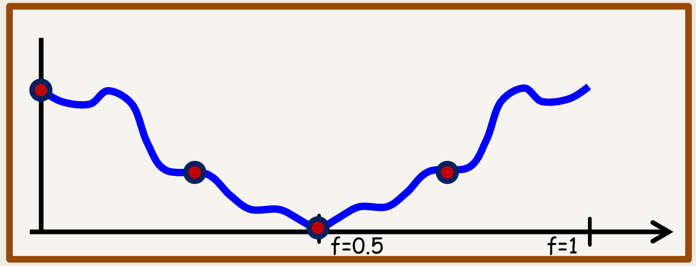
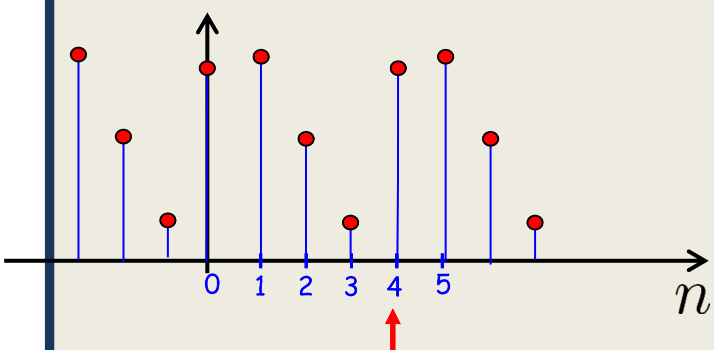


Assume that we take N=4 samples of the DTFT

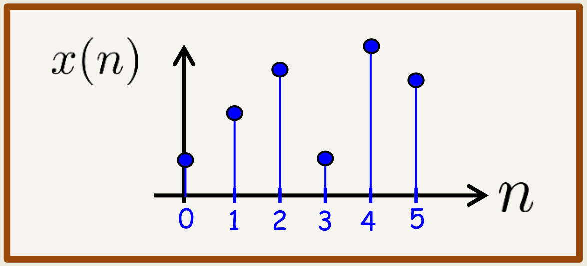


Periodic sequnce

Not equal to x(n)

$$x(n) \neq \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j 2\pi \cdot \frac{n}{N} \cdot k} = \sum_{m=-\infty}^{\infty} x(n - mN)$$

EITF75 Systems and Signals



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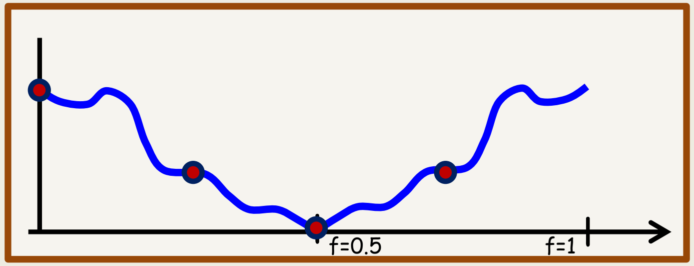
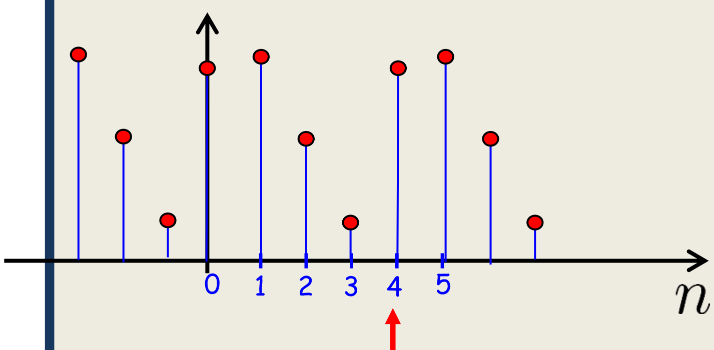


Assume that we take N=4 samples of the DTFT



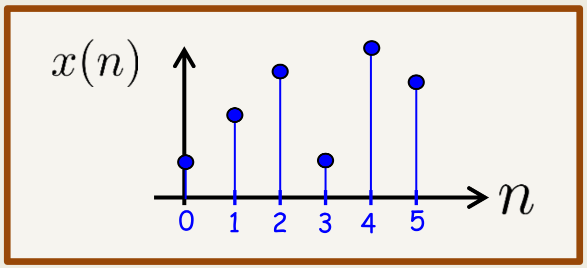
Periodic sequnce

Not equal to x(n)

Samples not enough to represent X(f) or X(n)

$$x(n) \neq \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j2\pi \cdot \frac{n}{N} \cdot k} = \sum_{m=-\infty}^{\infty} x(n - mN)$$

EITF75 Systems and Signals



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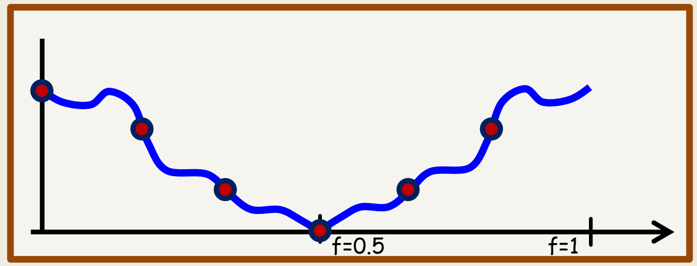
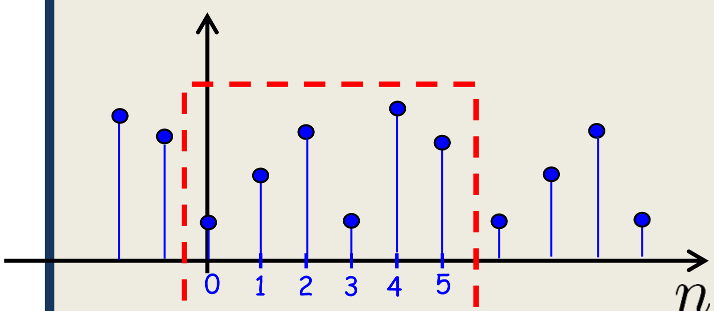


Assume that we take N=6 samples of the DTFT



Periodic sequnce

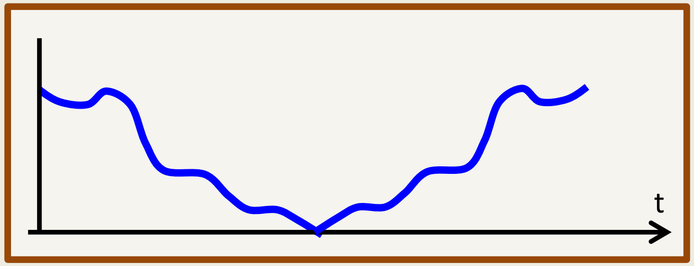
$$\frac{1}{N}\cdot\sum_{k=0}^{N-1}X(k)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{n}{N}\cdot k}=\sum_{m=-\infty}^{\infty}x(n-mN)$$

Samples enough to represent X(f) and X(n)

(m=0 only term)

EITF75 Systems and Signals

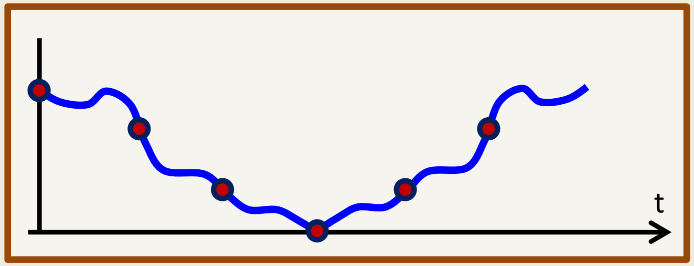
Not strange



Assume a time-signal

EITF75 Systems and Signals

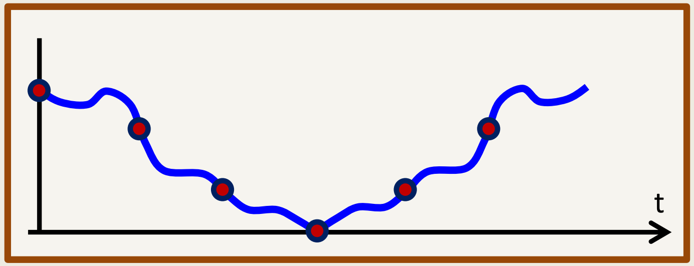
Not strange



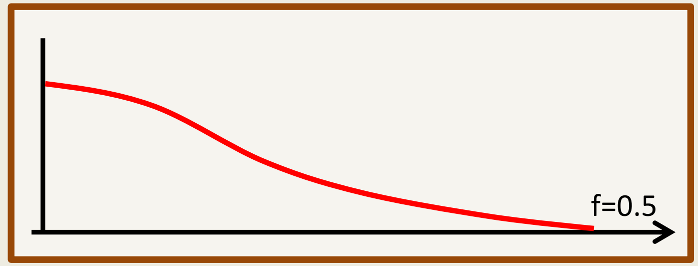
Sample it

EITF75 Systems and Signals

Not strange



Sample it

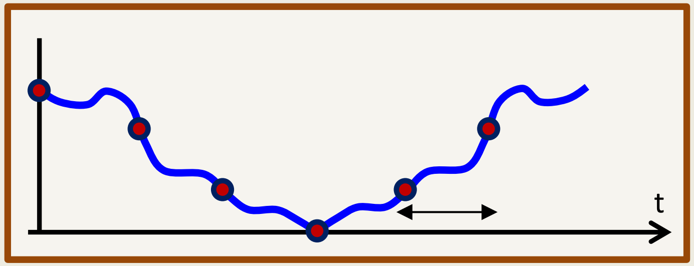


Compute the "other domain" representation from samples

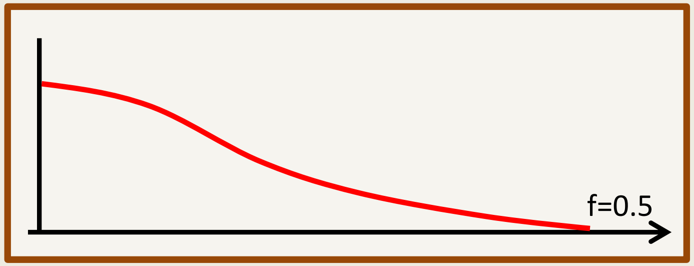
In this case, the Fourier domain

EITF75 Systems and Signals

Not strange



But if sample
spacing is too small...



