Solutions to Exercises

Digital Signal Processing

Exercise Problems and Solutions

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Autumn 2019

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Introduction

Solution 1.2

Periodically if there is a N so $\cos(\omega(n+N)) = \cos(\omega n)$ for all n. N is fundamental period if $\omega = 2\pi \frac{k}{N}$ where k and N are integers and should lack common factors.

a) $\cos(0.01\pi(n+N)) = cos(0.01\pi n)$ for this to be true, $0.01\pi N = 2\pi k$ and $\frac{N}{k} = 200$. So $x(n)$ is periodic and its period is $N = 200$.

b) Periodic, $N = 7$.

c) Periodic, $N = 2$.

d) $\sin(3n) = \sin(3(n+N))$ about $3N = 2\pi p$ and p integer. Then N cannot be integers so $\sin(3n)$ is not periodic.

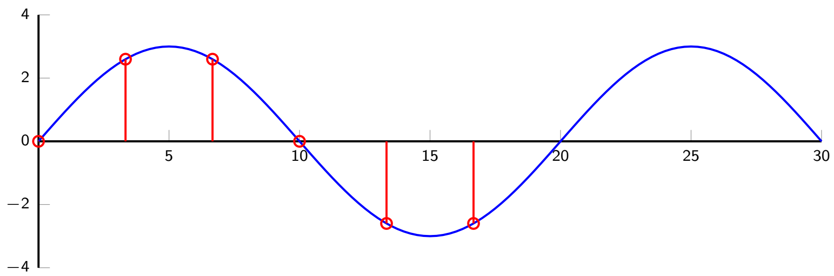
e) Periodic, $N = 10$.

Solution 1.5

a) See c).

b) $x(n) = x_a(nT) = 3\sin(2\pi nT \cdot 50) = 3\sin(2\pi n \cdot \frac{50}{300}) = 3\sin(2\pi n \cdot \frac{1}{6})$. Periodic with period $N = 6$ and frequency $f = 1/6$.

c) $N = 6$ and $T = 1/300$, so $T_p = NT = 0.02\,\mathrm{s}$.



d) $x(0) = 0$ and $x(1) = 3 = 3\sin\left(2\pi \cdot \frac{1}{4}\right) = 3\sin\left(2\pi \cdot \frac{50}{F_s}\right)$, so $\frac{50}{F_s} = \frac{1}{4}$ and thereby $F_s = 200\,\mathrm{Hz}$.

Solution 1.7

a) $F_s \geq 2F_{\max} = 2 \cdot 10\,\mathrm{kHz} = 20\,\mathrm{kHz}$.

b) $x(n) = x_a(nT) = \cos\left(2\pi n \cdot \frac{5000}{8000}\right) = \cos\left(2\pi n \cdot \frac{5}{8}\right)$ but since $\frac{5}{8} > \frac{1}{2}$ so $x(n) = \cos\left(2\pi n\left(1 - \frac{3}{8}\right)\right) = \cos\left(-2\pi n \cdot \frac{3}{8}\right) = \cos\left(2\pi n \cdot \frac{3}{8}\right)$ so $x_a(t) = \cos(2\pi t \cdot 3000)$, that is to say folding to $3\,\mathrm{kHz}$.

c) Folding to $1\,\mathrm{kHz}$.

Solution 1.8

a) Nyquist rate F_N is the lowest sample rate so that folding does not occur. $F_N = 2F_{\max} = 2 \cdot 100 = 200\,\mathrm{Hz}$.

b) $F_s = 250\,\mathrm{Hz}$ so $F_{\max} \leq 125\,\mathrm{Hz}$.

Solution 1.11

The analog singal is:

$$x_a(t) = 3\cos(2\pi t \cdot 50) + 2\sin(2\pi t \cdot 125) \tag{1}$$

Sample with $F_s = 200\,\text{Hz}$.

$$x(n) = x_a\left(n \cdot \frac{1}{200}\right) \tag{2}$$

$$= 3\cos\left(2\pi n \cdot \frac{50}{200}\right) + 2\sin\left(2\pi n \cdot \frac{125}{200}\right) \tag{3}$$

$$= 3\cos\left(2\pi n \cdot \frac{1}{4}\right) + 2\sin\left(2\pi n \cdot \frac{5}{8}\right) \tag{4}$$

$$= 3\cos\left(2\pi n \cdot \frac{1}{4}\right) + 2\sin\left(-2\pi n \cdot \frac{3}{8}\right) \tag{5}$$

Reconstruct with $F_{\text{rek}} = 1000\,\text{Hz}$.

$$y_a(t) = 3\cos\left(2\pi t \cdot \frac{1000}{4}\right) + 2\sin\left(-2\pi t \cdot \frac{3 \cdot 1000}{8}\right) \tag{6}$$

$$= 3\cos(2\pi t \cdot 250) - 2\sin(2\pi t \cdot 375) \tag{7}$$

MATLAB-code illustrating the example:

```
>> t = (0:10000-1)/1000;
>> x = 3*sin(100*pi*t)+2*sin(250*pi*t);
>> y = x(1:5:end);
>> subplot(2,1,1); plot(t(1:100),x(1:100));
>> subplot(2,1,2); plot(t(1:100),y(1:100));
```

Then listen to the signals:

```
>> soundsc(x, 1000);
>> soundsc(y, 1000);
```

Solution 1.13

The number of levels is $L = \frac{x_{\max} - x_{\min}}{\Delta} + 1$ and the number of bits are $b = \log_2 L$.

a) $b = 7$

b) $b = 10$

Solution 1.14

$Bitrate = numberofbitsXsamplingfrequency = nF_s$, so rate $= 20\,\text{Hz} \cdot 8\,\text{bitar} = 160\,\text{bitar/s}$. $F_{\max} = \frac{F_s}{2} = 10\,\text{Hz}$. $\Delta = \frac{DynamicRange}{2^n - 1} = 1/255\,\text{V}$.

Solution E1.1

Simulation in Matlab.

Solution New-1

The Euler's formula:

$$\cos(2\pi n) = \frac{1}{2}e^{j2\pi n} + \frac{1}{2}e^{-j2\pi n} \quad (8)$$

$$\sin(2\pi n) = \frac{1}{2j}e^{j2\pi n} - \frac{1}{2j}e^{-j2\pi n} \quad (9)$$

The drivative of $\cos(2\pi n)$ is as follows:

$$\begin{aligned}
\frac{d}{dn}\cos(2\pi n) &= \frac{d}{dn}(\frac{1}{2}e^{j2\pi n} + \frac{1}{2}e^{-j2\pi n}) && (10)\\
&= \frac{1}{2}(j2\pi)e^{j2\pi n} + \frac{1}{2}(-j2\pi)e^{-j2\pi n} && (11)\\
&= (-2\pi)(\frac{-j}{2}e^{j2\pi n} + \frac{j}{2}e^{-j2\pi n}) && (12)\\
&= (-2\pi)(\frac{1}{2j}e^{j2\pi n} - \frac{1}{2j}e^{-j2\pi n}) && (13)\\
&= (-2\pi)\sin(2\pi n) && (14)\\
& && (15)
\end{aligned}$$

The drivative of $\sin(2\pi n)$ is as follows:

$$\begin{aligned}
\frac{d}{dn}\sin(2\pi n) &= \frac{d}{dn}(\frac{1}{2j}e^{j2\pi n} - \frac{1}{2j}e^{-j2\pi n}) && (16)\\
&= \frac{1}{2j}(j2\pi)e^{j2\pi n} - \frac{1}{2}(-j2\pi)e^{-j2\pi n} && (17)\\
&= (2\pi)(\frac{1}{2}e^{j2\pi n} + \frac{1}{2}e^{-j2\pi n}) && (18)\\
&= (2\pi)\cos(2\pi n) && (19)\\
& && (20)
\end{aligned}$$

Solution New-2

See the formulas for the sum of geometric series from Lecture 1 slides.

a) $\sum_{n=0}^{N} 2^n = \frac{1-2^{N+1}}{1-2} = -1 + 2^{N+1}$

b) $\sum_{n=0}^{\infty} 0.5^n = \frac{1}{1-0.5} = 2$

Solution New-3

The Nyquist rate(NR) is twice the maximum signal frequency available in the signal.

a) $f_{max} = 75\,\text{Hz}, NR = 150\,\text{Hz}$.

b) $f_{max} = 200\,\text{Hz}, NR = 400\,\text{Hz}$.

c) $x(t) = 3\sin(100\pi t)\cos(250\pi t) = \frac{3}{2}(sin(350\pi t) - sin(150\pi t))$. so,$f_{max} = 175\,\text{Hz}, NR = 350\,\text{Hz}$.

Solution New-4

a) $x(n) = nu(n) = 1\delta(n-1) + 2\delta(n-2) + 3\delta(n-3) + 4\delta(n-4) + ... = \sum_{k=1}^{\infty} k\delta(n-k)$.

b) $x(n) = u(n+1) = \delta(n+1) + \delta(n) + \delta(n-1) + \delta(n-1) + .. = \sum_{k=-1}^{\infty} \delta(n-k)$

c) $x(n) = u(n-1) = \delta(n-1) + \delta(n-2) + \delta(n-3) + .. = \sum_{k=1}^{\infty} \delta(n-k)$

Solution New-5

Causal signal means that the signal is zero before n=0.

a) $x(n) = u(n+1)$, is non zero before $n = 0$ so it is noncausal.

b) $x(n) = u(n-1)$, is zero before $n = 0$ so it is causal.

Solution New-6

a) $x(n+1) = \{1,2,1,-3,5,8\}$, the corresponding $n = \{-1,0,1,2,3,4\}$

b) $x(n-1) = \{1,2,1,-3,5,8\}$, the corresponding $n = \{1,2,3,4,5,6\}$

c) $x(2n) = \{1,1,5\}, interpolation$, the corresponding $n = \{0,1,2\}$

d) $x(0.5n) = \{1,2,1,-3,5,8\}$,the corresponding $n = \{0,2,4,6,8,10\}$.

Solution New-7

Hint: For the signal to be energy signal, energy should be finite and power should be zero. If the energy is infinite and power is zero then it is power The signal.

Suggestion: The signals with decaying amplitude are energy signal and signals having constant amplitude are power signals. The signals whose amplitude increases are neither energy nor power signal.

a) $x(n) = u(n), n \geq 0$

Energy of the signal, $E = \sum_{n=0}^{\infty} \mid u(n) \mid^2 = \sum_{n=0}^{N-1} (1)^2 = \infty$.
Power of the signal,$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mid u(n) \mid^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (1)^2 = \lim_{N\to\infty} \frac{N}{N} = 1$.
Hence it is power signal.

b) $x(n) = nu(n), n \geq 0$

Energy of the signal, $E = \sum_{n=0}^{\infty} \mid nu(n) \mid^2 = \sum_{n=0}^{\infty} n^2 = \infty$.
Power of the signal,$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mid nu(n) \mid^2 = \lim_{N\to\infty} \frac{1^2+2^2+...(N-1)^2}{N} = \infty$.
It is neither energy and power signal.

c) $x(n) = (0.5)^n u(n)$

Energy of the signal, $E = \sum_{n=0}^{\infty} \mid (0.5)^n \mid^2 = \sum_{n=0}^{\infty} (0.25)^n = \frac{1}{1-0.25} = \frac{4}{3}$.
Power of the signal,$P = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \mid (0.5)^n u(n) \mid^2 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (0.25)^n = \lim_{N\to\infty} \frac{1}{N} \frac{1-o.25^{N+1}}{1-0.25} = 0$.
Hence it is Energy signal.

Solution New-8

Even signal if $x(-n) = x(n)$, odd signal if $x(-n) = -x(n)$.

a) $sin(-n) = \frac{1}{2j}e^{-jn} - \frac{1}{2j}e^{jn} = -(\frac{1}{2j}e^{jn} - \frac{1}{2j}e^{-jn}) = -sin(n)$, hence Odd signal.

b) $cos(-n) = \frac{1}{2}e^{-jn} + \frac{1}{2}e^{jn} = \frac{1}{2}e^{jn} + \frac{1}{2}e^{-} = cos(n)$, hence Even signal.

c) $cot(-n) = \frac{cos(-n)}{sin(-n)} = \frac{cos(n)}{-sin(n)} = -cot(n)$, hence Odd signal.

Solution New-9

Linearity: Linear system should satisfy homogeneity(if $y(n) = f(x(n))$ then $ay(n) = af(x(n))$) and additivity property (if $y_1(n) = f(x_1(n)), y_2(n) = f(x_2(n))$ then $y_1(n) + y_2(n) = f(x_1(n) + x_2(n))$.

Time Invariance: If $y(n) = f(x(n))$ then $y(n-D) = f(x(n-D))$.

a) Linearity check:

Homogeneity: For the input $ax_1(n)$, output is $y_1(n) = ax_1(n-1)$ so $y(n) = ay_1(n)$. Hence Satisfied.

Additive:$y_1(n) = x_1(n-1), y_2(n) = x_2(n-1)$. For the input,$x_1(n)+x_2(n)$,output is $y(n) = x_1(n-1)+x_2(n-1) = y_1(n) + y_2(n)$. Additive Property is satisfied.

Hence it is Linear.

Time Invariance check:

$y(n) = x(n-1)$ and $y(n-D) = x(n-1-D)$. Let us delay the input by D, now the output is $y(n) = x(n-1-D)$, which is same as $y(n-D)$ hence Time Invariant.

b) Linearity check:

Homogeneity: For the input $ax_1(n)$, output is $y_1(n) = ax_1(n^2)$ so $y(n) = ay_1(n)$. Hence Satisfied.

Additive:$y_1(n) = x_1(n^2), y_2(n) = x_2(n^2)$. For the input,$x_1(n) + x_2(n)$,output is $y(n) = x_1(n^2) + x_2(n^2) = y_1(n) + y_2(n)$. Additive Property is satisfied.

Hence Linear.

Time Invariance check:

$y(n) = x(n^2)$ and $y(n-D) = x(n^2 - D)$. Let us delay the input by D, now the output is $y(n) = x((n-D)^2)$, which is not same as $y(n-D)$ hence Not Time Invariant.

c) Linearity check:

Homogeneity: For the input $ax_1(n)$, output is $y_1(n) = a^2 x_1(n)^2$ so $y(n) \neq ay_1(n)$. Hence Not Satisfied.

Additive:$y_1(n) = x_1(n)^2, y_2(n) = x_2(n)^2$. For the input,$x_1(n) + x_2(n)$,output is $y(n) = (x_1(n) + x_2(n))^2 \neq y_1(n) + y_2(n)$.Additive Property is not satisfied.

Hence Not Linear. Even if one of the property is not satisfied, it will be non linear.

Time Invariance check:

$y(n) = x(n)^2$ and $y(n-D) = x(n-D)^2$. Let us delay the input by D, now the output is $y(n) = x(n-D)^2$, which is same as $y(n-D)$ hence Time Invariant.

Solution New-10

The analog signal is $x_a(t) = 3\cos(500\pi t) + 2\sin(1000\pi t)$.

a) The maximum signal frequency component $F_m ax = 500\,\text{Hz}$ so the Nyquist rate is $1000\,\text{Hz}$. So the sampling frequency should be at least higher than nyquist rate for ideal reconstruction.

b) For $F_s = 400\,\text{Hz}$,

$$x(n) = 3\cos(2\pi n\frac{250}{400}) + 2\sin(2\pi n\frac{500}{400}) \tag{21}$$
$$= 3\cos(2\pi n\frac{5}{8}) + 2\sin(2\pi n\frac{5}{4}) \tag{22}$$
$$= 3\cos(2\pi n\frac{3}{8}) + 2\sin(2\pi n\frac{1}{4}) \tag{23}$$

now reconstructing with $F_s = 400\,\text{Hz}$

$$x(t) = 3\cos(2\pi n\frac{3}{8}400) + 2\sin(2\pi n\frac{1}{4}400) \tag{24}$$
$$= 3\cos(300\pi t) + 2\sin(200\pi t) \tag{25}$$

Here the sampling frequency is higher very low compared to Nyquist rate, signal cannot be recovered after sampling.
For $F_s = 600\,\text{Hz}$,

$$x(n) = 3\cos(2\pi n\frac{250}{600}) + 2\sin(2\pi n\frac{500}{600}) \tag{26}$$
$$= 3\cos(2\pi n\frac{5}{12}) + 2\sin(2\pi n\frac{5}{6}) \tag{27}$$
$$= 3\cos(2\pi n\frac{5}{12}) - 2\sin(2\pi n\frac{1}{6}) \tag{28}$$
$$\tag{29}$$

Now reconstructing with $F_s = 600\,\text{Hz}$,

$$x(t) = 3\cos(2\pi t\frac{5}{12}600) - 2\sin(2\pi t\frac{1}{6}600) \tag{30}$$
$$= 3\cos(500\pi t) - 2\sin(200\pi t) \tag{31}$$
$$\tag{32}$$

Now reconstructing with $F_s = 600\,\text{Hz}$,. As the frequency signal is higher than the Nyquist rate of 250 Hz only this signal is recovered.
For $F_s = 1100\,\text{Hz}$

$$x(n) = 3\cos(2\pi n\frac{250}{1100}) + 2\sin(2\pi n\frac{500}{1100}) \tag{33}$$
$$= 3\cos(2\pi n\frac{5}{22}) + 2\sin(2\pi n\frac{5}{11}) \tag{34}$$

Now reconstructing with $F_{=}1100\,\text{Hz}$

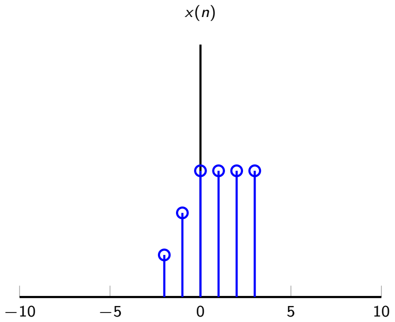
$$x(t) = 3\cos(2\pi t\frac{5}{22}1100) + 2\sin(2\pi t\frac{5}{11}1100) \tag{35}$$
$$= 3\cos(500\pi t) + 2\sin(10000\pi t) \tag{36}$$
$$\tag{37}$$

All frequency components are recovered as the sampling frequency is higher than Nyquist rate.

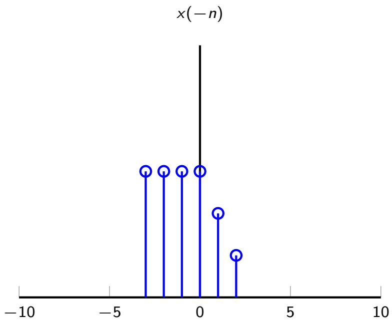
Impulse response and Convolution, chapter 2

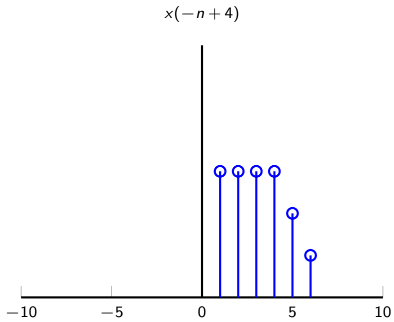
Solution 2.1

a)

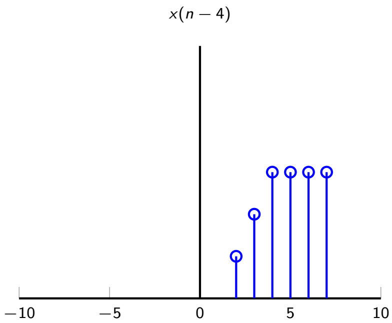


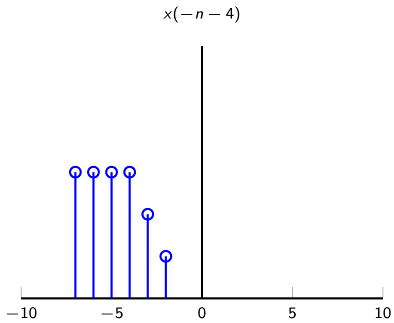
b) i)



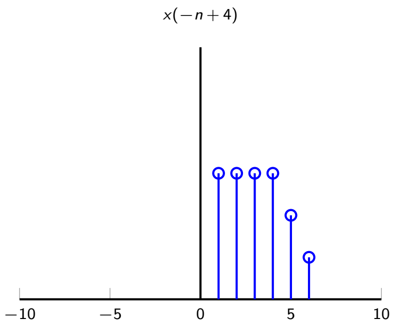


ii)





c)



d) First fold $x(n)$ and then delay the resulting signal by k sampels.

e) $x(n) = 1/3\ \delta(n+2) + 2/3\ \delta(n+1) + u(n) - u(n-4)$

Solution 2.7

a) output at n^{th} instant that means $y(n)$ only depends on the input at the same n^{th} time instant $x(n)$. So this is static system and causal
$\cos(a_1 x_1(n) + a_2 x_2(n)) \neq a_1 cos(x_1(n)) + a_2 cos(x_2(n))$. it is a non linear system.
$y(i) = \cos(x(i))$ if $i = n-k$ so $y(n-k) = \cos(x(n-k)$ system is time invariant.
$-1 \leq cos(x(n)) \leq +1$ any input produces a bounded output.
Static, nonlinear, time invariant, causal ,stable

b) Here output at n^{th} instant depends on input at k^{th} instant, where $k = -\infty, ..., n-1, n, n+1$. output depends on present input as well as past and future inputs..So the system is dynamic.

$$y_1(n) = \sum_{k=-\infty}^{n+1} x_1(k) \qquad y_2(n) = \sum_{k=-\infty}^{n+1} x_2(k) \qquad x(n) = a_1x_1(n) + a_2x_2(n)$$

$$y(n) = \sum_{k=-\infty}^{n+1} (a_1x_1(k) + a_2x_2(k)) = \sum_{k=-\infty}^{n+1} a_1x_1(k) + \sum_{k=-\infty}^{n+1} a_2x_2(k)$$
$$= a_1 \sum_{k=-\infty}^{n+1} x_1(k) + a_2 \sum_{k=-\infty}^{n+1} x_2(k) = a_1y_1(n) + a_2y_2(n)$$

system is linear.

$$y(n-i) = \sum_{k=-\infty}^{n-i+1} x(k) = \sum_{k^*=-\infty}^{n+1} x(k^* - i)$$
$$= \sum_{k=-\infty}^{n+1} x(k-i)$$

system is time invariant
output at time n depends on the input at t $n+1$ time (future) . So system is noncausal.
suppose $x(n) = a$ where $\leq a \leq \infty$ so $x(n)$ is bounded we have $y(n) = \sum_{k=-\infty}^{n+1} a \to \infty$ is unstable
Dynamic, Linear, time invariant, non causal, unstable.

c) output at n that means $y(n)$ depends on the input at the same n^{th} time instant $x(n)$. So this is static system and causal

$$y_1(n) = x_1(n)\cos(\omega_0 n) \qquad y_2(n) = x_2(n)\cos(\omega_0 n) \qquad x(n) = a_1x_1(n) + a_2x_2(n)$$
$$y(n) = x(n)\cos(\omega_0 n) = (a_1x_1(n) + a_2x_2(n))cos(\omega_0 n)$$
$$= a_1x_1(n)cos(\omega_0 n) + a_2x_2(n)cos(\omega_0 n) = a_1y_1(n) + a_2y_2(n)$$

system is linear.
suppose the system input is $x(n-i)$ so the output is $y^*(n) = x(n-i)\cos(\omega_0 n) \neq y(n-i)$
suppose that $x(n)$ is bounded since we know that $\cos(\omega_0 n)$ is bounded so $y(n)$ is bounded. system is stable.

Static, linear, time variant, causal ,stable

e) output at time n only depends on the input at the same n^{th} time instant $x(n)$. So this is static system and causal
suppose $x_1(n) = 2.3$ and $x_2(n) = 2.8$, so $y_1(n) = tran(x_1(n)) = 2$ and $y_2(n) = tran(x_2(n)) = 2$. $y_1(n) + y_2(n) = 4 \neq trun(x_1(n) + x_1(n)) = 5$ nonlinear system
if $x(n)$ is bounded the integer part of $x(n)$ is bounded, BIBO stable

Static, nonlinear, time invariant, causal ,stable

h) output at time n only depends on the input at the same n^{th} time instant $x(n)$. So this is static system and causal
suppose $x_d(n) = x(n-k) \to y_d(n) = x_d(n)u(n) = x(n-k)u(n)$ now delay the output by k, $y(n-k) = x(n-k)u(n-k)$ as we see $y_d(n) \neq y(n-k)$ system is time variant.
if $x(n)$ is bounded $y(n) = x(n)$ where $n > 0$ is bounded, BIBO stable

Static, linear, time variant, causal ,stable

j) Dynamic, Linear, time variant, non causal, stable. suppose $x_d(n) = x(n-k) \to y_d(n) = x_d(2n) = x(2n-k)$ now delay the output by k, $y(n-k) = x(2n-2k)$ as we see $y_d(n) \neq y(n-k)$ system is time variant.

n) Static, linear, time invariant, causal ,stable

Solution 2.13

a)

$$\sum_n y(n) = \sum_n \sum_k h(n-k)x(k) = \sum_k x(k) \sum_n h(n-k) = \sum_k x(k) \sum_l h(l) \tag{38}$$

b) (1) Sufficient: Suppose that $\sum_n |h(n)| = M_h < \infty$. Then apply with limited input signal, $\sum_n |x(n)| \leq M_x < \infty$,by using the fundamental properties of absolute values (Multiplicativity ,Subadditivity)

$$\left|y(k)\right| = \left|\sum_{n=-\infty}^{\infty} h(n)x(k-n)\right| \leq \sum_{n=-\infty}^{\infty} |h(n)| \cdot |x(k-n)| \leq M_h \cdot M_x < \infty \tag{39}$$

Shows that, the system is BIBO stable.

(2) Necessary: Assume that $\sum_n |h(n)| = \infty$. Form the input signal

$$x(n) = \begin{cases} h^*(-n)/|h(-n)| & h(-n) \neq 0, \\ 0 & h(-n) = 0. \end{cases} \tag{40}$$

$x(n)$ is limited for $|x(n)| \leq 1$. We get

$$y(0) = \sum_{n=-\infty}^{\infty} h(n)x(-n) = \sum_{n=-\infty}^{\infty} \frac{h(n)h^*(n)}{|h(n)|} = \sum_{n=-\infty}^{\infty} \frac{|h(n)|^2}{|h(n)|} = \sum_{n=-\infty}^{\infty} |h(n)| = \infty, \tag{41}$$

that is, a limited input signal provides an unlimited output and the system is not BIBO stable. Thus, if the system is BIBO -stable,$\sum_n |h(n)| < \infty$ must be BIBO-stable.

