EITF75 Systems and Signals

Summary of Course

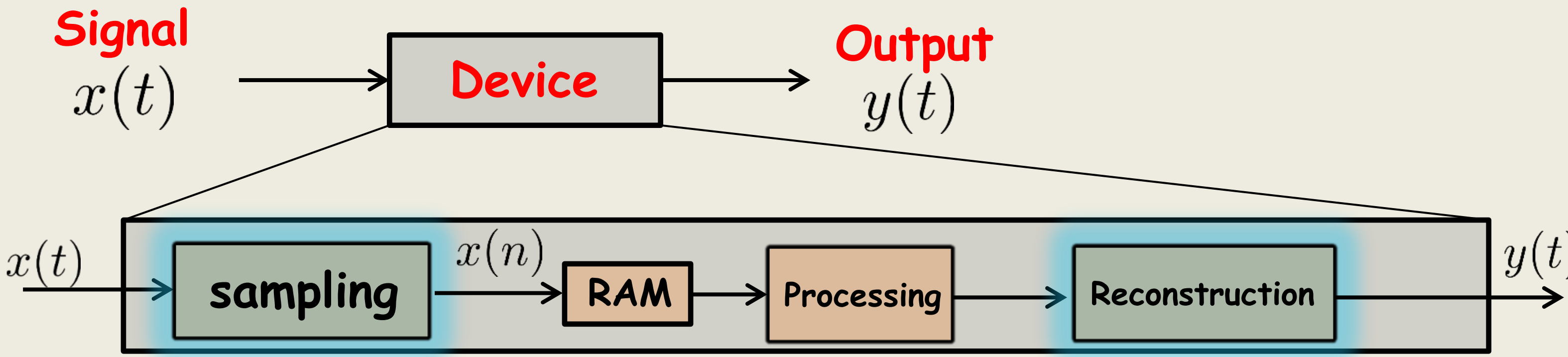
Fredrik Rusek

EITF75 Systems and Signals

Sampling and reconstruction

EITF75 Systems and Signals

A/D and D/A



To process a signal digitally, we need to first convert an analog signal to a discrete one **(Sampling)**

Then we often need to convert it back to analog **(Reconstruction)**

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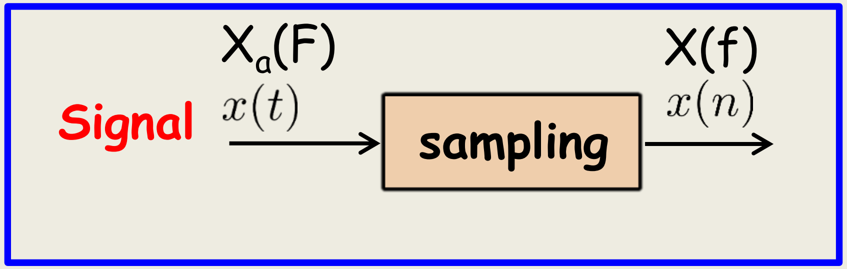
A/D and D/A



Key step is to understand what X(f) looks like in terms of $X_a(F)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

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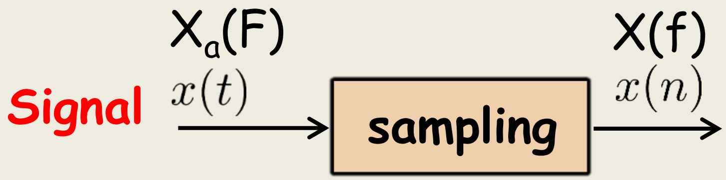
A/D and D/A



Key step is to understand what $X(f)$ looks like in terms of $X_a(F)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

If sampling is to sparse, there is aliasing.
We find $X(f)$ by the "folding technique"

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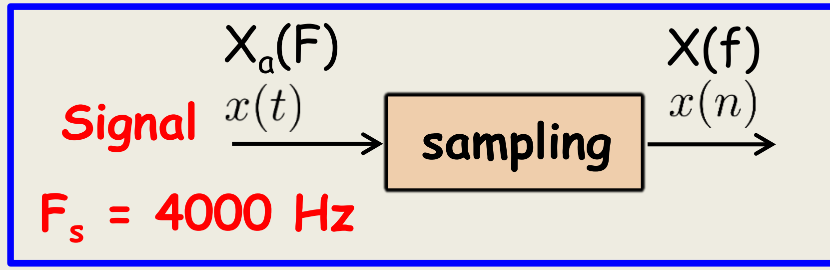
Example: Folding



Folding

Step 1: Identify $F_s/2$
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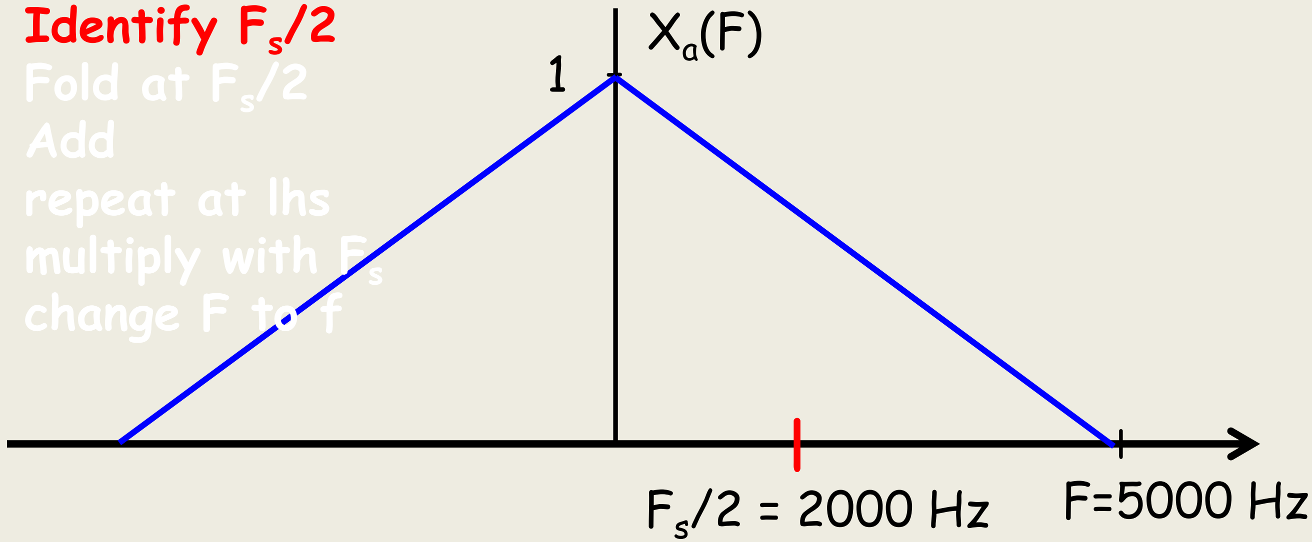
Step 4: repeat at lhs
Step 5: multiply with F_s
Step 6: change F to f



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Example: Folding

$F_s = 4000$ Hz

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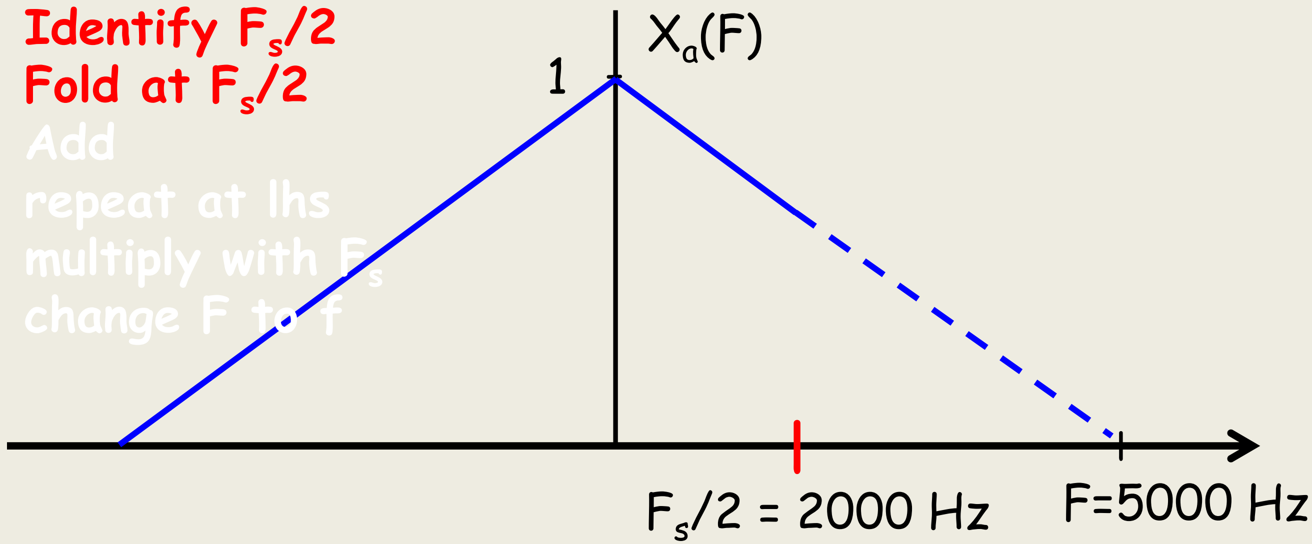

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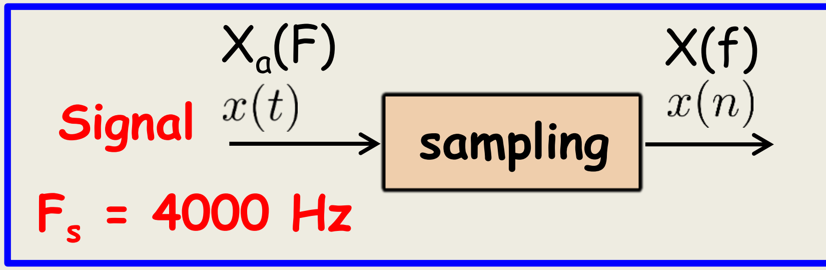

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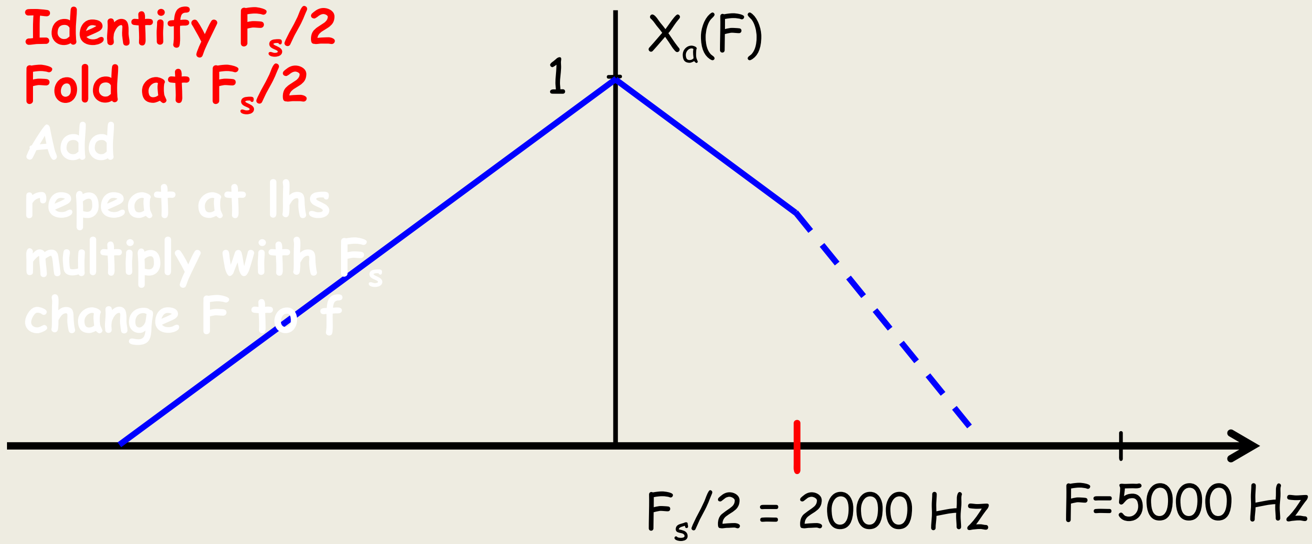

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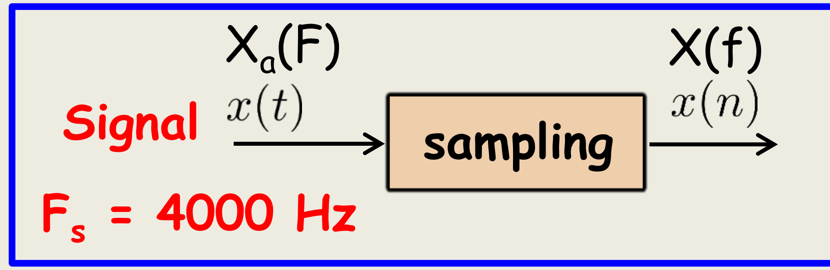

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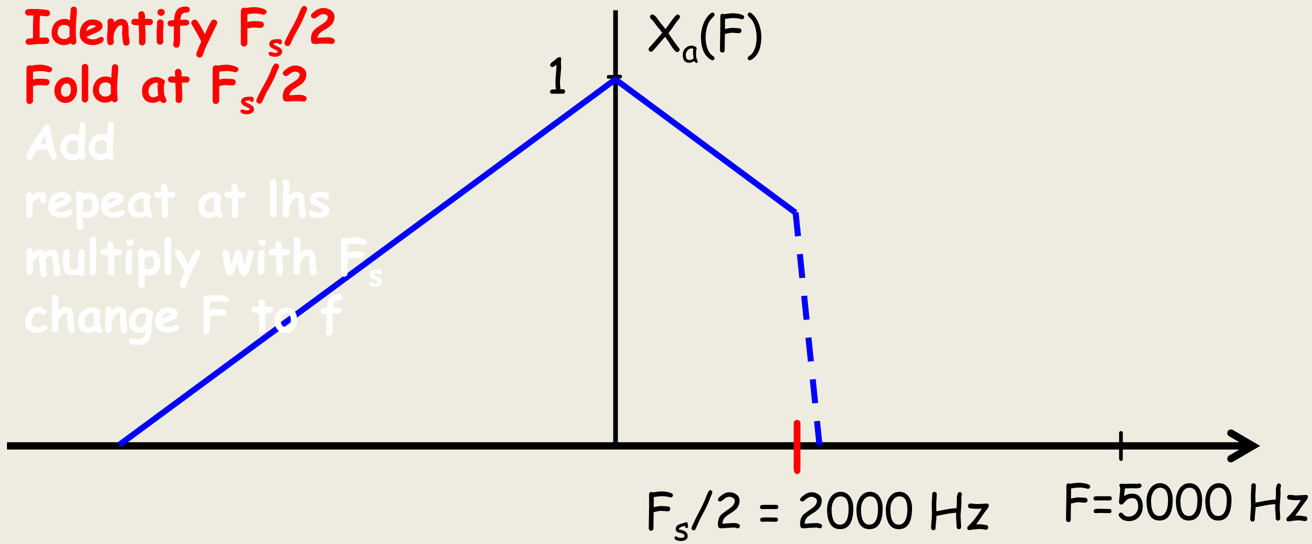

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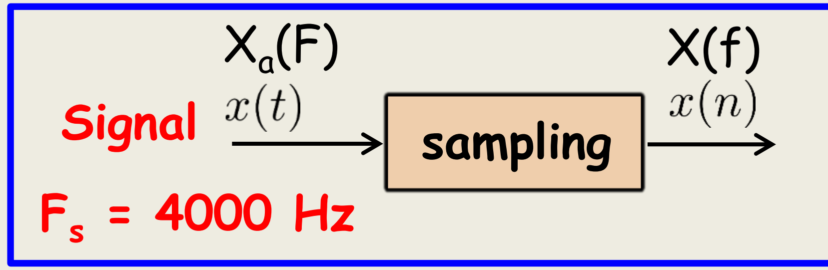