Compute the "other domain"
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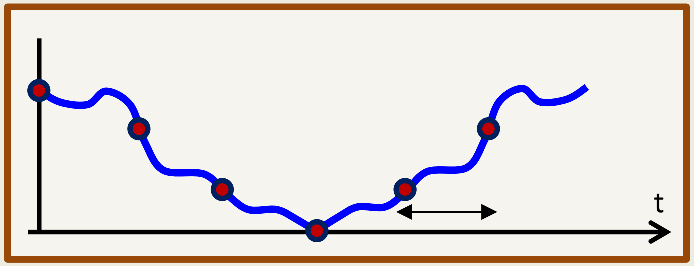
In this case, the Fourier domain

EITF75 Systems and Signals

Not strange



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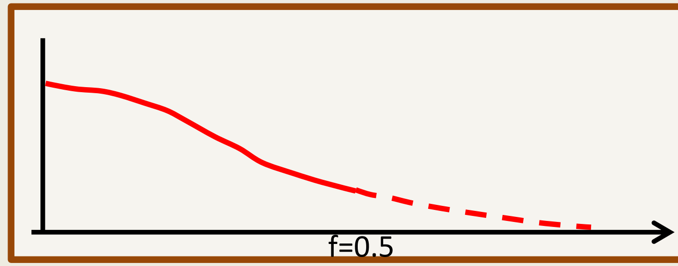
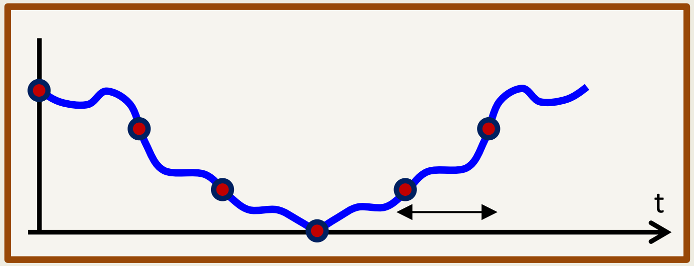


There is aliasing

EITF75 Systems and Signals

Not strange



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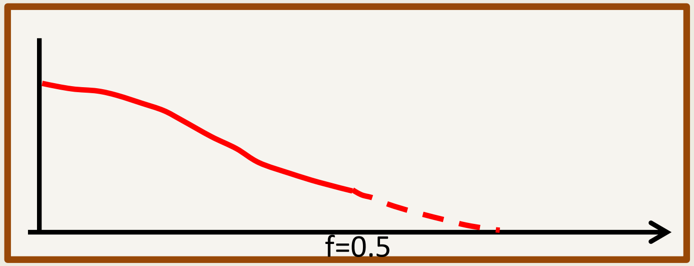

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EITF75 Systems and Signals

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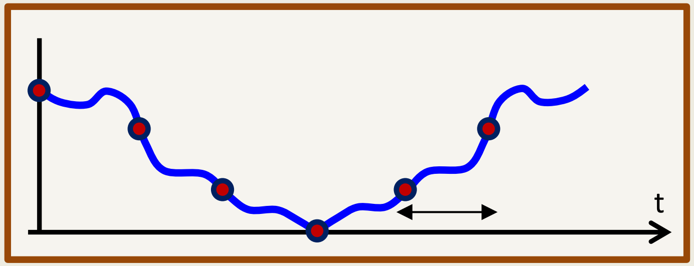

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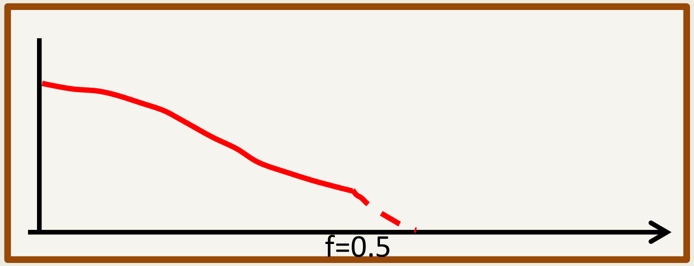

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EITF75 Systems and Signals

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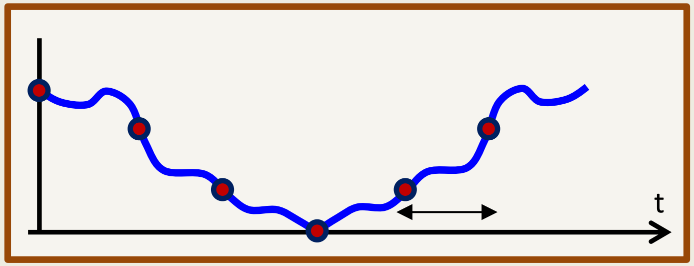

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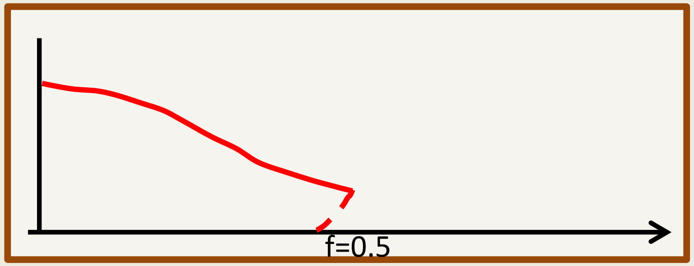

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EITF75 Systems and Signals

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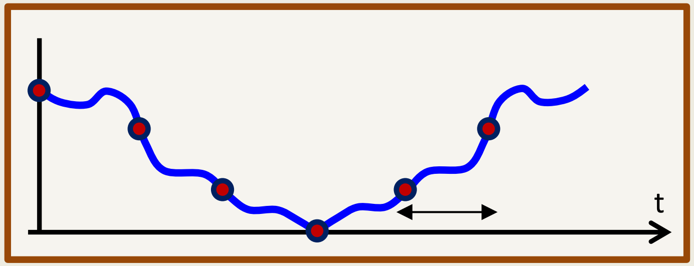

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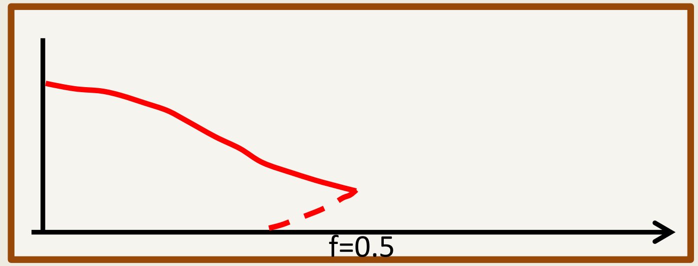

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EITF75 Systems and Signals

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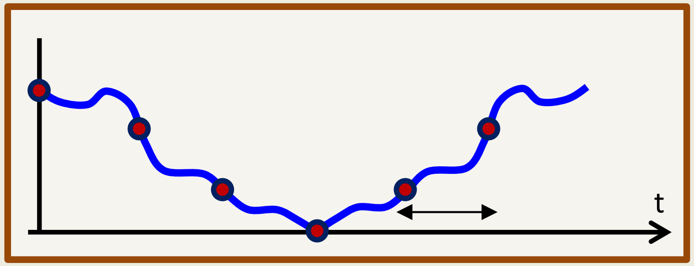

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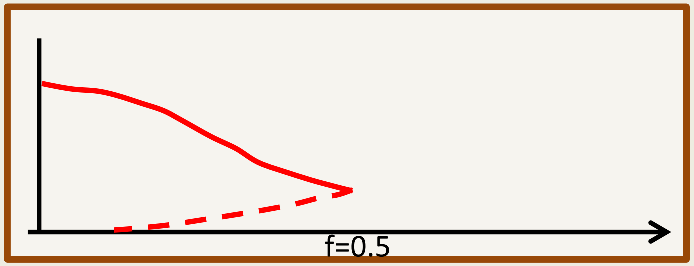
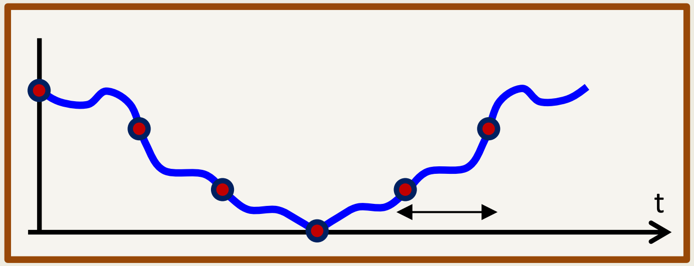


There is aliasing

EITF75 Systems and Signals

Not strange



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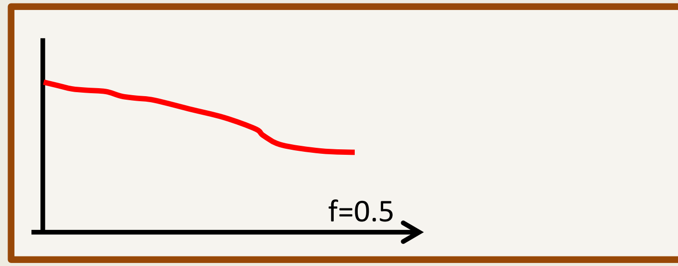

There is aliasing

Aliasing

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

No aliasing

$$X(f) = F_s X_a(fF_s)$$

EITF75 Systems and Signals

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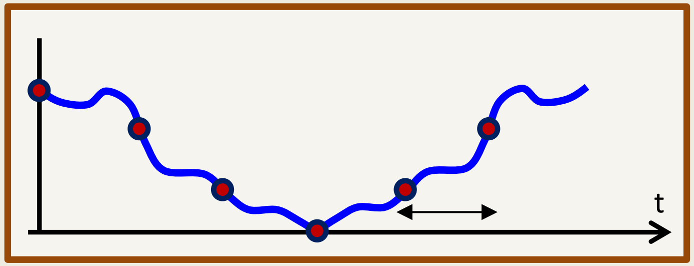

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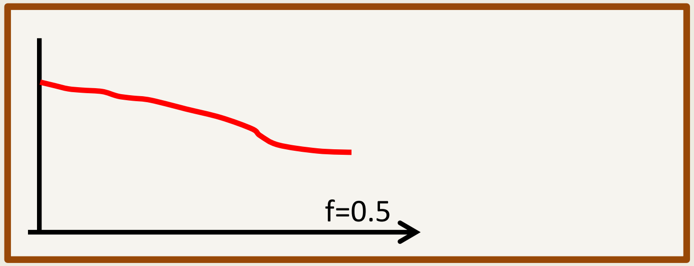