Solution 2.4

1)

$$y(0) = x(0) \tag{42}$$

$$y(1) = x(0) + x(1) \tag{43}$$

$$y(2) = x(0) + x(1) + x(2) \tag{44}$$

$$y(n) = x(0) + x(1) + x(2) + ... + x(n) \tag{45}$$

$$= \sum_{k=0}^{n} x(k) \tag{46}$$

$$= u(n) * x(n) \tag{47}$$

$$\text{since} \sum_{n=-\infty}^{\infty} |u(n)| = \infty, \qquad y(n) \qquad \text{is unstable.} \tag{48}$$

2)

$$y(n) = x(n) * (\delta(n) - \delta(n-1) - \delta(n-4)) \tag{49}$$

$$h(n) = \delta(n) - \delta(n-1) - \delta(n-4) \tag{50}$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = 3, \qquad y(n) \qquad \text{is stable.} \tag{51}$$

Solution 2.16

1) Graphic solution:

$h(0-k)$	4	2	$\underline{1}$							
$x(k)$			$\underline{1}$	1	1	1	1			
$h(0-k)x(k)$			$\underline{1}$							$\Sigma = 1 = y(0)$
$h(1-k)$		4	2	$\underline{1}$						
$x(k)$			$\underline{1}$	1	1	1	1			
$h(1-k)x(k)$			2	1						$\Sigma = 3 = y(1)$
$h(2-k)$			4	2	$\underline{1}$					
$x(k)$			$\underline{1}$	1	1	1	1			
$h(2-k)x(k)$			4	2	1					$\Sigma = 7 = y(2)$
$h(3-k)$				4	2	$\underline{1}$				
$x(k)$			$\underline{1}$	1	1	1	1			
$h(3-k)x(k)$				4	2	1				$\Sigma = 7 = y(3)$
$h(5-k)$					4	2	$\underline{1}$			
$x(k)$			$\underline{1}$	1	1	1	1			
$h(4-k)x(k)$						4	2			$\Sigma = 6 = y(5)$
$h(6-k)$							4	2	$\underline{1}$	
$x(k)$			$\underline{1}$	1	1	1	1			
$h(6-k)x(k)$							4			$\Sigma = 4 = y(6)$

$y(n) = \left\{ \underline{1} \quad 3 \quad 7 \quad 7 \quad 7 \quad 6 \quad 4 \right\}$

2) Graphic solution:

$h(0-k)$	-1	2	$\underline{1}$				
$x(k)$			$\underline{1}$	2	-1		
$h(k) * x(k)$			$\underline{1}$	4	2	-4	1

$y(n) = \left\{ 1 \quad 4 \quad 2 \quad -4 \quad 1 \right\}$

3) Graphic solution:

	1	2	3	$\underline{4}$	5
1	1	2	3	4	5
1	1	2	3	4	5

$y(n) = \left\{ 1 \quad 2 \quad 3 \quad 4 \quad 5 \right\}$

4) Grafisk lösning:

	1	-2	-3	$\underline{4}$
1	1	-2	-3	4
1	1	-2	-3	4
$\underline{0}$	0	0	0	$\underline{0}$
1	1	-2	-3	4
1	1	-2	-3	4

Summera antidiagonalerna,

$$y(n) = \left\{ \begin{array}{cccccccc} 1 & -1 & -5 & 2 & 3 & \underline{-5} & 1 & 4 \end{array} \right\}, \tag{52}$$

$$\sum_n y(n) = 0 = \sum_k x(k) \sum_l h(l) = 0 \cdot 4 \tag{53}$$

(5) Formel-lösning:

$$y(n) = \sum_{k=-\infty}^{\infty} \left(\frac{1}{2}\right)^k u(k) \left(\frac{1}{4}\right)^{n-k} u(n-k) \tag{54}$$

$$= \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} u(n-k) \tag{55}$$

$$= \sum_{k=0}^{n} \left(\frac{1}{2}\right)^k \left(\frac{1}{4}\right)^{n-k} \tag{56}$$

$$= \left(\frac{1}{4}\right)^n \sum_{k=0}^{n} 2^k \tag{57}$$

$$= \left(\frac{1}{4}\right)^n \frac{1-2^{n+1}}{1-2} \quad n \geq 0 \tag{58}$$

$$= \left[2 \cdot \left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n \right] u(n) \tag{59}$$

$$\sum_n y(n) = \frac{8}{3} = \sum_k x(k) \sum_l h(l) = 2 \cdot \frac{4}{3} \tag{60}$$

Solution 2.17

a)

	$\underline{6}$	5	4	3	2	1	0	0
$\underline{1}$	6	5	4	3	2	1	0	0
1	6	5	4	3	2	1	0	0
1	6	5	4	3	2	1	0	0
1	6	5	4	3	2	1	0	0

$y(n) = \left\{ \begin{array}{ccccccccc} \underline{6} & 11 & 15 & 18 & 14 & 10 & 6 & 3 & 1 \end{array} \right\}$

b)

	6	5	4	$\underline{3}$	2	1	0	0
$\underline{1}$	6	5	4	3	2	1	0	0
1	6	5	4	3	2	1	0	0
1	6	5	4	3	2	1	0	0
1	6	5	4	3	2	1	0	0

$y(n) = \left\{ \begin{array}{ccccccccc} 6 & 11 & 15 & \underline{18} & 14 & 10 & 6 & 3 & 1 \end{array} \right\}$

c) $y(n) = \left\{ \begin{array}{ccccc} 1 & \underline{2} & 2 & 2 & 1 \end{array} \right\}$

d) $y(n) = \left\{ \begin{array}{ccccc} \underline{1} & 2 & 2 & 2 & 1 \end{array} \right\}$

Solution 2.21

a) Om $a \neq b$:

$$y(n) = \sum_{k=-\infty}^{\infty} b^k u(k) a^{n-k} u(n-k) \tag{61}$$

$$= a^n \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k \tag{62}$$

$$= a^n \frac{1 - \left(\frac{b}{a}\right)^{n+1}}{1 - \frac{b}{a}} \tag{63}$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b} \quad n \geq 0 \tag{64}$$

Om $a = b$:

$$y(n) = a^n \sum_{k=0}^{n} 1^k = a^n \cdot (n+1) \quad n \geq 0 \tag{65}$$

b) $y(n) = \left\{ \begin{array}{ccccccccc} 1 & 1 & \underline{-1} & 0 & 0 & 3 & 3 & 2 & 1 \end{array} \right\}$

Solution 2.35

a) Parallel and serial connection:

$$h(n) = h_1(n) * [h_2(n) - h_3(n) * h_4(n)] \tag{66}$$

b)

$$h_3(n) * h_4(n) = (n-1) \cdot u(n-2) \tag{67}$$

$$h_2(n) - h_3(n) * h_4(n) = \delta(n) + 2u(n-1) \tag{68}$$

$$h(n) = \frac{1}{2}\delta(n) + \frac{5}{4}\delta(n-1) + 2\delta(n-2) + \frac{5}{2}u(n-3) \tag{69}$$

c) Falta med en komponent av $x(n)$ i taget.

$h(n) * \delta(n+2)$	$\frac{1}{2}$	$\frac{5}{4}$	$\underline{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\frac{5}{2}$	$\dots$
$h(n) * 3\delta(n-1)$	0	0	$\underline{0}$	$\frac{3}{2}$	$\frac{15}{4}$	6	$\frac{15}{2}$	$\frac{15}{2}$	$\frac{15}{2}$	$\dots$
$h(n) * (-4\delta(n-3))$	0	0	$\underline{0}$	0	0	-2	-5	-8	-10	-10
$\sum$	$\frac{1}{2}$	$\frac{5}{4}$	$\underline{2}$	4	$\frac{25}{4}$	$\frac{13}{2}$	5	2	0	0

The output signal becomes

$$y(n) = \left\{ \begin{array}{cccccccc} \frac{1}{2} & \frac{5}{4} & \underline{2} & 4 & \frac{25}{4} & \frac{13}{2} & 5 & 2 \end{array} \right\} \tag{70}$$

d)

$$h(n) = \frac{1}{2}\delta(n) + \frac{5}{4}\delta(n-1) + 2\delta(n-2) + \frac{5}{2}u(n-3) \tag{71}$$

$$\sum_{n=-\infty}^{\infty} |h(n)| = \frac{1}{2} + \frac{5}{4} + \frac{5}{2}\sum_{n=0}^{\infty} 1 = \infty \quad \text{system is unstable} \tag{72}$$

Solution 2.61

- For $r_{xx}(l)$:

$$r_{xx}(l) = \sum_n x(n)x(n-l) = \sum_{n=n_0-N}^{n_0+N} x(n-l) = \sum_{n=n_0-N+l}^{n_0+N} 1 = 2N - l + 1 \quad l \geq 0 \tag{73}$$

$$r_{xx}(l) = \sum_n x(n)x(n-l) = \sum_{n=n_0-N}^{n_0+N} x(n-l) = \sum_{n=n_0-N}^{n_0+N+l} 1 = 2N + l + 1 \quad l \leq 0 \tag{74}$$

$$r_{xx}(l) = 2N - |l| + 1 \tag{75}$$

- For $r_{xy}(l)$:

$$y(n) = x(n+n_0) \tag{76}$$

$$r_{xy}(l) = \sum_n x(n)y(n-l) = \sum_n x(n)x(n+n_0-l) \tag{77}$$

Lös grafiskt, dvs

$$r_{xy}(l) = r_{xx}(l-n_0) \tag{78}$$

Solution 2.62

a) $r_{xx}(l) = x(n) * x(-n)$
$x(n) = \left\{ 1 \quad 2 \quad 1 \quad 1 \right\}$ and $x(-n) = \left\{ 1 \quad 1 \quad 2 \quad 1 \right\}$
$r_{xx}(l) = \left\{ 1 \quad 3 \quad 5 \quad \underline{7} \quad 5 \quad 3 \quad 1 \right\}$

b) $r_{yy}(l) = y(n) * y(-n)$
$y(n) = \left\{ 1 \quad 1 \quad 2 \quad 1 \right\}$ and $y(-n) = \left\{ 1 \quad 2 \quad 1 \quad 1 \right\}$
$r_{yy}(l) = \left\{ 1 \quad 3 \quad 5 \quad \underline{7} \quad 5 \quad 3 \quad 1 \right\}$

Solution 2.64

$$x(n) = s(n) + r_1 s(n-k_1) + r_2 s(n-k_2) \tag{79}$$

$$r_{xx}(l) = \sum_n x(n)x(n-l) = \tag{80}$$

$$= \sum_n [s(n) + r_1 s(n-k_1) + r_2 s(n-k_2)] \cdot [s(n-l) + r_1 s(n-k_1-l) + r_2 s(n-k_2-l)] \tag{81}$$

$$= r_{ss}(l) + r_1 r_{ss}(l+k_1) + r_2 r_{ss}(l+k_2) \tag{82}$$

$$+ r_1 r_{ss}(l-k_1) + r_1^2 r_{ss}(l) + r_1 r_2 r_{ss}(l+k_2-k_1) \tag{83}$$

$$+ r_2 r_{ss}(l-k_2) + r_1 r_2 r_{ss}(l+k_1-k_2) + r_2^2 r_{ss}(l) \tag{84}$$

Låt $r_1 \ll 1$ och $r_2 \ll 1$:

$$r_{xx}(l) \approx r_{ss}(l) + r_1 r_{ss}(l+k_1) + r_2 r_{ss}(l+k_2) + r_1 r_{ss}(l-k_1) + r_2 r_{ss}(l-k_2) \tag{85}$$

Solution N3.1

a) Utilizing the linearity and time shifting properties of the z-transform, in the z-domain the difference equation becomes

$$Y(z) + z^{-1}Y(z) = X(z) \tag{86}$$

which can be written as

$$Y(z) = X(z)\frac{1}{1+z^{-1}} \tag{87}$$

Performing a z-transform of $x(n)$ via table (you can try with the definition of the z-transform as well)

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{88}$$

yields

$$X(z) = \frac{5}{1-z^{-1}} \tag{89}$$

The final result of $Y(z)$ becomes

$$Y(z) = \frac{5}{1-z^{-1}}\frac{1}{1+z^{-1}} = 5\frac{1}{1-z^{-2}} \tag{90}$$

b) To perform an inverse z-transform, Y(z) should be expressed in individual easy-to-transform terms. This can be done via a partial fraction. First, the fraction is extended by z^2, after which both sides are divided by z:

$$\frac{Y(z)}{z} = 5\frac{z}{z^2-1} \tag{91}$$

The poles of $Y(z)$ are given by setting the denominator to 0:

$$z^2 - 1 = 0 \tag{92}$$

$$z_1 = 1 \tag{93}$$

$$z_2 = -1 \tag{94}$$

Thus, we can form the expression

$$\frac{Y(z)}{z} = 5\frac{z}{(z-1)(z+1)} = \frac{A_1}{z-1} + \frac{A_2}{z+1} \tag{95}$$

The middle and right side is multiplied with $(z-1)(z+1)$:

$$5z = A_1(z+1) + A_2(z-1) \tag{96}$$

To find A_1 and A_2, the poles are inserted in turn:

$$z = 1: \tag{97}$$

$$5 = A_1(1+1) + A_2(1-1) \tag{98}$$

$$A_1 = \frac{5}{2} \tag{99}$$

$$z = -1: \tag{100}$$

$$-5 = A_1(-1+1) + A_2(-1-1) \tag{101}$$

$$A_2 = \frac{5}{2} \tag{102}$$

$Y(z)/z$ becomes

$$\frac{Y(z)}{z} = \frac{\frac{5}{2}}{z-1} + \frac{\frac{5}{2}}{z+1} \tag{103}$$

By multiplying both sides with z, followed by reducing both fractions with z, we get

$$Y(z) = \frac{\frac{5}{2}}{1-z^{-1}} + \frac{\frac{5}{2}}{1+z^{-1}} = \frac{5}{2}\left(\frac{1}{1-z^{-1}} + \frac{1}{1+z^{-1}}\right) \tag{104}$$

These terms conform to tabular inverse z-transform as such:

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{105}$$

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{106}$$

This gives the final result

$$y(n) = \frac{5}{2}\Big(u(n) + (-1)^n u(n)\Big) = \frac{5}{2}\Big(1 + (-1)^n\Big)u(n) \tag{107}$$

Try to check the value of $y(n)$ for the first few n:s, so that the original differential equation and the final equation are equivalent.

Solution N3.2

a) Utilizing the linearity and time shifting properties of the z-transform, in the z-domain the difference equation becomes

$$Y(z) - \frac{2}{3}z^{-1}Y(z) - \frac{1}{3}z^{-2}Y(z) = X(z) \tag{108}$$

Solving for $Y(z)$ yields

$$Y(z) = \frac{1}{1 - \frac{2}{3}z^{-1} - \frac{1}{3}z^{-2}}X(z) = H(z)X(z) \tag{109}$$

b) Isolated, $H(z)$ looks like

$$H(z) = \frac{1}{1 - \frac{2}{3}z^{-1} - \frac{1}{3}z^{-2}} \tag{110}$$

This expression should be expressed as individual easy-to-transform terms in order to determine $h(n)$. This can be done with a partial-fraction expansion. First, the poles of $H(z)$ should be found. The expression is extended by z^2, and both sides are divided by z:

$$\frac{H(z)}{z} = \frac{z}{z^2 - \frac{2}{3}z - \frac{1}{3}} \tag{111}$$

The poles are found by setting the denominator in the right-hand expression to zero.

$$z^2 - \frac{2}{3}z - \frac{1}{3} = 0 \tag{112}$$

Solving for z yields

$$z_1 = 1 \tag{113}$$

$$z_2 = -\frac{1}{3} \tag{114}$$

Thus

$$\frac{H(z)}{z} = \frac{z}{(z-1)(z+\frac{1}{3})} \tag{115}$$

Which should be expressed in the form of

$$\frac{H(z)}{z} = \frac{z}{(z-1)(z+\frac{1}{3})} = \frac{A_1}{z-1} + \frac{A_2}{z+\frac{1}{3}} \tag{116}$$

Multiplying both sides with the denominator of the middle yields

$$z = A_1(z+\frac{1}{3}) + A_2(z-1) \tag{117}$$

Inserting the pole values readily produces A_1 and A_2:

$$z = 1: \quad 1 = A_1(1+\frac{1}{3}) + A_2(1-1) \tag{118}$$

$$A_1 = \frac{3}{4} \tag{119}$$

$$z = -\frac{1}{3}: \quad -\frac{1}{3} = A_1(-\frac{1}{3}+\frac{1}{3}) + A_2(-\frac{1}{3}-1) \tag{120}$$

$$A_2 = \frac{1}{4} \tag{121}$$

Having A_1 and A_2, we move back to the partial-fraction expression

$$\frac{H(z)}{z} = \frac{A_1}{z-1} + \frac{A_2}{z+\frac{1}{3}} \tag{122}$$

To achieve a transformable form, both sides are first multiplied with z, both fractions are reduced by z, and lastly A_1 and A_2 are moved in front of their respective terms:

$$H(z) = A_1 \frac{1}{1-z^{-1}} + A_2 \frac{1}{1+\frac{1}{3}z^{-1}} \tag{123}$$

Due to the linearity of the z-transform, the right-hand terms can be transformed individually. Both correspond to the table transformation:

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{124}$$

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{125}$$

Thus

$$h(n) = A_1 u(n) + A_2 \left(\frac{1}{3}\right)^n u(n) \tag{126}$$

Inserting A_1 and A_2 and a slight restructuring yields the final result:

$$h(n) = \left(\frac{3}{4} + \frac{1}{4}\left(\frac{1}{3}\right)^n\right) u(n) \tag{127}$$

Solution 3.1

a) The z-transform is defined as

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{128}$$

The signal is finite, starting at $n = -5$ and ending at $n = 2$. The z-transform can thus be written as

$$X(z) = \sum_{n=-5}^{2} x(n) z^{-n} \tag{129}$$

Compute all individual terms:

n	-5	-4	-3	-2	-1	0	1	2
$x(n)$	3	0	0	0	0	$\underline{6}$	1	-4
$x(n)z^{-n}$	$3z^5$	0	0	0	0	$6z^0 = 6$	z^{-1}	$-4z^{-2}$

(130)

Add the terms together to produce the final result:

$$X(z) = 3z^5 + 6 + z^{-1} + -4z^{-2} \tag{131}$$

b) Utilizing a step function, the signal can be expressed as

$$x(n) = (\frac{1}{2})^n u(n-5) \tag{132}$$

Plugging this into the z-transform definition yields

$$X(z) = \sum_{n=5}^{\infty} (\frac{1}{2})^n z^{-n} \tag{133}$$

Simplifying it gives

$$X(z) = \sum_{n=5}^{\infty} (\frac{1}{2} z^{-1})^n \tag{134}$$

(The original signal description can also be plugged into the definition, case by case, yielding the same: $X(z) = \sum_{n=-\infty}^{4} 0^n z^{-n} + \sum_{n=5}^{\infty} (\frac{1}{2})^n z^{-n} = \sum_{n=5}^{\infty} (\frac{1}{2} z^{-1})^n$)
As the signal has no upper bound, the individual terms cannot be manually added together. However, due to the geometrical fraction $(\frac{1}{2} z^{-1})^n$, it is recognized as a geometric sum. Additionally, since the fraction is between -1 and 1, it converges.
Writing out the terms gives

$$X(z) = (\frac{1}{2} z^{-1})^5 + (\frac{1}{2} z^{-1})^6 + (\frac{1}{2} z^{-1})^7 + \ldots \tag{135}$$

A geometric sum can be solved by finding a similar geometric sum – with an identical fraction but a different starting term – and then reducing the original sum with the latter sum. Finding such a sum can be done by simply multiplying the original sum with its geometrical fraction, as such:

$$X(z)(\frac{1}{2} z^{-1}) = (\frac{1}{2} z^{-1})^6 + (\frac{1}{2} z^{-1})^7 + (\frac{1}{2} z^{-1})^8 + \ldots \tag{136}$$

Then, taking the difference, gives

$$X(z) - X(z)(\frac{1}{2} z^{-1}) = (\frac{1}{2} z^{-1})^5 \tag{137}$$

as all other terms cancel each other out (which works due to that the fraction approaches zero as n moves to infinity, if it were a finite signal the very last term of the second sum would remain as well).
Breaking out $X(z)$ gives the final result:

$$X(z) = \frac{(\frac{1}{2} z^{-1})^5}{1 - \frac{1}{2} z^{-1}} \tag{138}$$

Solution 3.2

a) Since the z-transform enjoys the linearity property, the terms of $x(n)$ can be transformed separately. Expanding $x(n)$ gives

$$x(n) = u(n) + nu(n) \tag{139}$$

Via table:

$$u(n) \leftrightarrow \frac{1}{1 - z^{-1}} \tag{140}$$

and

$$na^n u(n) \leftrightarrow \frac{az^{-1}}{(1 - az^{-1})^2} \tag{141}$$

The z-transform can thus be written as:

$$X(z) = \frac{1}{1 - z^{-1}} + \frac{z^{-1}}{(1 - z^{-1})^2} \tag{142}$$

(For a non-formulaic transformation, the first term can readily be transformed using the geometric sum method used in 3.1b; how to transform the second term is shown in example 2.7 in the book.)
Simplifying the expression gives the z-transform result:

$$X(z) = \frac{1}{(1-z^{-1})^2} \tag{143}$$

To retrieve the poles and zeros, the denominator is first expanded, as such

$$X(z) = \frac{1}{1-2z^{-1}+z^{-2}} \tag{144}$$

Then, the expression is extended according to the polynomial order:

$$X(z) = \frac{z^2}{z^2-2z+1} \tag{145}$$

Set the numerator and denominator to zero, to obtain the zeros and poles, respectively:

$$z^2 = 0 \Rightarrow \text{There are two zeros at } z = 0 \tag{146}$$

$$z^2 - 2z + 1 = 0 \Rightarrow \text{There are two poles at } z = 1 \tag{147}$$

c) Simplifying $x(n)$ gives

$$x(n) = (-\frac{1}{2})^n u(n) \tag{148}$$

Via table:

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{149}$$

The z-transform becomes:

$$X(z) = \frac{1}{1+\frac{1}{2}z^{-1}} \tag{150}$$

Again, extend the expression to procure the poles and zeros:

$$X(z) = \frac{z}{z+\frac{1}{2}} \tag{151}$$

Setting the numerator and denominator to zero, gives that there is a zero at $z = 0$ and a pole at $z = -\frac{1}{2}$

f) In order to perform a formulaic z-transform of $x(n)$ the phase shift is removed using the trigonometric formula

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{152}$$

$x(n)$ becomes

$$x(n) = Ar^n(\cos(\omega_0 n)\cos\phi - \sin(\omega_0 n)\sin\phi)u(n) \tag{153}$$

$$= A(r^n\cos(\omega_0 n)u(n)\cos\phi - r^n\sin(\omega_0 n)u(n)\sin\phi) \tag{154}$$

The cos- and sin-term are transformed individually per the formulas:

$$a^n\cos(\omega_0 n)u(n) \leftrightarrow \frac{1-az^{-1}\cos\omega_0}{1-2az^{-1}\cos\omega_0 - a^2z^{-2}} \tag{155}$$

$$a^n\sin(\omega_0 n)u(n) \leftrightarrow \frac{az^{-1}\sin\omega_0}{1-2az^{-1}\cos\omega_0 - a^2z^{-2}} \tag{156}$$

$X(z)$ then becomes (A, $\cos\phi$ and $\sin\phi$ are carried through unchanged):

$$X(z) = A\frac{(1-rz^{-1}\cos\omega_0)\cos\phi - rz^{-1}\sin\omega_0\sin\phi}{1-2rz^{-1}\cos\omega_0 + r^2z^{-2}} \tag{157}$$

To find the zeros and poles, the expression is extended with z^2

$$X(z) = A\frac{(z^2-rz\cos\omega_0)\cos\phi - rz\sin\omega_0\sin\phi}{z^2-2rz\cos\omega_0+r^2} \tag{158}$$

The numerator is set to zero to find the zeros:

$$0 = (z^2 - rz\cos\omega_0)\cos\phi - rz\sin\omega_0\sin\phi \tag{159}$$

$$= z\Big((z - r\cos\omega_0)\cos\phi - r\sin\omega_0\sin\phi\Big) \tag{160}$$

$$= z\Big(z\cos\phi - r(\cos\omega_0\cos\phi - \sin\omega_0\sin\phi)\Big) \tag{161}$$

The trigonometric formula used earlier can be used again in reverse on the second term:

$$z\Big(z\cos\phi - r\cos(\omega_0 + \phi)\Big) = 0 \tag{162}$$

Zeros are at $z = 0$ and $z = \frac{r\cos(\omega_0+\phi)}{\cos\phi}$
The denominator is set to zero to find the poles:

$$0 = z^2 - 2rz\cos\omega_0 + r^2 \tag{163}$$

$$= z^2 - rz(e^{j\omega_0} + e^{-j\omega_0}) + r^2 \tag{164}$$

$$= (z - re^{j\omega_0})(z - re^{-j\omega_0}) \tag{165}$$

Poles are at $z = re^{\pm j\omega_0}$

h) Plug $x(n)$ into the z-transform definition:

$$X(z) = \sum_{n=0}^{9} \left(\frac{1}{2}\right)^n z^{-n} \tag{166}$$

Simplify as a geometric sum to get the final z-transform:

$$X(z) = \frac{1 - \left(\frac{1}{2}z^{-1}\right)^{10}}{1 - \frac{1}{2}z^{-1}} = \frac{1 - \left(\frac{1}{2}\right)^{10} z^{-10}}{1 - \frac{1}{2}z^{-1}} \tag{167}$$

Extend with z^{10} to find zeros and poles:

$$X(z) = \frac{z^{10} - \left(\frac{1}{2}\right)^{10}}{z^{10} - \frac{1}{2}z^9} \tag{168}$$

Set numerator to zero to get zeros:

$$0 = z^{10} - \left(\frac{1}{2}\right)^{10} \tag{169}$$

Zeros are at $z = \frac{1}{2}e^{j2\pi k/10}$, for $k = 0...9$
Set denominator to zero to get poles:

$$0 = z^9 - \frac{1}{2}z^9 \tag{170}$$

$$= z^9(z - \frac{1}{2}) \tag{171}$$

There are 9 poles at $z = 0$ and 1 pole at $z = \frac{1}{2}$. Note however that the zero of $z = \frac{1}{2}$ and the pole of $z = \frac{1}{2}$ cancel each other out. Thus, zeros are at $z = \frac{1}{2}e^{j2\pi k/10}$, for $k = 1...9$ and 9 poles are at $z = 0$.