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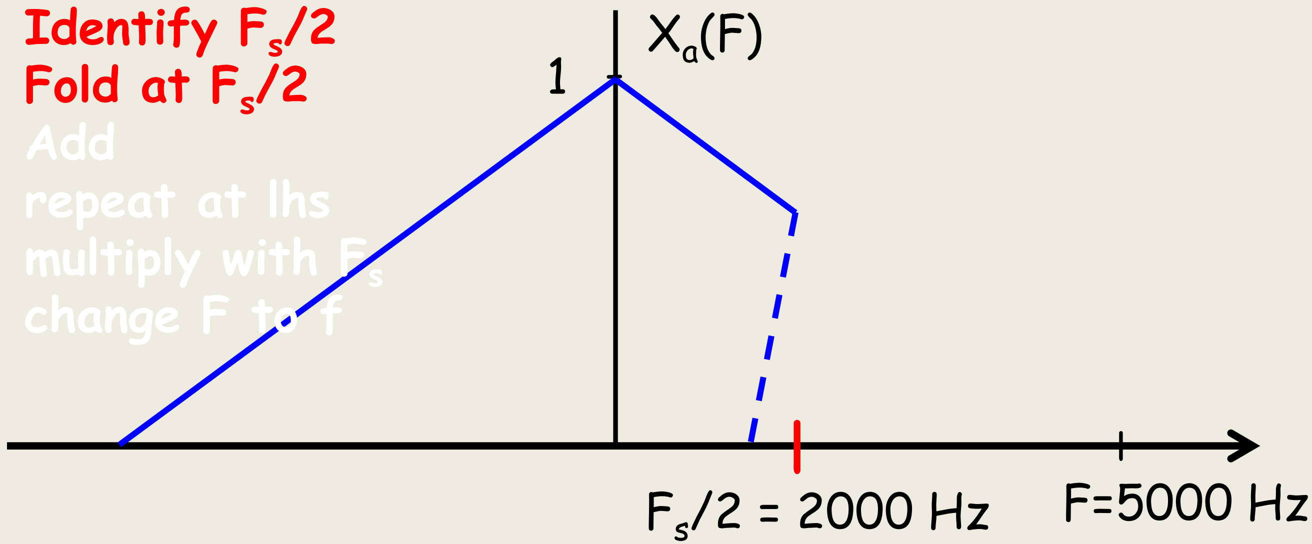

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Example: Folding

Signal

F_s = 4000 Hz

Folding

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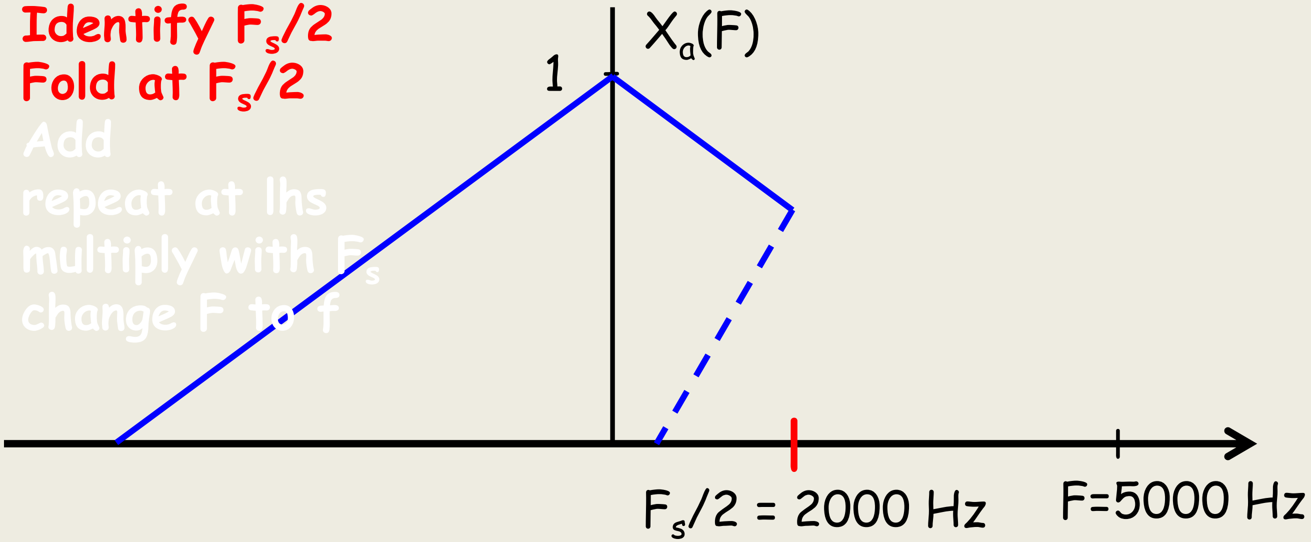

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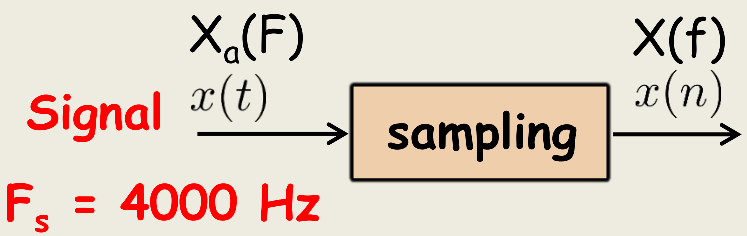

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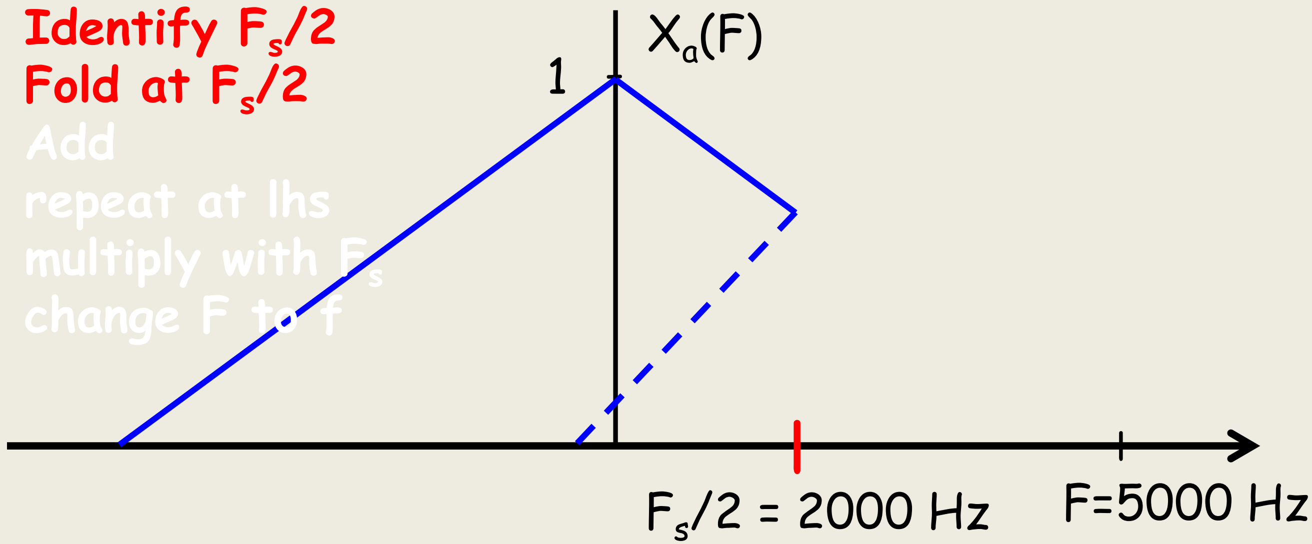

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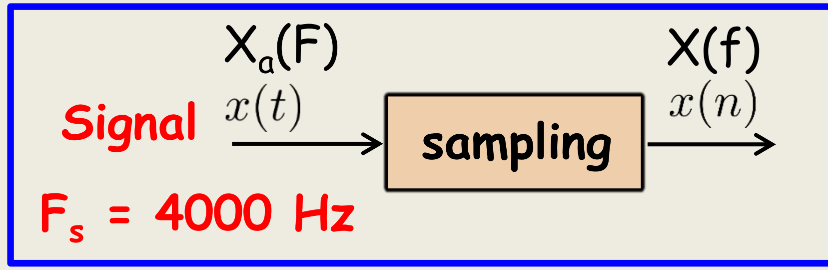

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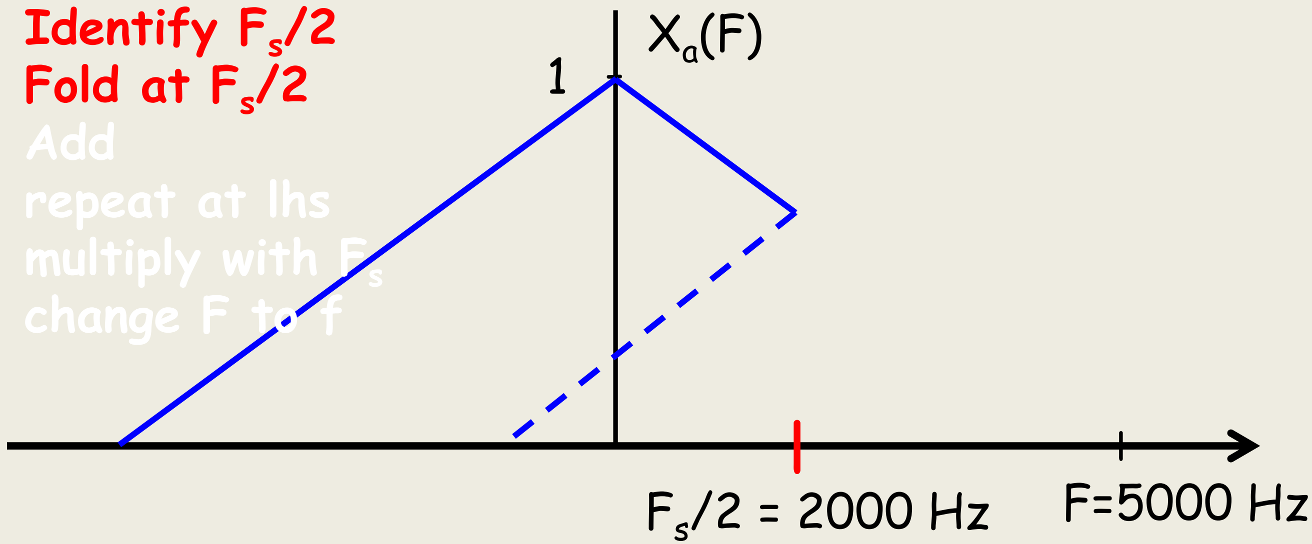

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Example: Folding

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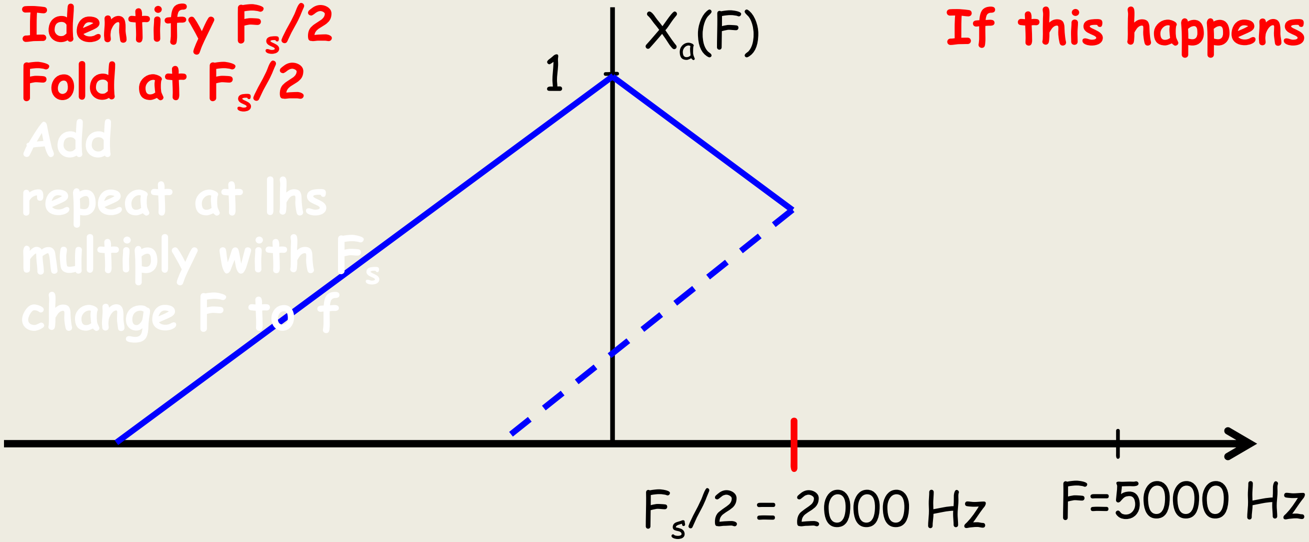

If this happens:

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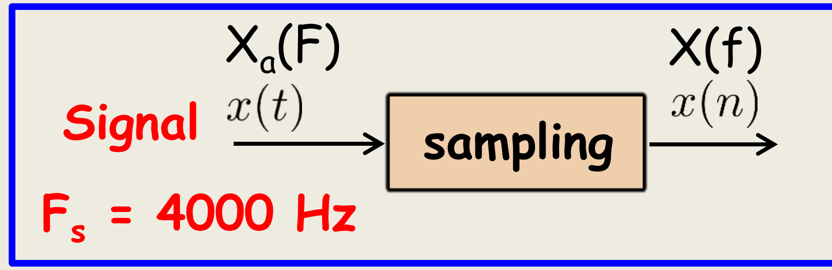
Example: Folding



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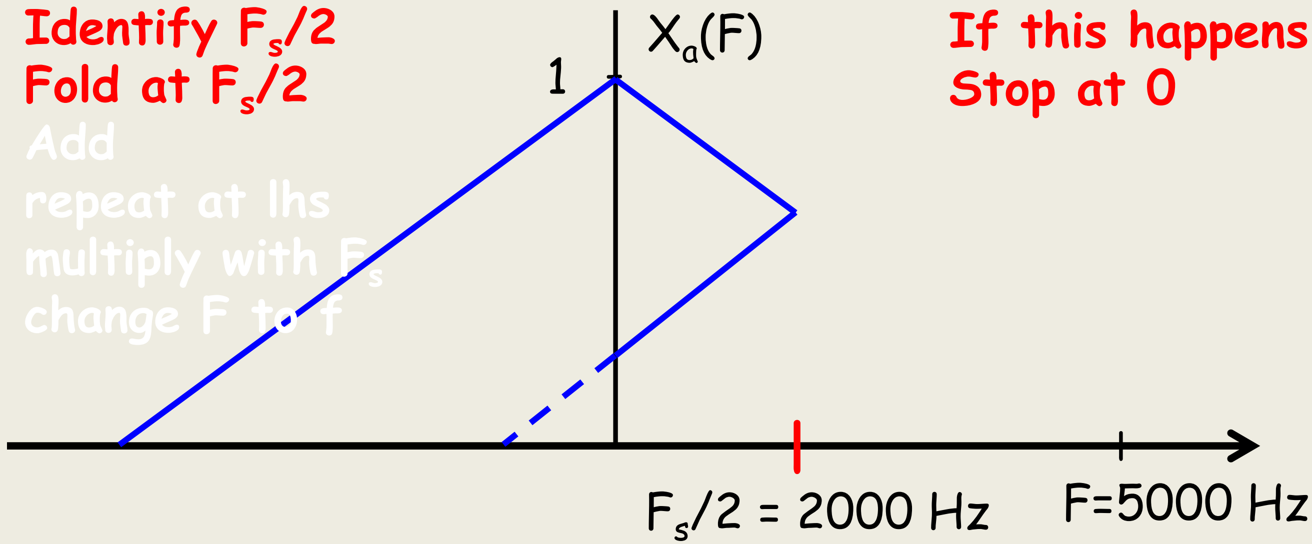