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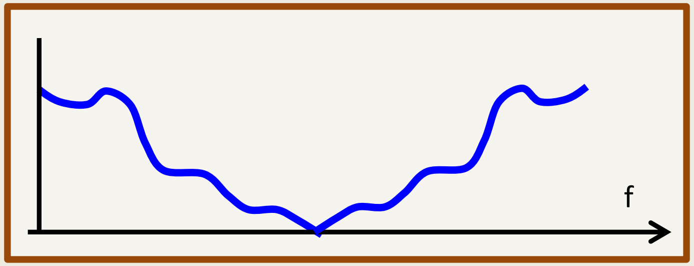
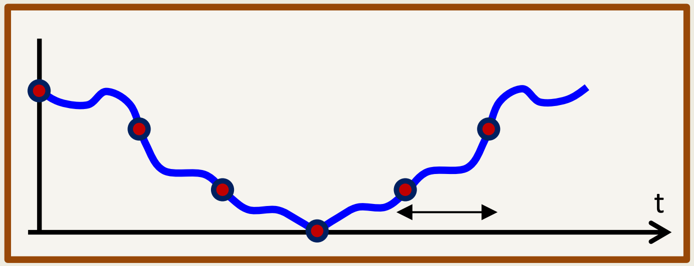


Assume a frequency-signal

EITF75 Systems and Signals

Not strange



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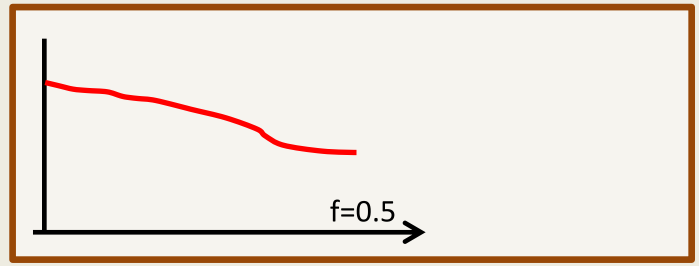

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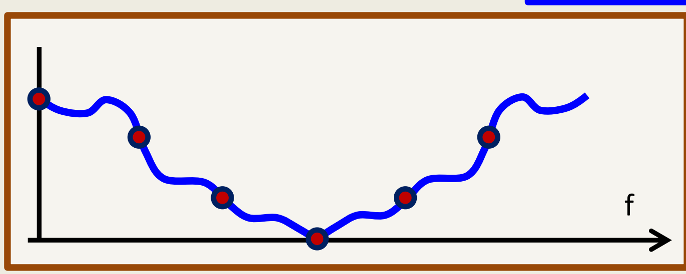
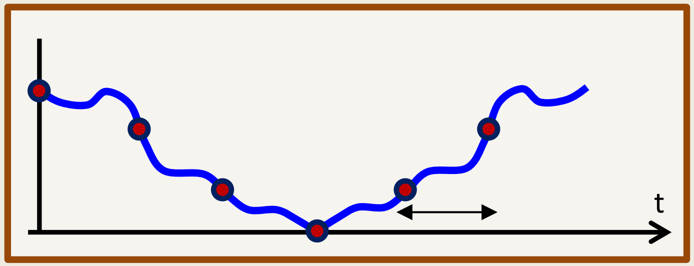


Sample it

EITF75 Systems and Signals

Not strange



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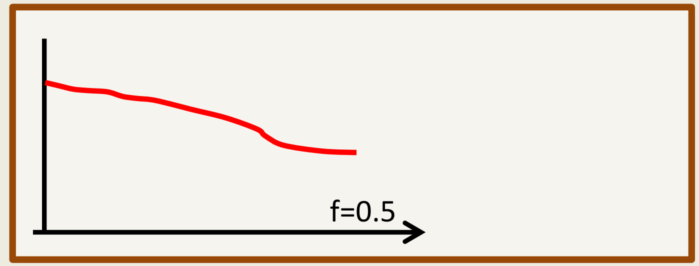

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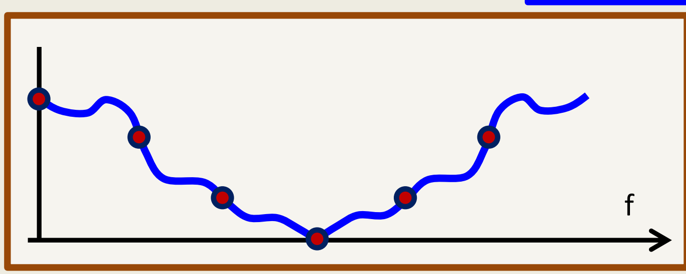

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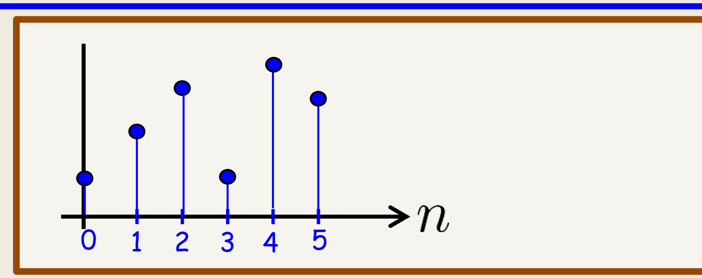
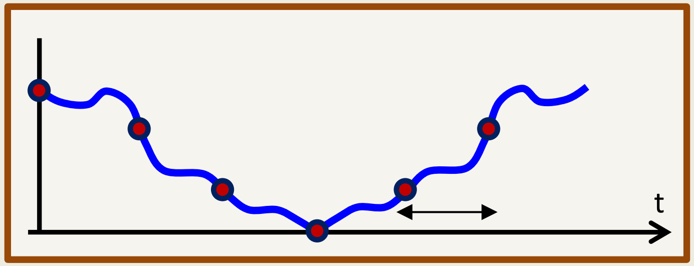


Compute the "other domain" representation
from samples. In this case, the time domain

EITF75 Systems and Signals

Not strange



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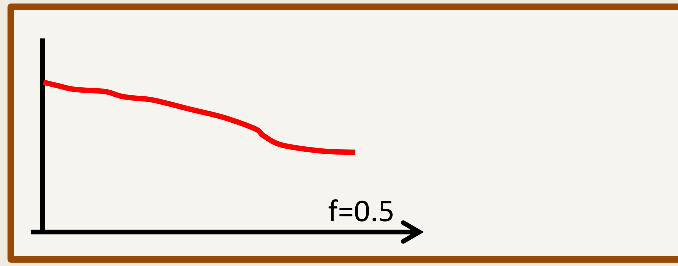

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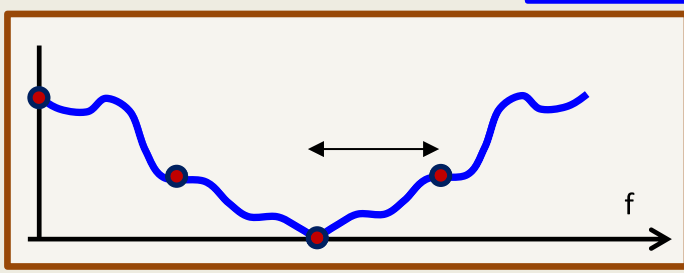

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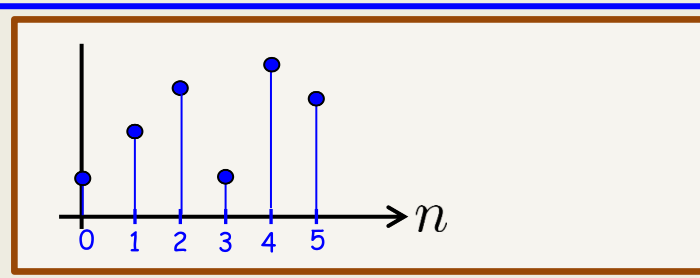

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EITF75 Systems and Signals

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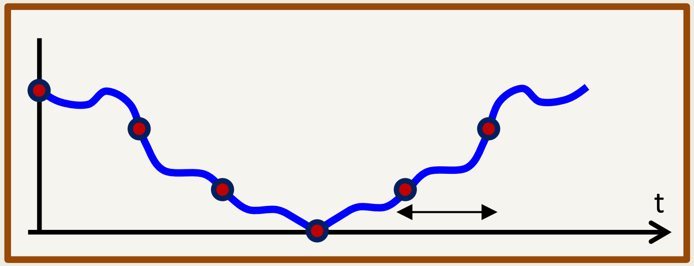

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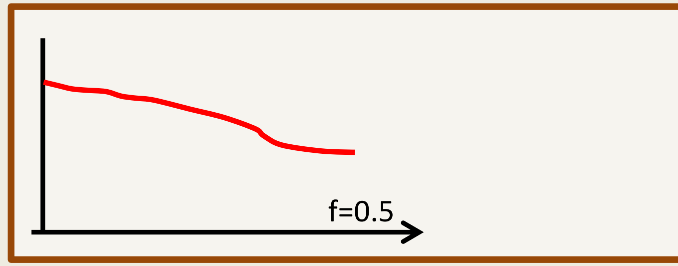

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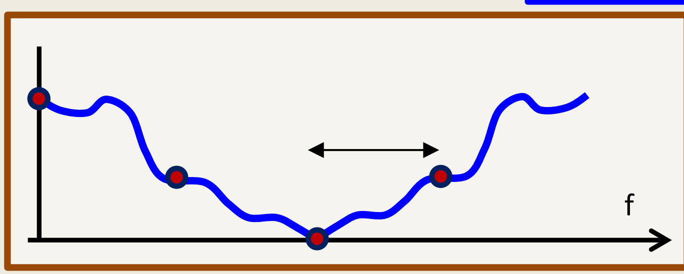

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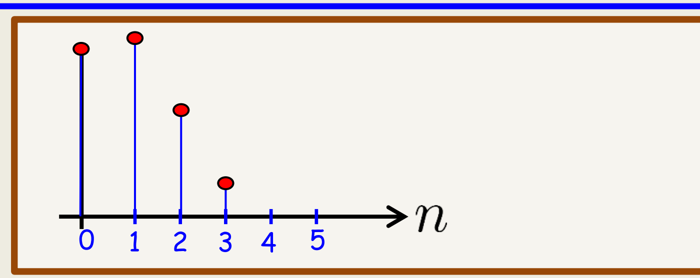

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EITF75 Systems and Signals

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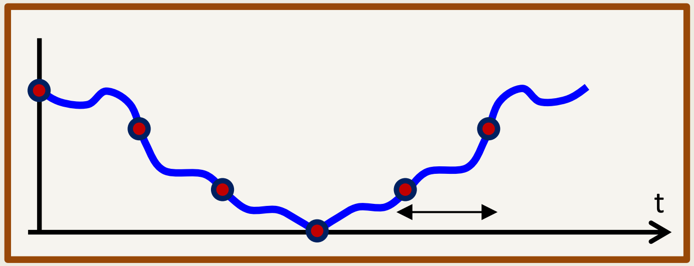