Solution 3.8

a) That a signal output is the sum of all previous inputs of the signal, can be expressed as a convolution of the signal with a unit step:

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) = x(n) * u(n) \tag{172}$$

Thus, by using the convolution property of the z-transform, we get

$$Y(z) = X(z)U(z) \tag{173}$$

Since

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{174}$$

the final result is

$$Y(z) = X(z)\frac{1}{1-z^{-1}} \tag{175}$$

b) Convolve $u(n) * u(n)$

$$u(n) * u(n) = \sum_{k=-\infty}^{\infty} u(k)u(n-k) \tag{176}$$

Since

$$u(k) = 0 \quad k < 0 \tag{177}$$

$$u(n-k) = 0 \quad n < k \tag{178}$$

We get

$$u(n) * u(n) = \sum_{k=0}^{n} u(k)u(n-k) = \sum_{k=0}^{n} 1 \tag{179}$$

Which is equal to $x(n) = (n+1)u(n)$.
Using the convolution property of the z-transform we have

$$Y(z) = U(z)U(z) \tag{180}$$

Since

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{181}$$

the z-transform becomes

$$Y(z) = \frac{1}{1-z^{-1}}\frac{1}{1-z^{-1}} \tag{182}$$

$$= \frac{1}{1-2z^{-1}+z^{-2}} \tag{183}$$

Solution 3.9

Given a pole or a zero at $z = re^{j\omega_1}$, it has its complex-conjugate paired pole or zero at $z = re^{-j\omega_1}$.
Scaling in the z-domain via table:

$$a^n x(n) \leftrightarrow X(a^{-1}z) \tag{184}$$

gives that the new poles or zeros are given by

$$ze^{-j\omega_0} = re^{j\omega_1} \text{ and } ze^{-j\omega_0} = re^{-j\omega_1} \tag{185}$$

Solving for z gives

$$z = re^{j\omega_1}e^{j\omega_0} = re^{j(\omega_1+\omega_0)} \text{ and } z = re^{-j\omega_1}e^{j\omega_0} = re^{j(-\omega_1+\omega_0)} \tag{186}$$

The poles or zeros thus are not complex-conjugate pairs anymore.

Solution 3.14

a) $X(z)$ is a proper rational transform and a partial-fraction can be determined as-is. First it is multiplied with z^2.

$$X(z) = \frac{z^2 + 3z}{z^2 + 3z + 2} \tag{187}$$

This is followed by dividing both sides with z (this helps due to that the individual fraction terms later will have a maximum power of z; thus it will aid in making the terms conform to the z-transform formulas) :

$$\frac{X(z)}{z} = \frac{z+3}{z^2 + 3z + 2} \tag{188}$$

Next the poles are found by setting the denominator to zero:

$$0 = z^2 + 3z + 2 \tag{189}$$

which gives $p_1 = -1$ and $p_2 = -2$. The transform is written as:

$$\frac{X(z)}{z} = \frac{z+3}{z^2 + 3z + 2} = \frac{z+3}{(z+1)(z+2)} = \frac{A_1}{z+1} + \frac{A_2}{z+2} \tag{190}$$

A_1 and A_2 can be found by first multiplying with the denominator $(z+1)(z+2)$

$$z + 3 = A_1(z+2) + A_2(z+1) \tag{191}$$

A_1 and A_2 are then found by setting z as the poles in turn, thus eliminating either A_1 or A_2:

$$z = p_1 = -1 : \tag{192}$$

$$-1 + 3 = A_1(-1+2) + A_2(-1+1) \tag{193}$$

$$A_1 = 2 \tag{194}$$

$$z = p_2 = -2 : \tag{195}$$

$$-2 + 3 = A_1(-2+2) + A_2(-2+1) \tag{196}$$

$$A_2 = -1 \tag{197}$$

Inserting A_1 and A_2 in the transform yields

$$\frac{X(z)}{z} = \frac{2}{z+1} + \frac{-1}{z+2} \tag{198}$$

Multiply both sides with z^1

$$X(z) = \frac{2z}{z+1} + \frac{-z}{z+2} \tag{199}$$

Extend the right terms with z^{-1}

$$X(z) = \frac{2}{1+z^{-1}} + \frac{-1}{1+2z^{-1}} \tag{200}$$

These terms can readily be inverse z-transformed individually. Via table:

$$a^n u(n) \leftrightarrow \frac{1}{1 - az^{-1}} \tag{201}$$

Thus

$$x(n) = 2(-1)^n u(n) - (-2)^n u(n) \tag{202}$$

b) X(z) is a proper rational transform and so a partial-fraction can be produced. The fraction is extended by z^2, and both sides are divided by z:

$$\frac{X(z)}{z} = \frac{z}{z^2 - z + 0.5} \tag{203}$$

The right-hand denominator is set to zero to find the poles:

$$z^2 - z + 0.5 = 0 \tag{204}$$

$$z = \frac{1}{2} \pm j\frac{1}{2} \tag{205}$$

(note that the poles are complex-conjugate). Thus we have

$$\frac{X(z)}{z} = \frac{z}{z^2 - z + 0.5} = \frac{z}{(z - (\frac{1}{2} + j\frac{1}{2}))(z - (\frac{1}{2} - j\frac{1}{2}))} \tag{206}$$

A partial-fraction is sought as

$$\frac{z}{(z - (\frac{1}{2} + j\frac{1}{2}))(z - (\frac{1}{2} - j\frac{1}{2}))} = \frac{A_1}{z - (\frac{1}{2} + j\frac{1}{2})} + \frac{A_2}{z - (\frac{1}{2} - j\frac{1}{2})} \tag{207}$$

In order to find A_1 and A_2, both sides are first multiplied with $(z - (\frac{1}{2} + j\frac{1}{2}))(z - (\frac{1}{2} - j\frac{1}{2}))$

$$z = A_1\left(z - \left(\frac{1}{2} - j\frac{1}{2}\right)\right) + A_2\left(z - \left(\frac{1}{2} + j\frac{1}{2}\right)\right) = A_1\left(z - \frac{1}{2} + j\frac{1}{2}\right) + A_2\left(z - \frac{1}{2} - j\frac{1}{2}\right) \tag{208}$$

Then, the pole-values are inserted individually:

$$z = \frac{1}{2} + j\frac{1}{2}: \tag{209}$$

$$\frac{1}{2} + j\frac{1}{2} = A_1\left(\frac{1}{2} + j\frac{1}{2} - \frac{1}{2} + j\frac{1}{2}\right) + A_2\left(\frac{1}{2} + j\frac{1}{2} - \frac{1}{2} - j\frac{1}{2}\right) \tag{210}$$

$$A_1 = \frac{\frac{1}{2} + j\frac{1}{2}}{j} = \frac{1}{2} - j\frac{1}{2} \tag{211}$$

$$z = \frac{1}{2} - j\frac{1}{2}: \tag{212}$$

$$\frac{1}{2} - j\frac{1}{2} = A_1\left(\frac{1}{2} - j\frac{1}{2} - \frac{1}{2} + j\frac{1}{2}\right) + A_2\left(\frac{1}{2} - j\frac{1}{2} - \frac{1}{2} - j\frac{1}{2}\right) \tag{213}$$

$$A_2 = \frac{\frac{1}{2} - j\frac{1}{2}}{j} = \frac{1}{2} + j\frac{1}{2} \tag{214}$$

(In general, it holds that a complex-conjugate pair of poles $p_1 = p_2^*$ will result in complex-conjugate coefficients $A_1 = A_2^*$). We now have that

$$\frac{X(z)}{z} = \frac{\frac{1}{2} - j\frac{1}{2}}{z - (\frac{1}{2} + j\frac{1}{2})} + \frac{\frac{1}{2} + j\frac{1}{2}}{z - (\frac{1}{2} - j\frac{1}{2})} \tag{215}$$

Multiplying both sides with z, followed by reducing the fractions with z, yields

$$X(z) = \frac{\frac{1}{2} - j\frac{1}{2}}{1 - (\frac{1}{2} + j\frac{1}{2})z^{-1}} + \frac{\frac{1}{2} + j\frac{1}{2}}{1 - (\frac{1}{2} - j\frac{1}{2})z^{-1}} \tag{216}$$

Inverse z-transforming according to

$$a^n u(n) \leftrightarrow \frac{1}{1 - az^{-1}} \tag{217}$$

yields

$$y(n) = \left(\frac{1}{2} - j\frac{1}{2}\right)\left(\frac{1}{2} + j\frac{1}{2}\right)^n u(n) + \left(\frac{1}{2} + j\frac{1}{2}\right)\left(\frac{1}{2} - j\frac{1}{2}\right)^n u(n) \tag{218}$$

$$= \left[\left(\frac{1}{2} - j\frac{1}{2}\right)\left(\frac{1}{2} + j\frac{1}{2}\right)^n + \left(\frac{1}{2} + j\frac{1}{2}\right)\left(\frac{1}{2} - j\frac{1}{2}\right)^n\right] u(n) \tag{219}$$

It is assumed that $x(n)$ is real, and so the complex terms should be reducible to real ones. First they are expressed in polar form:

$$y(n) = \left[\frac{1}{\sqrt{2}} e^{-j\pi/4}\left(\frac{1}{\sqrt{2}} e^{j\pi/4}\right)^n + \frac{1}{\sqrt{2}} e^{j\pi/4}\left(\frac{1}{\sqrt{2}} e^{-j\pi/4}\right)^n\right] u(n) \tag{220}$$

Then, the expression is rearranged to isolate the imaginary parts:

$$y(n) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^n \left(e^{-j\pi/4} e^{jn\pi/4} + e^{j\pi/4} e^{-jn\pi/4}\right) u(n) \tag{221}$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^n \left(e^{j(n-1)\pi/4} + e^{-j(n-1)\pi/4}\right) u(n) \tag{222}$$

According to Euler:

$$\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2} \tag{223}$$

Thus

$$y(n) = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right)^{n} 2\cos\left((n-1)\frac{\pi}{4}\right)u(n) \tag{224}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n-1} \cos\left((n-1)\frac{\pi}{4}\right)u(n) \tag{225}$$

c)

$$X(z) = \frac{z^{-6} + z^{-7}}{1 - z^{-1}} \tag{226}$$

By inspection it is seen that the denominator already conforms to the formulaic z-transform

$$u(n) \leftrightarrow \frac{1}{1 - z^{-1}} \tag{227}$$

Thus, the fraction is split according to the individual terms of the numerator:

$$X(z) = \frac{z^{-6} + z^{-7}}{1 - z^{-1}} = \frac{z^{-6}}{1 - z^{-1}} + \frac{z^{-7}}{1 - z^{-1}} \tag{228}$$

By using the z-transform formula above, and taking into account the time-shifting property of the z-transform, $x(n)$ becomes

$$x(n) = u(n-6) + u(n-7) \tag{229}$$

In general, improper rational transform (the numerator is a same or higher order polynomial than the denominator) such as this X(z) can always be expressed as a polynomial added with a proper rational transform (which then can be expressed as a partial fraction, and then can be inverse z-transformed, along with the polynomial). Such a form can be reached using a long division, as follows:

$$\begin{array}{r|cccccccc}
 & -z^{-6} & -2z^{-5} & -2z^{-4} & -2z^{-3} & -2z^{-2} & -2z^{-1} & -2 & \\
\hline
-z^{-1}+1 & z^{-7} & +z^{-6} & & & & & & \\
- & z^{-7} & -z^{-6} & & & & & & \\
\cline{2-3}
 & & 2z^{-6} & & & & & & \\
 & - & 2z^{-6} & -2z^{-5} & & & & & \\
\cline{3-4}
 & & & 2z^{-5} & & & & & \\
 & & - & 2z^{-5} & -2z^{-4} & & & & \\
\cline{4-5}
 & & & & 2z^{-4} & & & & \\
 & & & - & 2z^{-4} & -2z^{-3} & & & \\
\cline{5-6}
 & & & & & 2z^{-3} & & & \\
 & & & & - & 2z^{-3} & -2z^{-2} & & \\
\cline{6-7}
 & & & & & & 2z^{-2} & & \\
 & & & & & - & 2z^{-2} & -2z^{-1} & \\
\cline{7-8}
 & & & & & & & 2z^{-1} & \\
 & & & & & & - & 2z^{-1} & -2 \\
\cline{8-9}
 & & & & & & & & 2
\end{array} \tag{230}$$

Resulting in

$$X(z) = -2 - 2z^{-1} - 2z^{-2} - 2z^{-3} - 2z^{-4} - 2z^{-5} - z^{-6} + 2\frac{1}{1 - z^{-1}} \tag{231}$$

(Note the form of a polynomial and a proper rational transform). These are readily inverse z-transformed as:

$$x(n) = -2\delta(n) - 2\delta(n-1) - 2\delta(n-2) - 2\delta(n-3) - 2\delta(n-4) - 2\delta(n-5) - \delta(n-6) + 2u(n) \tag{232}$$

Or in the form of samples

$$x(n) = \{\underline{0}, 0, 0, 0, 0, 0, 1, 2, 2, 2, \ldots\} \tag{233}$$

Which can be expressed as:

$$x(n) = u(n-6) + u(n-7) \tag{234}$$

d) The transform is improper and an easy modification that would help is hard to discern. Thus we perform a long division:

$$\begin{array}{rrr} & 2 & \\ \cline{2-3} z^{-2}+1 \big(& 2z^{-2} & +1 \\ - & 2z^{-2} & +2 \\ \cline{2-3} & & -1 \end{array} \tag{235}$$

Thus

$$X(z) = 2 - \frac{1}{1+z^{-2}} \tag{236}$$

The rightmost term is now a proper rational and transform a partial-fraction expansion will be helpful. The roots of the denominator should be found and first we expand the fraction with z^{-2}, and remove a z to the front of the expression.

$$\frac{1}{1+z^{-2}} = \frac{z^2}{z^2+1} = z\frac{z}{z^2+1} \tag{237}$$

The roots of the denominator are $z = \pm j$, and we have

$$z\frac{z}{z^2+1} = z\frac{z}{(z+j)(z-j)} \tag{238}$$

The fraction (excluding the prior z) should be expanded as such:

$$\frac{z}{(z-j)(z+j)} = \frac{A_1}{z-j} + \frac{A_2}{z+j} \tag{239}$$

Multiplying with the left-hand side denominator yields

$$z = A_1(z+j) + A_2(z-j) \tag{240}$$

Inserting the roots individually:

$$z = j: \tag{241}$$

$$j = A_1(j+j) + A_2(j-j) \tag{242}$$

$$A_1 = \frac{1}{2} \tag{243}$$

$$z = -j: \tag{244}$$

$$-j = A_1(-j+j) + A_2(-j-j) \tag{245}$$

$$A_2 = \frac{1}{2} \tag{246}$$

$X(z)$ becomes

$$X(z) = 2 - z\left(\frac{\frac{1}{2}}{z-j} + \frac{\frac{1}{2}}{z+j}\right) \tag{247}$$

Inserting the prior z and reducing both fractions with z produce

$$X(z) = 2 - \frac{1}{2}\frac{1}{1-jz^{-1}} - \frac{1}{2}\frac{1}{1+jz^{-1}} \tag{248}$$

Via table:

$$\delta(n) \leftrightarrow 1 \tag{249}$$

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{250}$$

$x(n)$ becomes

$$x(n) = 2\delta(n) - \frac{1}{2}(j)^n u(n) - \frac{1}{2}(-j)^n u(n) \tag{251}$$

Isolating the complex terms and writing them in polar form:

$$x(n) = 2\delta(n) - \frac{1}{2}\left(e^{jn\pi/2} - e^{-jn\pi/2}\right)u(n) \tag{252}$$

According to Euler:

$$\cos\phi = \frac{e^{j\phi} + e^{-j\phi}}{2} \tag{253}$$

Thus

$$x(n) = 2\delta(n) - \frac{1}{2}\left(2\cos(n\frac{\pi}{2})\right)u(n) = 2\delta(n) - \cos(n\frac{\pi}{2})u(n) \tag{254}$$

g) The rational transform is improper and there is no easily discernible simple solution. A long division is utilized until the remainder forms a proper rational transform (numerator polynom is of a lower order than the denominator polynom):

$$\begin{array}{r|lll} & \frac{1}{4} & & \\ \hline 4z^{-2}+4z^{-1}+1 & z^{-2} & +2z^{-1} & +1 \\ - & z^{-2} & +z^{-1} & +\frac{1}{4} \\ \hline & & z^{-1} & +\frac{3}{4} \end{array} \tag{255}$$

Thus

$$X(z) = \frac{1}{4} + \frac{\frac{3}{4} + z^{-1}}{1 + 4z^{-1} + 4z^{-2}} = \frac{1}{4} + \frac{\frac{3}{4}}{1 + 4z^{-1} + 4z^{-2}} + \frac{z^{-1}}{1 + 4z^{-1} + 4z^{-2}} \tag{256}$$

The fractions could individually be partial-fraction expanded as usual, however the denominator polynoms have double zeros (which, if not seen initially, becomes apparent when starting the partial-fraction expansion). I.e.

$$X(z) = \frac{1}{4} + \frac{\frac{3}{4}}{(1+2z^{-1})^2} + \frac{z^{-1}}{(1+2z^{-1})^2} \tag{257}$$

All the terms should be modifiable to fit the formulaic transformations

$$\delta(n) \leftrightarrow 1 \tag{258}$$

$$na^n u(n) \leftrightarrow \frac{z^{-1}}{(1-az^{-1})^2} \tag{259}$$

as such:

$$X(z) = \frac{1}{4} + -\frac{3}{8}z\frac{-2z^{-1}}{(1-(-2)z^{-1})^2} + -\frac{1}{2}\frac{-2z^{-1}}{(1-(-2)z^{-1})^2} \tag{260}$$

Taking into account time scaling, this gives

$$x(n) = \frac{1}{4}\delta(n) + \left(-\frac{3}{8}\right)(n+1)(-2)^{(n+1)}u(n+1) + \left(-\frac{1}{2}\right)n(-2)^n u(n) \tag{261}$$

$$= \frac{1}{4}\delta(n) + \frac{3}{4}(n+1)(-2)^n u(n+1) + n(-2)^{n-1}u(n) \tag{262}$$

which is a valid transformation. However, it can be simplified significantly. We start by trying to make the unit step functions alike. Investigating for $n = -1$, shows

$$x(0) = 0 + 0 + 0 = 0 \tag{263}$$

Which should be the case of a causal signal. It also follows that

$$\frac{3}{4}(n+1)(-2)^n u(n+1) = \frac{3}{4}(n+1)(-2)^n u(n) \tag{264}$$

Investigating for $n = 0$, further shows that

$$x(0) = \frac{1}{4} + \frac{3}{4} + 0 = 1 \tag{265}$$

Thus,

$$\frac{3}{4}(n+1)(-2)^n u(n) = \frac{3}{4}\delta(n) + \frac{3}{4}(n+1)(-2)^n u(n-1), \tag{266}$$

$$\text{and} \quad n(-2)^{n-1}u(n) = n(-2)^{n-1}u(n-1) \tag{267}$$

We then have that

$$x(n) = \delta(n) + \left(\frac{3}{4}(n+1)(-2)^n + n(-2)^{n-1}\right)u(n-1) \tag{268}$$

Simplify to final result:

$$x(n) = \delta(n) + \left(\frac{3}{4}(n+1)(-2)^n + n(-2)^{n-1}\right)u(n-1) \tag{269}$$

$$= \delta(n) + \left(\frac{3}{4}(n+1)(-2)^n + n(-2)^n(-2)^n\right)u(n-1) \tag{270}$$

$$= \delta(n) + (-2)^n\left(\frac{3}{4}(n+1) - \frac{n}{2}\right)u(n-1) \tag{271}$$

$$= \delta(n) + (n+3)(-2)^{n-2}u(n-1) \tag{272}$$

Solution 3.16

a) As a convolution in the time domain becomes a simple multiplication in the z-domain, it is advantageous to perform it there. First the individual signals are z-transformed. First they are modified to fit a formulaic transform. $x_1(n)$ becomes

$$x_1(n) = \left(\frac{1}{4}\right)^n u(n-1) = \left(\frac{1}{4}\right)^{n-1+1} u(n-1) = \frac{1}{4}\left(\frac{1}{4}\right)^{n-1} u(n-1) \tag{273}$$

Which fits with

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{274}$$

if simple scaling and a time shift of z^{-1} is used. Meanwhile, $x_2(n)$ can be written as

$$x_2(n) = \left(1 + \left(\frac{1}{2}\right)^n\right)u(n) = u(n) + \left(\frac{1}{2}\right)^n u(n) \tag{275}$$

Which works directly with

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{276}$$

$$\text{and} \quad a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{277}$$

Thus,

$$X_1(z) = \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}, \tag{278}$$

$$\text{and} \quad X_2(z) = \frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} \tag{279}$$

Our sought signal $x(n) = x_1(n) * x_2(n) \leftrightarrow X(z) = X_1(z)X_2(z)$.

$$X(z) = \frac{\frac{1}{4}z^{-1}}{1-\frac{1}{4}z^{-1}}\left(\frac{1}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}}\right) \tag{280}$$

This should be expressed in readily inverse z-transformable terms. First, the multiplication is performed

$$X(z) = \frac{\frac{1}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-z^{-1})} + \frac{\frac{1}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \tag{281}$$

These terms are individually partial-fraction expanded:

$$\begin{array}{l|l} X_a(z)=\frac{\frac{1}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-z^{-1})} & X_b(z)=\frac{\frac{1}{4}z^{-1}}{(1-\frac{1}{4}z^{-1})(1-\frac{1}{2}z^{-1})} \\ X_a(z)=z\frac{\frac{1}{4}}{(z-\frac{1}{4})(z-1)} & X_b(z)=z\frac{\frac{1}{4}}{(z-\frac{1}{4})(z-\frac{1}{2})} \\ \frac{X_a(z)}{z}=\frac{\frac{1}{4}}{(z-\frac{1}{4})(z-1)}=\frac{A_1}{z-\frac{1}{4}}+\frac{A_2}{z-1} & \frac{X_b(z)}{z}=\frac{\frac{1}{4}}{(z-\frac{1}{4})(z-\frac{1}{2})}=\frac{B_1}{z-\frac{1}{4}}+\frac{B_2}{z-\frac{1}{2}} \\ \frac{1}{4}=A_1(z-1)+A_2(z-\frac{1}{4}) & \frac{1}{4}=B_1(z-\frac{1}{2})+B_2(z-\frac{1}{4}) \\ z=1: \quad A_2=\frac{1}{3} & z=\frac{1}{2}: \quad B_2=1 \\ z=\frac{1}{4}: \quad A_1=-\frac{1}{3} & z=\frac{1}{4}: \quad B_1=-1 \\ X_a(z)=z\left(\frac{-\frac{1}{3}}{z-\frac{1}{4}}+\frac{\frac{1}{3}}{z-1}\right) & X_b(z)=z\left(\frac{-1}{z-\frac{1}{4}}+\frac{1}{z-\frac{1}{2}}\right) \end{array} \tag{282}$$

$X(z)$ can thus be expressed as

$$X(z)=X_a(z)+X_b(z)=z\left(\frac{-\frac{1}{3}}{z-\frac{1}{4}}+\frac{\frac{1}{3}}{z-1}\right)+z\left(\frac{-1}{z-\frac{1}{4}}+\frac{1}{z-\frac{1}{2}}\right) \tag{283}$$

$$=\frac{-\frac{1}{3}}{1-\frac{1}{4}z^{-1}}+\frac{\frac{1}{3}}{1-z^{-1}}+\frac{-1}{1-\frac{1}{4}z^{-1}}+\frac{1}{1-\frac{1}{2}z^{-1}} \tag{284}$$

$$\tag{285}$$

Which is inverse z-transformable according to

$$u(n) \leftrightarrow \frac{1}{1-z^{-1}} \tag{286}$$

$$\text{and} \quad a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{287}$$

giving

$$x(n)=-\frac{1}{3}\left(\frac{1}{4}\right)^n u(n)+\frac{1}{3}u(n)-\left(\frac{1}{4}\right)^n u(n)+\left(\frac{1}{2}\right)^n u(n) \tag{288}$$

$$=\left(-\frac{1}{3}\left(\frac{1}{4}\right)^n+\frac{1}{3}-\left(\frac{1}{4}\right)^n+\left(\frac{1}{2}\right)^n\right)u(n) \tag{289}$$

$$=\left(\frac{1}{3}-\frac{1}{3}\left(\frac{1}{4}\right)^{n-1}+\left(\frac{1}{2}\right)^n\right)u(n) \tag{290}$$

and since

$$\begin{array}{rcccccccc} n= & 0 & 1 & 2 & 3 & 4 & 5 & \dots \\ x(n)= & 0 & 0.5 & 0.5 & 0.4375 & 0.39.. & 0.36.. & \dots \end{array} \tag{291}$$

the signal can be expressed as

$$x(n)=\left(\frac{1}{3}-\frac{1}{3}\left(\frac{1}{4}\right)^{n-1}+\left(\frac{1}{2}\right)^n\right)u(n-1) \tag{292}$$

c) The signals are directly transformable according to

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}}, \quad \text{and} \tag{293}$$

$$\cos(\omega_0 n)u(n) \leftrightarrow \frac{1-z^{-1}\cos(\omega_0)}{1-2z^{-1}\cos(\omega_0)+z^{-2}} \tag{294}$$

$X_1(z)$ and $X_2(z)$ become

$$X_1(z)=\frac{1}{1-0.5z^{-1}} \tag{295}$$

$$X_2(z)=\frac{1-z^{-1}\cos(\pi)}{1-2z^{-1}\cos(\pi)+z^{-2}}=\frac{1+z^{-1}}{1+2z^{-1}+z^{-2}} \tag{296}$$