If this happens:
Stop at 0

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Example: Folding

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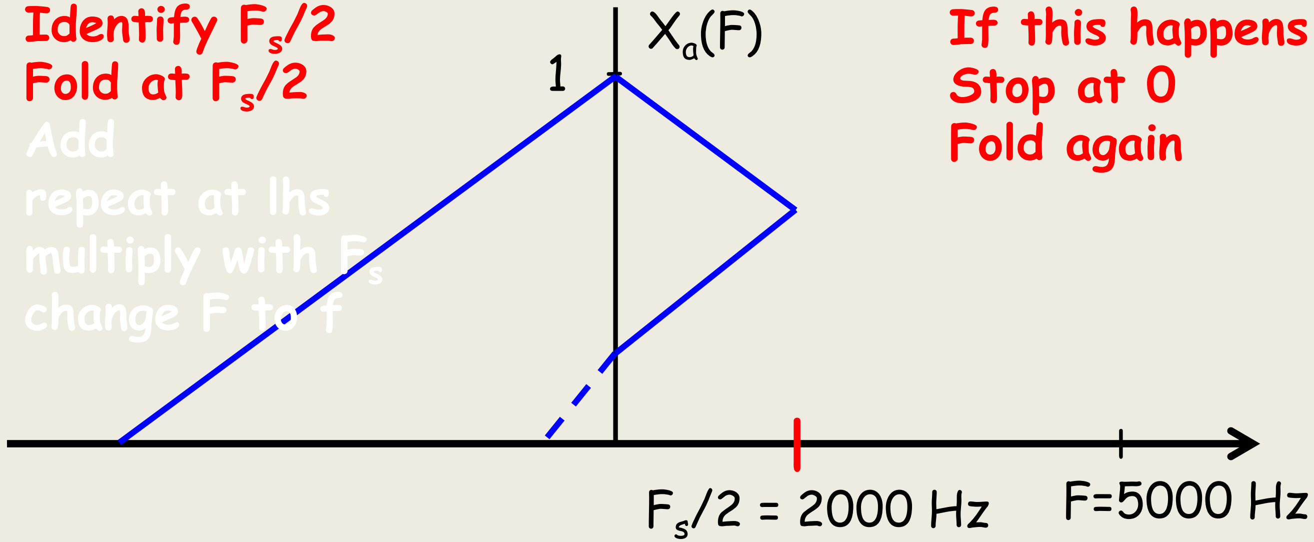

If this happens:
Stop at 0
Fold again

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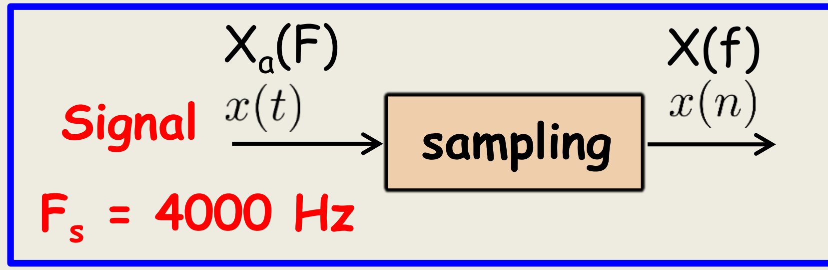
Example: Folding



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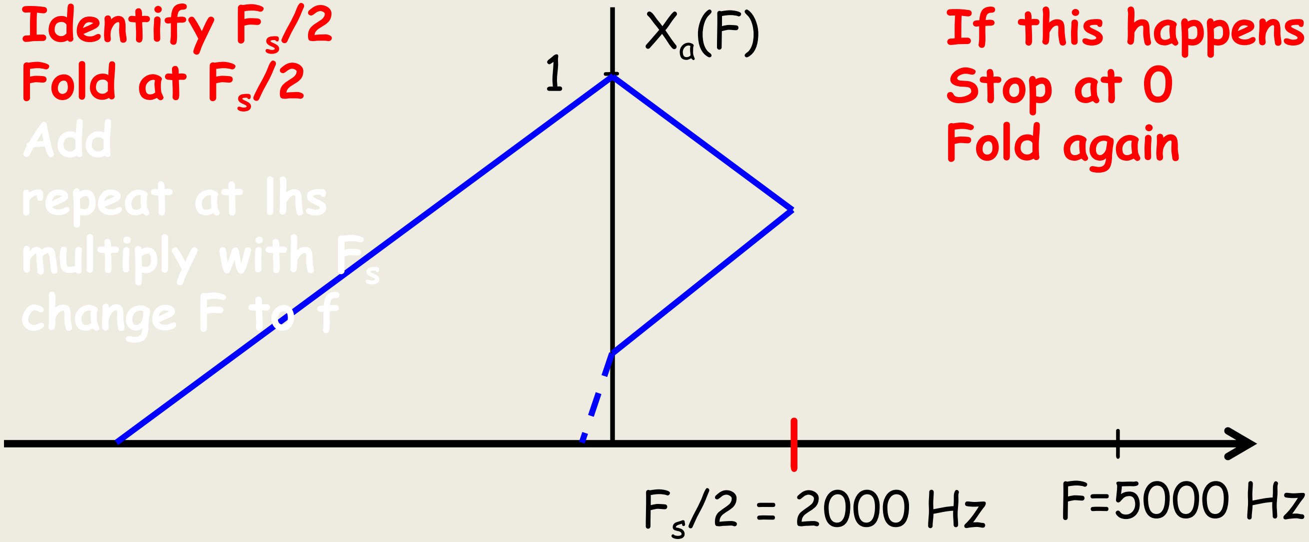

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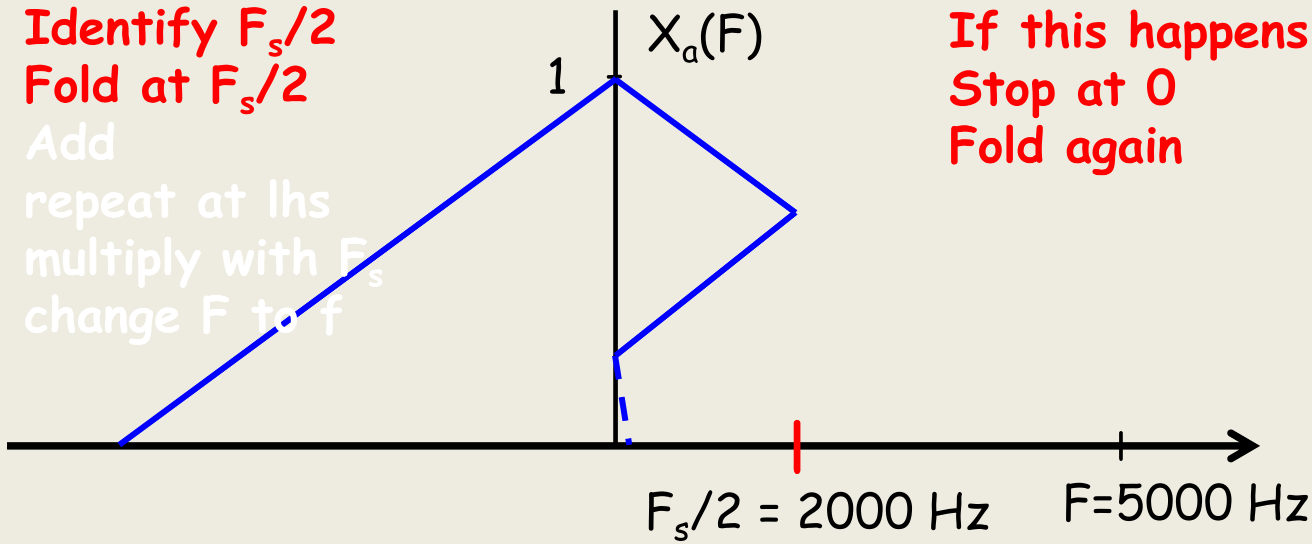

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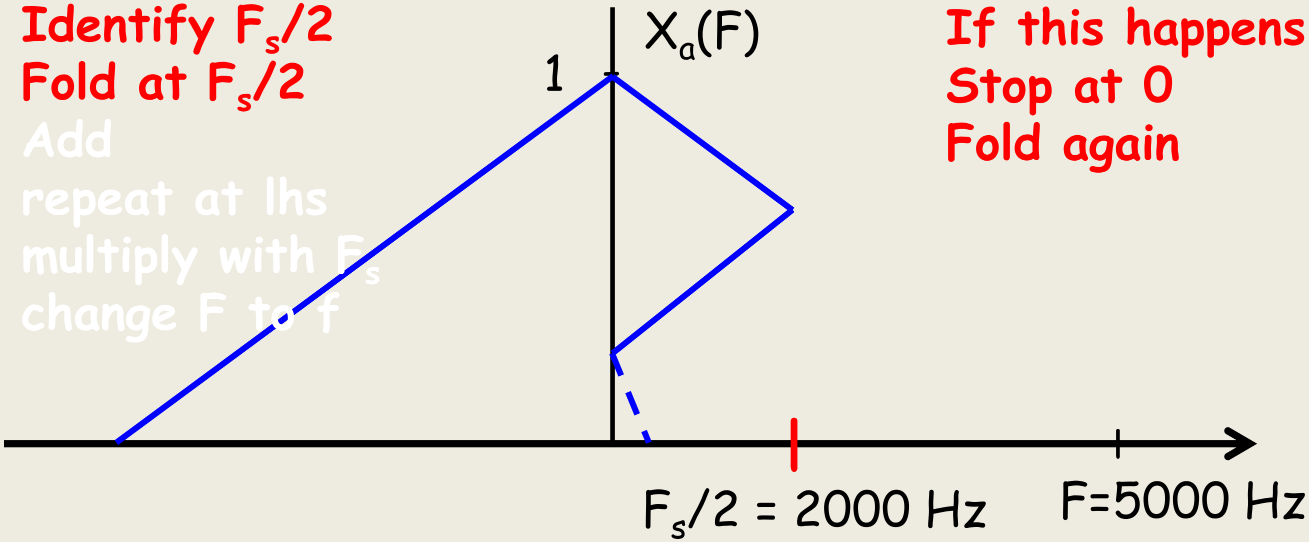

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EITF75 Systems and Signals

Example: Folding

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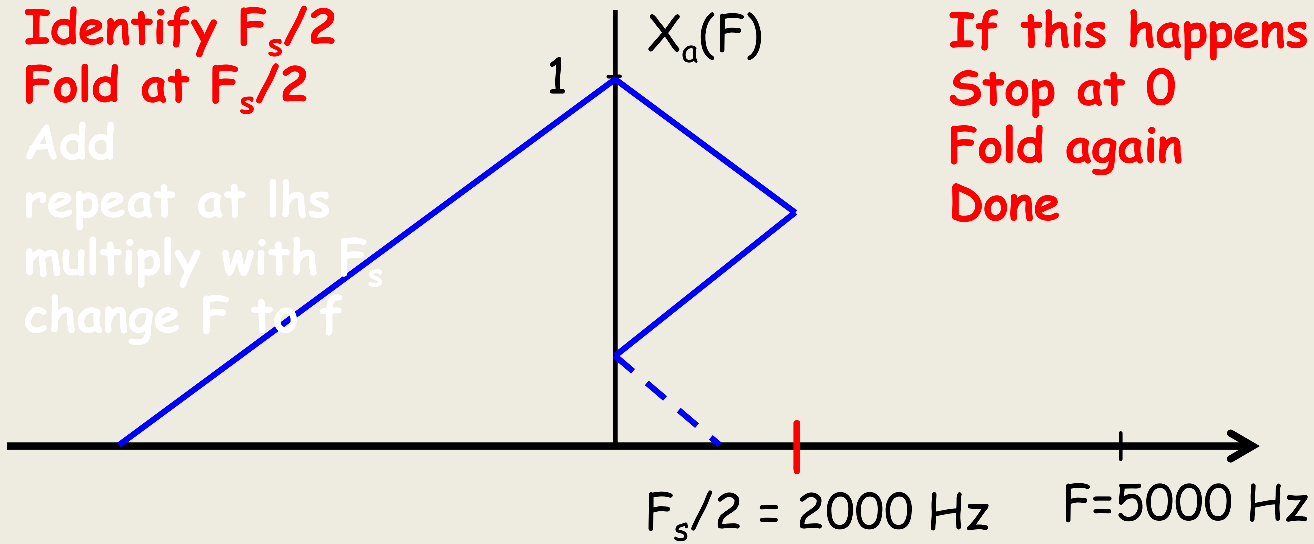
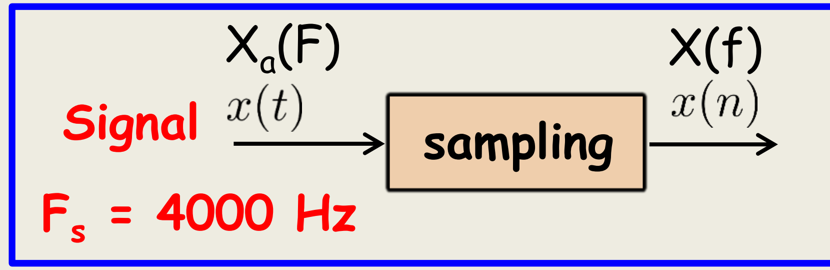


If this happens:
Stop at 0
Fold again
Done

EITF75 Systems and Signals

Example: Folding



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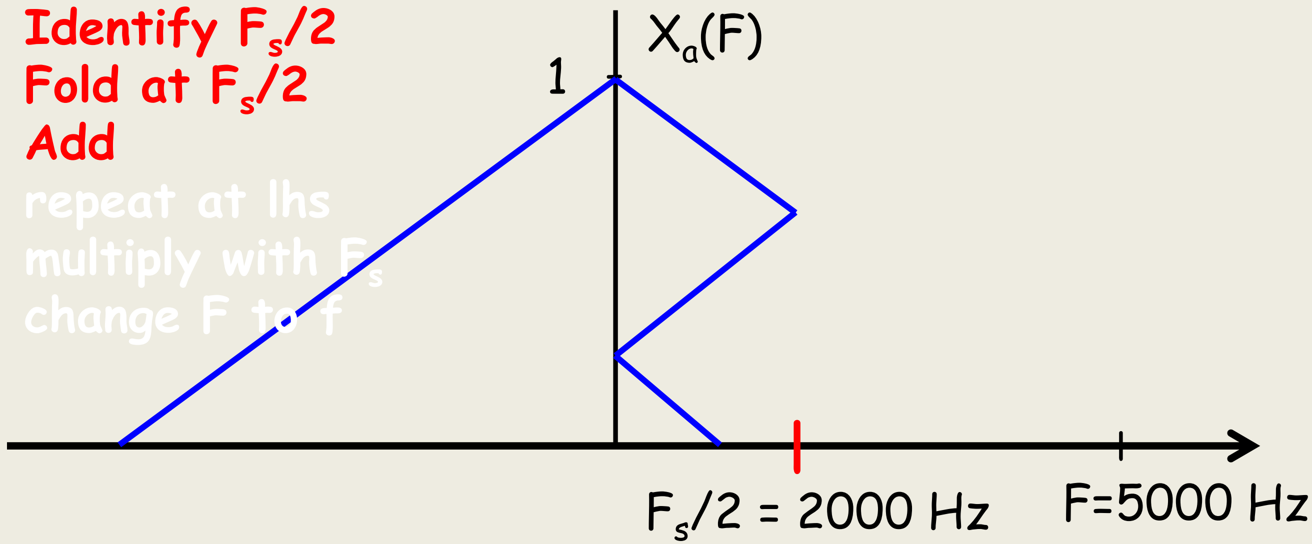