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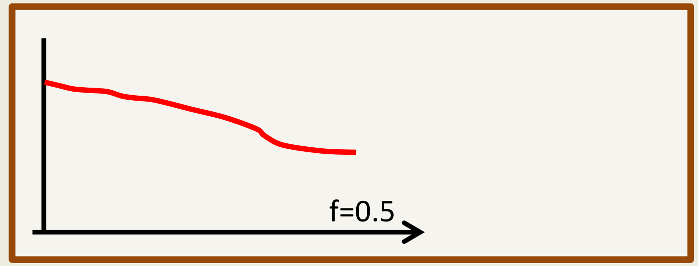

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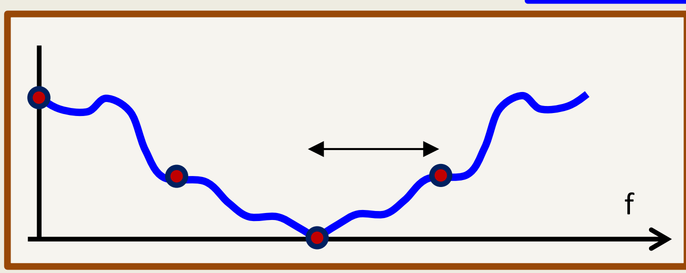

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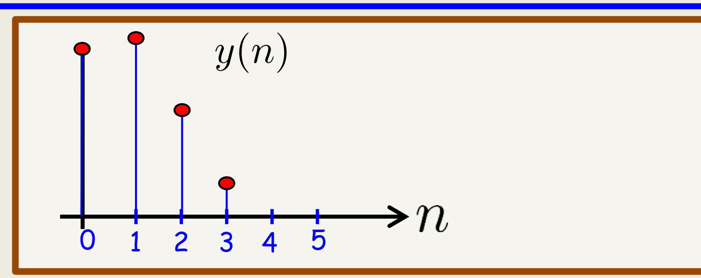

There is aliasing

Periodically extended

Aliasing $y(n) = \sum_{m=-\infty}^{\infty} x(n-mN)$

No aliasing $y(n) = x(n)$

EITF75 Systems and Signals

Formal definition

For a sequence $x(n)$ of arbitrary length, the N-point DFT is defined as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

and the inverse transform (IDFT) as

$$x_{\mathrm{IDFT}}(n) = \frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) e^{j2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } n = 0, 1, \ldots, N-1$$

Result

if the length of $x(n)$ is N, then

$x_{\mathrm{IDFT}}(n) = x(n)$ and $X_{\mathrm{DFT}}(k) = X(f \mid f = k/N)$

EITF75 Systems and Signals

Obvious Question

Why would anybody want to compute a DFT of less length than that of $x(n)$?

Result

if the length of $x(n)$ is N, then

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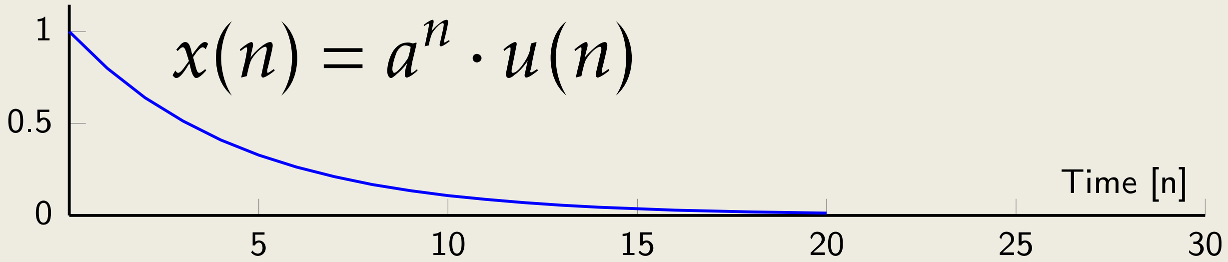
EITF75 Systems and Signals

Less obvious answer

Signals with infinite time-support requires this



$$x(n) = a^n \cdot u(n)$$

Obvious Question

Why would anybody want to compute a DFT of less length than that of $x(n)$?

Result

if the length of $x(n)$ is N, then

$$x_{\mathrm{IDFT}}(n) = x(n) \quad \text{and} \quad X_{\mathrm{DFT}}(k) = X(f \mid f = k/N)$$

EITF75 Systems and Signals

Less obvious answer

Signals with infinite time-support requires this

$$x(n) = a^n \cdot u(n)$$

We have two options, a computer has one

1. Compute the N-point DFT according to

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

2. Compute $X(f)$, and sample it to obtain $X(k)$

(computer cannot do 2 if it just encounters a signal. 2 requires us to know its math formula)

EITF75 Systems and Signals

Less obvious answer

Signals with infinite time-support requires this

$$x(n) = a^n \cdot u(n)$$

Let us invert 1&2, and compare

$$\frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k}$$

1. Compute the N-point DFT as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

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EITF75 Systems and Signals

Less obvious answer

Signals with infinite time-support requires this

$$x(n) = a^n \cdot u(n)$$

Inversion of 1: we know the result

$a^n \cdot u(n) \quad n = 0, 1, \ldots, N-1$

Let us invert 1&2, and compare

$$\frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k}$$

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2. Compute $X(f)$, and sample it to obtain $X(k)$

EITF75 Systems and Signals

Less obvious answer

Signals with infinite time-support requires this

$$x(n) = a^n \cdot u(n)$$

Inversion of 2: we know the result

$$\sum_{m=-\infty}^{\infty} x(n - mN) \quad n = 0, 1, \ldots, N-1$$

$$\frac{1}{N} \cdot \sum_{k=0}^{N-1} X(k) \mathrm{e}^{\mathrm{j}2\pi \cdot \frac{n}{N} \cdot k}$$

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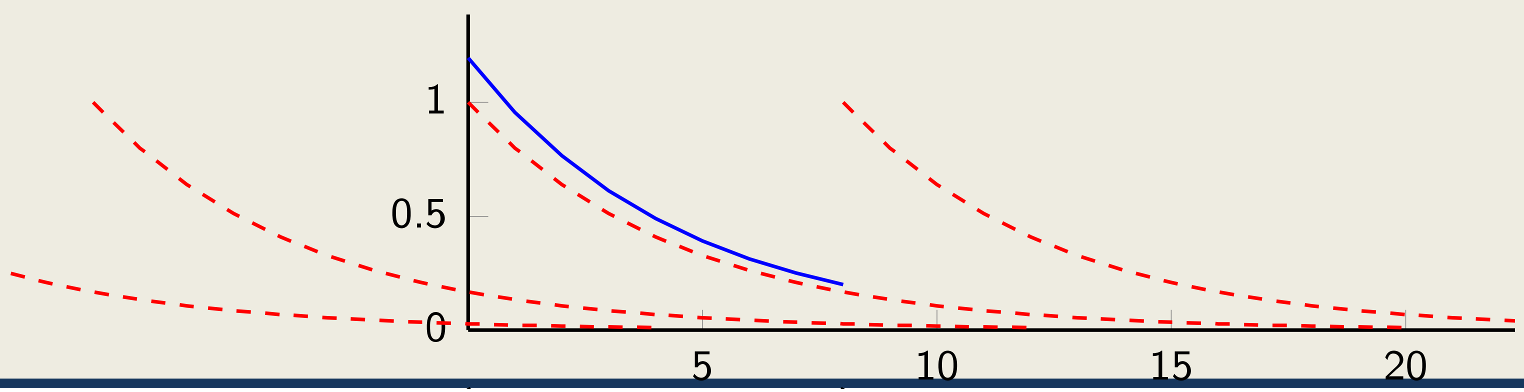

EITF75 Systems and Signals

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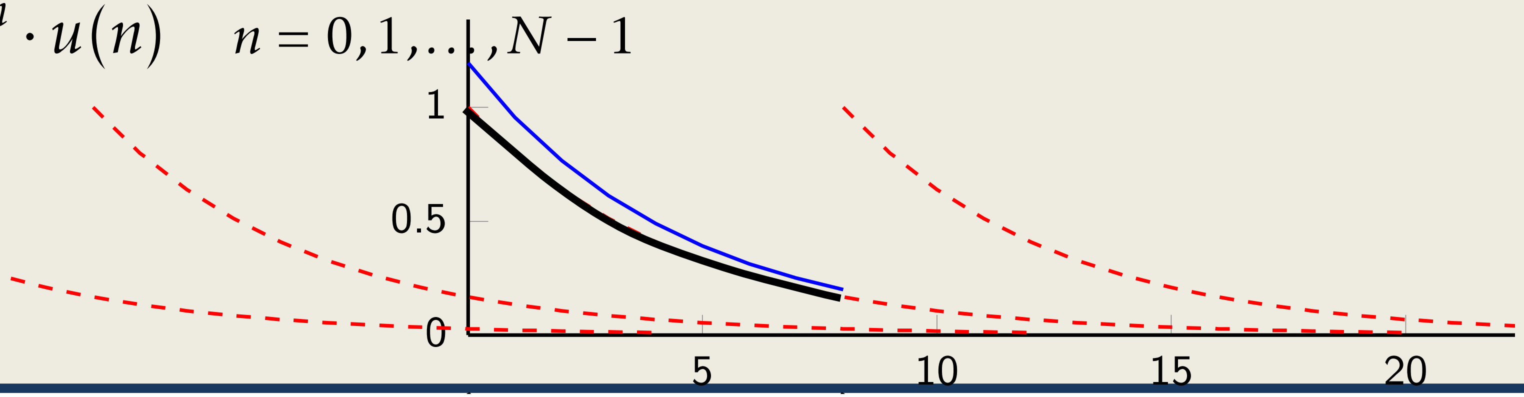

EIT F75 Systems and Signals

Computational complexity

DFT defined as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Number of operations needed:

Computational complexity

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Number of operations needed:

N values $X_{\mathrm{DFT}}(k)$ to be computed

EITF75 Systems and Signals

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Number of operations needed:

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Each value requires N multiplications $x(n) \cdot \mathrm{e}^{-j2\pi kn/N}$

EITF75 Systems and Signals

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Number of operations needed:

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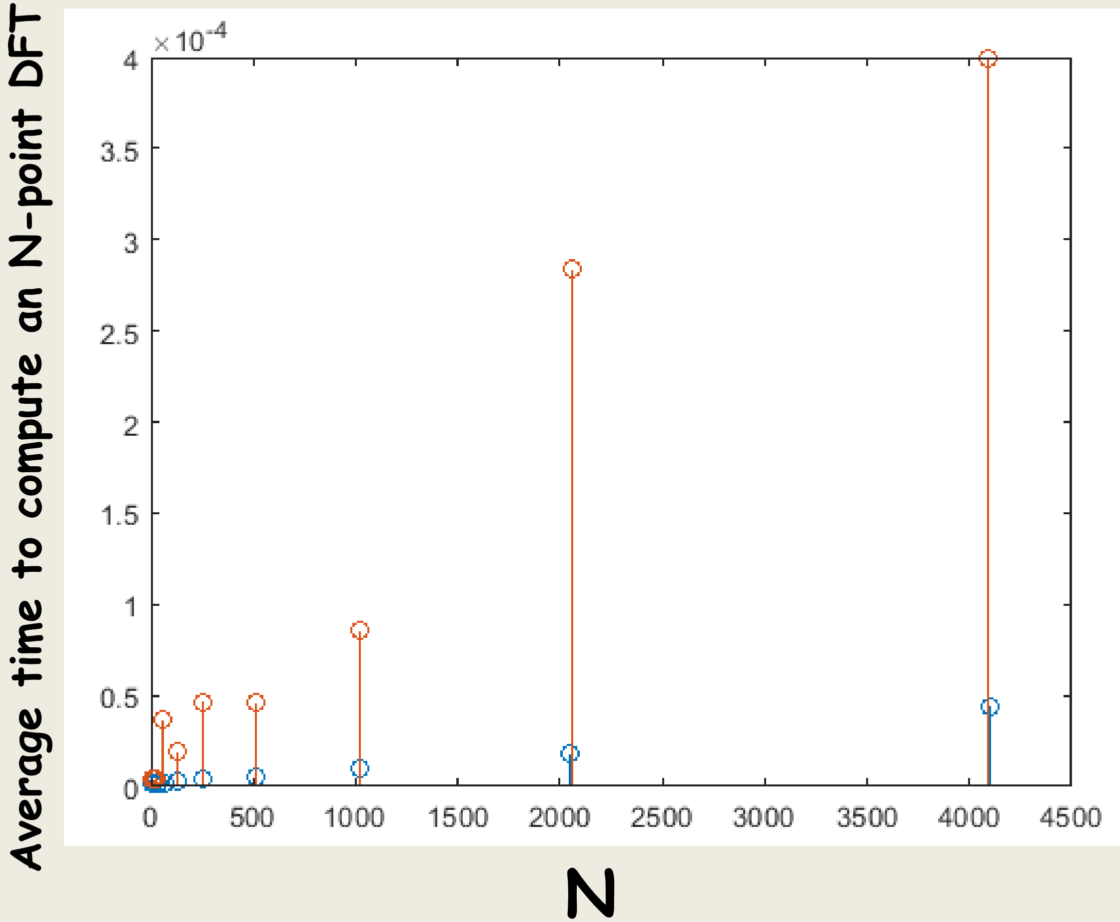
Total complexity N^2