Multiplication in the z-domain is equivalent to convolution in the time domain:

$$X(z) = X_1(z)X_2(z) = \frac{1}{1-0.5z^{-1}}\frac{1+z^{-1}}{1+2z^{-1}+z^{-2}} \tag{297}$$

$$= \frac{1+z^{-1}}{(1-0.5z^{-1})(1+2z^{-1}+z^{-2})} \tag{298}$$

By finding the roots of the second denominator polynomial, we factorize and get

$$X(z) = \frac{1+z^{-1}}{(1-0.5z^{-1})(1+z^{-1})^2} \tag{299}$$

One zero and one node cancel each other out:

$$X(z) = \frac{1}{(1-0.5z^{-1})(1+z^{-1})} \tag{300}$$

Partial-fraction expansion is performed as

$$X(z) = \frac{z^2}{(z-0.5)(z+1)} \tag{301}$$

$$\frac{X(z)}{z} = \frac{z}{(z-0.5)(z+1)} = \frac{A_1}{(z-0.5)} + \frac{A_2}{(z+1)} \tag{302}$$

$$z = A_1(z+1) + A_2(z-0.5) \tag{303}$$

$$z = -1: \quad A_2 = \frac{2}{3} \tag{304}$$

$$z = 0.5: \quad A_1 = \frac{1}{3} \tag{305}$$

$X(z)$ becomes

$$\frac{X(z)}{z} = \frac{A_1}{(z-0.5)} + \frac{A_2}{(z+1)} \tag{306}$$

$$X(z) = \frac{A_1}{(1-0.5z^{-1})} + \frac{A_2}{(1+z^{-1})} \tag{307}$$

These terms are transformable according to:

$$a^n u(n) \leftrightarrow \frac{1}{1-az^{-1}} \tag{308}$$

Resulting in

$$x(n) = A_1\left(\frac{1}{2}\right)^n u(n) + A_2(-1)^n u(n) \tag{309}$$

$$= \left(A_1\left(\frac{1}{2}\right)^n + A_2(-1)^n\right)u(n) \tag{310}$$

Inserting the coefficient values:

$$= \left(\frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}(-1)^n\right)u(n) \tag{311}$$

Which can be written as

$$= \left(\frac{1}{3}\left(\frac{1}{2}\right)^n + \frac{2}{3}\cos(\pi n)\right)u(n) \tag{312}$$

Solution 3.35

a) $y_{zs}(n) = h * x(n)$ when the system is at *rest*, that is,

$$y(-\ell) = 0 \quad \ell = 1 \dots N \tag{313}$$

$$Y_{zs}(z) = H(z) \cdot X(z) = \frac{1}{1 - \frac{1}{3}z^{-1}} \cdot \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} \tag{314}$$

$$= \left[\frac{A}{1 - \frac{1}{3}z^{-1}} + \frac{B + Cz^{-1}}{1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}} \right] \tag{315}$$

$$= \frac{A - \frac{1}{2}Az^{-1} + \frac{1}{4}Az^{-2} + B + Cz^{-1} - \frac{1}{3}Bz^{-1} - \frac{1}{3}Cz^{-2}}{\left(1 - \frac{1}{3}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2}\right)} \tag{316}$$

Identification of coefficients gives

$$\left.\begin{aligned} A + B = 1 \\ -\frac{1}{2}A - \frac{1}{3}B + C = -\frac{1}{4} \\ \frac{1}{4}A - \frac{1}{3}C = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} A &= \frac{1}{7} \\ B &= \frac{6}{7} \\ C &= \frac{3}{28} \end{aligned} \tag{317}$$

$$Y_{zs}(z) = \frac{1/7}{1 - 1/3\ z^{-1}} + \frac{6/7 + 3/28\ z^{-1}}{(1 - 1/2\ z^{-1} + 1/4\ z^{-2})} \tag{318}$$

$$= \frac{1/7}{1 - 1/3\ z^{-1}} + \frac{6}{7} \cdot \frac{1 + 1/8\ z^{-1}}{1 - 1/2\ z^{-1} + 1/4\ z^{-2}} \tag{319}$$

$$= \frac{1}{7} \cdot \frac{1}{1 - 1/3\ z^{-1}} + \frac{6}{7} \cdot \frac{1 - 1/4\ z^{-1}}{1 - 1/2\ z^{-1} + 1/4\ z^{-2}} + \frac{3\sqrt{3}}{7} \cdot \frac{\sqrt{3}/4\ z^{-1}}{1 - 1/2\ z^{-1} + 1/4\ z^{-2}} \tag{320}$$

$$Y_{zs}(z) \xrightarrow{z^{-1}} y_{zs}(n) = \left[\frac{1}{7}\left(\frac{1}{3}\right)^n + \frac{6}{7}\left(\frac{1}{2}\right)^n \cos\left(n\frac{\pi}{3}\right) + \frac{3\sqrt{3}}{7}\left(\frac{1}{2}\right)^n \sin\left(n\frac{\pi}{3}\right) \right] u(n) \tag{321}$$

d)

$$\left.\begin{aligned} y(n) &= \frac{1}{2} \cdot x(n) - \frac{1}{2} \cdot x(n-1) \\ x(n) &= 10\cos\left(\frac{\pi}{2}n\right)u(n) \end{aligned}\right\} \Rightarrow y_{zs}(n) = 5\cos\left(\frac{\pi}{2}n\right)u(n) - 5\cos\left(\frac{\pi}{2}(n-1)\right)u(n-1) \tag{322}$$

Solution 3.40

A causal LTI system

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} \cdot u(n-1) \tag{323}$$

$$y(n) = \left(\frac{1}{3}\right)^n \cdot u(n) \tag{324}$$

a) Causal in, causal out: initial value $= 0$.

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} = \frac{1 - \frac{1}{4}z^{-1}}{1 - \frac{1}{2}z^{-1}} \tag{325}$$

$$Y(z) = \frac{1}{1-\frac{1}{3}z^{-1}} \tag{326}$$

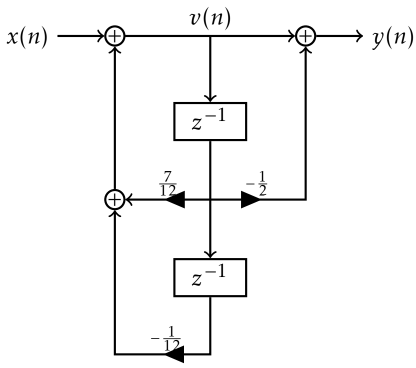
$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\frac{1}{3}z^{-1}} \cdot \frac{1-\frac{1}{2}z^{-1}}{1-\frac{1}{4}z^{-1}} = \tag{327}$$

$$= \frac{1-\frac{1}{2}z^{-1}}{1-\frac{7}{12}z^{-1}+\frac{1}{12}z^{-2}} = \frac{-2}{1-\frac{1}{3}z^{-1}} + \frac{3}{1-\frac{1}{4}z^{-1}} \tag{328}$$

$$H(z) \xrightarrow{\text{z}^{-1}} h(n) = \left[-2\left(\frac{1}{3}\right)^n + 3\left(\frac{1}{4}\right)^n\right] \cdot u(n) \tag{329}$$

b) $y(n) - \frac{7}{12}y(n-1) + \frac{1}{12}y(n-2) = x(n) - \frac{1}{2}x(n-1)$

c) Realization:



d)

$$\left.\begin{matrix} |\text{poles}| < 1 \\ h(n) \text{ causal} \end{matrix}\right\} \Rightarrow \text{stable} \tag{330}$$

Solution 3.49

b)

$$Y^+(z) - 1.5\left[z^{-1}Y^+(z) + 1\right] + 0.5\left[z^{-2}Y^+(z) + z^{-1}\right] = 0 \tag{331}$$

$$Y^+(z) - 1.5z^{-1}Y^+(z) + 0.5z^{-2}Y^+(z) = 1.5 - 0.5z^{-1} \tag{332}$$

$$Y^+(z) = \frac{1.5-0.5z^{-1}}{1-1.5z^{-1}+0.5z^{-2}} = \frac{2}{1-z^{-1}} + \frac{-1/2}{1-1/2\;z^{-1}} \tag{333}$$

$$Y^+(z) \xrightarrow{\text{z}^{-1}} y_{zi}(n) = \left(2 - \frac{1}{2}\left(\frac{1}{2}\right)^n\right) \cdot u(n) \tag{334}$$

c)

$$Y^+(z) = \frac{1}{2}\;Y^+(z)z^{-1} + \frac{1}{2}\;y(-1) + \frac{1}{1-\frac{1}{3}z^{-1}} \tag{335}$$

$$Y^+(z) = \frac{1}{2}\;z^{-1}Y^+(z) + \frac{1}{2} + \frac{1}{1-\frac{1}{3}z^{-1}} \tag{336}$$

$$Y^+(z) = \frac{1}{2\left(1-\frac{1}{2}z^{-1}\right)} + \frac{1}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{2}z^{-1}\right)} \tag{337}$$

$$= \frac{1-\frac{1}{3}z^{-1}+2}{2\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = \frac{7/2}{1-\frac{1}{2}z^{-1}} + \frac{-2}{1-\frac{1}{3}z^{-1}} \tag{338}$$

$$Y^+(z) \xrightarrow{\text{z}^{-1}} y(n) = \frac{7}{2}\left(\frac{1}{2}\right)^n \cdot u(n) - 2\left(\frac{1}{3}\right)^n \cdot u(n) \tag{339}$$

d)

$$Y^+(z) = \frac{1}{4}Y^+(z)z^{-2} + \frac{1}{4}z^{-1}y(-1) + \frac{1}{4}y(-2) + \frac{1}{1-z^{-1}} \tag{340}$$

$$Y^+(z) = \frac{1}{4}z^{-2}Y^+(z) + \frac{1}{4} + \frac{1}{1-z^{-1}} \tag{341}$$

$$Y^+(z) = \frac{1}{4\left(1-\frac{1}{4}z^{-2}\right)} + \frac{1}{\left(1-z^{-1}\right)\left(1-\frac{1}{4}z^{-2}\right)} \tag{342}$$

$$= \frac{1-z^{-1}+4}{4\left(1-\frac{1}{4}z^{-2}\right)\left(1-z^{-1}\right)} \tag{343}$$

$$= \frac{-3/8}{1-\frac{1}{2}z^{-1}} + \frac{7/24}{1+\frac{1}{2}z^{-1}} + \frac{4/3}{1-z^{-1}} \tag{344}$$

$$Y^+(z) \xrightarrow{\text{z}^{-1}} y(n) = -\frac{3}{8}\left(\frac{1}{2}\right)^n \cdot u(n) + \frac{7}{24}\left(-\frac{1}{2}\right)^n \cdot u(n) + \frac{4}{3} \cdot u(n) \tag{345}$$

Solution E3.1

A-III, B-I, C-II

The longer the distance from a pole to the unit circle, the more damped the impulse response will be. A double pole on the unit circle gives an unstable system, which is the reason that poles should never be allowed on the unit circle, even if the system is not explicitly unstable. An input signal pole at the same location will give an unlimited output signal.

Solution E3.2

a)

$$x(n) = 3\sin\left(\frac{\pi}{2}n\right)u(n) \xrightarrow{\text{z}} X(z) = 3\frac{z^{-1}}{1+z^{-2}} \tag{346}$$

$$y(n) = -\frac{1}{2}y(n-1) + \frac{1}{5}x(n-1) \quad \text{där } y(-1) = \frac{1}{3} \tag{347}$$

$$Y^+(z) = -\frac{1}{2}z^{-1}\left[Y^+(z) + y(-1)\cdot z\right] + \frac{1}{5}z^{-1}X(z) \tag{348}$$

$$Y^+(z) = \frac{-\frac{1}{6}}{1+\frac{1}{2}z^{-1}} + \frac{\frac{3}{5}z^{-2}}{\left(1+\frac{1}{2}z^{-1}\right)(1+z^{-2})} \tag{349}$$

$$= \frac{-\frac{1}{6}}{1+\frac{1}{2}z^{-1}} + \frac{3}{5}z^{-1}\left[\frac{2}{5}\frac{1+2z^{-1}}{1+z^{-2}} - \frac{2}{5}\frac{1}{1+\frac{1}{2}z^{-1}}\right] \tag{350}$$

$$y(n) = -\frac{1}{6}\left(-\frac{1}{2}\right)^n u(n) + \frac{6}{25}\left[\cos\left(\frac{\pi}{2}(n-1)\right) + 2\sin\left(\frac{\pi}{2}(n-1)\right) - \left(-\frac{1}{2}\right)^{n-1}\right]u(n-1) \tag{351}$$

b) $y(n) = 0$, that is, a zero at $\omega_0 = 2\pi f_0$.

$$T(z) = b_0 + b_1 z^{-1} + b_2 z^{-2} = 2\left(1 + z^{-1} + z^{-2}\right) = 0 \tag{352}$$

$$z_{1,2} = \frac{-1 \pm \mathrm{j}\sqrt{3}}{2} = \mathrm{e}^{\pm \mathrm{j}2\pi\ 1/3} \quad \text{gives } f_0 = \frac{1}{3} \tag{353}$$

c) A pole on the unit circle at f_1.

$$N(z) = 1 + a_1 z^{-1} + a_2 z^{-2} = 1 - z^{-1} + z^{-2} = 0 \tag{354}$$

$$z_{1,2} = \frac{1 \pm \mathrm{j}\sqrt{3}}{2} = \mathrm{e}^{\pm \mathrm{j}2\pi\ 1/6} \quad \text{gives } f_1 = \frac{1}{6} \tag{355}$$

Solution E3.3

$$y(n) - y(n-1) + \frac{3}{16}\, y(n-2) = x(n) \tag{356}$$

$$Y(z)\left(1 - z^{-1} + \frac{3}{16} z^{-2}\right) = X(z) \tag{357}$$

$$Y(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{3}{4} z^{-1}\right)} \cdot X(z) \tag{358}$$

Poles:

$$p_{1,2} = \begin{cases} 1/4 \\ 3/4 \end{cases} \tag{359}$$

$$Y(z) = \frac{1}{\left(1 - \frac{1}{4} z^{-1}\right)\left(1 - \frac{3}{4} z^{-1}\right)} \cdot X(z) \tag{360}$$

Let $x(n) = x_1(n) + x_2(n)$ where

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n) \tag{361}$$

$$x_2(n) = \sin\left(2\pi \frac{1}{4} n\right) \tag{362}$$

$$X_1(z) = \frac{1}{1 - \frac{1}{2}\, z^{-1}} \Rightarrow Y_1(z) = \frac{\frac{1}{2}}{1 - \frac{1}{4} z^{-1}} + \frac{\frac{9}{2}}{1 - \frac{3}{4} z^{-1}} - \frac{4}{1 - \frac{1}{2} z^{-1}} \tag{363}$$

$$H\left(\omega = 2\pi \frac{1}{4}\right) = \frac{1}{1 - \mathrm{e}^{-\mathrm{j}2\pi \frac{1}{4}} + \frac{3}{16} \mathrm{e}^{-\mathrm{j}2\frac{\pi}{4}2}} = \frac{1}{\frac{13}{16} + \mathrm{j}} = 0.776\mathrm{e}^{-\mathrm{j}0.888} \tag{364}$$

$$y(n) = \left(\frac{1}{2}\left(\frac{1}{4}\right)^n + \frac{9}{2}\left(\frac{3}{4}\right)^n - 4\left(\frac{1}{2}\right)^n\right) \cdot u(n) + 0.776 \sin\left(2\pi \frac{1}{4} n - 0.888\right) \tag{365}$$

Solution E3.4

$$H(z) = \frac{1}{1 - z^{-1} + 0.5 z^{-2}} \tag{366}$$

Poles: $p_{1,2} = \frac{1}{\sqrt{2}} \mathrm{e}^{\pm \mathrm{j}\frac{\pi}{4}}$

a)

$$y_{zi}(n) = -\left(\frac{1}{\sqrt{2}}\right)^n \left[\cos(\frac{\pi}{4} n) + \sin(\frac{\pi}{4} n)\right] u(n) \tag{367}$$

b)

$$y_{zs}(n) = z^{-1} \cdot H(z) \cdot X(z) \tag{368}$$

$$= z^{-1}\left[\frac{0.5z^{-1} - (1-0.5z^{-1})}{1 - z^{-1} + 0.5z^{-2}} + \frac{2}{1-z^{-1}}\right] \tag{369}$$

$$= \left(\frac{1}{\sqrt{2}}\right)^n \cdot \left(\sin\frac{\pi}{4}n - \cos\frac{\pi}{4}n\right) \cdot u(n) + 2u(n) \tag{370}$$

c)

$$y_{tr} = \left(\frac{1}{\sqrt{2}}\right)^{n-1} \cos\left(\frac{\pi}{4}n + \frac{3\pi}{4}\right) + \left(\frac{1}{\sqrt{2}}\right)^n \cdot \sin\frac{\pi}{4}n - \cos\frac{\pi}{4}n \tag{371}$$

$$= -(\frac{1}{\sqrt{2}})^n 2\cos\frac{\pi}{4}n,\ n \geq 0. \tag{372}$$

d)

$$y_{ss} = 2u(n) \tag{373}$$

Solution E3.5

$$y(n) - \frac{1}{4}\, y(n-1) = x(n) \tag{374}$$

$$Y(z)\left(1 - \frac{1}{4}z^{-1}\right) = X(z) \quad \text{where } p = \frac{1}{4} \tag{375}$$

$$H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} \quad \Rightarrow \quad H(\omega) = \frac{1}{1 - \frac{1}{4}\,\mathrm{e}^{-\mathrm{j}\omega}} \tag{376}$$

For $n < 0$:

$$H\left(2\pi\frac{1}{4}\right) = \frac{1}{1 - \frac{1}{4}\mathrm{e}^{-\mathrm{j}2\pi\frac{1}{4}}} = \frac{1}{1 + \frac{1}{4}\mathrm{j}} = \frac{4}{\sqrt{17}}\mathrm{e}^{-\mathrm{j}\arctan\frac{1}{4}} = 0.97\mathrm{e}^{-\mathrm{j}0.24} \tag{377}$$

$$y(n) = 0.97\sin\left(2\pi\frac{1}{4}n - 0.24\right) \quad \Rightarrow \quad y(-1) = -\frac{4}{\sqrt{17}} \cdot \frac{4}{\sqrt{17}} = -\frac{16}{17} = 0.94 \tag{378}$$

For $n \geq 0$:

$$Y^+(n) = \frac{1}{1 - 1/4\, z^{-1}} \cdot \frac{1}{4}y(-1) = -\frac{1}{4} \cdot \frac{1}{1 - 1/4\, z^{-1}} \quad \Rightarrow \quad y(n) = -\frac{4}{17}\left(\frac{1}{4}\right)^n \cdot u(n) \tag{379}$$

Fourier Transform, LTI Systems, and Sampling, Chapters 4, 5, and 6

Solution N1

Here the input signal can be divided as two parts, the sinusoid part and the unit step part. Since the given system is shown as the LTI system, we can consider this two parts separately. In other words, the response to each part should be calculated separately and combined together later.
The input signal can be divided as:

$$x(n) = x_1(n) + x_2(n); \quad x_1(n) = u(n-1) \quad x_2(n) = sin(2\pi\frac{1}{4}n + \frac{\pi}{4})$$

Then the output signal can be represented as

$$y(n) = y_1(n) + y_2(n) \tag{380}$$

where $y_1(n)$ represents the response of input $x_1(n)$ while $y_2(n)$ represents $x_2(n)$. Regarding to $y_1(n)$, we can calculate it by the z transform

$$\begin{aligned} y_1(n) &= \mathcal{Z}^{-1}\Big(X_1(z)H(z)\Big) \\ &= \mathcal{Z}^{-1}\Big[\frac{z^{-1}}{(1-0.5z^{-1})(1+0.5z^{-1})}\Big] \\ &= \mathcal{Z}^{-1}\Big[\frac{1}{1-0.5z^{-1}} - \frac{1}{1+0.5z^{-1}}\Big] \\ &= (0.5)^n - (-0.5)^n \end{aligned} \tag{381}$$

Regarding to the $y_2(n)$, we can leverage the frequency response of the system. Since the input signal is sinusoid, the amplitude and phase response can be calculated directly.
The frequency response of the system can be calculated as:

$$H(e^{jw}) = \frac{1-e^{-j\omega}}{(1-0.5e^{-j\omega})(1+0.5e^{j\omega})} \tag{382}$$

when $w = \frac{\pi}{2}$, we can calculate the system response as:

$$H(e^{jw})|_{w=\frac{\pi}{2}} = \frac{1-e^{-j\frac{\pi}{2}}}{(1-0.5e^{-j\frac{\pi}{2}})(1+0.5e^{j\frac{\pi}{2}})} = 0.8 + j0.8 = 1.1314\angle\frac{\pi}{4} \tag{383}$$

Therefore, the second part of the output signal $y_2(n)$ can be represented as

$$y_2(n) = 1.1314\, sin(2\pi\frac{1}{4}n + \frac{\pi}{2}) \tag{384}$$

Finally the output signal is

$$y(n) = y_1(n) + y_2(n) = (0.5)^n - (-0.5)^n + 1.1314\, sin(2\pi\frac{1}{4}n + \frac{\pi}{2}) \tag{385}$$

Solution N2

By applying similar method, we divide the input signal as:

$$x(n) = x_1(n) + x_2(n); \quad x_1(n) = 0.5^{n-1}u(n-1) \quad x_2(n) = sin(2\pi\frac{1}{3}n)$$

Then the output signal can be represented as

$$y(n) = y_1(n) + y_2(n) \tag{386}$$

where $y_1(n)$ represents the response of input $x_1(n)$ while $y_2(n)$ represents $x_2(n)$. Regarding to $y_1(n)$, we can calculate it by the z transform

$$\begin{aligned} y_1(n) &= \mathcal{Z}^{-1}\Big(X_1(z)H(z)\Big) \\ &= \mathcal{Z}^{-1}\Big[\frac{z^{-1}}{1-0.5z^{-1}}\frac{1-0.5z^{-1}}{1-z^{-1}}\Big] \\ &= \mathcal{Z}^{-1}\Big[\frac{z^{-1}}{1-z^{-1}}\Big] \\ &= u(n-1) \end{aligned} \tag{387}$$

The frequency response of the system can be calculated as:

$$H(e^{jw}) = \frac{1-0.5e^{-j\omega}}{1-e^{-j\omega}} \tag{388}$$

when $w = \frac{2\pi}{3}$, we can calculate the system response as:

$$H(e^{jw})|_{w=\frac{2\pi}{3}} = \frac{1-0.5e^{-j\frac{2\pi}{3}}}{1-e^{-j\frac{2\pi}{3}}} = 0.75 - j0.1443 = 0.7638\angle-0.19 \tag{389}$$

Therefore, the second part of the output signal $y_2(n)$ can be represented as

$$y_2(n) = 0.7638\ sin(2\pi \frac{2}{3} n - 0.19) \tag{390}$$

Finally the output signal is

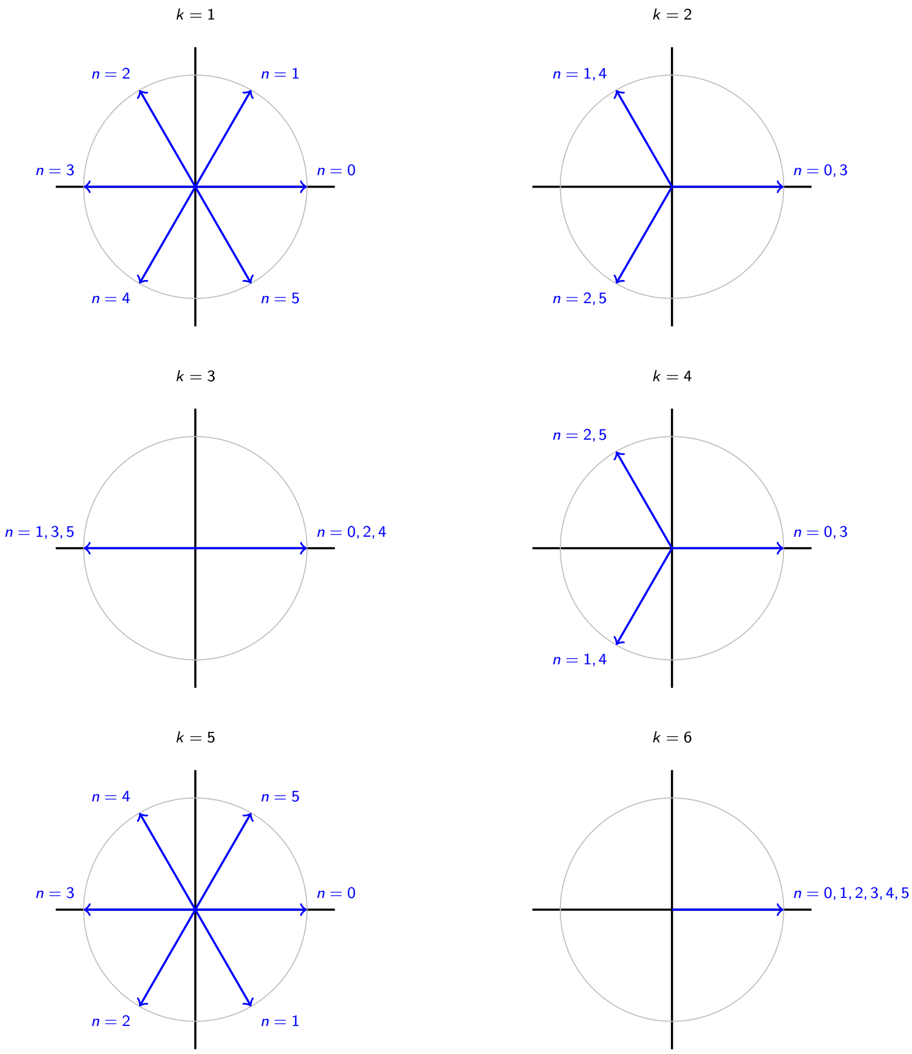
$$y(n) = y_1(n) + y_2(n) = u(n-1) + 0.7638\ sin(2\pi \frac{2}{3} n - 0.19) \tag{391}$$

Solution 4.8

a) This part can be solved by using the geometry sum, but we need to analyze this problem by two cases since $e^{j2\pi k/N} = 1$ when $k = 0, \pm N, \pm 2N, \cdots = \ell N,\ \ell \in \mathbf{Z}$.

$$\sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{j}\pi kn/N} = \begin{cases} \dfrac{1 - \mathrm{e}^{\mathrm{j}2\pi k/N \cdot N}}{1 - \mathrm{e}^{\mathrm{j}2\pi k/N}} = 0 & \text{for } k \neq 0, \pm N, \pm 2N, \ldots \\ \displaystyle\sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi \ell N n/N} = \sum_{n=0}^{N-1} 1 = N & \text{for } k = 0, \pm N, \pm 2N, \cdots = \ell N,\ \ell \in \mathbf{Z} \end{cases} \tag{392}$$

b) Here note that when k is fixed, the signal regarding to n can be plotted on the complex plane because each n will result in a complex number.