EITF75 Systems and Signals

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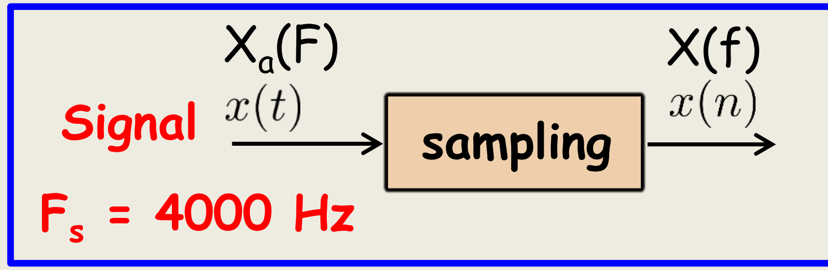

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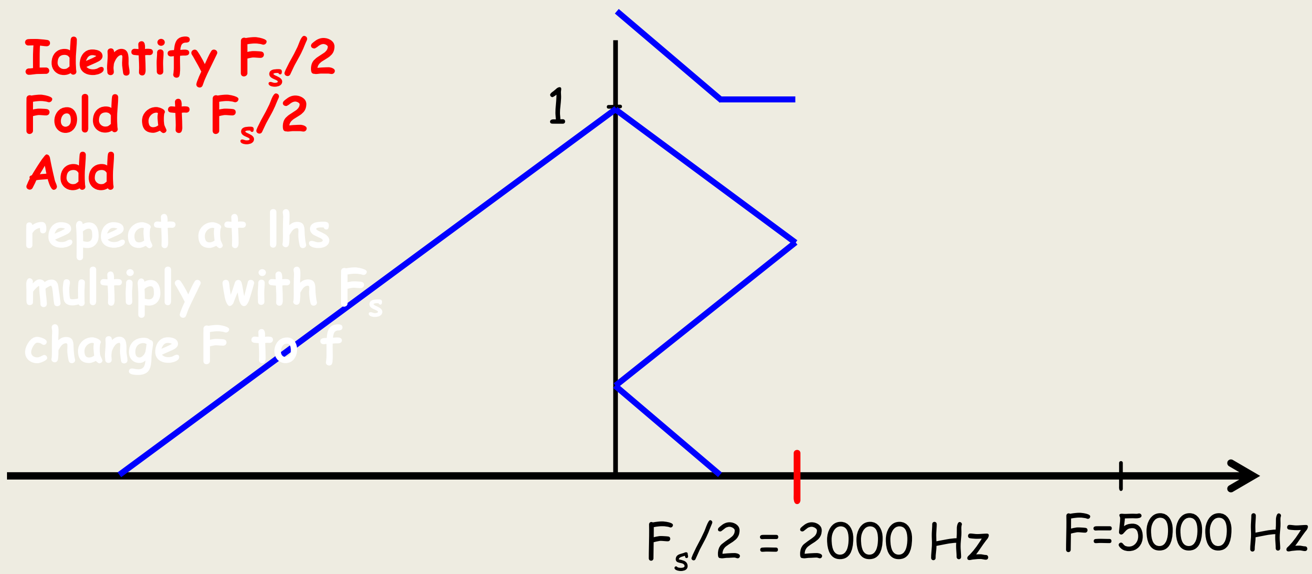

EITF75 Systems and Signals

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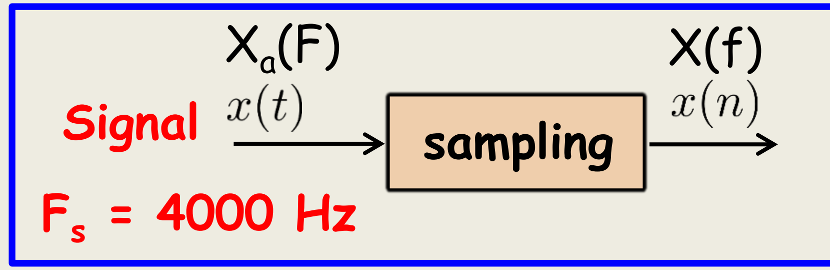

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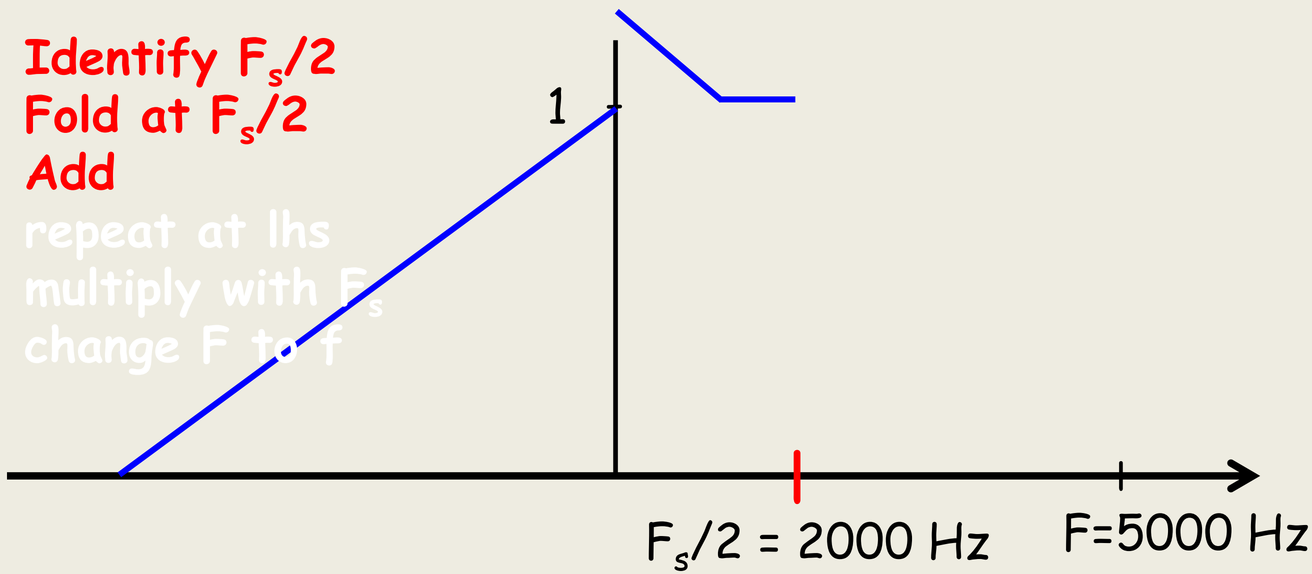

EITF75 Systems and Signals

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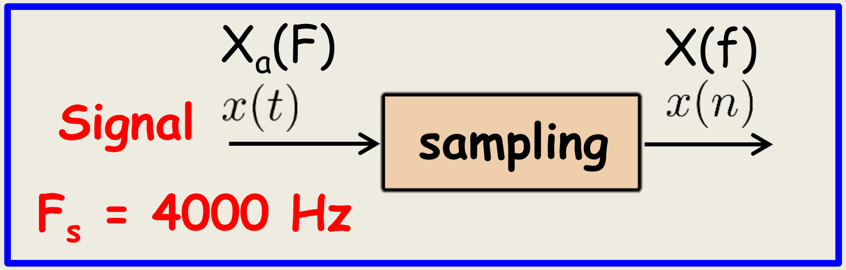

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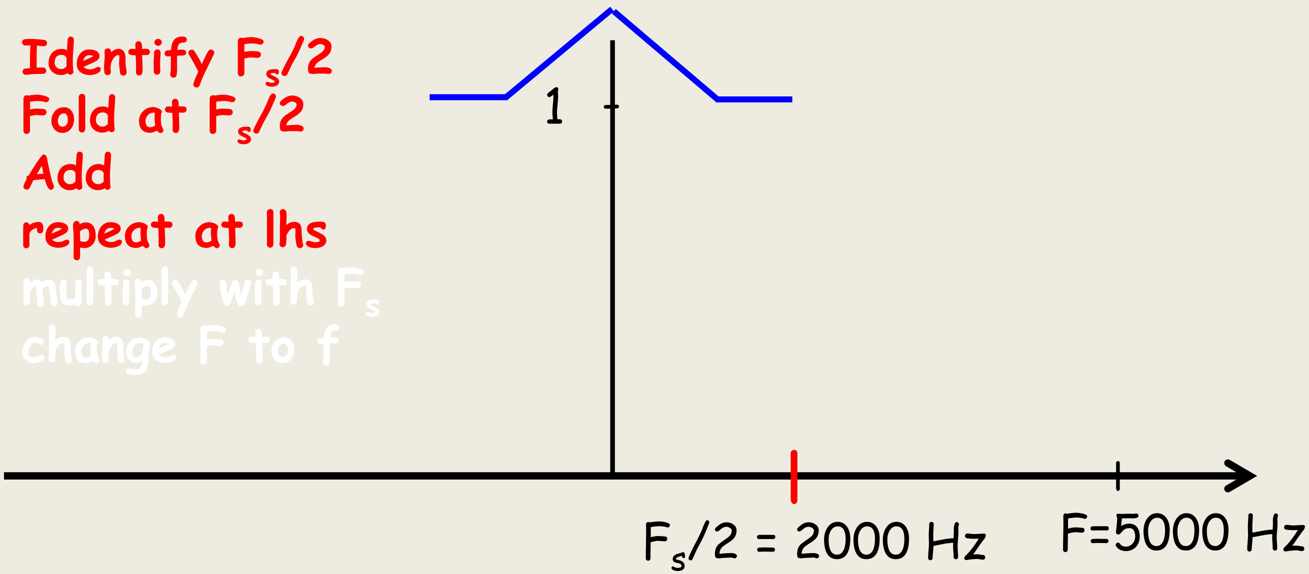

EITF75 Systems and Signals

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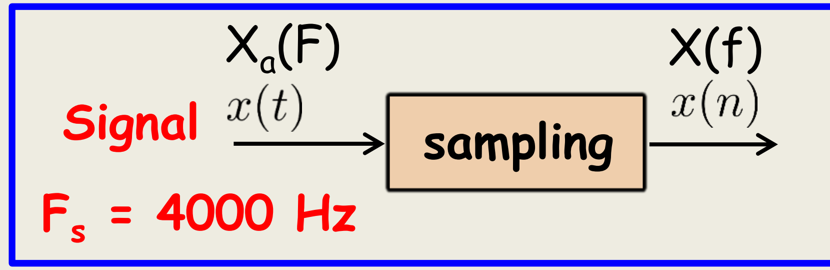

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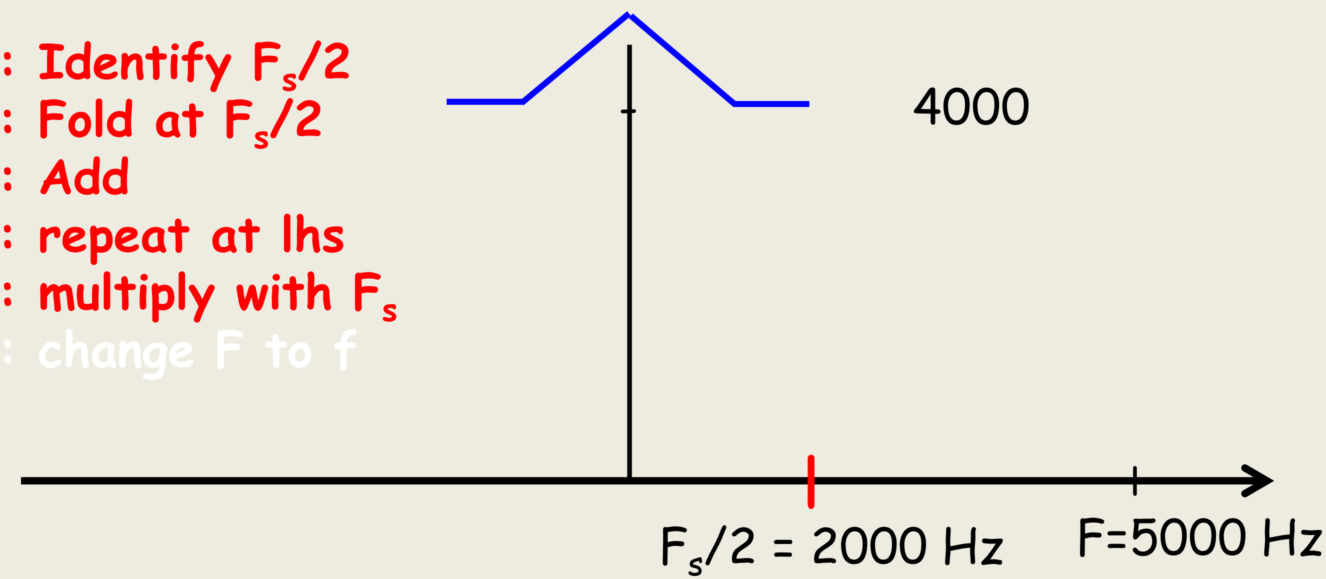

EITF75 Systems and Signals

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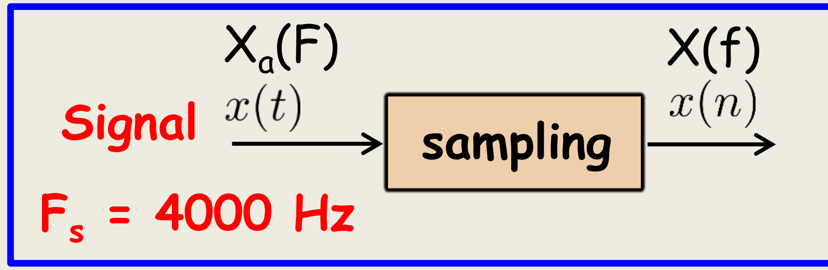

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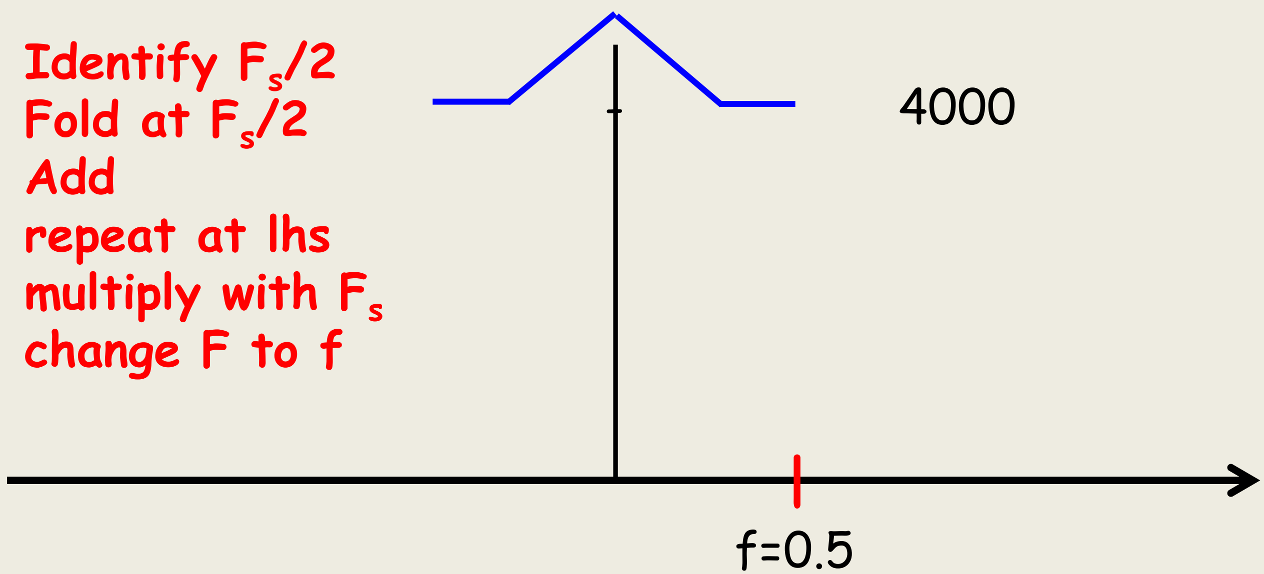
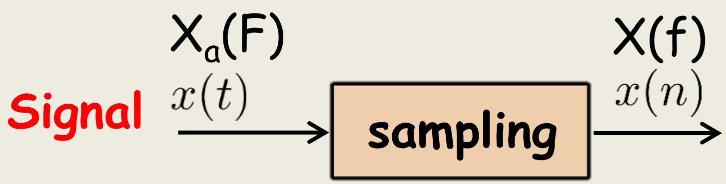