EITF75 Systems and Signals

Computational complexity

Test in Matlab

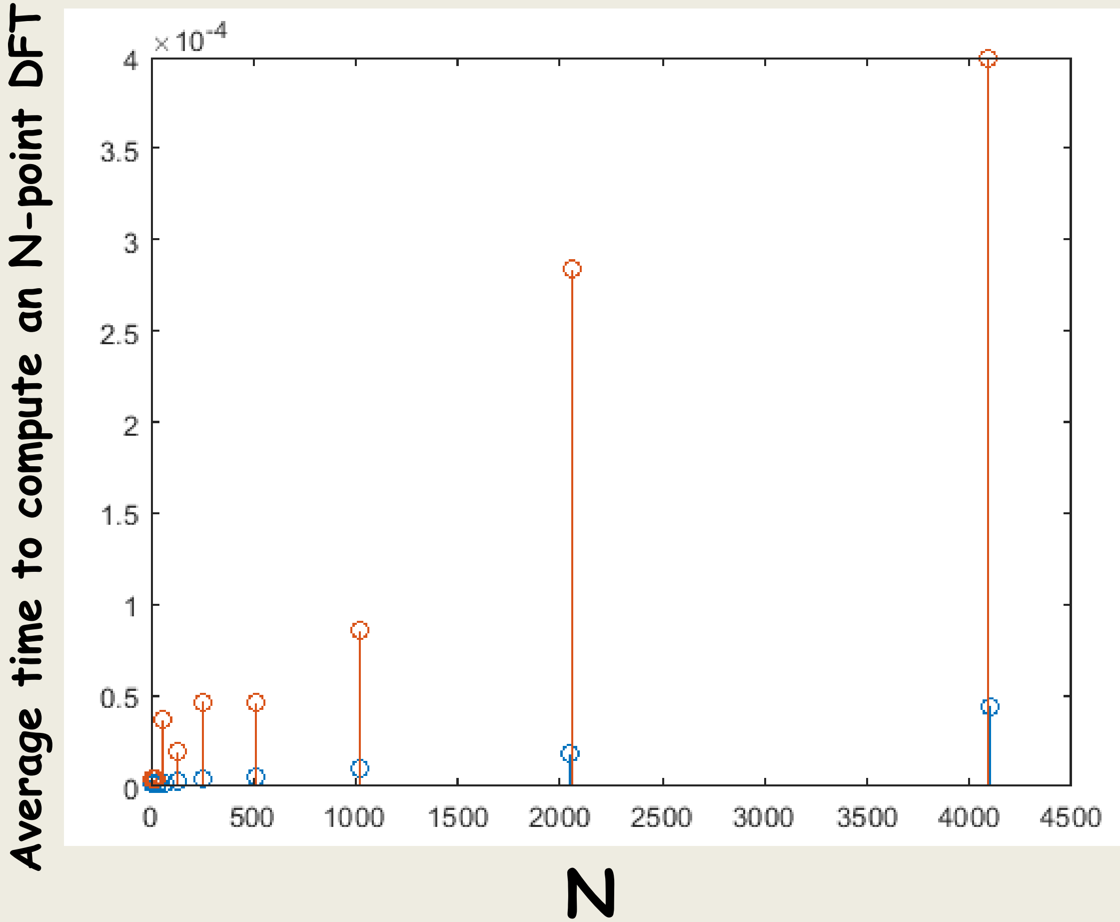


EITF75 Systems and Signals

Computational complexity

Test in Matlab

We see that for some values of N, much less time is needed



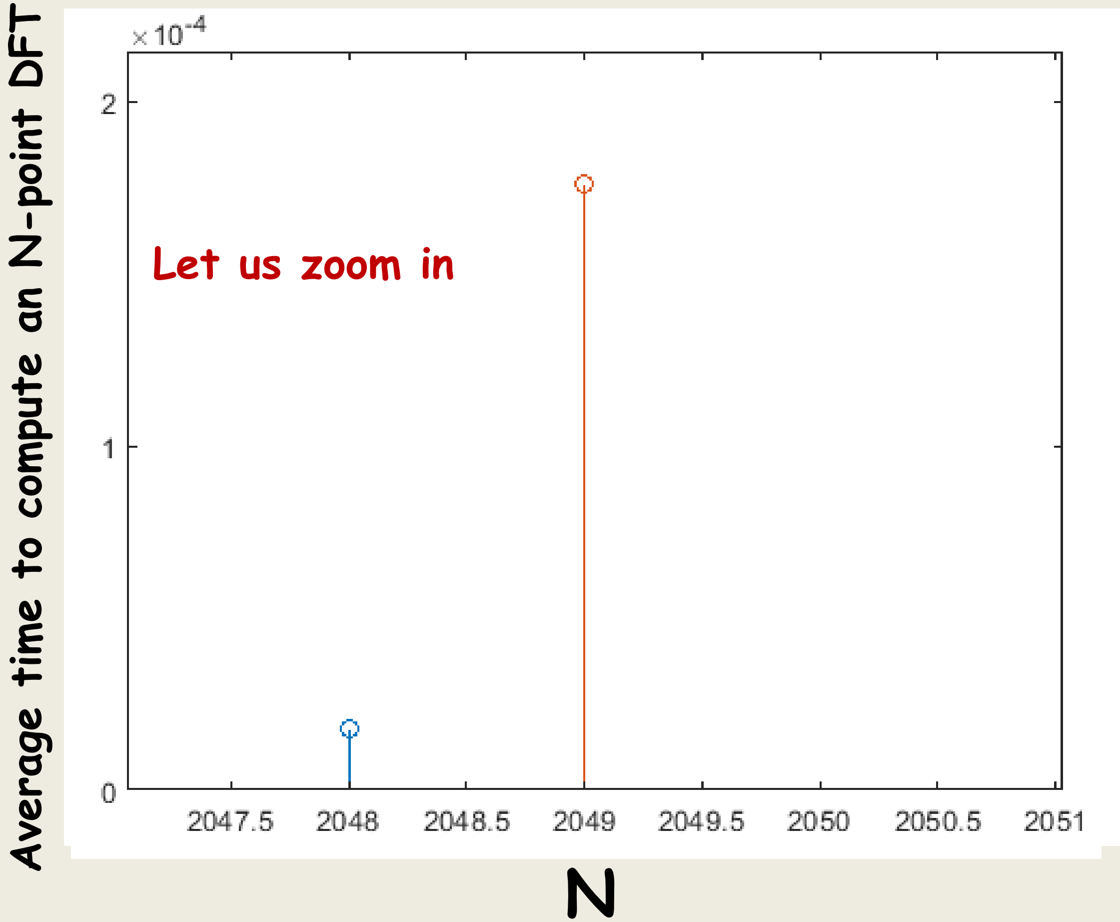
EITF75 Systems and Signals

Computational complexity

Test in Matlab

2048 is 2^{11}

2049 not power of 2



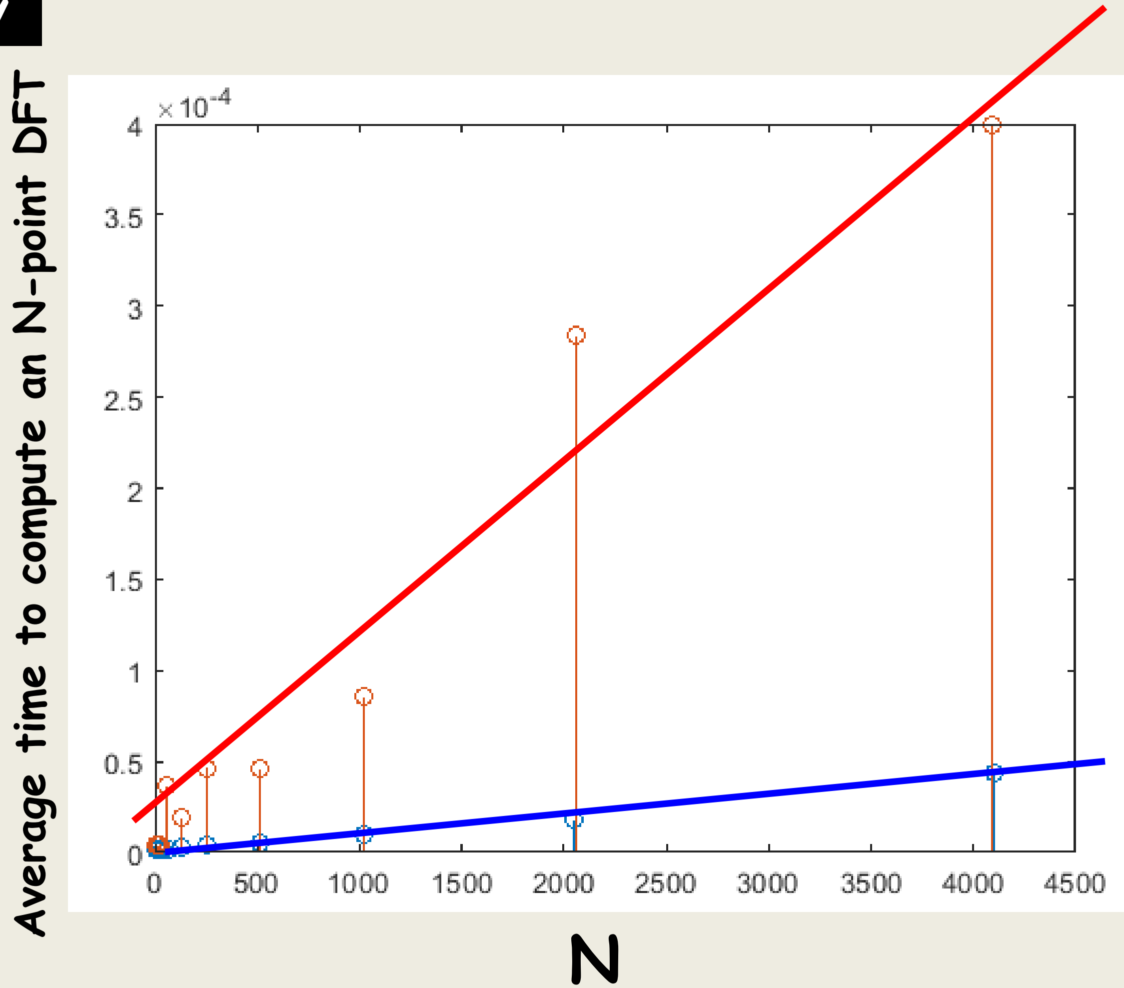
EITF75 Systems and Signals

Computational complexity

Test in Matlab

2048 is 2^{11}

Significant speed-up possible for $N=2^k$



EITF75 Systems and Signals

Computational complexity

FFT not included in course, but good to know about

Test in Matlab

Fast Fourier transform (FFT)

If $N=2^k$, then $N \log_2(N)$ complexity to compute

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Made possible by some algebraic manipulations and tricks.

Cooley and Tukey 1965

Method known to, and used by, Gauss in 1805

EITF75 Systems and Signals

Computational complexity

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Made possible by some algebraic manipulations and tricks.

The importance of the FFT cannot be underestimated. WIFI and 4G, etc could not been implemented without the FFT

For a computer,

1. It can avoid the continuous DTFT
2. It can compute the DFT extremely fast

EITF75 Systems and Signals

Properties

For DTFTs, we have

$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0}$$

Still true ? I.e.

$$x(n) \star y(n) \leftrightarrow X(k)Y(k)$$

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EITF75 Systems and Signals

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Assume length N sequences

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EITF75 Systems and Signals

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Assume length N sequences. Follows that DFTs also length N

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EITF75 Systems and Signals

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But this is length 2N-1

$$x(n) \star y(n) \leftrightarrow X(k)Y(k)$$

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Properties

For DTFTs, we have

$$\boxed{\begin{array}{l} x(n) \star y(n) \leftrightarrow X(f)Y(f) \\ x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0} \end{array}}$$

Still true ? I.e.

Assume length N sequences. Follows that DFTs also length N
But this is length 2N-1. So its DFT must be length 2N-1

$$\boxed{\begin{array}{l} \cancel{x(n) \star y(n) \leftrightarrow X(k)Y(k)} \\ x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(k)\mathrm{e}^{-i2\pi k n_0/N} \end{array}}$$

EITF75 Systems and Signals

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Assume length N sequences. Follows that DFTs also length N
But this is length 2N-1. So its DFT must be length 2N-1

Assume length N. Ex {1 2 3 4}

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(k)\mathrm{e}^{-i2\pi k n_0/N}$$

EITF75 Systems and Signals

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Assume length N. Ex {1 2 3 4}

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Also length N. Becomes {10 2+2i 2 2-2i}

EITF75 Systems and Signals

Properties

For DTFTs, we have

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$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0}$$

Still true ? I.e.

Assume length N sequences. Follows that DFTs also length N
But this is length 2N-1. So its DFT must be length 2N-1

Assume length N. Ex {1 2 3 4}

Length $N+n_0$. Ex {0 1 2 3 4}

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(k)\mathrm{e}^{-i2\pi k n_0/N}$$

Also length N. Becomes {10 2+2i 2 2-2i}

EITF75 Systems and Signals

Properties

For DTFTs, we have

$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

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Also length N. Becomes {10 2+2i 2 2-2i}

Still length N_0

Makes no sense…

EITF75 Systems and Signals

Properties

For DTFTs, we have

$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

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Still true ? NO

EITF75 Systems and Signals

Properties

For DTFTs, we have

$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0}$$

For DFTs, we have

$$x_1(n) \otimes x_2(n) \leftrightarrow X(k)Y(k)$$

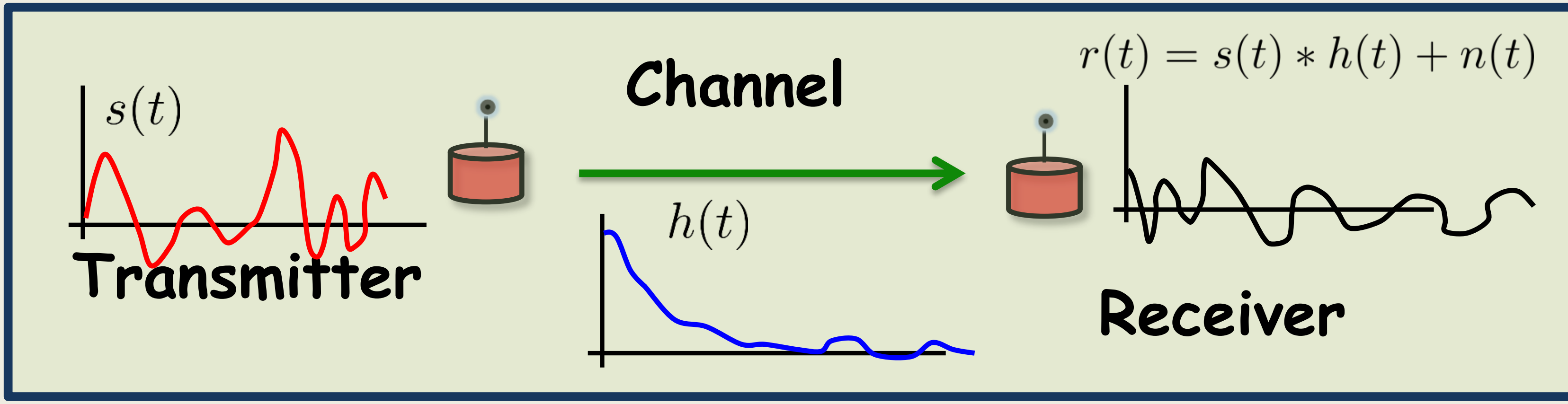
$$x(n-n_0 \bmod N) \leftrightarrow X(k)\mathrm{e}^{-i2\pi k n_0/N}$$

where $x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k)x_2(n-k \bmod N)$

Circular convolution

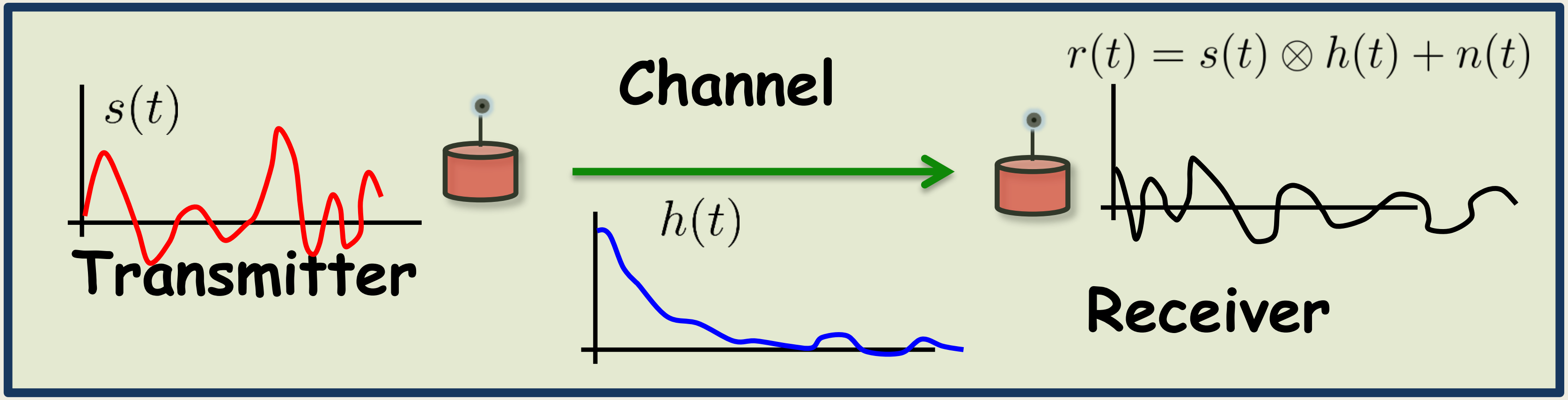
EITF75 Systems and Signals

Interlude



EITF75 Systems and Signals

Interlude



In WIFI, 4G, 5G etc etc, we give up 5-10% of the possible data rate in order to turn channel convolution into a circular convolution.
Then we can use the FFT and save complexity

EITF75 Systems and Signals

Example

Circular convolution

$$x(n) = \{\underline{1}\ \ 2\ \ 3\ \ 4\}$$

$$h(n) = \{\underline{2}\ \ 2\ \ 1\ \ 1\}$$

$$y_C(n) = x(n) \otimes h(n)$$

EITF75 Systems and Signals

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$$x(n) = \{\underline{1}\ 2\ 3\ 4\}$$

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Method 1

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$					
$x(k)$	2	3	4	$\underline{1}$	2	3	4	1	2	3
$y_C(k)$				$\underline{15}$	13	15	17			