c) Here we need to be aware that the harnomic signal can be represented by $e^{j(2\pi/N)\ell n}$, where $l = mk$ and $m = 2, 3, ...N$, then by using the geometric sum, we can get:

$$\sum_{n=N_1}^{N_1+N-1} e^{j(2\pi/N)kn} \cdot e^{-j(2\pi/N)\ell n} = \sum_{n=N_1}^{N_1+N-1} e^{j(2\pi/N)(k-\ell)n} = \begin{cases} N & k-\ell = 0, \pm N, \pm 2N \\ 0 & \text{f.ö.} \end{cases} \tag{393}$$

$$\sum_{n=N_1}^{N_1+N-1} e^{j(2\pi/N)(k-\ell)n} = \begin{cases} N & k = \ell \text{ (ty } s_k(n) = s_{k+N}(n)) \\ 0 & \text{otherwise} \end{cases} \tag{394}$$

Solution 4.9

a)

$$X(\omega) = \sum_{n=-\infty}^{\infty} (u(n) - u(n-6))e^{-j\omega n} \tag{395}$$

$$= \sum_{n=0}^{5} e^{-j\omega n} = \frac{1 - e^{-j\omega 6}}{1 - e^{-j\omega}} = \frac{e^{-jw3}(e^{j\omega 3} - e^{-j\omega 3})}{e^{-j\frac{\omega}{2}}(e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}})} \tag{396}$$

$$= \frac{\sin(3\omega)}{\sin\left(\frac{\omega}{2}\right)} \cdot e^{-j\frac{5\omega}{2}} \tag{397}$$

b) Let $p = -n$, then $x(n) = x(-p) = 2^{-p}u(p)$, then we can leverage the z transform, such as

$$\mathscr{Z}\{x(-p)\} = \frac{1}{1 - \frac{1}{2}z^{-1}} \quad \mathscr{Z}\{x(p)\} \ = X(\frac{1}{z}) \ = \frac{1}{1 - \frac{1}{2}z} \tag{398}$$

Then we utilize the relationship between the z transform and discrete fourier transform $z = e^{jw}$, we can get easily

$$X(\omega) = \frac{1}{1 - \frac{1}{2}e^{j\omega}} \tag{399}$$

c) We first write the original signal as $x(n) = (0.25)^{-4}(0.25)^{n+4}u(n+4)$, then we can use the z transform so that:

$$X(z) = 256\frac{1}{1 - 0.25z^{-1}}z^4 \tag{400}$$

Then we use the same technology, we can get:

$$X(\omega) = \frac{256e^{j4\omega}}{1 - \frac{1}{4}e^{-j\omega}} \tag{401}$$

d)

$$X(\omega) = \sum_{n=-\infty}^{\infty} (\alpha^n \sin\omega_0 n) \cdot u(n) \cdot e^{-j\omega n} \tag{402}$$

$$= \sum_{n=0}^{\infty} \frac{\alpha^n e^{j\omega_0 n} - \alpha^n e^{-j\omega_0 n}}{2j} \cdot e^{-j\omega n} \tag{403}$$

$$= \frac{1}{2j(1 - \alpha e^{j\omega_0}e^{-j\omega})} - \frac{1}{2j(1 - \alpha e^{-j\omega_0 e^{-j\omega})}} \tag{404}$$

$$= \frac{1 - \alpha e^{-j\omega_0}e^{-j\omega} - 1 + \alpha e^{j\omega_0}e^{-j\omega}}{2j(1 - \alpha e^{-j\omega_0}e^{-j\omega} - \alpha e^{j\omega_0}e^{-j\omega} + \alpha^2 e^{-j2\omega})} \tag{405}$$

$$= \frac{\alpha \sin\omega_0 e^{-j\omega}}{1 - 2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} \tag{406}$$

Observe that one alternative way for achieving the result is to calculate the z transform and get the result immediately.

g)

$$X(\omega) = \sum_{n=-2}^{n=2} x(n)e^{j\omega n} \tag{407}$$

$$= -2e^{-j2\omega} - e^{-j\omega} + e^{j\omega} + 2e^{j2\omega} \tag{408}$$

$$= -2\mathrm{j}(\sin\omega + 2\sin 2\omega) \tag{409}$$

Solution 4.10

a)

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\mathrm{e}^{\mathrm{j}\omega n} d\omega = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)\cos(\omega n) + \mathrm{j}X(\omega)\sin(\omega n) d\omega \tag{410}$$

$$= \left[\mathrm{j}\int_{-\pi}^{\pi} X(\omega)\sin(\omega n) d\omega = 0 \quad \text{observe that } X(\omega) \text{ is even function and } \sin(\omega n) \text{ is odd function}\right] \tag{411}$$

$$= \frac{2}{2\pi}\int_{\omega_0}^{\pi} 1\cdot\cos(\omega n) d\omega = \frac{1}{\pi}\left[\frac{\sin(\omega n)}{n}\right]_{\omega_0}^{\pi} \tag{412}$$

$$= \frac{\sin(\pi n)}{\pi n} - \frac{\sin(\omega_0 n)}{\pi n} = \delta(n) - \frac{\sin(\omega_0 n)}{\pi n} \tag{413}$$

b)

$$X(\omega) = \cos^2\omega = \frac{1}{2} + \frac{1}{2}\cdot\cos 2\omega \tag{414}$$

$$= \frac{1}{2} + \frac{1}{4}\cdot\mathrm{e}^{\mathrm{j}\omega 2} + \frac{1}{4}\cdot\mathrm{e}^{-\mathrm{j}\omega 2} \tag{415}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} \quad \text{only the terms with } n = 0, -2, 2 \text{ exist} \tag{416}$$

$$x(n) = \frac{1}{2}\delta(n) + \frac{1}{4}\delta(n+2) + \frac{1}{4}\delta(n-2) \tag{417}$$

Solution 4.12

c) Multiplication with $\mathrm{e}^{\mathrm{j}\omega_c n}$ in the time domain gives a shift ω_c in the frequency domain.
Let $X_L(\omega)$ be an idela low pass filter with a cutoff frequency $W/2$ and amplitude 2.

$$\left.\begin{aligned} &x_L(n) = \frac{1}{2\pi}\int_{-W/2}^{W/2} 2\mathrm{e}^{\mathrm{j}\omega n} d\omega = 2\cdot\frac{\sin(\frac{W}{2}n)}{\pi n} \\ &2\cos\omega_c n = \mathrm{e}^{\mathrm{j}\omega_c n} + \mathrm{e}^{-\mathrm{j}\omega_c n} \end{aligned}\right\} \Rightarrow \mathcal{F}\{x_L(n)\cdot 2\cos\omega_c n\} = X(\omega) \tag{418}$$

This means

$$x(n) = 4\cdot\frac{\sin\left(\frac{W}{2}n\right)}{\pi n}\cdot\cos\omega_c n \tag{419}$$

Solution 4.14

a) This is the sum of all the signals, since we can get from the definition that $X(\omega) = \sum x(n)e^{-i\omega n}$, the result is therefore $X(0) = -1$.

b) This one I do not think it can completely avoid calculation, we can discuss this later, the result is $\arg X(\omega) = \pi \qquad [X(\omega) \text{ real and negative for all } \omega]$

c) Here since $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$ by definition, then let $n = 0$. Because we have already known the value $x(0)$, it is easy to get the result: $\int_{-\pi}^{\pi} X(\omega)d\omega = -6\pi$

d) By definition $X(\pi) = \sum_{n=0}^{+\infty} x(n)e^{j\omega n} = x(0) - x(1) + x(2) - x(-1) + x(-2)$. Then we can get result as: $X(\pi) = -9$

e) $\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega = 2\pi \sum_n |x(n)|^2 = 38\pi \qquad [see Parseval's formula]$

Solution 5.2

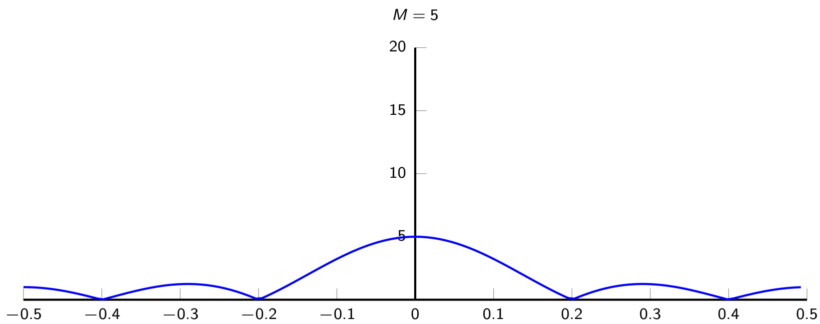
a)

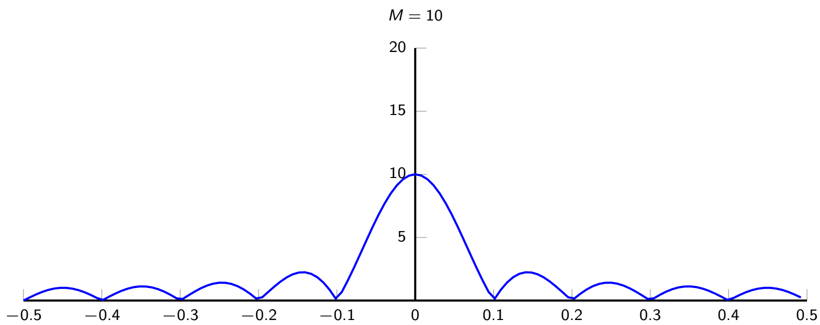
$$W_R(\omega) = \sum_{n=-\infty}^{\infty} w_R(n)\mathrm{e}^{-\mathrm{j}\omega n} \tag{420}$$

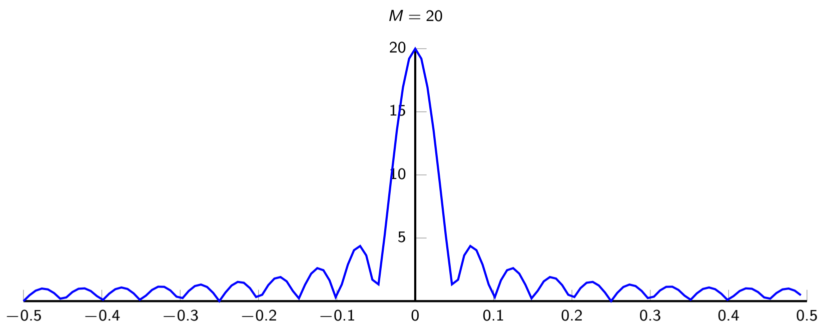
$$= \sum_{n=0}^{M} \mathrm{e}^{-\mathrm{j}\omega n} = \frac{1-\mathrm{e}^{-\mathrm{j}\omega(M+1)}}{1-\mathrm{e}^{-\mathrm{j}\omega}} \tag{421}$$

$$= \frac{\mathrm{e}^{-\mathrm{j}\omega(M+1)/2}}{\mathrm{e}^{-\mathrm{j}\omega/2}} \cdot \frac{\left(\mathrm{e}^{\mathrm{j}\omega(M+1)/2} - \mathrm{e}^{-\mathrm{j}\omega(M+1)/2}\right)}{\mathrm{e}^{\mathrm{j}\omega/2} - \mathrm{e}^{-\mathrm{j}\omega/2}} \tag{422}$$

$$= \mathrm{e}^{-\mathrm{j}\omega M/2} \cdot \frac{\sin \omega \frac{(M+1)}{2}}{\sin \frac{\omega}{2}} \tag{423}$$







b)

$$w_T(n) = w_R(n) * w_R(n-1) \tag{424}$$

where

$$w_R(n) = \begin{cases} 1 & n = 0 \ldots \frac{M}{2} - 1 \\ 0 & \text{f.ö.} \end{cases} \tag{425}$$

Then by using the result of (a) part, we can replace M in (32) by $\frac{M}{2} - 1$, we can get the frequency spectrum $W_R(\omega)$ as:

$$W_R(\omega) = e^{-j\omega(\frac{M}{2}-1)/2} \frac{sin\omega\frac{M}{4}}{sin(\frac{\omega}{2})} \tag{426}$$

$$W_T(\omega) = W_R(\omega) \cdot W_R(\omega)\, \mathrm{e}^{-\mathrm{j}\omega} = \mathrm{e}^{-\mathrm{j}\omega\frac{M}{2}} \frac{\sin^2 \omega\frac{M}{4}}{\sin^2 \frac{\omega}{2}} \tag{427}$$

Solution 5.17

a) According to the block diagram, the collection of formulas

$$y(n) = x(n) - 2\cos\omega_0 x(n-1) + x(n-2) \tag{428}$$

By using the definition of the impulse response, if we input a delta function to the system, the output gives

$$h(n) = \delta(n) - 2\cos\omega_0 \delta(n-1) + \delta(n-2) \tag{429}$$

b) Applying standard DTFT, we can get

$$H(\omega) = 1 - 2\cos\omega_0 \mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega} \tag{430}$$

$$= \mathrm{e}^{-\mathrm{j}2\omega}\left(\mathrm{e}^{\mathrm{j}2\omega} - 2\cos\omega_0 \mathrm{e}^{\mathrm{j}\omega} + 1\right) \tag{431}$$

$$= \mathrm{e}^{-\mathrm{j}2\omega}\left[e^{j2\omega} - (e^{j\omega_0} + e^{-j\omega_0})e^{j\omega} + 1\right] \tag{432}$$

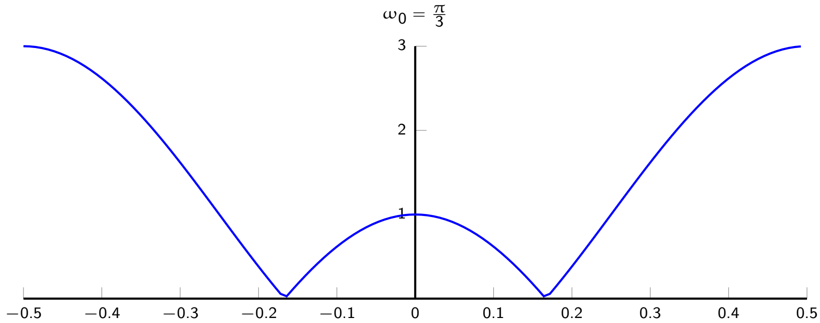
$$= \frac{\left(\mathrm{e}^{\mathrm{j}\omega} - \mathrm{e}^{\mathrm{j}\omega_0}\right)\left(\mathrm{e}^{\mathrm{j}\omega} - \mathrm{e}^{-\mathrm{j}\omega_0}\right)}{\mathrm{e}^{\mathrm{j}2\omega}} \tag{433}$$

$$|H(\omega)| = \left|\mathrm{e}^{\mathrm{j}\omega} - \mathrm{e}^{\mathrm{j}\omega_0}\right| \cdot \left|\mathrm{e}^{\mathrm{j}\omega} - \mathrm{e}^{-\mathrm{j}\omega_0}\right| = V_1(\omega)V_2(\omega) \tag{434}$$

$$\phi(\omega) = \arg\left[1 - 2\cos\omega_0 \mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega}\right] \tag{435}$$

$$= \arg\left(e^{-j\omega}\left(e^{j\omega} + e^{-j\omega} - 2cos\omega_0\right)\right) \tag{436}$$

$$= \arg\left[2\mathrm{e}^{-\mathrm{j}\omega}(\cos\omega - \cos\omega_0)\right] = \begin{cases} -\omega & \omega < \omega_0 \\ -\omega - \pi & \omega > \omega_0 \end{cases} \tag{437}$$



c)

$$x(n) = 3\cos\left(\frac{\pi}{3}\ n + \frac{\pi}{6}\right) \quad -\infty < n < \infty \tag{438}$$

Here keep in mind that the system response to sinosoid signals can be shown as the phase shift and the the scaling of the amplitude, which can be calculated from the transfer function.

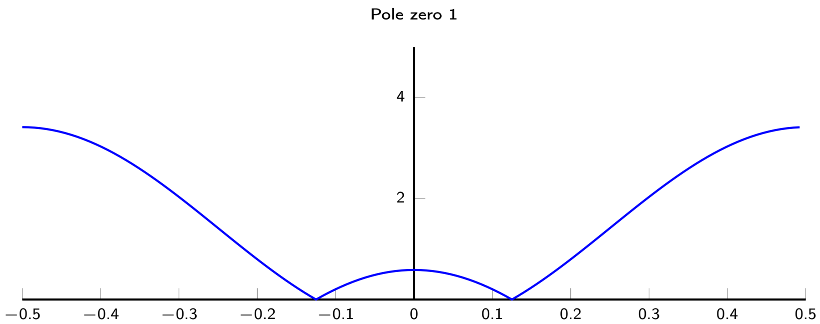
$$y(n) = 3 \cdot \left|H\left(\frac{\pi}{3}\right)\right| \cdot \cos\left(\frac{\pi}{3}n + \frac{\pi}{6} + \phi\left(\frac{\pi}{3}\right)\right) \quad -\infty < n < \infty \tag{439}$$

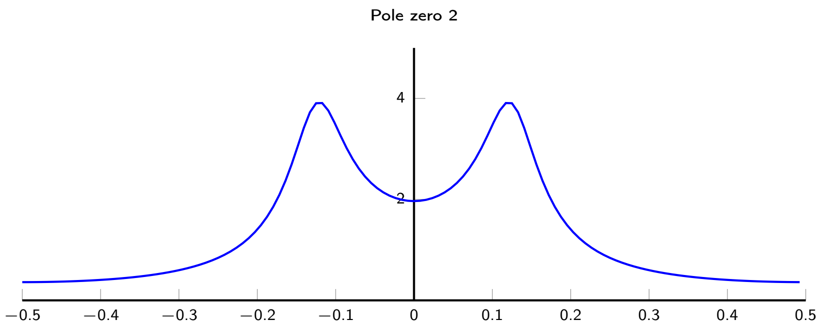
$$\omega_0 = \frac{\pi}{2} \quad \Rightarrow \quad \left|H\left(\frac{\pi}{3}\right)\right| = \left|1 + \mathrm{e}^{\mathrm{j}2\ \pi/3}\right| = \sqrt{\left(1 - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1 \tag{440}$$

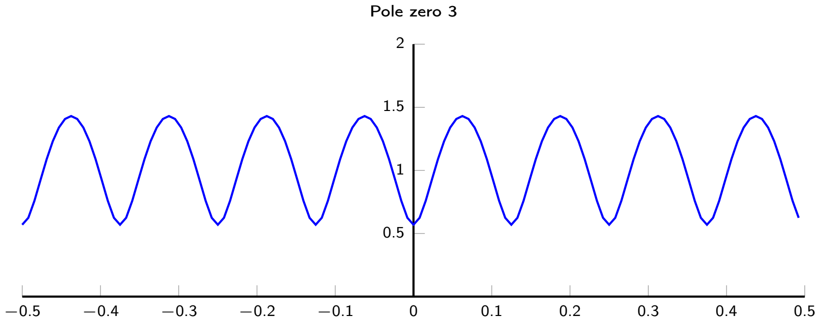
$$\phi\left(\frac{\pi}{3}\right) = -\frac{\pi}{3} \quad \Rightarrow \quad y(n) = 3\cos\left(\frac{\pi}{3}n - \frac{\pi}{6}\right) \quad -\infty < n < \infty \tag{441}$$

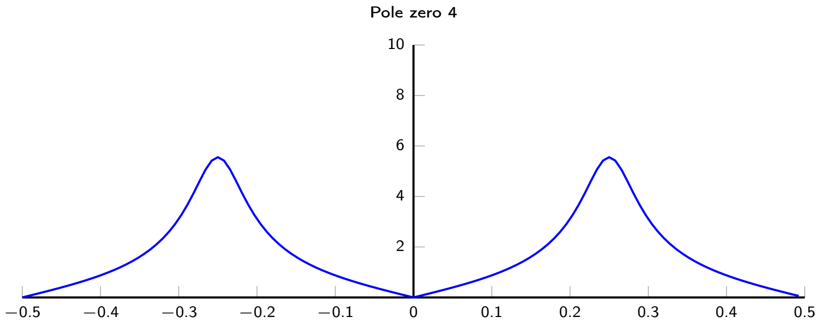
Solution 5.25

Plot $|X(f)|$ with MATLAB. Here keep in mind that the response will be boosted when the digital frequency is closed to poles and attenuated when the digital frequency is closed to zeros.









Solution 5.26

Here one way to design the transfer function for the system can be shown as:

$$\begin{aligned} W(z) &= \frac{(z-e^{-j\frac{\pi}{4}})(z-e^{j\frac{\pi}{4}})}{z^2} \\ &= (1-e^{-j\frac{\pi}{4}}z^{-1})(1-e^{j\frac{\pi}{4}}z^{-1}) \\ &= 1-\sqrt{2}z^{-1}+z^{-2} \end{aligned} \tag{442}$$

$$\omega_0 = \frac{\pi}{4} \Rightarrow h(n) = \left\{ \begin{array}{ccc} 1 & -\sqrt{2} & 1 \end{array} \right\} \tag{443}$$

$$x(n) = \left\{ \begin{array}{ccccccccc} 0 & \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -1 & -\frac{1}{\sqrt{2}} & \dots \end{array} \right\} \tag{444}$$

For example, a graphical convolution gives

$$y(n) = \left\{ \begin{array}{cccccc} 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \dots \end{array} \right\} \tag{445}$$

As the input signal starts in $n = 0$, there is a transient in the output signal.

Solution 5.35

I think here I want to suggest a better solution, which is easier to understand.
The transfer function can be shown as

$$W(z) = G\frac{(z-e^{-j\frac{3\pi}{4}})(z-e^{j\frac{3\pi}{4}})}{(z-\frac{1}{2})^2} \tag{446}$$

Since $W(0) = W(e^{j\omega})|_{\omega=0} = 1$, $G = \frac{1}{4(2+\sqrt{2})}$

Solution 5.39

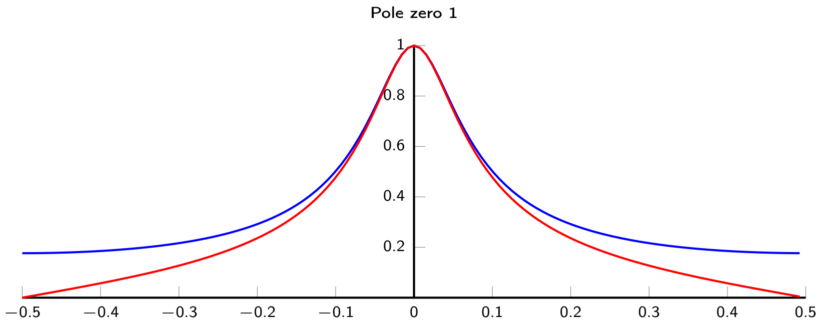
$$H_1(\omega) = \frac{1-a}{1-a\mathrm{e}^{-\mathrm{j}\omega}} \quad \Rightarrow \quad |H_1(0)| = \frac{1-a}{1-a} = 1 \tag{447}$$

$$|H_1(\omega_{3dB})| = |H(0)| \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tag{448}$$

$$|H_1(\omega_{3dB})|^2 = \frac{(1-a)^2}{|1-a\cos\omega_{3dB} + a\mathrm{j}\sin\omega_{3dB}|^2} = \frac{1}{2} \quad \Rightarrow \quad \omega_{3dB} = \arccos\frac{4a-a^2-1}{2a} \tag{449}$$

$$H_2(\omega) = \frac{1-a}{2} \cdot \frac{1+\mathrm{e}^{-\mathrm{j}\omega}}{1-a\mathrm{e}^{-\mathrm{j}\omega}} \quad \Rightarrow \quad |H_2(0)| = \frac{1-a}{2} \cdot \frac{2}{1-a} = 1 \tag{450}$$

$$|H_2(\omega)|^2 = \left(\frac{1-a}{2}\right)^2 \cdot \frac{|1+\cos\omega - \mathrm{j}\sin\omega|^2}{|1-a\cos\omega + a\mathrm{j}\sin\omega|^2} = \frac{1}{2} \quad \Rightarrow \quad \omega = \arccos\frac{2a}{1+a^2} \tag{451}$$



H_2 is best because of the zero in $z = -1$.

Solution E4.1

Choose zeros in $z = \mathrm{j}$, $z = -\mathrm{j}$, and $z = -1$. This gives $H(z) = b_0(1 + z^{-1} + z^{-2} + z^{-3})$. If the constant amlitude is 1, we get $b_0 = 1/4$ and $b_0 = b_1 = b_2 = b_3 = 0.25$.

$$|H(\omega)| = \frac{1}{4} \cdot \left|\frac{\sin(2\omega)}{\sin(\omega/2)}\right| = \frac{1}{2} \cdot \left|\cos\left(\frac{3}{2}\omega\right) + \cos\left(\frac{1}{2}\omega\right)\right| \tag{452}$$

$$\arg H(\omega) = -\frac{3}{2}\omega + \pi \quad \text{om } \pi/2 < \omega < \pi \tag{453}$$

$H(\omega)$ periodical.