EITF75 Systems and Signals

A/D and D/A



Key step is to understand what $X(f)$ looks like in terms of $X_a(F)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

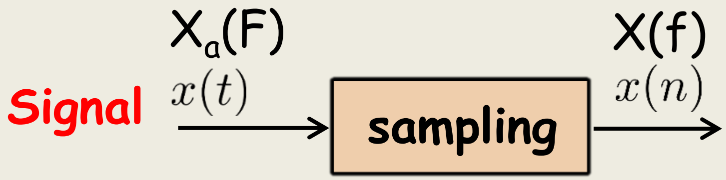
Sampling Theorem (Shannon 1948)

If $F_s > 2B$, where B is the highest frequency of the analog signal, then the analog signal can be recovered from its sampled version

If there is aliasing, we cannot, in general, recover $x(t)$ from $x(n)$

EITF75 Systems and Signals

A/D and D/A



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$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(f) = F_s X_a(f F_s)$$

k=0

If no aliasing (e.g., sampling Theorem fulfilled)

Sampling Theorem (Shannon 1948)

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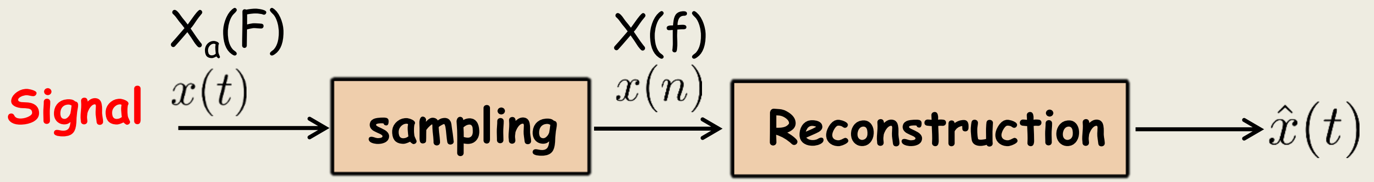
EITF75 Systems and Signals

A/D and D/A



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Reconstruction.

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)\mathrm{sinc}(F_s(t - n/F_s))$$

EITF75 Systems and Signals

A/D and D/A

$X_a(F)$ $\qquad$ $X(f)$

Signal $x(t) \longrightarrow$ **sampling** $\xrightarrow{x(n)}$ **Reconstruction** $\longrightarrow \hat{x}(t)$

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(f) = F_s X_a(fF_s)$$

k=0

Reconstruction.

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)\operatorname{sinc}(F_s(t-n/F_s))$$

No aliasing

$$\hat{x}(t) = x(t)$$

Aliasing

$$\hat{x}(t) \neq x(t)$$

EITF75 Systems and Signals

A/D and D/A

$X_a(F)$ $\qquad\qquad$ $X(f)$

Signal $x(t) \longrightarrow \boxed{\text{sampling}} \xrightarrow{x(n)} \boxed{\text{Reconstruction}} \longrightarrow \hat{x}(t)$

F_1 Hz $\qquad\qquad$ F_2 Hz

Note: sampling and reconstruction frequencies can differ. See lecture 9

$$X(f) = F_s \sum_{k=-\infty}^{\infty} X_a((f-k)F_s)$$

$$X(f) = F_s X_a(fF_s)$$

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Reconstruction.

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} x(n)\operatorname{sinc}(F_s(t - n/F_s))$$

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EITF75 Systems and Signals

LTI systems and z-transforms

EITF75 Systems and Signals

LTI systems

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Linear system

$$x(n) = \alpha x_1(n) + \beta x_2(n)$$
$$\Longleftrightarrow$$
$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

Time invariant system

$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

EITF75 Systems and Signals

LTI systems

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Linear system

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Time invariant system

$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

An LTI system is fully characterized by a difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

EITF75 Systems and Signals

LTI systems

$x(n) \rightarrow$ **system** $\rightarrow y(n)$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Assume that we turn on the circuit at n=0

System **at rest** if

$$y(-k) = 0, \; 1 \leq k \leq N$$

Not at rest if (has initial conditions)

$$\exists k, \; 1 \leq k \leq N, \; : \; y(-k) \neq 0$$

An LTI system is fully characterized by a difference equation

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LTI systems

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Impulse response $h(n)$

Output if input is

$$x(n) = \delta(n) = [1 \;\; 0 \;\; 0 \;\; \cdots]$$

and system at rest

EITF75 Systems and Signals

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

What is output for a given input

Found by z-transform

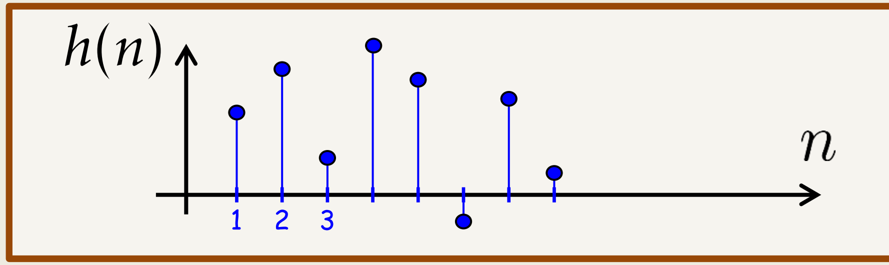
EITF75 Systems and Signals

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

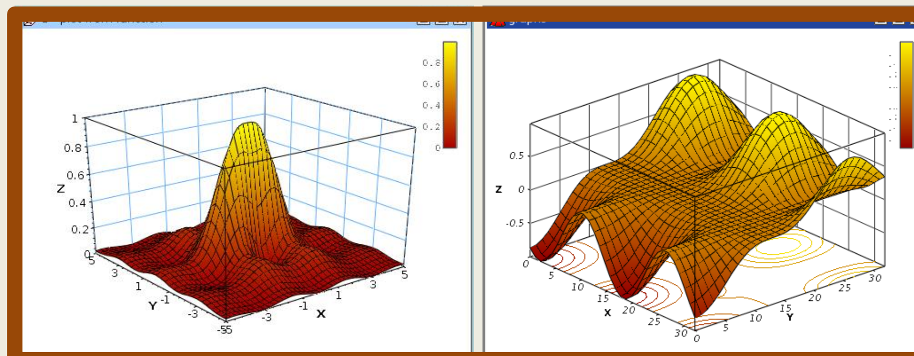
What is output for a given input
Found by z-transform

The z-transform of $h(n)$ is defined as

$$H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$$



z-transform



What is the z-transform?

- A map from sequences to complex valued functions

What is H(z)?

- A complex function of a complex number

If we want to plot H(z), we need 2 plots, one for the real part, one for the imaginary

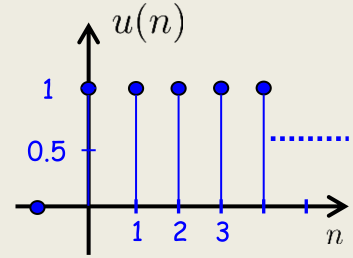
Z-transforms are not meant for "plotting and obtaining insights"

EITF75 Systems and Signals

An *important* example

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

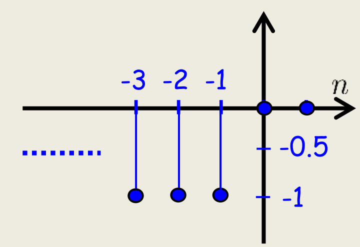
$$\text{ROC} \quad |z| > 1$$



$$\begin{aligned} H(z) &= \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n} \\ &= -\sum_{n=-\infty}^{-1} z^{-n} \;=\; -\sum_{n=0}^{\infty} z^{n} + 1 \\ &= -\frac{1}{1-z} + 1 \;=\; \ldots \;=\; \frac{-z}{1-z} \\ &= \frac{1}{1-z^{-1}} \end{aligned}$$

Anti-causal step

$$h(n) = -u(-n-1)$$



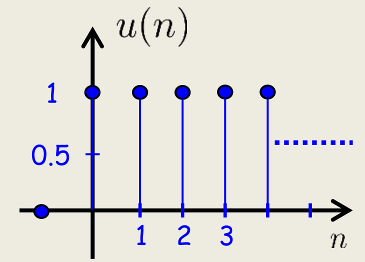
Different signals,
Same z-transform

EITF75 Systems and Signals

An *important* example

$$h(n) = u(n) \quad \Leftrightarrow \quad H(z) = \frac{1}{1 - z^{-1}}$$

$$\text{ROC} \quad |z| > 1$$



$$H(z) = \sum_{n=-\infty}^{\infty} -u(-n-1) z^{-n}$$

Let's specify the ROC

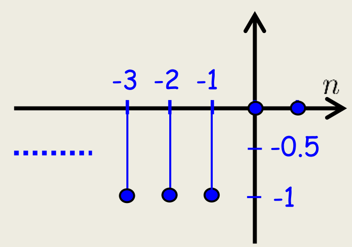
$$= -\sum_{n=-\infty}^{-1} z^{-n} = -\sum_{n=0}^{\infty} z^{n} + 1$$

$$= -\frac{1}{1-z} + 1 = \ldots = \frac{-z}{1-z}$$

$$= \frac{1}{1 - z^{-1}} \qquad \text{ROC} \quad |z| < 1$$

Anti-causal step

$$h(n) = -u(-n-1)$$



Different signals,
Same z-transform
Different ROC

EITF75, z-transform

Convention

If we are given an $X(z)$, and assume that the signal $x(n)$ is causal, then we can be a bit sloppy with the ROC

This is what we do in this (most) of this course

In other words. There are many $x(n)$ for the same $X(z)$, and the ROC specifies the particular one. However, there is only one that is causal.

EITF75, z-transform

Illustration

Sequence	Transform	ROC
$x_1(n)$		
$x_2(n)$		
$x_3(n)$		
$x_4(n)$		
$x_5(n)$		
$x_6(n)$		
$x_7(n)$		

Assume a bunch of *different* sequences

EITF75, z-transform

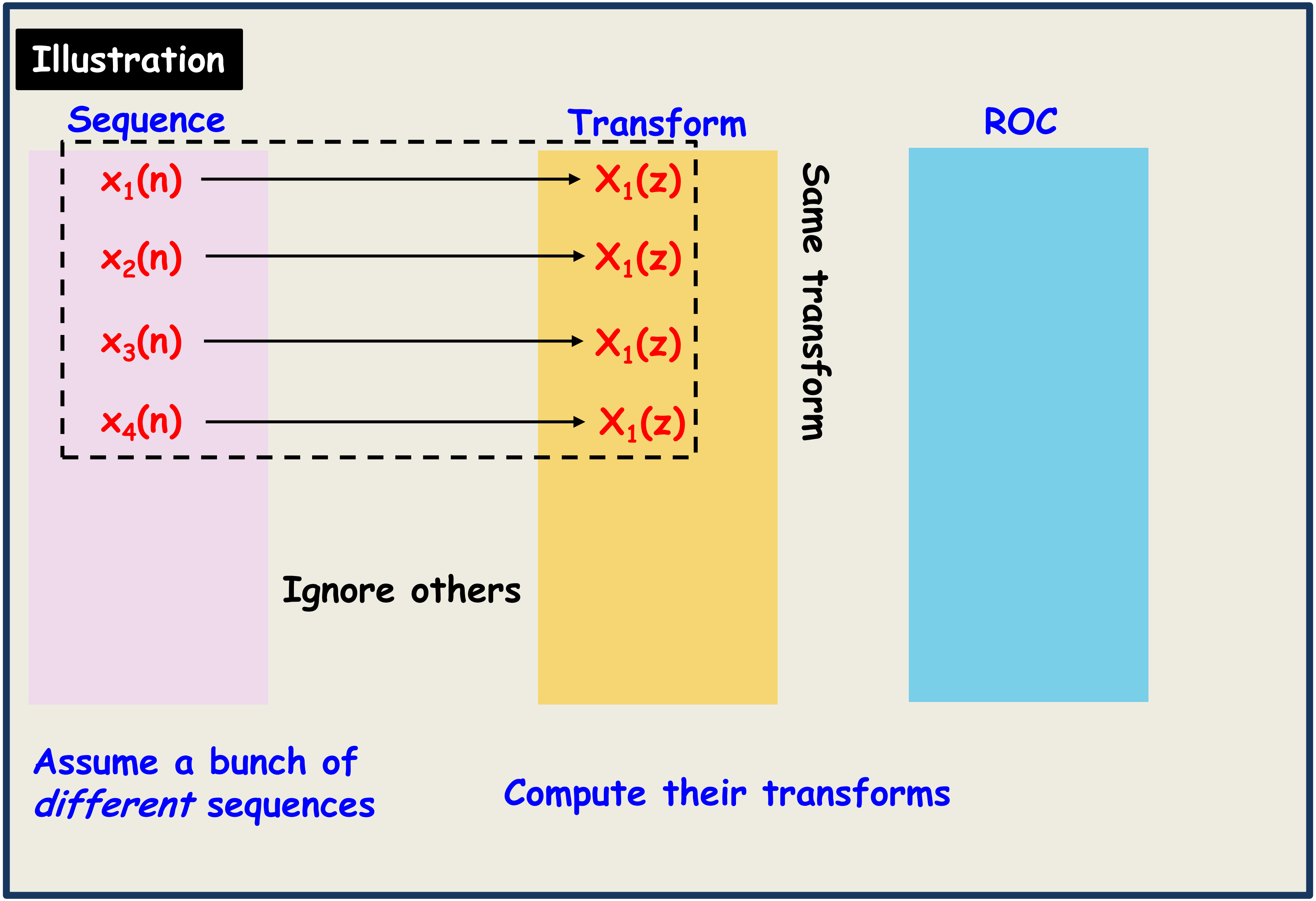
Illustration

Sequence		Transform	ROC
$x_1(n)$	$\rightarrow$	$X_1(z)$	
$x_2(n)$	$\rightarrow$	$X_1(z)$	
$x_3(n)$	$\rightarrow$	$X_1(z)$	
$x_4(n)$	$\rightarrow$	$X_1(z)$	
$x_5(n)$	$\rightarrow$	$X_2(z)$	
$x_6(n)$	$\rightarrow$	$X_2(z)$	
$x_7(n)$	$\rightarrow$	$X_2(z)$	