Method 2

	2	3	4	$\underline{1}$	2	3	4
				$\underline{15}$	13	15	17
$\underline{2}$	4	6	8	$\underline{2}$	4	6	8
2	4	6	8	2	4	6	8
1	2	3	4	1	2	3	4
1	2	3	4	1	2	3	4

Method 3

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$									
$x(k)$	0	0	0	$\underline{1}$	2	3	4	0	0	0	0	1	2	3
$y_L(k)$				$\underline{2}$	6	11	17	13	7	4	0			

EITF75 Systems and Signals

Example

Circular convolution

Method 1

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$					
$x(k)$	2	3	4	$\underline{1}$	2	3	4	1	2	3
$y_C(k)$				$\underline{15}$	13	15	17			

Method 4

```
>> x = [1 2 3 4];
>> h = [2 2 1 1];
>> yc = ifft(fft(x).*fft(h))
yc =
    15      13      15      17
```

Method 2

	2	3	4	$\underline{1}$	2	3	4
				$\underline{15}$	13	15	17
$\underline{2}$	4	6	8	$\underline{2}$	4	6	8
2	4	6	8	2	4	6	8
1	2	3	4	1	2	3	4
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Method 3

$h(k)$	1	1	2	$\underline{2}$	$\rightarrow$									
$x(k)$	0	0	0	$\underline{1}$	2	3	4	0	0	0	0	1	2	3
$y_L(k)$				$\underline{2}$	6	11	17	13	7	4	0			

EITF75 Systems and Signals

Example

Linear convolution computed via DFTs

Given: Two length N sequences, $x(n)$, $y(n)$

Task: Compute their linear convolution by
using DFT and its inverse IDFT

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>> x=[1 2 3 4];
>> y=[2 2 1 1];
>> yL=conv(x,y)
yL =
     2     6    11    17    13     7     4
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This is the result, But not computed via DFT

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```
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```

This fails

EITF75 Systems and Signals

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>> y=[2 2 1 1];
>> yL=conv(x,y)
yL =
     2     6    11    17    13     7     4
```

This is the result, But not computed via DFT