Solution E4.2

$$H(z) = 1 + z^{-D} + z^{-2D} = \frac{z^{2D} + z^D + 1}{z^{2D}} \tag{454}$$

$$|H(\omega)| = \left|1 + \mathrm{e}^{-\mathrm{j}\omega D} + \mathrm{e}^{-\mathrm{j}\omega 2D}\right| = |1 + 2\cos\omega D| \tag{455}$$

Poles: $D = 1000$ in origo.
Zeros: $z^{2D} + z^D + 1 = 0$ gives $z^D = 0.5\left(-1 \pm \mathrm{j}\sqrt{3}\right) = \mathrm{e}^{\mathrm{j}\pm 2\pi\frac{1}{3}} \cdot \mathrm{e}^{\mathrm{j}2\pi k}$ för $k = 0, 1, \ldots D-1$, and finally $z_k = \mathrm{e}^{\pm\mathrm{j}2\pi\frac{1}{3D}} \cdot \mathrm{e}^{\mathrm{j}2\pi k/D}$

Solution E4.3

a) $y(n) = x(n) + 0.9x(n-D)$ gives $h(n) = \delta(n) + 0.9\delta(n-D)$

b) $H(z) = 1 + 0.9z^{-D} = \frac{z^D+0.9}{Z^D}$ with $D = 500$ gives 500 poles in origo. Zeros:

$$z^{500} = -0.9 = 0.9\mathrm{e}^{\mathrm{j}2\pi k+\mathrm{j}\pi} \tag{456}$$

$$z_k = 0.9^{1/500}\mathrm{e}^{\mathrm{j}2\pi k/500+\mathrm{j}\pi/500} \quad \text{for } k = 0, 1, 2, ..., 499 \text{ (on a circle).} \tag{457}$$

Solution E4.4

$$|H(f)| = \left|\frac{\sin 4\omega}{\sin \omega/2}\right| \tag{458}$$

Multiplication with $\cos(2\pi n/8)$ moves the spectrum $\pm 1/8$. $H(f)$ blocks all frequencies except $f = 0$ (draw the spectra). This gives $y(t) = 4\cos(2\pi 1000t)$

Solution N3

a) 50 Hz in the time domain represents a frequency of $\frac{50}{44100} * 2\pi = \frac{2\pi}{882}$ rad in the z-domain. A notch filter can be described as such:

$$H(z) = \frac{(1-e^{j\omega}z^{-1})(1-e^{-j\omega}z^{-1})}{(1-re^{j\omega}z^{-1})(1-re^{-j\omega}z^{-1})} \tag{459}$$

Where ω is the frequency to be suppressed, and r is the radius of the poles. Thus our filter can be written as

$$H(z) = \frac{(1-e^{j\frac{2\pi}{882}}z^{-1})(1-e^{-j\frac{2\pi}{882}}z^{-1})}{(1-0.9e^{j\frac{2\pi}{882}}z^{-1})(1-0.9e^{-j\frac{2\pi}{882}}z^{-1})} \tag{460}$$

b) $\frac{3000}{44100} * 2\pi = \frac{2\pi}{14.7}$ rad. The new notch filter then becomes:

$$H_n(z) = \frac{(1-e^{j\frac{2\pi}{14.7}}z^{-1})(1-e^{-j\frac{2\pi}{14.7}}z^{-1})}{(1-0.9e^{j\frac{2\pi}{14.7}}z^{-1})(1-0.9e^{-j\frac{2\pi}{14.7}}z^{-1})} \tag{461}$$

Adding a filter to an existing filter is equivalent with multiplying them in the z-domain. The new filter becomes

$$H(z) = \frac{(1-e^{j\frac{2\pi}{882}}z^{-1})(1-e^{-j\frac{2\pi}{882}}z^{-1})}{(1-0.9e^{j\frac{2\pi}{882}}z^{-1})(1-0.9e^{-j\frac{2\pi}{882}}z^{-1})} \cdot \frac{(1-e^{j\frac{2\pi}{14.7}}z^{-1})(1-e^{-j\frac{2\pi}{14.7}}z^{-1})}{(1-0.9e^{j\frac{2\pi}{14.7}}z^{-1})(1-0.9e^{-j\frac{2\pi}{14.7}}z^{-1})} \tag{462}$$

Sampling

Solution E4.5

Observe that this problem shows the importance of applying antialiasing filter before sampling especially when we work with the signal has infinite frequency spectrum range, such as this signal.

a)

$$x_a(t) = \mathrm{e}^{-10t}u(t) \tag{463}$$

By using the standard analog fourier transform definition, we can show that the spectrum is:

$$X_a(F) = \int_0^\infty \mathrm{e}^{-(10+\mathrm{j}2\pi F)t}dt = \left[\frac{\mathrm{e}^{-(10+\mathrm{j}2\pi F)t}}{-(10+\mathrm{j}2\pi F)}\right]_0^\infty \tag{464}$$

$$= \frac{1}{10+\mathrm{j}2\pi F} \tag{465}$$

$$|X_a(F)|^2 = \frac{1}{10^2+(2\pi F)^2} \tag{466}$$

b) If we use a antialiasing filter observe that all of the frequency above 50Hz should be blocked, observe that we should calculate the energy at both negative frequency and positive frequency side, since the energy at negative frequency part still contributes the total signal energy according to pasaval theorem. Therefore, the blocked energy can be calculated as:

$$E_s = \int_{-\infty}^{-50} |X_a(F)|^2\, dF + \int_{50}^{\infty} |X_a(F)|^2\, \mathrm{d}F \tag{467}$$

$$= 2\int_{50}^{\infty} \frac{1}{10^2 + (2\pi F)^2}\, \mathrm{d}F \tag{468}$$

$$= 2\cdot \frac{1}{10\cdot 2\pi}\left[\arctan 2\pi \frac{F}{10}\right]_{50}^{\infty} \tag{469}$$

$$= \frac{1}{10\pi}\left[\frac{\pi}{2} - 1.539\right] \tag{470}$$

Total energy:

$$E_{tot} = \int_{-\infty}^{+\infty} |X_a(F)|^2\, dF = \frac{1}{10\pi}\cdot\frac{\pi}{2} \quad \Rightarrow \quad \text{the blocked fraction is } \frac{\frac{\pi}{2} - 1.539}{\frac{\pi}{2}} \approx 2\% \tag{471}$$

c) Without filter: After sampling we can get the signal,

$$y_a(nT_s) = e^{-10nT_s} u(n) \tag{472}$$

Applying standard DTFT to signal $y_a(n)$, we can get the spectrum $Y(f)$ as:

$$|Y(f)| = \left|\sum_{n=0}^{\infty} \mathrm{e}^{-10n/100} \mathrm{e}^{-\mathrm{j}2\pi f n}\right| = \left|\frac{1}{1 - \mathrm{e}^{-0.1}\mathrm{e}^{-\mathrm{j}2\pi f}}\right| \tag{473}$$

$$= \frac{1}{\sqrt{(1 - \mathrm{e}^{-0.1}\cos(2\pi f)^2) + (e^{-0.1})^2 \sin^2 2\pi f}} = \frac{1}{\sqrt{1.8187 - 1.8096\cos 2\pi f}} \tag{474}$$

With filter: Observe that here since the frequency components above 50Hz are removed, the formula (1.16) See course book page 399) holds, therefore:

$$\left|\tilde{Y}(f)\right| = F_s\, |X_a(F)| = F_s \cdot \frac{1}{\sqrt{100 + (2\pi f F_s)^2}} = \frac{1}{\sqrt{0.01 + (2\pi f)^2}}. \tag{475}$$

	$\left\lvert\tilde{Y}(f)\right\rvert$	$\lvert Y(f)\rvert$
$f = 0$	10	10.5
$f = 0.25$	0.64	0.74
$f = 0.5$	0.32	0.53

Solution E4.6

This problem gives the insight of viewing the sampling process at frequency domain. It is important to understand the relationship between analog frequency, sampling frequency and digital frequency.

First since the input signal has frequency component at $F_0 = 600Hz$, the spectrum can be plotted by Fig 1. After sampling, since here $F_s > 2 * F_0$, therefore all the analog frequency components above $\frac{F_s}{2} = 500Hz$ (below -500Hz) will be folded back with the centre of $\frac{F_s}{2} = 500Hz$, which plots figure 2.

Then the signal multiplies $(-1)^n$, which is equivalent to multiply $x_1(n) = \cos(\pi n)$, according to the modulation property (formula on Course book Page 296), the frequency spectrum is shifted and then we can get Fig 3.
Then the signal is upsampled by a factor of 2, which means that the spectrum is compressed with a factor of 2. Observe that the spectrum outside the range [-0.5 0.5] is visible after this compression, which generates Fig 4.
The final step we construct the signal with constructing frequency $F_s = 1000Hz$, which generates Fig 5.

Fig.1 Spectrum for the signal $x_a(t)$

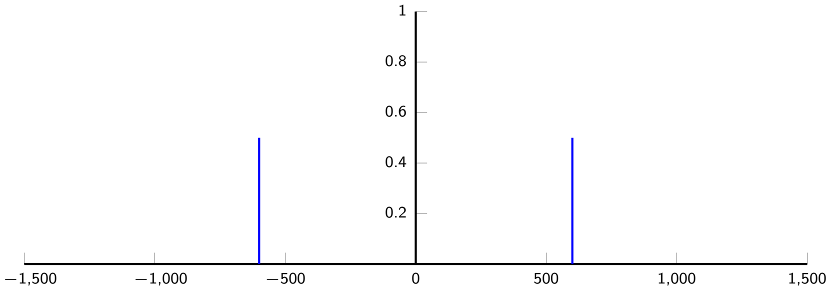


Fig.2 Spectrum for the signal $x(n)$

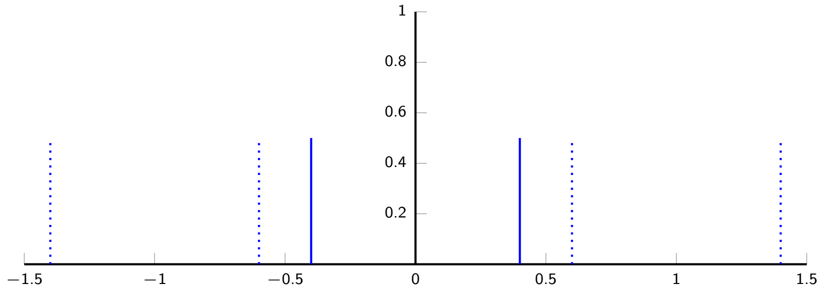


Fig 3 Spectrum for the signal $z(n)$ after modulation

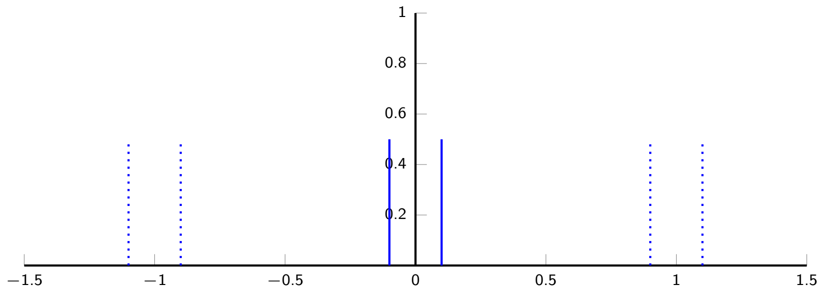


Fig 4 Spectrum for the signal $y(n)$ after upsampling

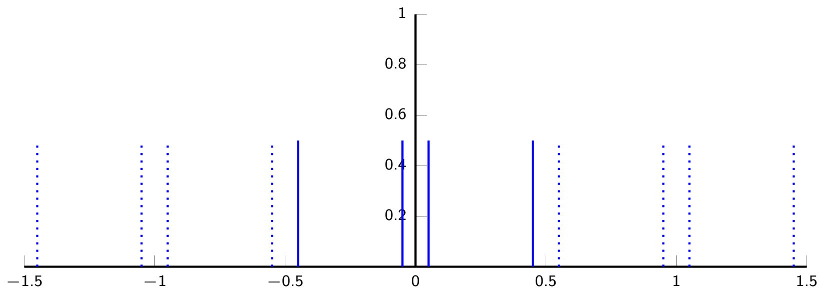
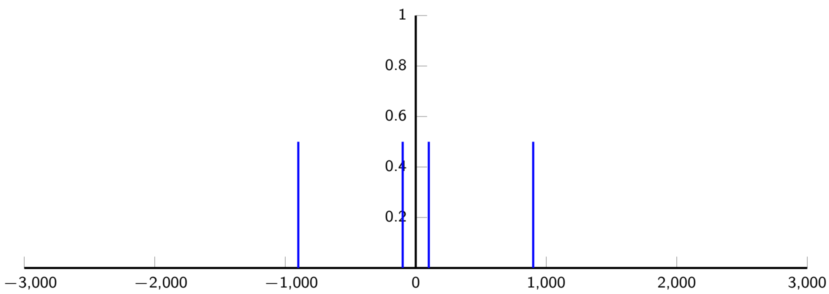


Fig 5 Spectrum for the signal $y_a(t)$ after reconstruction



Then our signal expression can be written as $y_a(t) = \cos(2\pi 100t) + \cos(2\pi 900t)$.

Solution 7.1

If $x(n)$ is real $|X(\omega)|$ is an even function, and $\arg(X(\omega))$ is an odd function. $X(\omega)$ is always a periodical function with the period 2π. This is true even when $X(\omega)$ is sampled to $X(k)$ and the period for discrete signal $X(k)$ is $N = 8$ (eight-point DFT) This means that when

$$X(k) = X^*(N-k) \tag{476}$$
$$X(0) = 0.25 \tag{477}$$
$$X(1) = 0.125 - \mathrm{j}0.3018 \tag{478}$$
$$X(2) = 0 \tag{479}$$
$$X(3) = 0.125 - \mathrm{j}0.0518 \tag{480}$$
$$X(4) = 0 \tag{481}$$

we have

$$X(5) = X^*(3) = 0.125 + \mathrm{j}0.0518 \tag{482}$$
$$X(6) = X^*(2) = 0 \tag{483}$$
$$X(7) = X^*(1) = 0.125 + \mathrm{j}0.3018 \tag{484}$$

Solution 7.2

a) $x(n) = x_1 \odot x_2 = \sum_{k=0}^{N-1} x_1(k)x_2(n-k \mod N)$where $N = 8$

$$y(0) = x_1(0)\cdot x_2(0) + x_1(1)\cdot x_2(N-1) + x_1(2)\cdot x_2(N-2) + x_1(3)\cdot x_2(N-3) + x_1(4)\cdot x_2(N-4) + \tag{485}$$
$$+ x_1(5)\cdot x_2(N-5) + x_1(6)\cdot x_2(N-6) + x_1(7)\cdot x_2(N-7) = 1.25 \tag{486}$$
$$y(1) = x_1(0)\cdot x_2(1) + x_1(1)\cdot x_2(0) + x_1(2)\cdot x_2(N-1) + x_1(3)\cdot x_2(N-2) + x_1(4)\cdot x_2(N-3) + \tag{487}$$
$$+ x_1(5)\cdot x_2(N-4) + x_1(6)\cdot x_2(N-5) + x_1(7)\cdot x_2(N-6) = 2.55 \tag{488}$$
$$y(2) = x_1(0)\cdot x_2(2) + x_1(1)\cdot x_2(1) + x_1(2)\cdot x_2(0) + x_1(3)\cdot x_2(N-1) + x_1(4)\cdot x_2(N-2) + \tag{489}$$
$$+ x_1(5)\cdot x_2(N-3) + x_1(6)\cdot x_2(N-4) + x_1(7)\cdot x_2(N-5) = 2.55 \tag{490}$$
$$y(3) = x_1(0)\cdot x_2(3) + x_1(1)\cdot x_2(2) + x_1(2)\cdot x_2(1) + x_1(3)\cdot x_2(0) + x_1(4)\cdot x_2(N-1) + \tag{491}$$
$$+ x_1(5)\cdot x_2(N-2) + x_1(6)\cdot x_2(N-3) + x_1(7)\cdot x_2(N-4) = 1.25 \tag{492}$$
$$y(4) = x_1(0)\cdot x_2(4) + x_1(1)\cdot x_2(3) + x_1(2)\cdot x_2(2) + x_1(3)\cdot x_2(1) + x_1(4)\cdot x_2(0) + \tag{493}$$
$$+ x_1(5)\cdot x_2(N-1) + x_1(6)\cdot x_2(N-2) + x_1(7)\cdot x_2(N-3) = 0.25 \tag{494}$$
$$y(5) = x_1(0)\cdot x_2(5) + x_1(1)\cdot x_2(4) + x_1(2)\cdot x_2(3) + x_1(3)\cdot x_2(2) + x_1(4)\cdot x_2(1) + \tag{495}$$
$$+ x_1(5)\cdot x_2(0) + x_1(6)\cdot x_2(N-1) + x_1(7)\cdot x_2(N-2) = -1.06 \tag{496}$$
$$y(6) = x_1(0)\cdot x_2(6) + x_1(1)\cdot x_2(5) + x_1(2)\cdot x_2(4) + x_1(3)\cdot x_2(3) + x_1(4)\cdot x_2(2) + \tag{497}$$
$$+ x_1(5)\cdot x_2(1) + x_1(6)\cdot x_2(0) + x_1(7)\cdot x_2(N-1) = -1.06 \tag{498}$$
$$y(7) = x_1(0)\cdot x_2(7) + x_1(1)\cdot x_2(6) + x_1(2)\cdot x_2(5) + x_1(3)\cdot x_2(4) + x_1(4)\cdot x_2(3) + \tag{499}$$
$$+ x_1(5)\cdot x_2(2) + x_1(6)\cdot x_2(1) + x_1(7)\cdot x_2(0) = 0.25 \tag{500}$$
$$\tag{501}$$

$$y(n) = \left\{ \begin{array}{cccccccc} 1.25 & 2.55 & 2.55 & 1.25 & 0.25 & -1.06 & -1.06 & 0.25 \end{array} \right\}.$$

Solution 7.3

$x(n)$ is low pass filtered when some values in $X(k)$ are set to zero, because the k-values between k_c and $N - k_c$ represent high frequencies from $\omega = \pi$ (the hightest frequency) and down to $2\pi k_c/N$. The frequencies $\omega = \pi$ up to $2\pi(N - k_c)/N$ represent the periodicity.

Solution 7.4

a) $\frac{N}{2} \cdot \sin\left(\frac{2\pi}{N} n\right)$

b) $-\frac{N}{2} \cdot \sin\left(\frac{2\pi}{N} \ell\right)$

c) $\frac{N}{2} \cdot \cos\left(\frac{2\pi}{N} \ell\right)$

d) $\frac{N}{2} \cdot \cos\left(\frac{2\pi}{N} \ell\right)$

Solution 7.7

1) $X_c(k) = \frac{1}{2}[X((k - k_0))_N + X((k + k_0))_N]$

2) $X_s(k) = \frac{1}{2j}[X((k - k_0))_N - X((k + k_0))_N]$

Solution 7.8

Circular convolution: make one signal periodic and compute the convolution as usual.

$$x_1 = \left\{ \ldots \quad \underline{1} \quad 2 \quad 3 \quad 1 \quad 1 \quad 2 \quad 3 \quad 1 \quad \ldots \right\} \tag{502}$$

$$x_2 = \left\{ \underline{4} \quad 3 \quad 2 \quad 2 \right\} \tag{503}$$

	2	3	1	<u>1</u>	2	3	1
<u>4</u>	8	12	4	<u>4</u>	8	12	4
3	6	9	3	3	6	9	3
2	4	6	2	2	4	6	2
2	4	6	2	2	4	6	2

Then $y = x_1 \odot x_2$:

$$y(0) = 4 \cdot 1 + 3 \cdot 1 + 2 \cdot 3 + 2 \cdot 2 = 17 \tag{504}$$

$$y(1) = 4 \cdot 2 + 3 \cdot 1 + 2 \cdot 1 + 2 \cdot 3 = 19 \tag{505}$$

$$y(2) = 4 \cdot 3 + 3 \cdot 2 + 2 \cdot 1 + 2 \cdot 1 = 22 \tag{506}$$

$$y(3) = 4 \cdot 1 + 3 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 = 19 \tag{507}$$

$$y_c = \left\{ \underline{17} \quad 19 \quad 22 \quad 19 \right\} \tag{508}$$

For linear convolution The convolution between $x_1(n)$ and $x_2(n)$ yields $y_l(n)$of length 7,
$y_l = x_1 * x_2$:

$$x_1 = \left\{ \underline{1} \quad 2 \quad 3 \quad 1 \right\} \tag{509}$$

$$x_2 = \left\{ \underline{4} \quad 3 \quad 2 \quad 2 \right\} \tag{510}$$

$$\begin{array}{c|cccc} & \underline{1} & 2 & 3 & 1 \\ \hline \underline{4} & \underline{4} & 8 & 12 & 4 \\ 3 & 3 & 6 & 9 & 3 \\ 2 & 2 & 4 & 6 & 2 \\ 2 & 2 & 4 & 6 & 2 \end{array}$$

$$y(0) = 4 \tag{511}$$

$$y(1) = 4 \cdot 2 + 3 \cdot 1 = 11 \tag{512}$$

$$y(2) = 4 \cdot 3 + 3 \cdot 2 + 2 \cdot 1 = 20 \tag{513}$$

$$y(3) = 4 \cdot 1 + 3 \cdot 3 + 2 \cdot 2 + 2 \cdot 1 = 19 \tag{514}$$

$$y(4) = 3 \cdot 1 + 2 \cdot 3 + 2 \cdot 2 = 13 \tag{515}$$

$$y(5) = 2 \cdot 1 + 2 \cdot 3 = 8 \tag{516}$$

$$y(6) = 2 \cdot 1 = 2 \tag{517}$$

$$y_l = \left\{ \begin{array}{ccccccc} \underline{4} & 11 & 20 & 19 & 13 & 8 & 2 \end{array} \right\} \tag{518}$$

The linear convolution is related to the circular convolution.

$$y_c = \left\{ \begin{array}{cccc} \underline{17} & 19 & 22 & 19 \end{array} \right\} \tag{519}$$

$$y_l = \left\{ \begin{array}{ccccccc} \underline{4} & 11 & 20 & 19 & 13 & 8 & 2 \end{array} \right\} \tag{520}$$

$$y_c = \left\{ \begin{array}{cccc} y_l(0) + y_l(4) & y_l(1) + y_l(5) & y_l(2) + y_l(6) & y_l(3) + y_l(7) \end{array} \right\} \tag{521}$$

Solution 7.9

$$x_1(n) = \left\{ \begin{array}{cccc} \underline{1} & 2 & 3 & 1 \end{array} \right\} \tag{522}$$

$$x_2(n) = \left\{ \begin{array}{cccc} \underline{4} & 3 & 2 & 2 \end{array} \right\} \tag{523}$$

$$x_3(n) = x_1(n) \odot x_2(n) \tag{524}$$

$$X(k) = \mathrm{DFT}\{x_n\} = \sum_{n=0}^{3} x_n \mathrm{e}^{-\mathrm{j}2\pi nk/4} \quad k = 0 \ldots 3 \tag{525}$$

$$X_1(0) = 7 \qquad X_2(0) = 11 \tag{526}$$

$$X_1(1) = -2 - \mathrm{j} \qquad X_2(1) = 2 - \mathrm{j} \tag{527}$$

$$X_1(2) = 1 \qquad X_2(2) = 1 \tag{528}$$

$$X_1(3) = X_1^*(1) = -2 + \mathrm{j} \qquad X_2(3) = X_2^*(1) = 2 + \mathrm{j} \tag{529}$$

$$X_3(k) = X_1(k) X_2(k) \tag{530}$$

$$X_3(0) = 77 \tag{531}$$

$$X_3(1) = -5 \tag{532}$$

$$X_3(2) = 1 \tag{533}$$

$$X_3(3) = -5 \tag{534}$$

$$x(n) = \mathrm{IDFT}\{X(k)\} = \frac{1}{4}\sum_{k=0}^{3} X(k)\mathrm{e}^{\mathrm{j}2\pi nk/4} \quad n = 0\dots 3 \tag{535}$$

$$x_3(0) = 17 \tag{536}$$

$$x_3(1) = 19 \tag{537}$$

$$x_3(2) = 22 \tag{538}$$

$$x_3(3) = 19 \tag{539}$$

Solution 7.10

$$x(n) = \cos\left(\frac{2\pi k_0 n}{N}\right) \quad 0 \le n \le N-1 \tag{540}$$

$$E_x = \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{n=0}^{N-1} \cos^2\left(\frac{2\pi k_0}{N}\ n\right) = \sum_{n=0}^{N-1} \frac{1+\cos\left(\frac{4\pi k_0}{N}\ n\right)}{2} \tag{541}$$

$$\tag{542}$$

For $k_0 = \frac{i}{2}$ where $i = 0, 1, 2, \ldots$:

$$\cos\left(\frac{4\pi k_0}{N}\ n\right) = \cos(2\pi i.n) = 1 \tag{543}$$

$$E_x = N \tag{544}$$

$$\tag{545}$$

Otherwise:

$$E_x = \sum_{n=0}^{N-1}\frac{1}{2} + \sum_{n=0}^{N-1}\frac{\cos\left(\frac{4\pi k_0}{N}\ n\right)}{2} \tag{546}$$

$$= \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1}\cos\left(\frac{4\pi k_0}{N}\ n\right) \tag{547}$$

$$= \frac{N}{2} + \frac{1}{2}\sum_{n=0}^{N-1}\frac{\mathrm{e}^{\mathrm{j}4\pi k_0 n/N} + \mathrm{e}^{-\mathrm{j}4\pi k_0 n/N}}{2} \tag{548}$$

$$= \frac{N}{2} + \frac{1}{4}(\sum_{n=0}^{N-1}\mathrm{e}^{\mathrm{j}4\pi k_0 n/N} + \sum_{n=0}^{N-1}\mathrm{e}^{-\mathrm{j}4\pi k_0 n/N}) \tag{549}$$

$$= \frac{N}{2} + \frac{1}{4}\cdot\left(\underbrace{\frac{1-\mathrm{e}^{\mathrm{j}4\pi k_0 N/N}}{1-\mathrm{e}^{\mathrm{j}4\pi k_0/N}}}_{=0} + \underbrace{\frac{1-\mathrm{e}^{-\mathrm{j}4\pi k_0 N/N}}{1-\mathrm{e}^{-\mathrm{j}4\pi k_0/N}}}_{=0}\right) = \frac{N}{2} \tag{550}$$

$$E_x = \frac{N}{2} + \frac{1}{4}\cdot\left(\underbrace{\frac{1-\mathrm{e}^{\mathrm{j}4\pi kN/N}}{1-\mathrm{e}^{\mathrm{j}4\pi k/N}}}_{=0} + \underbrace{\frac{1-\mathrm{e}^{-\mathrm{j}4\pi kN/N}}{1-\mathrm{e}^{-\mathrm{j}4\pi k/N}}}_{=0}\right) = \frac{N}{2} \tag{551}$$

Solution 7.11

$X(k)$ for $k = 0 \ldots 7$ is given.

$$\begin{cases} x_1(n) = x(n-5 \mod 8) \\ x_2(n) = x(n-2 \mod 8) \end{cases} \Rightarrow \begin{cases} X_1(k) = X(k)\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{8} \cdot 5} \\ X_2(k) = X(k)\mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{8} \cdot 2} \end{cases} \tag{552}$$

Solution 7.18

$Y(k) = H(f)$ where $f = \frac{k}{N}$.