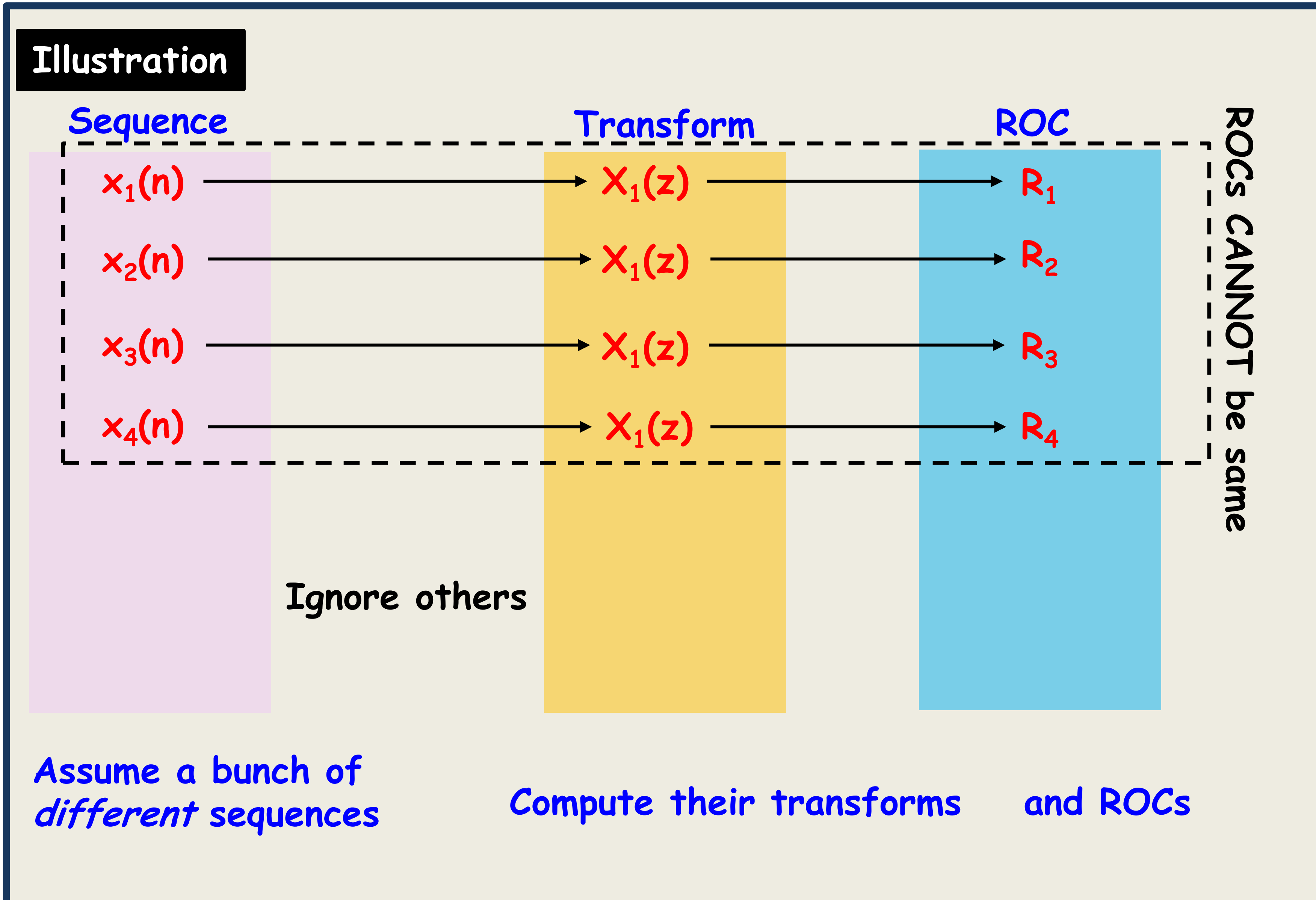
Assume a bunch of *different* sequences

Compute their transforms

EITF75, z-transform



EITF75, z-transform



EITF75, z-transform

Illustration

Sequence		Transform		ROC
$x_1(n)$	$\rightarrow$	$X_1(z)$	$\rightarrow$	R_1
$x_2(n)$	$\rightarrow$	$X_1(z)$	$\rightarrow$	R_2
$x_3(n)$	$\rightarrow$	$X_1(z)$	$\rightarrow$	R_3
$x_4(n)$	$\rightarrow$	$X_1(z)$	$\rightarrow$	R_4

Not causal

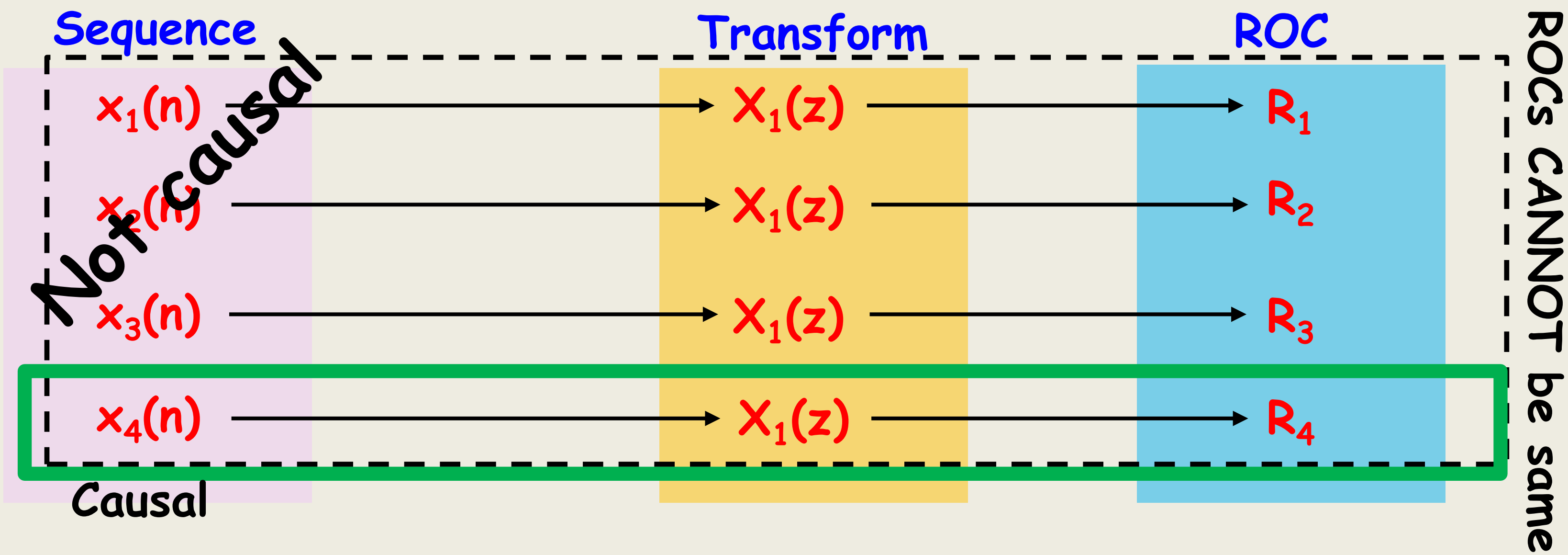
Causal

ROCs CANNOT be same

Exactly one of the ROCs corresponds to a causal signal

EITF75, z-transform

Illustration



Exactly one of the ROCs corresponds to a causal signal

So, if we know $X_1(z)$ and that we work with causal $x(n)$, we can establish $x_4(n)$ without knowing the ROC

EITF75 Systems and Signals

LTI systems

$x(n) \longrightarrow \boxed{\text{system}} \longrightarrow y(n)$

What is output for a given input

Found by z-transform

$$y(n)+\sum_{k=1}^{N} a_k y(n-k)=\sum_{k=0}^{M} b_k x(n-k)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Solution for general difference equation (at rest)

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) = \boxed{\frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}}} \cdot X(z)$$

$$H(z)$$

Step 1:

Change y(n-k) to z^{-k} Y(z)

Step 2:

Change x(n-k) to z^{-k} X(z)

Step 3:

Express Y(z) as H(z)X(z)

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k)$$

Solution for general difference equation (at rest)

$$Y(z)+\sum_{k=1}^{N}a_k z^{-k}Y(z)=\sum_{k=0}^{M}b_k z^{-k}X(z)$$

$$Y(z)=\boxed{\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}}\cdot X(z)$$

$H(z)$

Step 4:

Find the roots of the denominator and nominator of H(z). Roots should be in terms of z, not z^{-1}

$$Y(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0z^M+b_1z^{M-1}+\cdots+b_M}{z^N+a_1z^{N-1}+\cdots+a_N}X(z)$$

zeros

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

poles

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k)$$

Solution for general difference equation (at rest)

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$H(z)$

Step 4:

Find the roots of the denominator and nominator of H(z). Roots should be in terms of z, not z^{-1}

$$Y(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0 z^M+b_1 z^{M-1}+\cdots+b_M}{z^N+a_1 z^{N-1}+\cdots+a_N}X(z)$$

zeros

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

poles

If degree of numerator >= degree of denominator. Perform polynomial division

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n)+\sum_{k=1}^{N}a_k y(n-k)=\sum_{k=0}^{M}b_k x(n-k)$$

Step 4:

Find the roots of the denominator and nominator of H(z). Roots should be in terms of z, not z^{-1}

Solution for general difference equation (at rest)

$$Y(z)+\sum_{k=1}^{N}a_k z^{-k}Y(z)=\sum_{k=0}^{M}b_k z^{-k}X(z)$$

$$Y(z)=\boxed{\frac{b_0+b_1z^{-1}+\cdots+b_Mz^{-M}}{1+a_1z^{-1}+\cdots+a_Nz^{-N}}}\cdot X(z)$$

$$Y(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0z^{M}+b_1z^{M-1}+\cdots+b_M}{z^{N}+a_1z^{N-1}+\cdots+a_N}X(z)$$

zeros

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

poles

$H(z)$

Will turn up in the time-domain as a delay

(can be negative delay)

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n)+\sum_{k=1}^{N} a_k y(n-k)=\sum_{k=0}^{M} b_k x(n-k)$$

Assume $\boxed{X(z)=\frac{N(z)}{Q(z)}}$

$$Y(z)=\frac{\text{Some polynomial in } z}{(z-p_1)\cdots(z-p_N)\cdot(z-q_1)\cdots(z-q_L)}$$

Assuming all poles are real and distinct

Assuming deg(num) < deg(denom)

$$Y(z)=\sum_{k=1}^{N}\frac{A_k}{1-z^{-1}p_k}+\sum_{k=1}^{L}\frac{Q_k}{1-z^{-1}q_k}$$

$$Y(z)=\frac{z^{-M}}{z^{-N}}\frac{b_0 z^M+b_1 z^{M-1}+\cdots+b_M}{z^N+a_1 z^{N-1}+\cdots+a_N}X(z)$$

$$=b_0\cdot\frac{z^{-M}}{z^{-N}}\cdot\frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}\cdot X(z)$$

Perform partial fraction expansion

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Assume $X(z) = \frac{N(z)}{Q(z)}$

$$Y(z) = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N} X(z)$$

$$= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \cdot X(z)$$

$$Y(z) = \frac{\text{Some polynomial in } z}{(z-p_1)\cdots(z-p_N) \cdot (z-q_1)\cdots(z-q_L)}$$

Assuming all poles are real and distinct

Assuming deg(num) < deg(denom)

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - z^{-1} p_k} + \sum_{k=1}^{L} \frac{Q_k}{1 - z^{-1} q_k}$$

Invert

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Expression for general difference equation

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1 - z^{-1}p_k} + \sum_{k=1}^{L} \frac{Q_k}{1 - z^{-1}q_k}$$

This…

…generates that

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Assume $X(z) = \frac{N(z)}{Q(z)}$

$$Y(z) = \frac{\text{Some polynomial in } z}{(z-p_1)\cdots(z-p_N)\cdot(z-q_1)\cdots(z-q_L)}$$

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-z^{-1}p_k} + \sum_{k=1}^{L} \frac{Q_k}{1-z^{-1}q_k}$$

Important: poles in H(z) and in X(z) determines the output Structure

"You can never get a term in y(n) that doesn't exist in either X(z) or H(z)"

$$Y(z) = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N} X(z)$$

$$= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \cdot X(z)$$

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

Analyzing a general difference equation (at rest)

Assume $\boxed{X(z) = \frac{N(z)}{Q(z)}}$

$$Y(z) = \frac{}{(z-p_1)\cdots(z-p_N)} \cdot \frac{\text{Some polynomial in z}}{(z-q_1)\cdots(z-q_L)}$$

$$Y(z) = \sum_{k=1}^{N} \frac{A_k}{1-z^{-1}p_k} + \sum_{k=1}^{L} \frac{Q_k}{1-z^{-1}q_k}$$

Important: To get stable output, all poles must be inside the unit circle

$$Y(z) = \frac{z^{-M}}{z^{-N}} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^N + a_1 z^{N-1} + \cdots + a_N} X(z)$$

$$= b_0 \cdot \frac{z^{-M}}{z^{-N}} \cdot \frac{(z-z_1)\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)} \cdot X(z)$$

$$y(n) = \sum_{k=1}^{N} A_k (p_k)^n u(n) + \sum_{k=1}^{L} Q_k (q_k)^n u(n)$$

EITF75 Systems and Signals

A complex conjugated pair of poles

$$h(n) = r^n \cdot \sin(\omega n)u(n)$$

$$h(n) = r^n \cdot \cos(\omega n)u(n)$$

$$H(z) = \frac{r\sin(\omega)z^{-1}}{1-2r\cos(\omega)z^{-1}+r^2z^{-2}}$$

$$H(z) = \frac{1-r\cos(\omega)z^{-1}}{1-2r\cos(\omega)z^{-1}+r^2z^{-2}}$$

EITF75 Systems and Signals

A complex conjugated pair of poles

$$h(n) = r^n \cdot \sin(\omega n)u(n)$$

$$h(n) = r^n \cdot \cos(\omega n)u(n)$$

$$H(z) = \frac{r\sin(\omega)z^{-1}}{1-2r\cos(\omega)z^{-1}+r^2z^{-2}}$$

$$H(z) = \frac{1-r\cos(\omega)z^{-1}}{1-2r\cos(\omega)z^{-1}+r^2z^{-2}}$$

Polar coordinates: r is "length" and w is angle of the pole.
To get stable output: r<1 (poles inside the unit circle)

Example

Quite messy to invert a mixture of the two above: Make sure you know how to do that.