```
>> xp=[1 2 3 4 0 0 0 0];
>> yp=[2 2 1 1 0 0 0 0];
>> yL=ifft(fft(xp).*fft(yp))
yL =
    2.0000    6.0000   11.0000   17.0000   13.0000    7.0000    4.0000   -0.0000
```

Still a circular convolution carried out, but due to zero-padding, it behaves linear.

EITF75 Systems and Signals

More examples

$$x(n) = \{1 \ 1 \ 1 \ 1 \ 1 \ 1\}$$

Compute 16-point DFT (N=16) $X(k), \ k = 0, 1, \cdots, 15$

More examples

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EITF75 Systems and Signals

More examples

$$x(n) = \{1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1\}$$

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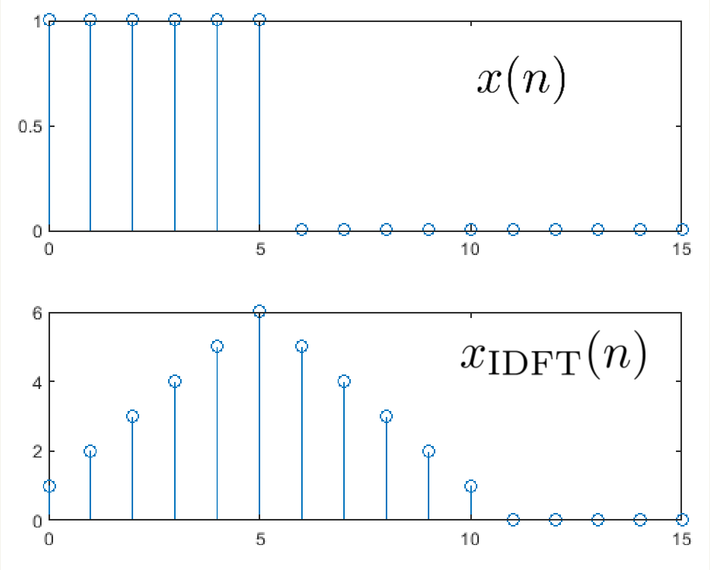

EITF75 Systems and Signals

More examples

Repeat for $x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\}$

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\}$$

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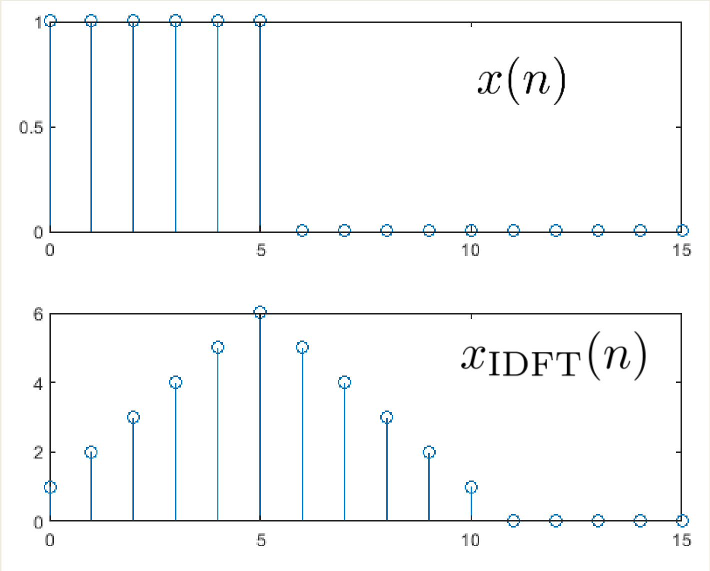

EITF75 Systems and Signals

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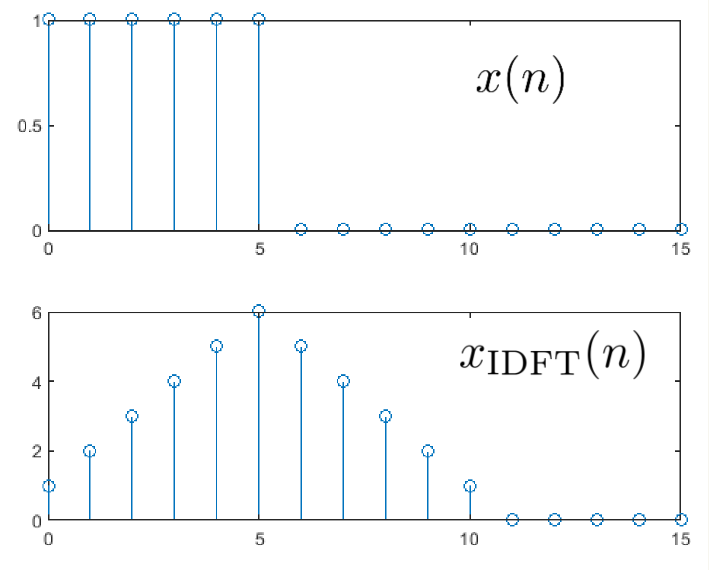
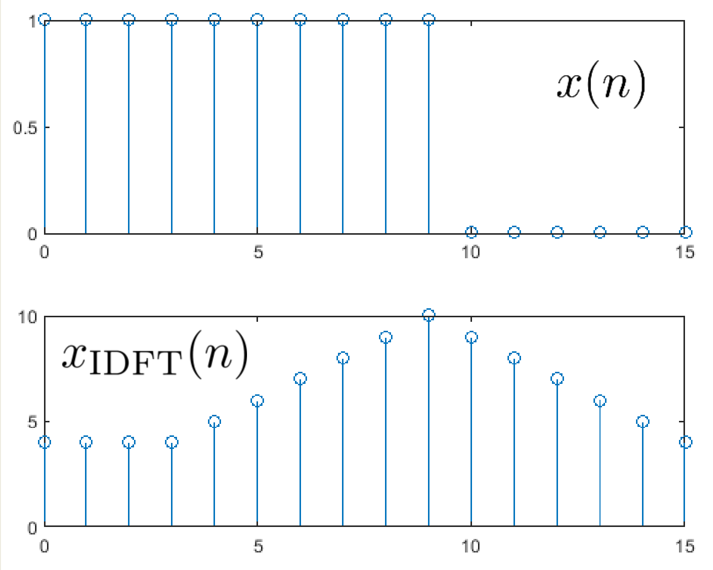


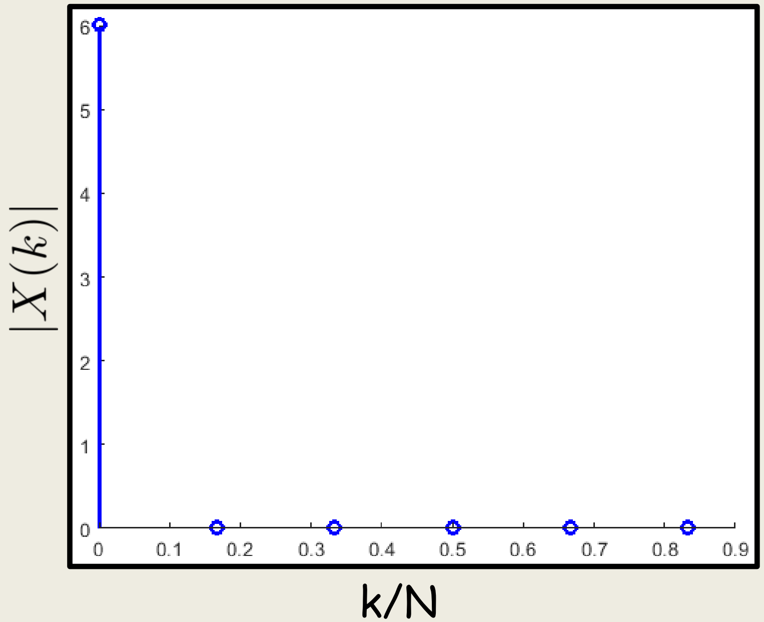


EITF75 Systems and Signals

More examples

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\}$$

Compute DFT (N=6)



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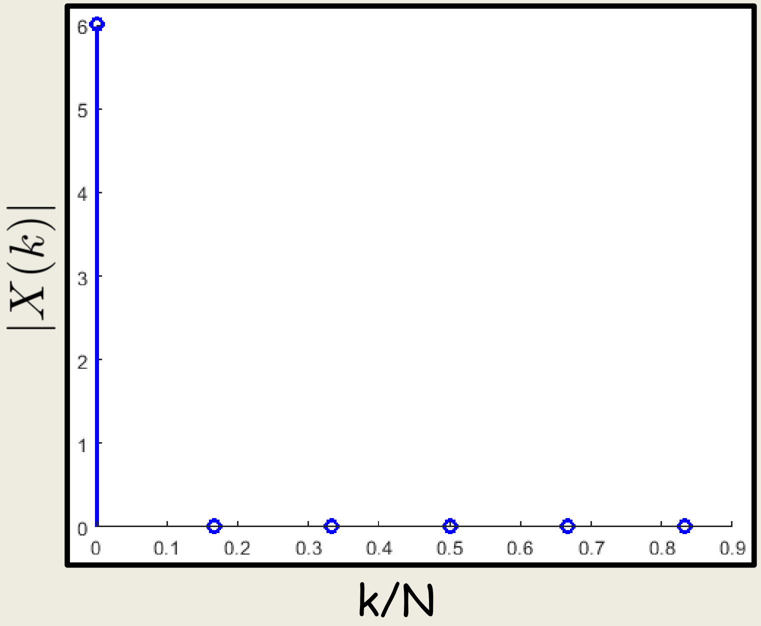
More examples

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\}$$

Compute DFT (N=8)

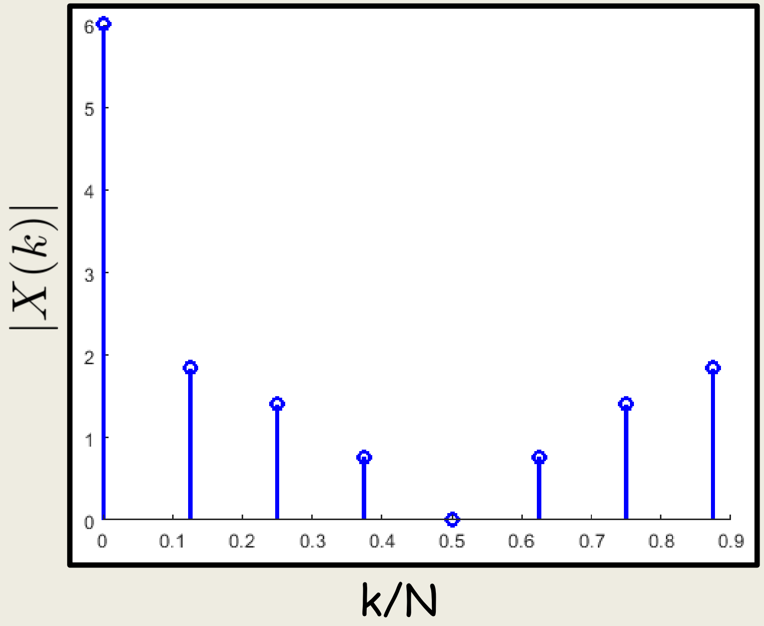


EITF75 Systems and Signals

More examples

$$x(n) = \{1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\}$$

Compute DFT (N=8)

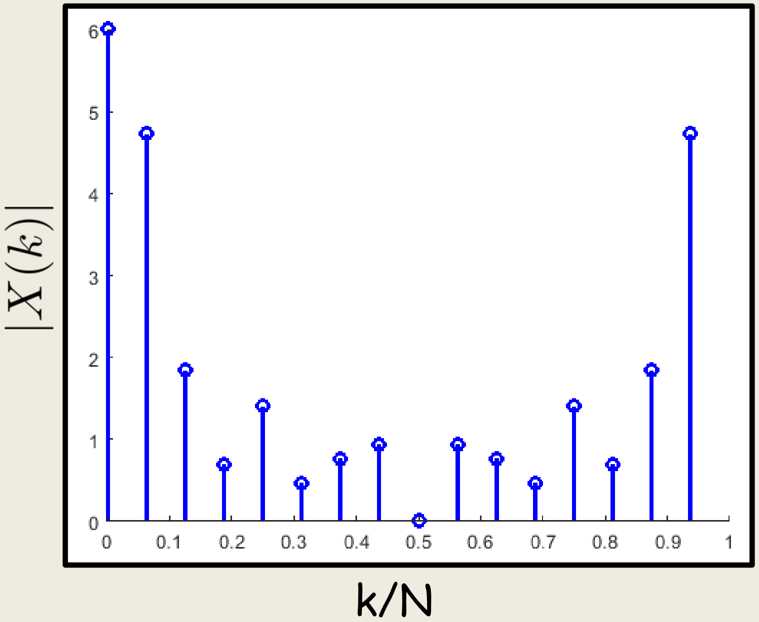


EITF75 Systems and Signals

More examples

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)

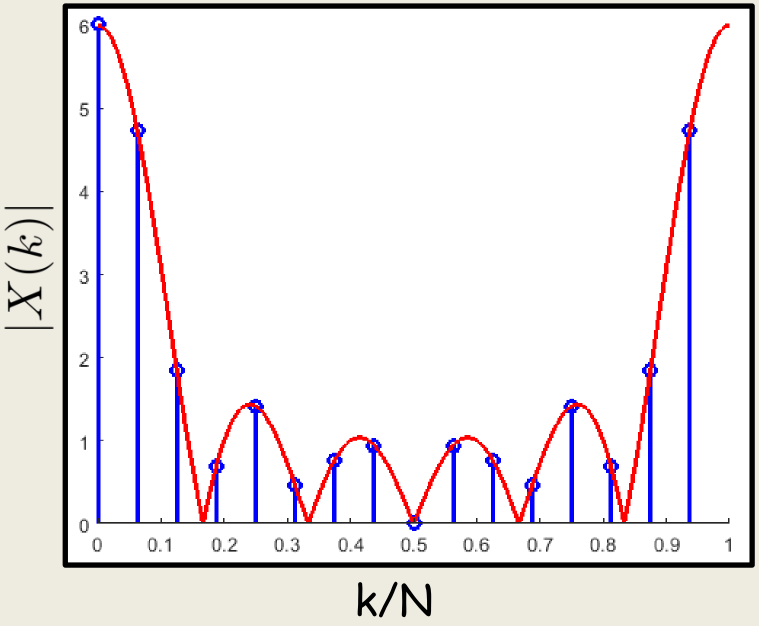


EITF75 Systems and Signals

More examples

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



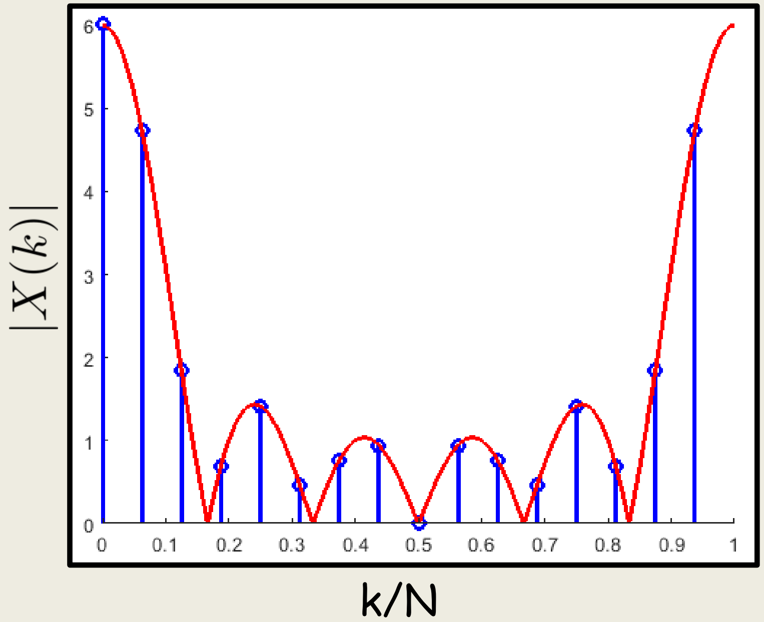
What is this line?

EITF75 Systems and Signals

More examples

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



What is this line?

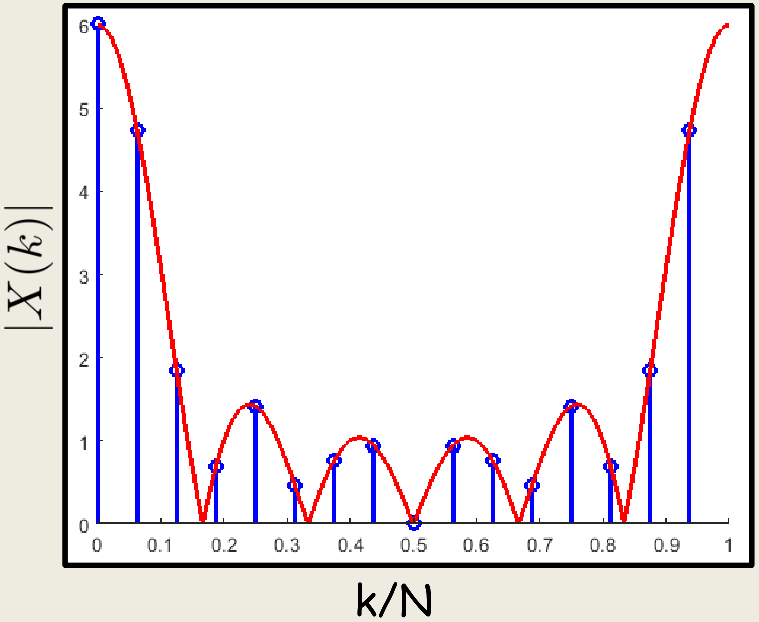
DFT size larger-or-equal to the length of x(n)

EITF75 Systems and Signals

More examples

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



What is this line?

DFT size larger-or-equal to the length of x(n)

Therefore, DFT samples of DTFT

As a matrix computation

$$X(k) = \sum_{n=0}^{N-1} x(n)\mathrm{e}^{-i2\pi kn/N}$$

We can write a DFT as a matrix multiplication

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \mathrm{e}^{-i2\pi/N} & \mathrm{e}^{-i2\pi 2/N} & \cdots & \mathrm{e}^{-i2\pi(N-1)/N} \\ 1 & \mathrm{e}^{-i2\pi 2/N} & \mathrm{e}^{-i2\pi 2\cdot 2/N} & \cdots & \mathrm{e}^{-i2\pi 2\cdot(N-1)/N} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \mathrm{e}^{-i2\pi(N-1)/N} & \mathrm{e}^{-i2\pi(N-1)\cdot 2/N} & \cdots & \mathrm{e}^{-i2\pi(N-1)\cdot(N-1)/N} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

EITF75 Systems and Signals

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$$\mathbf{X} = \mathbf{W}\mathbf{x}$$

Therefore

$$\mathbf{x} = \mathbf{W}^{-1}\mathbf{X}$$

Easy to show that W is unitary (orthogonal, but for complex matrices) $\mathbf{W}^{\mathrm{H}}\mathbf{W} = \mathbf{I}$