Solution 7.23

$X(k) = \sum_{n=0}^{N-1} x(n)\mathrm{e}^{-\mathrm{j}2\pi kn/N} \qquad k = 0, 1 \ldots . N-1$

a) $X(k) = \sum_{n=0}^{N-1} \delta(n)\mathrm{e}^{-\mathrm{j}2\pi kn/N} = \mathrm{e}^{-\frac{\mathrm{j}2\pi k \times 0}{N}} = 1$

b) $X(k) = \sum_{n=0}^{N-1} \delta(n-n_0)\mathrm{e}^{-\mathrm{j}2\pi kn/N} = \mathrm{e}^{-\mathrm{j}2\pi kn_0/N} \quad k = 0, \ldots, N-1$

c)

$$X(k) = \sum_{n=0}^{N-1} a^n \mathrm{e}^{-\mathrm{j}2\pi kn/N} \tag{553}$$

$$= \sum_{n=0}^{N-1} (a\mathrm{e}^{-\mathrm{j}2\pi k/N})^n \tag{554}$$

$$= \frac{1-(a)^N \mathrm{e}^{-\mathrm{j}2\pi kN/N}}{1-a\mathrm{e}^{-\mathrm{j}2\pi k/N}} \quad k = 1, \ldots, N-1 \tag{555}$$

$$= \frac{1-a^N}{1-a\mathrm{e}^{-\mathrm{j}2\pi k/N}} \tag{556}$$

d)

$$x(n) = \begin{cases} 1 & 0 \le n \le \frac{N}{2}-1 \\ 0 & \frac{N}{2} \le n \le N-1 \end{cases} \quad N \text{ even} \tag{557}$$

$$X(k) = \sum_{n=0}^{N/2-1} \mathrm{e}^{-\mathrm{j}2\pi kn/N} = \{k \neq 0\} \tag{558}$$

$$= \frac{1-\mathrm{e}^{-\mathrm{j}2\pi k\ N/2 \cdot N}}{1-\mathrm{e}^{-\mathrm{j}2\pi k/N}} = \frac{1-\mathrm{e}^{-\mathrm{j}2\pi k/2}}{1-\mathrm{e}^{-\mathrm{j}2\pi k/N}} = \tag{559}$$

$$= \frac{1-(-1)^k}{1-\mathrm{e}^{-\mathrm{j}2\pi k/N}} \quad k = 1, \ldots, N-1 \tag{560}$$

$$X(0) = \frac{N}{2} \tag{561}$$

e) $x(n) = \mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k_0 n} \quad \Rightarrow \quad X(k) = N \cdot \delta(k - k_0)$

$$X(k) = \sum_{n=0}^{N-1} (\mathrm{e}^{\mathrm{j}2\pi k_0 n/N} \times \mathrm{e}^{-\mathrm{j}2\pi k n/N}) = \tag{562}$$

$$= \sum_{n=0}^{N-1} \mathrm{e}^{\mathrm{j}2\pi(k_0-k)n/N} = \{k \neq k_0\} \tag{563}$$

$$= \frac{1 - \mathrm{e}^{-\mathrm{j}2\pi(k-k_0)\cdot N/N}}{1 - \mathrm{e}^{-\mathrm{j}2\pi(k-k_0)/N}} = \frac{1 - \mathrm{e}^{-\mathrm{j}2\pi k}}{1 - \mathrm{e}^{-\mathrm{j}2\pi k/N}} = \tag{564}$$

$$= \frac{1 - (1)^k}{1 - \mathrm{e}^{-\mathrm{j}2\pi k/N}} = 0 \quad k \neq k_0 \tag{565}$$

$$X(k_0) = N \tag{566}$$

$$X(k) = N\delta(k - k_0) \tag{567}$$

f) $x(n) = \cos\left(\frac{2\pi k_0}{N}\, n\right) = \frac{\mathrm{e}^{\mathrm{j}2\pi k_0 n/N} + \mathrm{e}^{-\mathrm{j}2\pi k_0 n/N}}{2}$

$$\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k_0 n} \quad \Rightarrow \quad N \cdot \delta(k - k_0) \tag{568}$$

$$\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k_0 n} \quad \Rightarrow \quad N \cdot \delta(k - (N - k_0)) \tag{569}$$

$$X(k) = \frac{N}{2}\, (\delta(k - k_0) + \delta(k - (N - k_0))) \tag{570}$$

g) $x(n) = \sin\left(\frac{2\pi k_0}{N}\, n\right) = \frac{\mathrm{e}^{\mathrm{j}2\pi k_0 n/N} - \mathrm{e}^{-\mathrm{j}2\pi k_0 n/N}}{2\mathrm{j}}$

$$\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}k_0 n} \quad \Rightarrow \quad N \cdot \delta(k - k_0) \tag{571}$$

$$\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}k_0 n} \quad \Rightarrow \quad N \cdot \delta(k - (N - k_0)) \tag{572}$$

$$X(k) = \frac{N}{2\mathrm{j}}\, (\delta(k - k_0) - \delta(k - (N - k_0))) \tag{573}$$

h) Given that N is even

$$x(n) = \begin{cases} 1 & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{574}$$

$$= \frac{1}{2}(1 + (-1)^n) = \frac{1}{2}(\mathrm{e}^{\mathrm{j}2\pi(0)n/N} + \mathrm{e}^{\mathrm{j}2\pi(\frac{N}{2})n/N}) \tag{575}$$

$$\mathrm{e}^{\mathrm{j}\frac{2\pi}{N}0n} \quad \Rightarrow \quad N \cdot \delta(k) \tag{576}$$

$$\mathrm{e}^{-\mathrm{j}\frac{2\pi}{N}\frac{N}{2}n} \quad \Rightarrow \quad N \cdot \delta(k - \frac{N}{2}) \tag{577}$$

$$X(k) = \frac{N}{2}\left(\delta(k) + \delta\left(k - \frac{N}{2}\right)\right) \tag{578}$$

Solution 7.24

$$X(k) = 1 + 2\mathrm{e}^{-\mathrm{j}\,\pi k/2} + 3\mathrm{e}^{-\mathrm{j}\pi k} + \mathrm{e}^{-\mathrm{j}\,3\pi k/2} = \left\{\begin{array}{cccc} 7 & -2-\mathrm{j} & 1 & -2+\mathrm{j} \end{array}\right\} \tag{579}$$

Solution 7.25

a)

$$x(n) = \left\{ \begin{array}{cccccc} 1 & 2 & \underline{3} & 2 & 1 & 0 \end{array} \right\} \tag{580}$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{e}^{-\mathrm{j}\omega n} = \mathrm{e}^{\mathrm{j}2\omega} + 2\mathrm{e}^{\mathrm{j}\omega} + 3 + 2\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega} = \tag{581}$$

$$= 3 + 4\cos\omega + 2\cos 2\omega \tag{582}$$

b)

$$v(n) = \left\{ \begin{array}{cccccc} 3 & 2 & 1 & 0 & 1 & 2 \end{array} \right\} \tag{583}$$

$$V_{DFT}(k) = \sum_{n=0}^{5} v(n)\mathrm{e}^{-\mathrm{j}2\pi\frac{k}{6}\ n} \tag{584}$$

$$= 3 + 2\mathrm{e}^{-\mathrm{j}\frac{\pi}{3}\ k} + \mathrm{e}^{-\mathrm{j}\frac{2\pi}{3}\ k} + \mathrm{e}^{-\mathrm{j}\frac{4\pi}{3}\ k} + 2\mathrm{e}^{-\mathrm{j}\frac{5\pi}{3}\ k} \tag{585}$$

c) $V_{DFT}(k) = 3 + 4\cos\frac{\pi}{3}k + 2\cos\frac{2\pi}{3}k$ for $k = 0\ldots5$. That is, $V_{DFT}(k) = X(\omega_k)$ where $\omega_k = 2\pi\frac{k}{6}$ för $k = 0\ldots5$.

Solution E5.1

Let $x_1(n) = \left\{ \begin{array}{cccccccc} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{array} \right\}$ and $x(n)$ is the circular shift of $x_1(n)$. A shift only affects the phase. Thus, $|X(k)| = |X_1(k)| = \left|\frac{\sin 2\pi\frac{k}{4}}{\sin 2\pi\frac{k}{16}}\right|$.

Solution E5.2

a) $y(n) = x(-n) = \left\{ \begin{array}{cccccccc} 0 & 2 & 2 & 7 & 8 & 3 & 1 & 1 \end{array} \right\}$.

b) $y(n) = x(n-4 \mod 8) = \left\{ \begin{array}{cccccccc} 8 & 7 & 2 & 2 & 0 & 1 & 1 & 3 \end{array} \right\}$.

Solution E5.3

$$H_{FIR}(f) = H_{IIR}(f) \quad \text{för } f = k\cdot\frac{1}{N} \text{ since the input signal is periodical} \tag{586}$$

$$\tag{587}$$

$$h_{FIR}(n) = \mathrm{IDFT}\left\{H_{FIR}\left(\frac{k}{N}\right)\right\} = \frac{1}{N}\sum_{k=0}^{N-1} H_{FIR}\left(\frac{k}{N}\right)\cdot\mathrm{e}^{\mathrm{j}2\pi\ kn/N} \tag{588}$$

where

$$H_{FIR}\left(\frac{k}{N}\right) = H_{IIR}\left(\frac{k}{N}\right) = \sum_{m=0}^{\infty} h_{IIR}(m)\mathrm{e}^{-\mathrm{j}2\pi\ km/N} \tag{589}$$

$$\tag{590}$$

$$h_{FIR}(n) = \frac{1}{N}\sum_{m=0}^{\infty} h_{IIR}(m)\sum_{k=0}^{N-1}\mathrm{e}^{-\mathrm{j}2\pi\ k/N\ (n-m)} = \sum_{\ell=-\infty}^{\infty} h_{IIR}(n-\ell N) \tag{591}$$

Compare with aliasing

$$h_{FIR}(n) = \sum_{\ell=-\infty}^{0} a^{n-\ell N} = a^n \frac{1}{1-a^N} \cdot [u(n) - u(n-N)] \tag{592}$$

Solution E5.4

$f_0 = \pm 138 + n \cdot 400 = \left\{ \begin{array}{ccc} 138 & 262 & \ldots \end{array} \right\}.$

Solution E5.5

1-C, 2-F, 3-G, 4-H.

Solution E5.6

a)

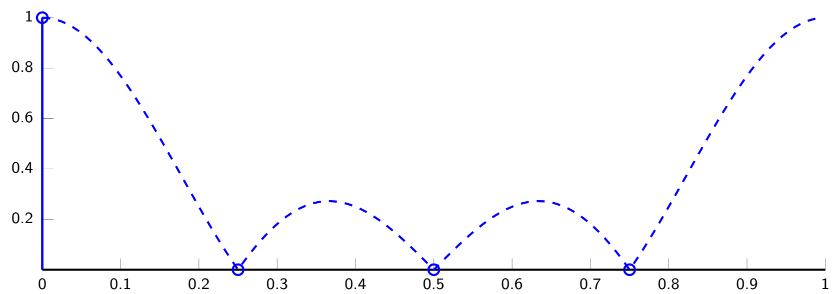
$$H(f) = \frac{1}{4} + \frac{1}{4}e^{-j2\pi f} + \frac{1}{4}e^{-j2\pi 2f} + \frac{1}{4}e^{-j2\pi 3f} \tag{593}$$

$$\tag{594}$$

$$H(k) = \frac{1}{4} + \frac{1}{4}e^{-j2\pi k/N} + \frac{1}{4}e^{-j2\pi 2k/N} + \frac{1}{4}e^{-j2\pi 3k/N} + 0 + 0 + \ldots + 0 \tag{595}$$

for $k = 0 \ldots N-1$. $H(k)$ are samples of $H(f)$ in the points $f = \frac{k}{N}$, $k = 0 \ldots N-1$.

b) $H(0) = 1$, $H(1) = 0$, $H(2) = 0$, and $H(3) = 0$. $h_p(n) = \frac{1}{4}$ for all values of n.



Solution E5.7

$Y(k/N) = X(k/N)$

Solution E5.8

a) $y(n) = \delta(n) + 2\,\delta(n-1) + 1.5\,\delta(n-2) + 0.5\,\delta(n-3)$

b) See above.

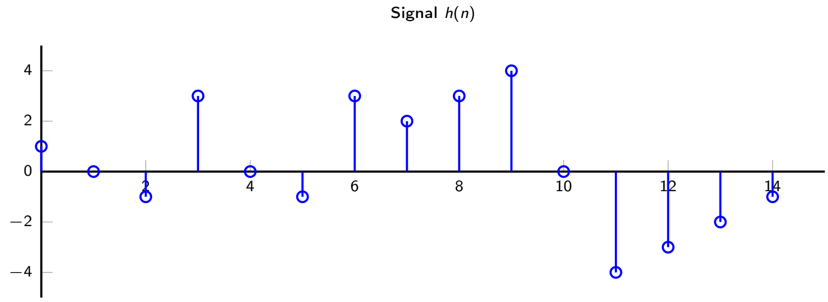
c) $M = 4$; generally $M = P + Q - 1$ where P is the length of the impulse response and Q is the length of the input signal.

Solution E5.9

$$y_p(n) = \sum_{l=0}^{L} b_l \quad \text{for } n = 0 \ldots N-1 \tag{596}$$

Solution E5.10

The method is called overlap-and-add if the computation is done with DFT.

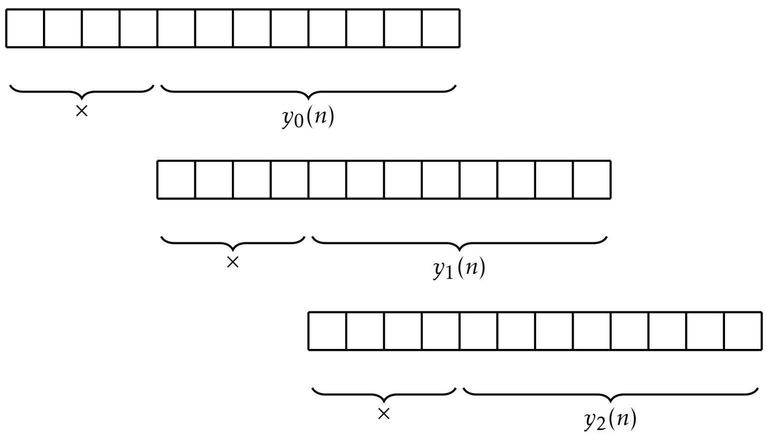


Solution E5.11

$$y(n) = \begin{cases} \frac{1}{1-(\frac{1}{2})^{10}}(\frac{1}{2})^n & 0 \le n \le 9 \\ 0 & \text{otherwise} \end{cases} \tag{597}$$

Solution E5.12

The figure shows how the split is done. Example with $M = 4$ and $N = 12$.



Realizations, Chapter 9

Solution 9.3

Choose $v(n)$ according to the delay element, which gives

$$v(n+1) = \frac{1}{2}\ v(n) + x(n) \tag{598}$$

$$y(n) = 2[v(n+1) + 3x(n)] + 2v(n) = \tag{599}$$

$$= v(n) + 2x(n) + 6x(n) + 2v(n) = \tag{600}$$

$$= 3v(n) + 8x(n) \tag{601}$$

and the state matrices $F = \frac{1}{2}$, $q = 1$, $g^T = 3$, and $d = 8$. The impulse response is

$$h(n) = 3\left(\frac{1}{2}\right)^{n-1} u(n-1) + 8\delta(n) \tag{602}$$

$$H(z) = \frac{3z^{-1}}{1 - \frac{1}{2}\ z^{-1}} + 8 = \frac{8 - z^{-1}}{1 - \frac{1}{2}\ z^{-1}} \tag{603}$$

Solution 9.9

a)

$$y(n) = \frac{3}{4}y(n-1) - \frac{1}{8}y(n-2) + x(n) + \frac{1}{3}x(n-1) \tag{604}$$

- Direct I: from the difference equation
- Direct II: from the collection of formulas and the difference equation
- Cascade: Z-transform of the difference equation

$$H(z) = \frac{1 + \frac{1}{3}z^{-1}}{\left(1 - \frac{1}{4}z^{-1}\right)\left(1 - \frac{1}{2}z^{-1}\right)} \tag{605}$$

- Parallel:

$$H(z) = \frac{\frac{10}{3}}{1 - \frac{1}{2}z^{-1}} - \frac{\frac{7}{3}}{1 - \frac{1}{4}z^{-1}} \tag{606}$$

f)

$$y(n) = y(n-1) - \frac{1}{2}y(n-2) + x(n) - x(n-1) + x(n-2) \tag{607}$$

The system has complex poles, that is, the Direct form II, the cascade, and the parallel forms are equivalent.

Solution 9.15

Note! There is an error in Proakis, Third Edition, 3: $a_2(2) = \frac{1}{3}$.

$$H(z) = A_2(z) = 1 + 2z^{-1} + \frac{1}{3}z^{-2} \tag{608}$$

$$B_2(z) = \frac{1}{3} + 2z^{-1} + z^{-2} \quad \text{ger} \quad K_2 = \alpha_2(2) = \frac{1}{3} \tag{609}$$

$$A_1(z) = \frac{A_2(z) - K_2 B_2(z)}{1 - K_2^2} = \frac{1 + 2z^{-1} + \frac{1}{3}z^{-2} - \frac{1}{3}\left(\frac{1}{3} + 2z^{-1} + z^{-2}\right)}{1 - \left(\frac{1}{3}\right)^2} \tag{610}$$

$$= 1 + \frac{4/3}{8/9}\ z^{-1} = 1 + \frac{3}{2}\ z^{-1} \tag{611}$$

$$B_1(z) = \frac{3}{2} + z^{-1} \quad \text{ger} \quad K_1 = \alpha_1(1) = \frac{3}{2} \tag{612}$$

$$A_0(z) = \frac{1 + \frac{3}{2}z^{-1} - \frac{3}{2}\left(\frac{3}{2} + z^{-1}\right)}{1 - \left(\frac{3}{2}\right)^2} = \frac{5/4}{5/4} = 1 \tag{613}$$

$$K_2 = \frac{1}{3} \quad K_1 = \frac{3}{2} \quad (K_0 = 1) \tag{614}$$

Solution 9.19

a)

$$K_1 = \frac{1}{2} \quad K_2 = -\frac{1}{3} \quad K_3 = 1 \tag{615}$$

$$A_0(z) = B_0(z) = 1 \tag{616}$$

$$A_1(z) = A_0(z) + K_1(z) \cdot z^{-1} \cdot B_0(z) = 1 + \frac{1}{2}z^{-1} \cdot 1 = 1 + \frac{1}{2}z^{-1} \tag{617}$$

$$B_1(z) = \frac{1}{2} + z^{-1} \tag{618}$$

$$\begin{cases} A_2(z) & = A_1(z) + K_2(z) \cdot z^{-1} \cdot B_1(z) = 1 + \frac{1}{2}z^{-1} - \frac{1}{3}z^{-1} \cdot \left(\frac{1}{2} + z^{-1}\right) = 1 + \frac{1}{3}z^{-1} - \frac{1}{3}z^{-2} \\ B_2(z) & = -\frac{1}{3} + \frac{1}{3}z^{-1} + z^{-2} \end{cases} \tag{619}$$

$$\begin{cases} A_3(z) & = 1 + \frac{1}{3}z^{-1} - \frac{1}{3} \cdot z^{-2} + z^{-1} \cdot \left(-\frac{1}{3} + \frac{1}{3}\, z^{-1} + z^{-2}\right) = 1 + z^{-3} \\ B_3(z) & = 1 + z^{-3} \end{cases} \tag{620}$$

Zeros:

$$1 + z^{-3} = 0 \tag{621}$$

$$z^{-3} = \mathrm{e}^{-\mathrm{j}\pi(2k+1)} \tag{622}$$

$$z = \mathrm{e}^{\mathrm{j}\pi(2k+1)/3} \quad \text{for } k = 0, 1, 2 \quad \left(z = \mathrm{e}^{\pm \mathrm{j}\cdot\pi/3} \quad z = -1\right) \tag{623}$$

b)

$$A_3(z) = 1 + \frac{1}{3}z^{-1} - \frac{1}{3}z^{-2} + (-1) \cdot z^{-1} \cdot \left(-\frac{1}{3} + \frac{1}{3}z^{-1} + z^{-2}\right) \tag{624}$$

$$= 1 + \frac{2}{3}z^{-1} - \frac{2}{3}z^{-2} - z^{-3} \tag{625}$$

$$B_3(z) = -1 - \frac{2}{3}\, z^{-1} + \frac{2}{3}\, z^{-2} + 1 \tag{626}$$

$$A_3(z) = (1 - z^{-1})\left(1 + \frac{5}{3}\, z^{-1} + z^{-2}\right) - \frac{5}{6} \pm \sqrt{\frac{25}{36} - 1} = \frac{-5 \pm \mathrm{j} \cdot \sqrt{11}}{6} \tag{627}$$

c) If the absolute value of the last reflection coefficient is one, all zeros are on the unit circle

d) For a):

$$H(z) = 1 + z^{-3} \tag{628}$$

$$H(\omega) = 1 + \mathrm{e}^{-\mathrm{j}\cdot 3\omega} = \mathrm{e}^{-\mathrm{j}\cdot 3\omega/2}\left(\mathrm{e}^{\mathrm{j}\cdot 3\omega/2} + \mathrm{e}^{-\mathrm{j}\cdot 3\omega/2}\right) = 2 \cdot cos\left(\frac{3\omega}{2}\right) \cdot \mathrm{e}^{-\mathrm{j}\; 3\omega/2} \tag{629}$$

$$0 \le \omega < \frac{\pi}{3} \quad \rightarrow \quad \theta(\omega) = -\frac{3\omega}{2} \tag{630}$$

$$\frac{\pi}{3} < \omega \le \pi \quad \rightarrow \quad \theta(\omega) = \pi - \frac{3\omega}{2} \tag{631}$$

Linear phase (symmetrical FIR)
For b):

$$H(z) = 1 + \frac{2}{3}z^{-1} - \frac{2}{3}z^{-2} - z^{-3} \tag{632}$$

$$H(\omega) = 1 + \frac{2}{3}\cdot \mathrm{e}^{-\mathrm{j}\omega} - \frac{2}{3}\cdot \mathrm{e}^{-\mathrm{j}2\omega} - \mathrm{e}^{-\mathrm{j}3\omega} = \mathrm{e}^{-\mathrm{j}\left(\frac{3\omega}{2}+\frac{\pi}{2}\right)}\cdot 2\left(\sin\left(\frac{3}{2}\omega\right) + \frac{2}{3}\sin\left(\frac{\omega}{2}\right)\right) \tag{633}$$

All zewros are on the unit circle which means linear phase.

Examples of Filter Design

Solution E8.1

a) From the realization, $y(n)$ becomes

$$y(n) = x(n) + x(n-1) + x(n-2) + x(n-3) \tag{634}$$

Performing the z-transform yields

$$Y(z) = X(z) + z^{-1}X(z) + z^{-2}X(z) + z^{-3}X(z) \quad = (1 + z^{-1} + z^{-2} + z^{-3})X(z) \tag{635}$$

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} \quad = \frac{z^3 + z^2 + z + 1}{z^3} \tag{636}$$

The three poles then are readily recognizable as

$$p_{1,2,3} = 0 \tag{637}$$

To find the zeros, one zero can be guessed as

$$z_1 = -1 \tag{638}$$

The others can then be found by a long division using this zero:

$$\begin{array}{r|rrrr} & z^2 & & +1 & \\ \hline z+1 & z^3 & +z^2 & +z & +1 \\ - & z^3 & z^2 & & \\ \hline & & & z & +1 \\ & & - & z & +1 \\ \hline & & & & 0 \end{array} \tag{639}$$

Thus

$$H(z) = \frac{z^3 + z^2 + z + 1}{z^3} = \frac{(z+1)(z^2+1)}{z^3} \tag{640}$$

And the zeroes are

$$z_1 = -1 \quad z_{2,3} = \pm j \tag{641}$$

b) + d) The magnitude of the system response is asked for, thus the Fourier transform of $h(n)$ should be used.

$$H(z) = 1 + z^{-1} + z^{-2} + z^{-3} \tag{642}$$

$$H(f)\left[= H(z = e^{jf})\right] = 1 + e^{-jf} + e^{-j2f} + e^{-j3f} \quad \text{[rewrite to conform with Euler]} \tag{643}$$

$$= e^{-j3/2f}\left(e^{j3/2f} + e^{j1/2f} + e^{-j1/2f} + e^{-j3/2f}\right) \tag{644}$$

$$= e^{-j3/2f}\left(e^{j1/2f} + e^{-j1/2f} + e^{j3/2f} + e^{-j3/2f}\right) \tag{645}$$

$$= 2e^{-j3/2f}\left(\cos\frac{1}{2}f + \cos\frac{3}{2}f\right) \tag{646}$$

The magnitude becomes

$$|H(f)| = 2\left|\cos\frac{1}{2}f + \cos\frac{3}{2}f\right| \tag{647}$$

With this form it is easy to see for instance that $|H(0)| = 4$. However, seeing where for instance $|H(f)| = 0$ is trickier. Another form which might make it easier can be found via geometric sum:

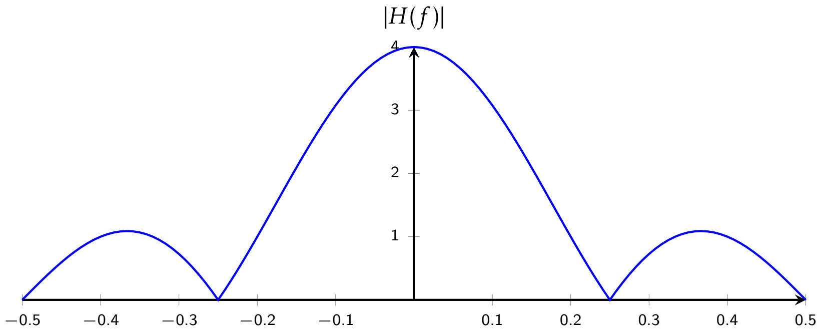
$$H(f) = 1 + e^{-jf} + e^{-j2f} + e^{-j3f} \quad = \sum_{n=0}^{3} e^{-jnf} \tag{648}$$

$$\tag{649}$$

Which can be plugged into the geometric sum formula, however the result can easily be shown as follows:

$$\begin{aligned}
H(f) &= \sum_{n=0}^{3} e^{-jnf} && (650)\\
&= 1 + e^{-jf} + e^{-j2f} + e^{-j3f} && (651)\\
e^{-jf} * H(f) &= e^{-jf} + e^{-j2f} + e^{-j3f} + e^{-j4f} && (652)\\
H(f) - e^{-jf} * H(f) &= 1 - e^{-j4f} && (653)\\
H(f)(1 - e^{-jf}) &= 1 - e^{-j4f} && (654)\\
H(f) &= \frac{1 - e^{-j4f}}{1 - e^{-jf}} \quad \text{[rewrite to conform with Euler]} && (655)\\
&= \frac{e^{-j2f}(e^{j2f} - e^{-j2f})}{e^{-j1/2f}(e^{j1/2f} - e^{-j1/2f})} \quad = \frac{2je^{-j2f}\sin 2f}{2je^{-j1/2f}\sin\frac{1}{2}f} \quad = e^{-j3/2f}\frac{\sin 2f}{\sin\frac{1}{2}f} && (656)\\
|H(f)| &= \left|\frac{\sin 2f}{\sin\frac{1}{2}f}\right| && (657)\\
& && (658)
\end{aligned}$$

Thus $|H(f)| = 0$ at $f = \pm\frac{1}{4}, \pm\frac{1}{2}$.
Plotting (any version of) $|H(f)|$ yields:

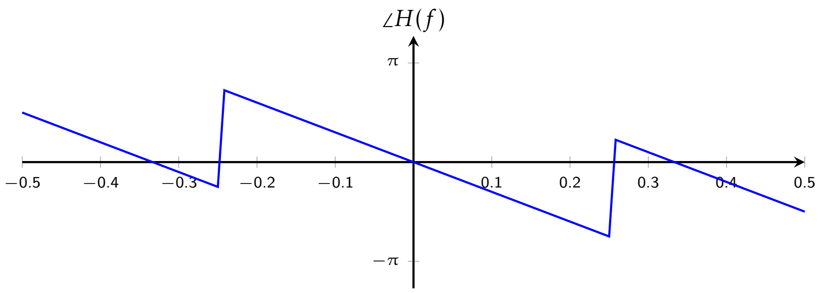


c)
The phase function is given by $\angle H(f)$:

$$\angle H(f) = \angle e^{-j3/2f}\frac{\sin 2f}{\sin\frac{1}{2}f} \quad = \angle e^{-j3/2f} + \angle\frac{\sin 2f}{\sin\frac{1}{2}f} \quad = -3/2f + \angle\frac{\sin 2f}{\sin\frac{1}{2}f} \tag{659}$$

$$\text{[note that real positive values have argument 0 and real negative values argument } \pi\text{]} \tag{660}$$

$$= \left\{\begin{array}{ll} -3/2f - \pi, & -\frac{1}{2} < f < -\frac{1}{4} \\ -3/2f, & -\frac{1}{4} < f < \frac{1}{4} \\ -3/2f + \pi, & \frac{1}{4} < f < \frac{1}{2} \end{array}\right\}, -\frac{1}{2} < f < \frac{1}{2} \tag{661}$$



e) The filter attenuates (reduces) high frequency components, letting low frequency components "pass", thus it is a low-pass filter.