Invert $H(z) = z^{-1} \cdot \dfrac{1-z^{-1}}{1-1.27z^{-1}+0.81z^{-2}}$

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform $$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform $$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

End result: The solution at rest + contribution from initial conditions

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform $$X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

End result: The solution at rest + contribution from initial conditions

$$Y(z) = H(z)X(z) \qquad = \frac{B(z)}{A(z)} X(z)$$

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform $X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

End result: The solution at rest + contribution from initial conditions

$$Y(z) = H(z)X(z) + \frac{N_0(z)}{A(z)} = \frac{B(z)}{A(z)}X(z) + \frac{N_0(z)}{A(z)}$$

$$N_0(z) = -\sum_{k=1}^{N} a_k z^{-k} \sum_{n=1}^{k} y(-n) z^n$$

N: highest power of z^{-1} in $A(z)$

EITF75 Systems and Signals

Systems not at rest

Use the one-sided z-transform $X^+(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

End result: The solution at rest + contribution from initial conditions

$$Y(z) = H(z)X(z) + \frac{N_0(z)}{A(z)} = \frac{B(z)}{A(z)}X(z) + \frac{N_0(z)}{A(z)}$$

EITF75 Systems and Signals

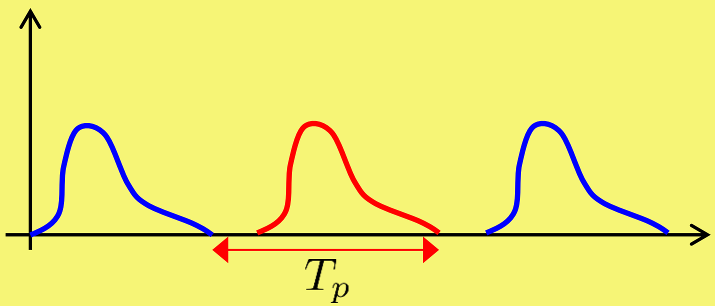
Fourier analysis. 4 cases

Periodic/aperiodic signal
Continuous/discrete signal

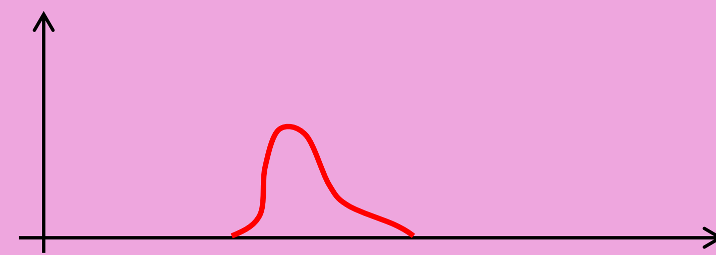
EITF75, Fourier transforms

4 different type of signals Time signals shown, not Fourier transforms

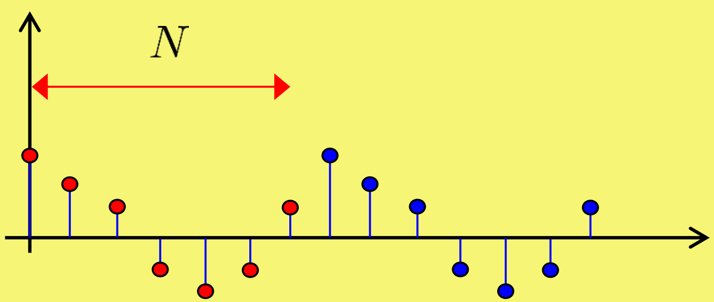
Continuous and periodic



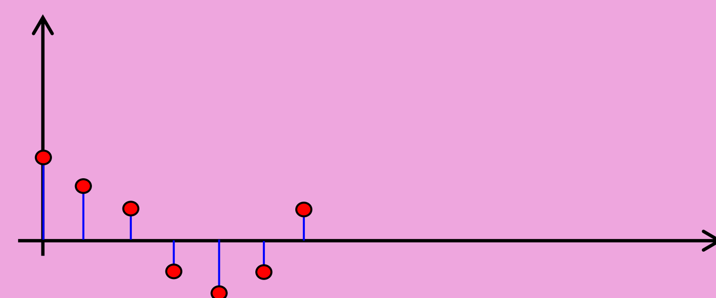
Continuous and aperiodic



Discrete and periodic



Discrete and aperiodic



Discrete spectra

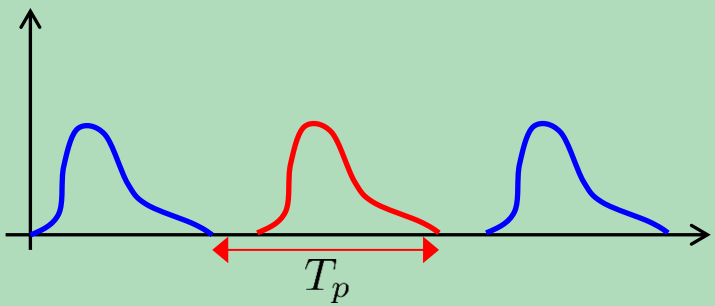
Continuous spectra

EITF75, Fourier transforms

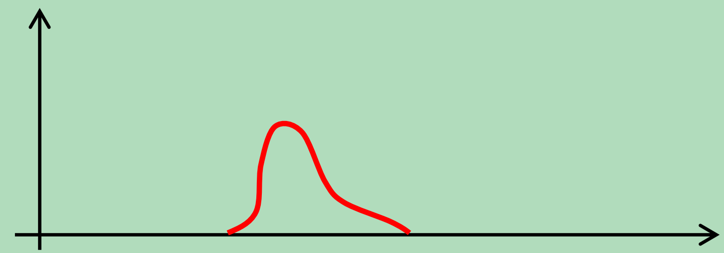
4 different type of signals

Aperiodic spectra

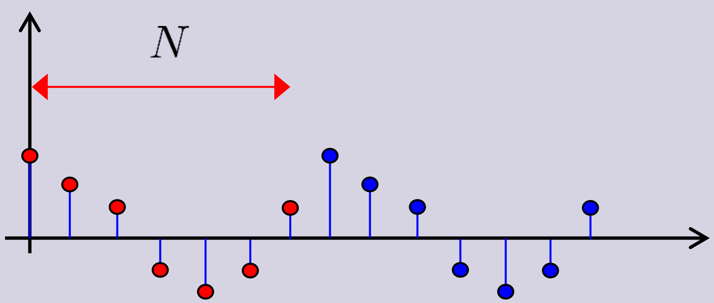
Continuous and periodic



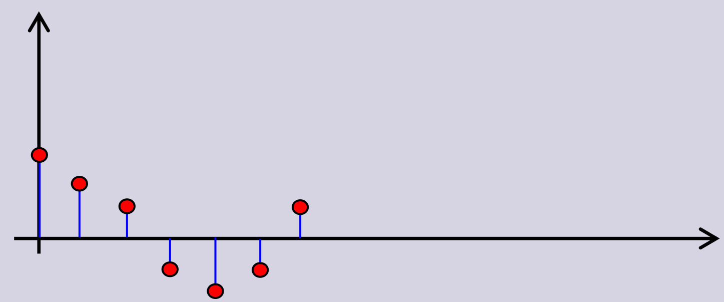
Continuous and aperiodic



Discrete and periodic

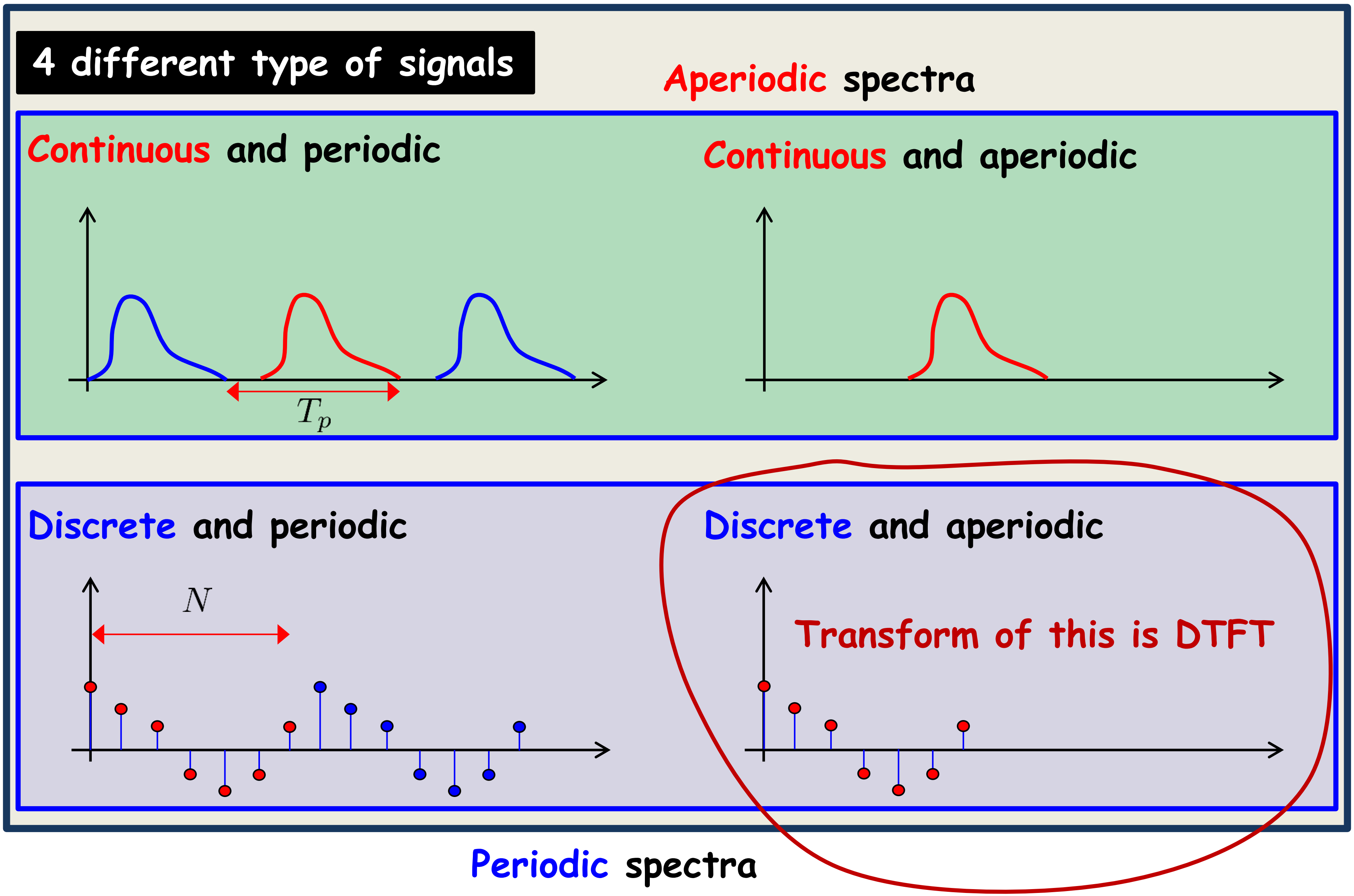


Discrete and aperiodic



Periodic spectra

EITF75, Fourier transforms



EITF75 Systems and Signals

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

DTFT

(discrete time Fourier transform)

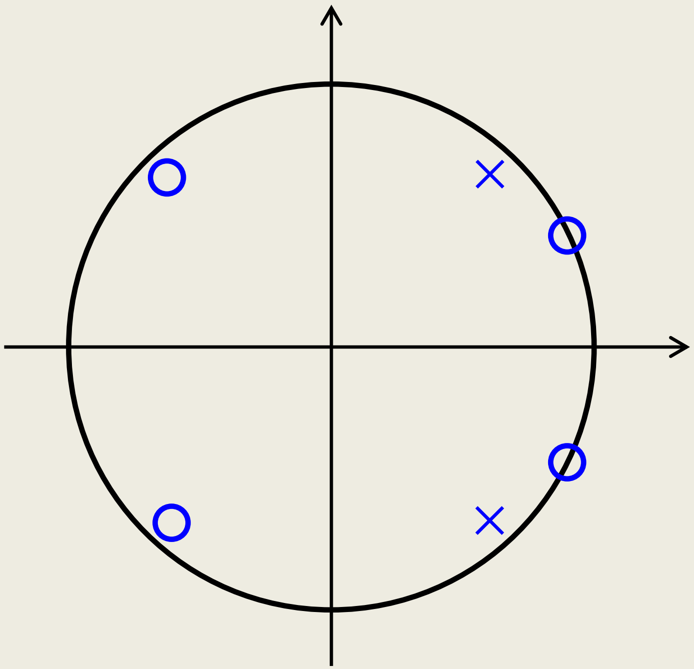
$$X(f) = \sum_{n=-\infty}^{\infty} x(n) \exp(-i2\pi n f)$$

$$= X(z|z = \exp(i2\pi f))$$

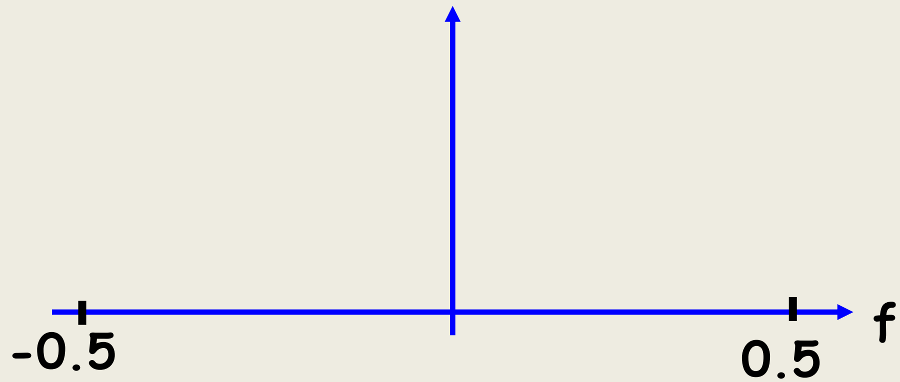
Important: DTFT is z-transform evaluated at unit circle

EITF75, DTFT

Pole-zero plot



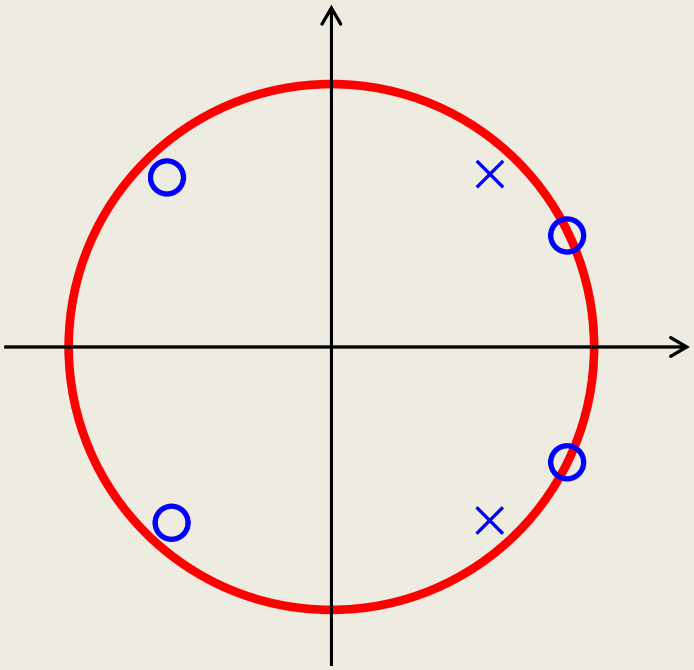
DTFT



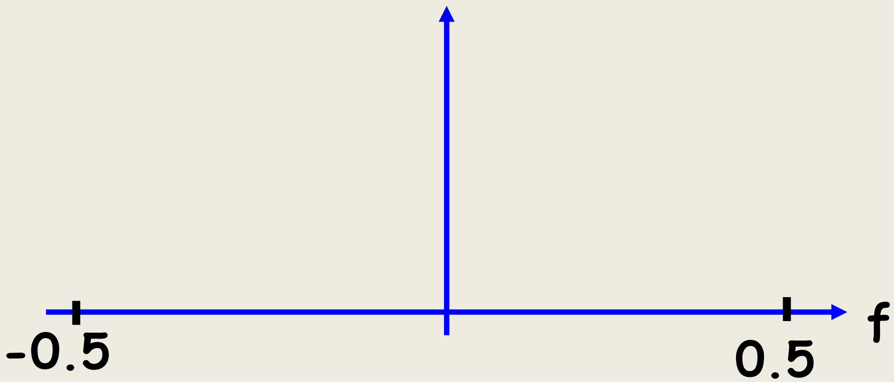
Book makes a big deal out of this. But quite easy….