Solution E8.3

a)

$$y(n) = \frac{1}{5}\left(x(n) + x(n-1) + x(n-2) + x(n-3) + x(n-4)\right) \tag{662}$$

$$Y(z) = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right)X(z) \tag{663}$$

$$H(z) = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right) \tag{664}$$

$$h(n) = \frac{1}{5}\left[\delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3) + \delta(n-4)\right] \tag{665}$$

b)

$$H(z) = \frac{1}{5}\left(1 + z^{-1} + z^{-2} + z^{-3} + z^{-4}\right) \tag{666}$$

$$H(f)\left[= H(z = e^{jf})\right] = \frac{1}{5}\left(1 + e^{-jf} + e^{-j2f} + e^{-j3f} + e^{-j4f}\right) \tag{667}$$

This can be simplified by either method employed in E8.1. The geometric sum method yields:

$$H(f) = \frac{1}{5}\left(1 + e^{-jf} + e^{-j2f} + e^{-j3f} + e^{-j4f}\right) \tag{668}$$

$$e^{-jf}H(f) = \frac{1}{5}\left(e^{-jf} + e^{-j2f} + e^{-j3f} + e^{-j4f} + e^{-j5f}\right) \tag{669}$$

$$H(f) - e^{-jf}H(f) = \frac{1}{5}\left(1 - e^{-j5f}\right) \tag{670}$$

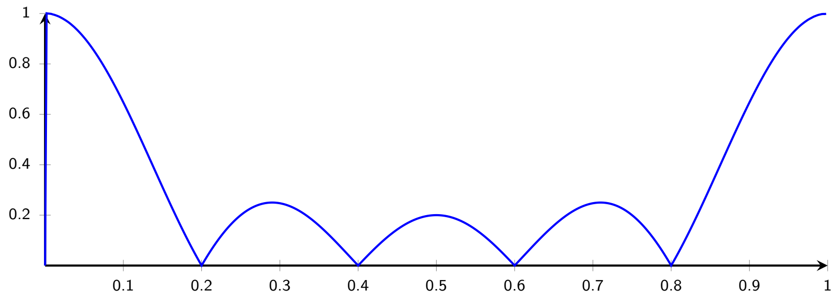
$$H(f) = \frac{1}{5}\frac{1 - e^{-j5f}}{1 - e^{-jf}} \quad \text{[rewrite to conform with Euler]} \quad = \frac{1}{5}\frac{e^{-j5/2f}(e^{j5/2f} - e^{-j5/2f})}{e^{-j1/2f}(e^{-j1/2f} - e^{-j1/2f})} \tag{671}$$

$$= \frac{1}{5}\frac{2je^{-j5/2f}\sin\frac{5}{2}}{2je^{-j1/2f}\sin\frac{1}{2}} \quad = \frac{1}{5}e^{-j3f}\frac{\sin\frac{5}{2}}{\sin\frac{1}{2}} \tag{672}$$

$$|H(f)| = \left|\frac{1}{5}e^{-j3f}\frac{\sin\frac{5}{2}}{\sin\frac{1}{2}}\right| \quad = \frac{1}{5}\left|\frac{\sin\frac{5}{2}}{\sin\frac{1}{2}}\right| \tag{673}$$

$|H(0)| = 1$
$|H(f)| = 0$ at $f = \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$

$|H(f)|$



Solution E8.4

Try a first order FIR filter:

$$H(z) = b_0 + b_1 z^{-1} \tag{674}$$

$$H(\omega)\Big[= H(z = e^{j\omega})\Big] = b_0 + b_1 e^{-j\omega} \tag{675}$$

We know that $H(\omega)|_{\omega=\pi/2} = \mathrm{e}^{\mathrm{j}\pi/3}$, thus

$$H(\omega)|_{\omega=\pi/2} = b_0 + b_1 e^{-j\pi/2} = e^{j\pi/3} \tag{676}$$

$$b_0 - jb_1 = \frac{1}{2} + j\frac{\sqrt{3}}{2} \quad \text{[noting that } b_0 \text{ is real and } b_1 \text{ imaginary allows the identification:]} \tag{677}$$

$$b_0 = \frac{1}{2} \quad b_1 = -\frac{\sqrt{3}}{2} \tag{678}$$

The filter thus is given by $H(z) = \frac{1}{2} - \frac{\sqrt{3}}{2} z^{-1}$.

Solution E8.5

Since we should have linear phase, $H(z)$ must be 1. an FIR filter, and 2. have a symmetric or antisymmetric impulse response $h(n)$. Further, since $h(n)$ should be real, all complex zeros must have a conjugate pair. Given that $H(\omega = 2/5) = H(\omega = -2/5) = 0$, we can begin building our filter as (note that the resulting $h(n)$ would become symmetrical):

$$H_1(z) = (1 - e^{-j2\pi 2/5} z^{-1})(1 - e^{j2\pi 2/5} z^{-1}) \tag{679}$$

The specification states that $|H(\omega = 0)| = 1$. Evaluation at $\omega = 0$ yields

$$|H_1(\omega = 0)| = 1 = \left|(1 - e^{-j2\pi 2/5} e^{-j2\pi * 0})(1 - e^{j2\pi 2/5} e^{-j2\pi * 0})\right| \quad = \left|(1 - e^{-j2\pi 2/5})(1 - e^{j2\pi 2/5})\right| \tag{680}$$

$$= \left|2je^{-j2\pi 1/5} \sin 2\pi 1/5 * -2je^{j2\pi 1/5} \sin 2\pi 1/5\right| \quad = 4\sin^2 2\pi 1/5 \tag{681}$$

As the equality does not hold, $H_1(z)$ must be appended with a gain, $g_1 = \frac{1}{4\sin^2 2\pi 1/5}$, which would yield:

$$H_2(z) = g_1(1 - e^{-j2\pi 2/5} z^{-1})(1 - e^{j2\pi 2/5} z^{-1}) \quad = \frac{1}{4\sin^2 2\pi 1/5}(1 - e^{-j2\pi 2/5} z^{-1})(1 - e^{j2\pi 2/5} z^{-1}) \tag{682}$$

However, the specification further states that $|H(\omega = 1/5)| = 1$, and evaluation at $\omega = 1/5$ yields

$$H_2(z) = 1 = \left|\frac{1}{4\sin^2 2\pi 1/5}(1 - e^{-j2\pi 1/5} e^{-j2\pi 2/5})(1 - e^{j2\pi 2/5} e^{-j2\pi 1/5})\right| \quad = 0.618... \tag{683}$$

Thus, we instead will add more zeros to satisfy the specifications. We try to remove the gain add a conjugate pair of zeros with radius r and phase α:

$$H_3(z) = (1 - e^{-j2\pi 2/5} z^{-1})(1 - e^{j2\pi 2/5} z^{-1})(1 - re^{-j2\pi\alpha} z^{-1})(1 - re^{j2\pi\alpha} z^{-1}) \tag{684}$$

$$= (1 - e^{j2\pi 2/5} z^{-1} - e^{-j2\pi 2/5} z^{-1} + z^{-2})(1 - re^{j2\pi\alpha} z^{-1} - re^{-j2\pi\alpha} z^{-1} + r^2 z^{-2}) \tag{685}$$

$$= (1 - 2\cos 2\pi 2/5\ z^{-1} + z^{-2})(1 - 2r\cos 2\pi\alpha\ z^{-1} + r^2 z^{-2}) \tag{686}$$

$$[\mathbf{c}_1 = 2\cos 2\pi 2/5,\ \mathbf{c}_\alpha = 2\cos 2\pi\alpha] \tag{687}$$

$$= (1 - c_1 z^{-1} + z^{-2})(1 - rc_\alpha z^{-1} + r^2 z^{-2}) \tag{688}$$

$$= 1 - rc_\alpha z^{-1} + r^2 z^{-2} - c_1 z^{-1} + rc_1 c_\alpha z^{-2} - r^2 c_1 z^{-3} + z^{-2} - rc_\alpha z^{-3} + r^2 z^{-4} \tag{689}$$

$$= 1 - (rc_\alpha + c_1) z^{-1} + (r^2 + rc_1 c_\alpha + 1) z^{-2} - (r^2 c_1 + rc_\alpha) z^{-3} + r^2 z^{-4} \tag{690}$$

However, for any $r \neq 1$, $h(n)$ lose symmetry, and thus we set $r = 1$. However, since we probably (as we have seen) need two variables to fulfill both $|H(\omega = 0)| = |H(\omega = 1/5)| = 1$, we also reintroduce a gain g_2. We have:

$$H_4(z) = g_2(1 - e^{-j2\pi 2/5} z^{-1})(1 - e^{j2\pi 2/5} z^{-1})(1 - e^{-j2\pi\alpha} z^{-1})(1 - e^{j2\pi\alpha} z^{-1}) \tag{691}$$

$$= g_2(1 - (c_\alpha + c_1) z^{-1} + (2 + c_1 c_\alpha) z^{-2} - (c_1 + c_\alpha) z^{-3} + z^{-4}) \tag{692}$$

In order to find suitable r and α we first utilize $|H(\omega = 0)| = 1$:

$$|H_4(\omega = 0)| = 1 = |g_2(1 - (c_\alpha + c_1) + (2 + c_1 c_\alpha) - (c_1 + c_\alpha) + 1)| \tag{693}$$

$$1 = |g_2 * (4 - c_\alpha(2 - c_1) - 2c_1)| \tag{694}$$

$$\begin{array}{l|l} case1 & case2 \\ \frac{1}{g_2} = 4 - c_\alpha(2 - c_1) - 2c_1 & -\frac{1}{g_2} = 4 - c_\alpha(2 - c_1) - 2c_1 \\ c_\alpha = -\frac{\frac{1}{g_2} - 4 + 2c_1}{(2 - c_1)} & c_\alpha = -\frac{-\frac{1}{g_2} - 4 + 2c_1}{(2 - c_1)} \end{array} \tag{695}$$

Next, $|H(\omega = 1/5)| = 1$ is utilized:

$$|H_4(\omega = 1/5)| = 1 = \left|g_2(1 - e^{-j2\pi 2/5}e^{-j2\pi 1/5})(1 - e^{j2\pi 2/5}e^{-j2\pi 1/5})(1 - e^{-j2\pi\alpha}e^{-j2\pi 1/5})(1 - e^{j2\pi\alpha}e^{-j2\pi 1/5})\right| \tag{696}$$

$$= \left|g_2(1 - (c_\alpha + c_1)e^{-j2\pi 1/5} + (2 + c_1 c_\alpha)e^{-j2\pi 2/5} - (c_1 + c_\alpha)e^{-j2\pi 3/5} + e^{-j2\pi 4/5})\right| \tag{697}$$

$$= \left|g_2 e^{-j2\pi 2/5}(e^{j2\pi 2/5} - (c_\alpha + c_1)e^{j2\pi 1/5} + (2 + c_1 c_\alpha) - (c_1 + c_\alpha)e^{-j2\pi 1/5} + e^{-j2\pi 2/5})\right| \tag{698}$$

$$= \left|g_2 e^{-j2\pi 2/5}(2\cos 2\pi 2/5 \ - 2(c_\alpha + c_1)\cos 2\pi 1/5 \ + 2 + c_1 c_\alpha)\right| \tag{699}$$

$$= |g_2(2\cos 2\pi 2/5 \ - 2(c_\alpha + c_1)\cos 2\pi 1/5 \ + 2 + c_1 c_\alpha)| \tag{700}$$

$$[\mathbf{c}_1 = 2\cos 2\pi 2/5, \ \mathbf{c}_2 = 2\cos 2\pi 1/5] \tag{701}$$

$$= |g_2(c_1 - (c_\alpha + c_1)c_2 + 2 + c_1 c_\alpha)| \quad = |g_2(c_1 - c_\alpha c_2 - c_1 c_2 + 2 + c_1 c_\alpha)| \tag{702}$$

$$= |g_2(c_1 + c_\alpha(-c_2 + c_1) - c_1 c_2 + 2)| \quad \left[\text{insert case 1 above } c_\alpha = -\frac{\frac{1}{g_2} - 4 + 2c_1}{(2 - c_1)}\right] \tag{703}$$

$$= \left|g_2 c_1 - \frac{1 - 4g_2 + 2c_1 g_2}{(2 - c_1)}(-c_2 + c_1) - c_1 c_2 g_2 + 2g_2\right| \tag{704}$$

$$= \left|\frac{1}{2 - c_1}(g_2 c_1(2 - c_1) - (1 - 4g_2 + 2c_1 g_2)(-c_2 + c_1) - c_1 c_2 g_2(2 - c_1) + 2g_2(2 - c_1))\right| \tag{705}$$

$$= \left|\frac{1}{2 - c_1}(g_2 c_1(2 - c_1) - (-c_2 + c_1) + 4g_2(-c_2 + c_1) - 2c_1 g_2(-c_2 + c_1) - c_1 c_2 g_2(2 - c_1) + 2g_2(2 - c_1))\right| \tag{706}$$

$$= \left|\frac{1}{2 - c_1}(g_2(c_1(2 - c_1) + 4(-c_2 + c_1) - 2c_1(-c_2 + c_1) - c_1 c_2(2 - c_1) + 2(2 - c_1)) + c_2 - c_1)\right| \tag{707}$$

$$= \left|g_2 \frac{1}{2 - c_1}(c_1(2 - c_1) + 4(-c_2 + c_1) - 2c_1(-c_2 + c_1) - c_1 c_2(2 - c_1) + 2(2 - c_1)) + \frac{1}{2 - c_1}(c_2 - c_1)\right| \tag{708}$$

$$= \left|g_2 \frac{-4c_2 + c_1(4 - 3c_1 + c_1 c_2) + 4}{2 - c_1} + \frac{c_2 - c_1}{2 - c_1}\right| \tag{709}$$

$$\begin{array}{l|l} case1.1 & case1.2 \\ g_2 = \frac{2 - c_2}{-4c_2 + c_1(4 - 3c_1 + c_1 c_2) + 4} & g_2 = \frac{-2 + 2c_1 - c_2}{-4c_2 + c_1(4 - 3c_1 + c_1 c_2) + 4} \end{array} \tag{710}$$

$$\left[\text{for } c_\alpha = -\frac{-\frac{1}{g_2} - 4 + 2c_1}{(2 - c_1)} \text{ , we instead get}\right] \tag{711}$$

$$\begin{array}{l|l} case2.1 & case2.2 \\ g_2 = \frac{2 - 2c_1 + c_2}{-4c_2 + c_1(4 - 3c_1 + c_1 c_2) + 4} & g_2 = \frac{-2 + c_2}{-4c_2 + c_1(4 - 3c_1 + c_1 c_2) + 4} \end{array} \tag{712}$$

We can thus write the final filter H_4 as:

$$H_4(z) = g_2(1 - (c_\alpha + c_1)z^{-1} + (2 + c_1 c_\alpha)z^{-2} - (c_1 + c_\alpha)z^{-3} + z^{-4}) \tag{713}$$

For any case:

$$\begin{array}{lllll} & case1.1 & case1.2 & case2.1 & case2.2 \\ g_2 = & -0.1236 & 0.5236 & -0.5236 & 0.1236 \\ c_\alpha = & 4.2361 & 1.4721 & 1.4721 & 4.2361 \end{array} \tag{714}$$

Choosing for instance $g_2 = 0.5236$ and $c_\alpha = 1.4721$ (yielding a real angle α) gives a valid filter

$$H_4(z) = 0.5236 + 0.0764z^{-1} - 0.2z^{-2} + 0.0764z^{-3} + 0.5236z^{-4} \tag{715}$$

Solution E8.7

An FIR filter with linear phase and 50 dB damping means a Hamming window. -6 dB at $f = f_1 = 0.1$ measn that at -50 dB and $f = 0.15$ we get $(0.15 - 0.1) \cdot M = 1.62$

$$M = \frac{1.62}{0.05} = 32.4 \quad \Rightarrow \quad M = 33 \tag{716}$$

$$h(n) = \begin{cases} 2 \cdot f_1 \operatorname{sinc}(2f_1(n-16)) \cdot \left[0.54 + 0.46\cos\frac{2\pi(n-16)}{32}\right] & 0 \le n \le 32 \\ 0 & \text{f.ö.} \end{cases} \tag{717}$$

Solution E8.8

$$\begin{cases} H_{dB}(0.2) = -3\,\text{dB} & \text{gives } (0.2 - f_c) \cdot M = -0.40 \\ H_{dB}(0.25) = -40\,\text{dB} & \text{gives } (0.25 - f_c) \cdot M = 1.49 \end{cases} \tag{718}$$

$$M = \frac{1.49 + 0.40}{0.25 - 0.20} = 37.8 \quad \Rightarrow \quad M = 39 \quad \text{(odd)} \tag{719}$$

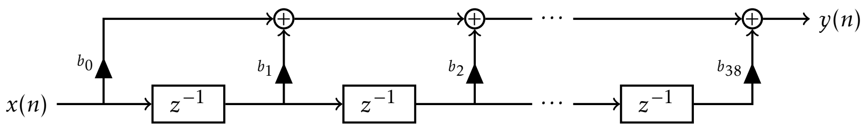
$$\begin{cases} f_c = 0.2 + \frac{0.40}{39} = 0.2103 \\ f_c = 0.25 - \frac{1.49}{39} = 0.2118 \end{cases} \tag{720}$$

Choose $f_c = 0.2103$, but any f_c in the interval $0.2103 \le f_c \le 0.2118$ works.

$$b_n = \left[h(n) = \hat{h}(n-19) = h_d(n-19)w_H(n-19)\right] \tag{721}$$

$$= \frac{\sin(2\pi \cdot 0.2103(n-19))}{\pi(n-19)} \cdot \left(0.54 + 0.46\cos\frac{2\pi(n-19)}{38}\right) \tag{722}$$

for $n = 0, \ldots, 38$.



Solution E8.9

From the Hamming window diagram:

$$\begin{cases} -0.4 = -(0.16 - f_c)M \\ 1.49 = -(0.10 - f_c)M \end{cases} \quad \Rightarrow \quad \begin{cases} M = 33 \\ f_c = 0.1479 \end{cases} \tag{723}$$

$$w_H(n) = 0.54 + 0.46\cos\frac{2\pi n}{32} \tag{724}$$

$$h_d(n) = \delta(n) - 2f_c \operatorname{sinc}(2f_c n) \tag{725}$$

$$\hat{h}(n) = h_d(n-16)\, w_H(n-16) \quad 0 \le n \le 32 \tag{726}$$

Solution E8.10

Start from an ideal band pass filter:

$$h_d(n) = 4f_c \operatorname{sinc}(2f_c n) \cdot \cos(2\pi f_0 n) \tag{727}$$

Truncate $h_d(n)$ using $w_H(n)$. A Hamming works becaise we only need 40 dB dampning.

$$w_H(n) = \begin{cases} 0.54 + 0.46 \cos \frac{2\pi n}{M-1} & -\frac{M-1}{2} \leq n \leq \frac{M-1}{2} \\ 0 & \text{f.ö.} \end{cases} \tag{728}$$

Write

$$\hat{h}(n) = h_d(n) \cdot w_H(n) \tag{729}$$

Translate $\hat{h}(n)$ until it becomes causal:

$$h(n) = \hat{h}\left(n - \frac{M-1}{2}\right) \tag{730}$$

WE also need to compute M, f_0, and f_1. First we look at the left side. The collection of formulas gives

$$\begin{cases} -(0.05 - (f_0 - f_1))M = 1.49 & (-40\,\text{dB}) \quad (1) \\ -(0.10 - (f_0 - f_1))M = -0.4 & (-3\,\text{dB}) \quad (2) \end{cases} \quad \Rightarrow \quad 0.05 \cdot M = 1.89 \quad \Rightarrow \quad M = 37.8 \tag{731}$$

Then we look at the right side:

$$\begin{cases} (0.25 - (f_0 + f_1))M = -0.4 & (-3\,\text{dB}) \quad (3) \\ (0.275 - (f_0 + f_1))M = 0.91 & (-20\,\text{dB}) \quad (4) \end{cases} \quad \Rightarrow \quad 0.025 \cdot M = 1.31 \quad \Rightarrow \quad M = 52.4 \tag{732}$$

To fulfill the demand at both sides, we need $M > \max(37.8,\ 52.4)$: choose $M = 53$. Theer is also the deamdns that at the freqeuncies 0.10 and 0.25 the damping should be 3 dB, whcih measdn that when we solve for f_0 and f_1 we must use Equations (2) and (3). The M value to be used is $M = 53$, in both equations

$$\begin{cases} \text{ekv (2)} & \Rightarrow & (0.10 - (f_0 - f_1)) \cdot 53 = 0.4 \\ \text{ekv (3)} & \Rightarrow & (0.25 - (f_0 + f_1)) \cdot 53 = -0.4 \end{cases} \quad \Rightarrow \quad \begin{cases} f_0 = 0.175 \\ f_1 = 0.0825 \end{cases} \tag{733}$$

Thus:

$$h_d(n) = 0.3302 \cdot \operatorname{sinc}(0.1652n) \cdot \cos(2\pi \cdot 0.175n) \tag{734}$$

$$w_H(n) = \begin{cases} 0.54 + 0.46 \cos \frac{2\pi n}{52} & -26 \leq n \leq 26 \\ 0 & \text{otherwise} \end{cases} \tag{735}$$

and

$$h(n) = h_d(n-26) \cdot w_H(n-26) \tag{736}$$

$$= \begin{cases} [0.3302 \cdot \operatorname{sinc}(0.1652(n-26)) \cdot \cos(2\pi \cdot 0.175(n-26))] \times \left[0.54 + 0.46 \cos \frac{2\pi(n-26)}{52}\right] & 0 \leq n \leq 52 \\ 0 & \text{otherwise} \end{cases} \tag{737}$$

Solution E8.11

The center frequency is $f_0 = 0.2096$, $f_c = 0.1233$, and $M = 41$ (from the expression). This is used in

$$(f_1 - (f_0 + f_c))M = -0.4 \quad \rightarrow \quad f_1 = 0.3231 \tag{738}$$

for the right side (low pass) and in

$$-(f_2 - (f_0 - f_c))M = -0.4 \quad \rightarrow \quad f_2 = 0.0960 \tag{739}$$

for the left side (high pass). The bandwidth is $\Delta f = f_1 - f_2 = 0.2270$.

Solution E8.12

$H(z)$ shoudl have linear phase. Let $H_2(z) = a + bz^{-1} + cz^{-2}$ since first order is not enough. Calcuating $H(f)$ we get linear phase if

$$a = cr^2 \tag{740}$$

$$b - 2ar\cos(\theta) = br^2 - 2cr\cos(\theta) \tag{741}$$

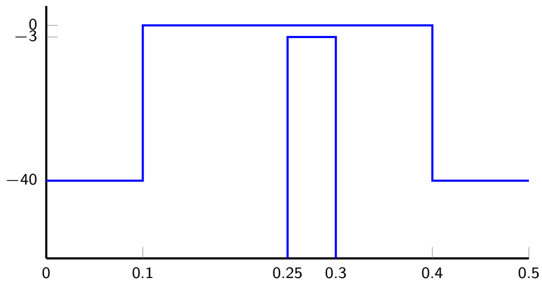
Combined with the demand on the stationary amplification this gives

$$H_2(z) = r^2 - 2r\cos(\theta)z^{-1} + z^{-2} \tag{742}$$

$H(z)$ is order 4 and has a symmetrical impulse response. This gives $\arg(H(f)) = -4\pi f$.

Solution E8.13

Sampling with 10 MHz gives aliasing. The overtoneds appear in 4 MHz to 5 MHz and the disturbances appear in 0 MHz to 1 MHz. This gives a band pass filter with the following demands.



A Hamming window gives $L = 19$, $f_0 = 0.275$, and $f_1 = 0.0461$. The impulse response is

$$h(n) = \left(0.54 + 0.46\cos\left(\frac{2\pi(n-9)}{18}\right)\right) \cdot 4f_1 \operatorname{sinc}(2f_1(n-9))\cos(2\pi f_0(n-9)) \quad 0 \le n \le 18 \tag{743}$$