EITF75, DTFT

Pole-zero plot



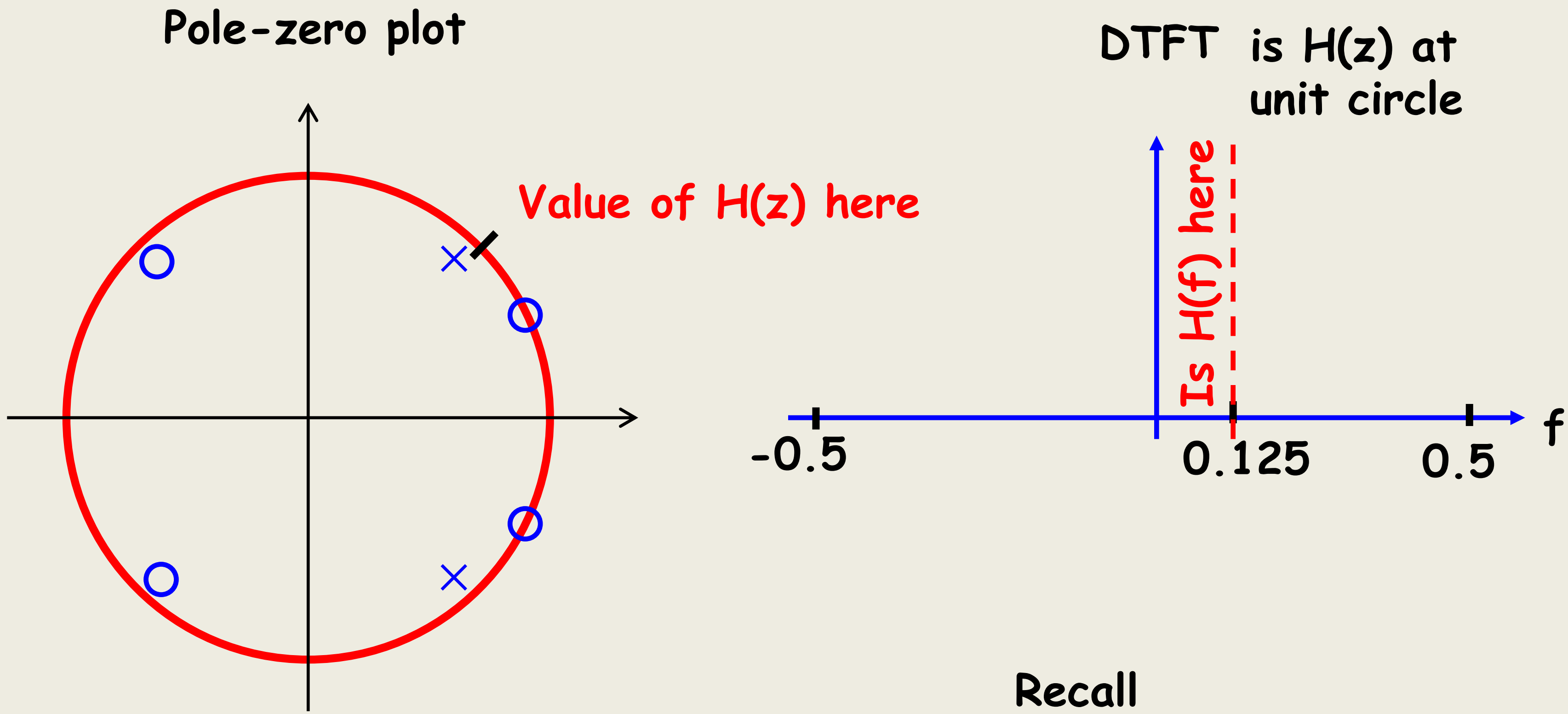
DTFT is H(z) at unit circle



Recall

$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

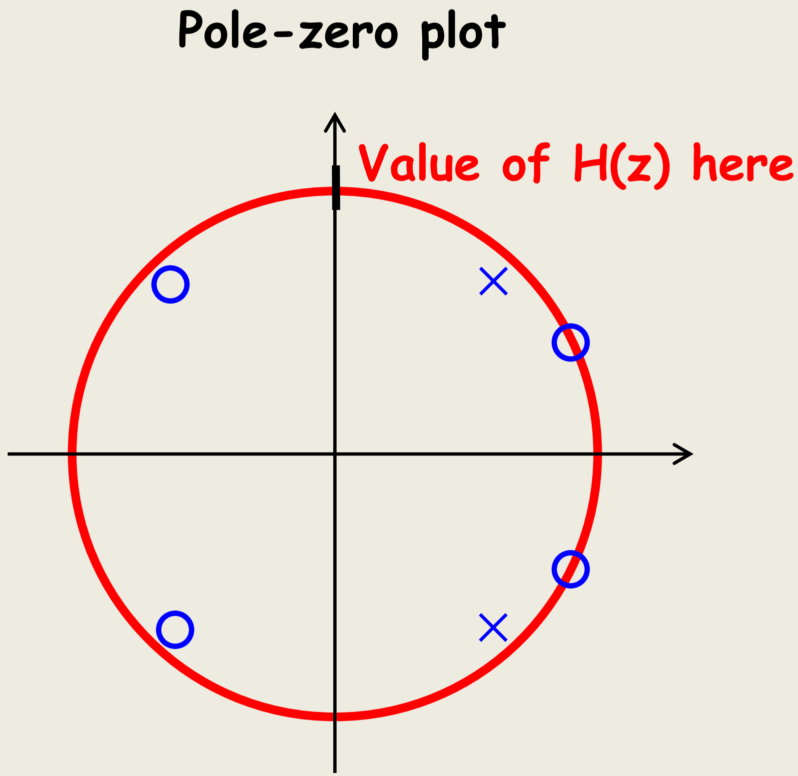
EITF75, DTFT

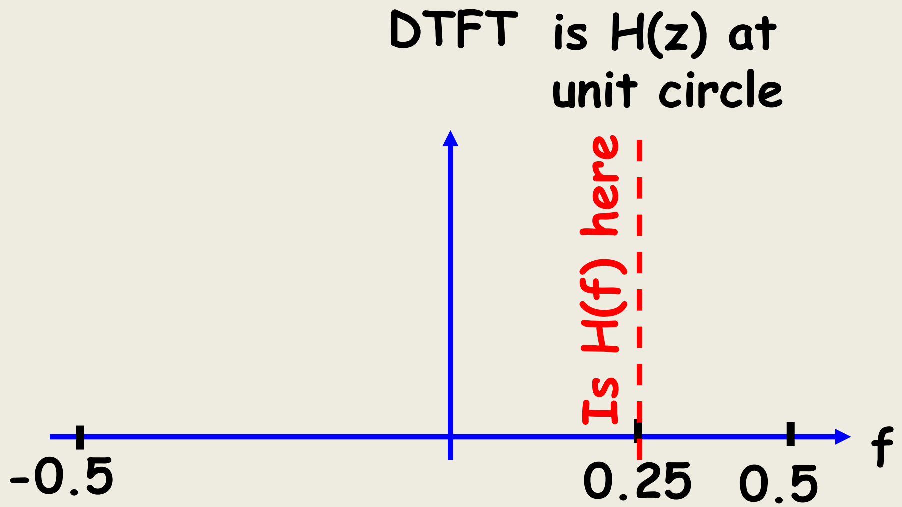


Recall

$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

EITF75, DTFT





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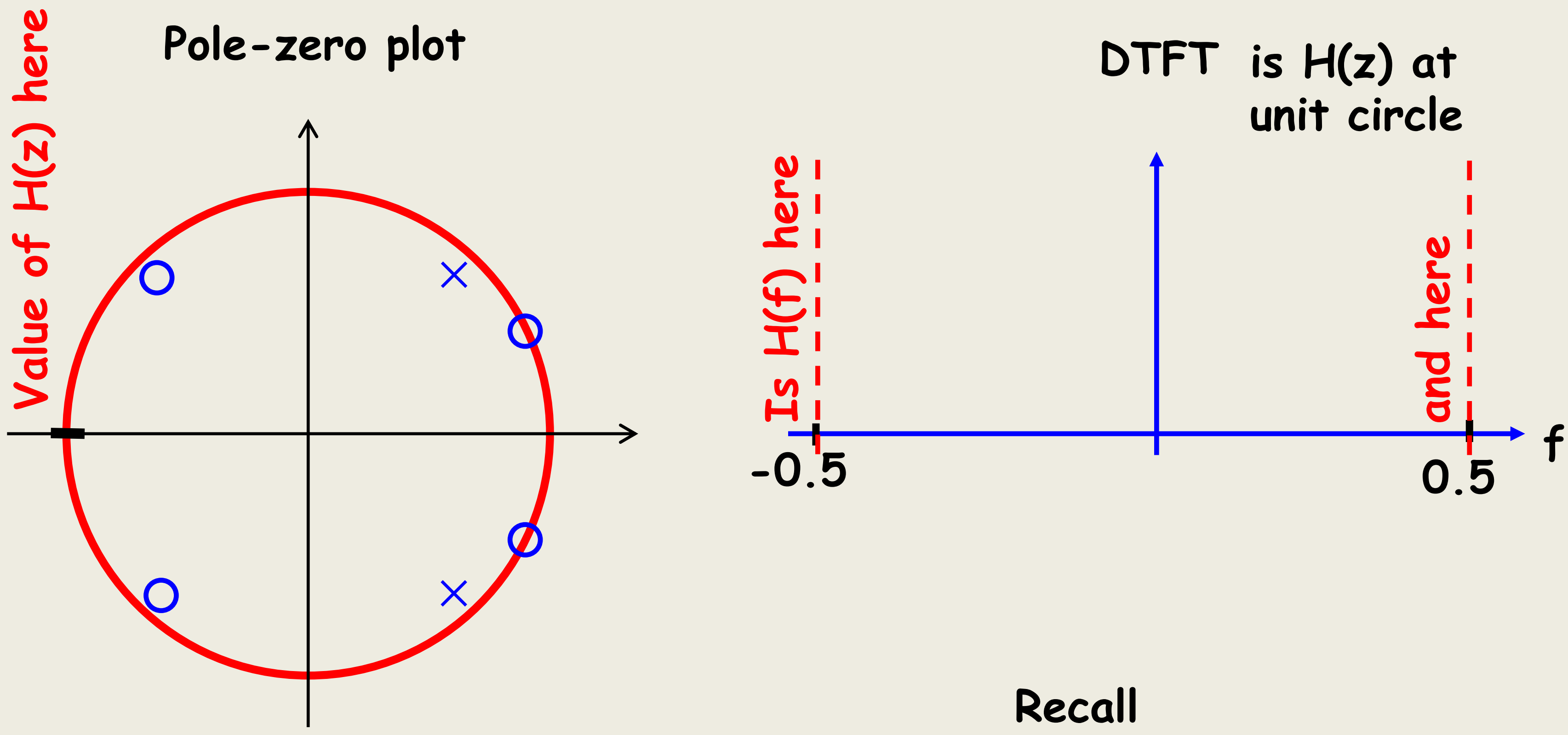
EITF75, DTFT



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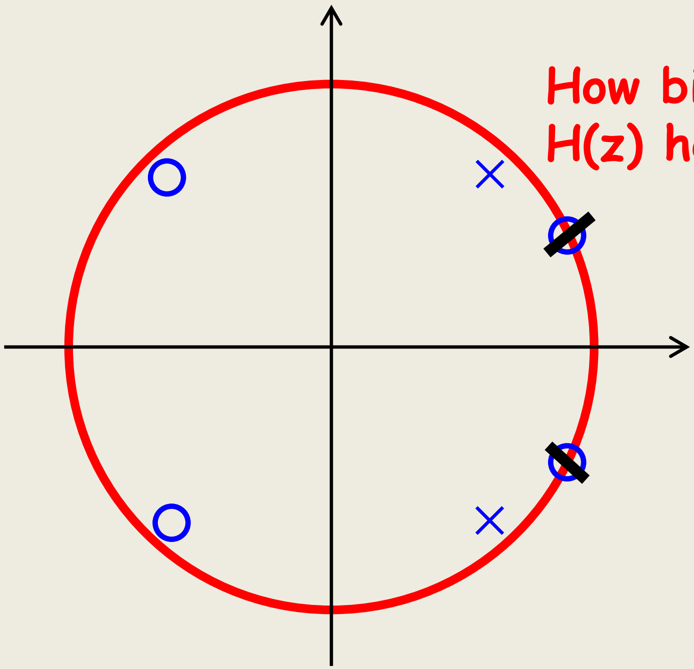
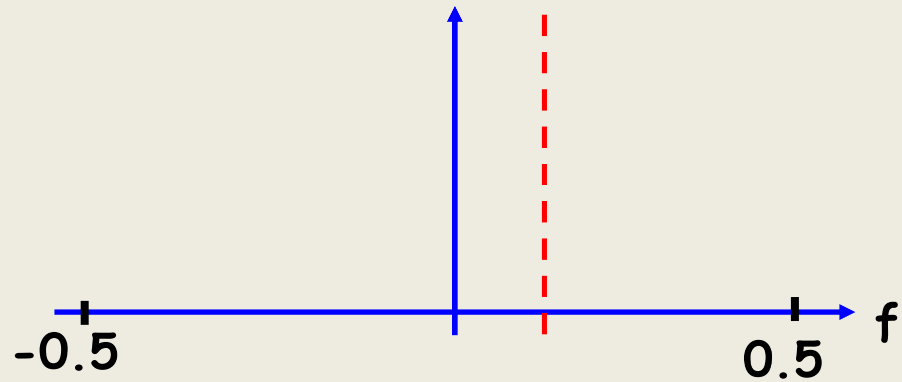
EITF75, DTFT

Pole-zero plot



DTFT is H(z) at unit circle



Recall

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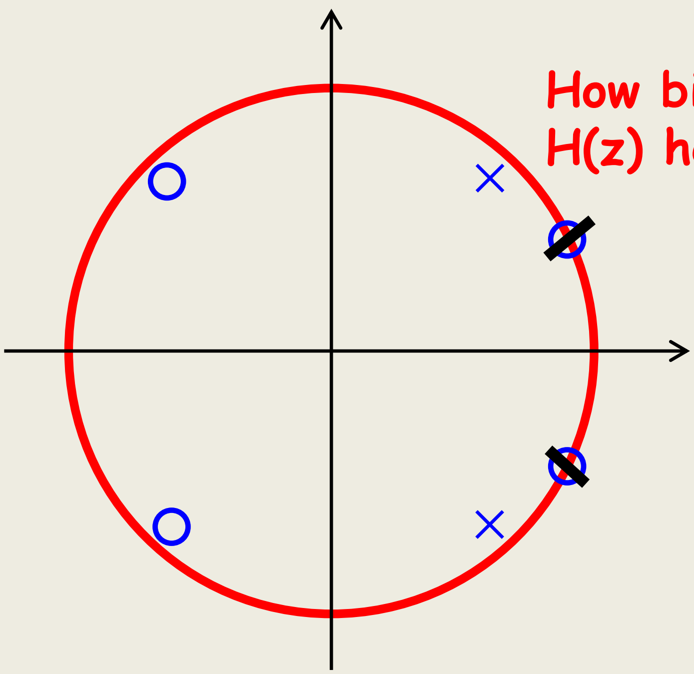
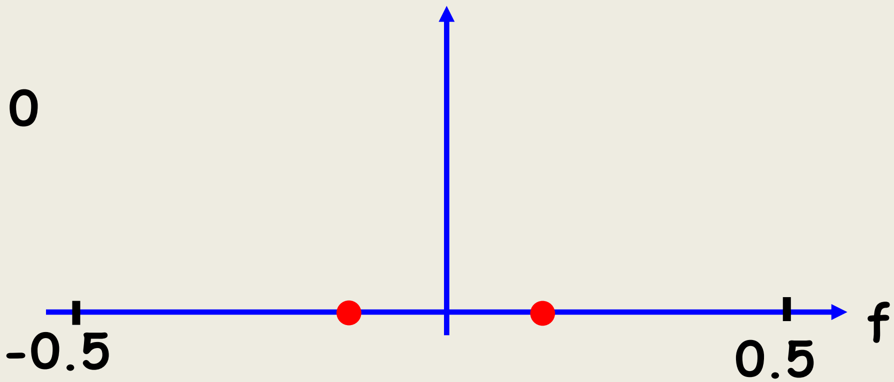
EITF75, DTFT

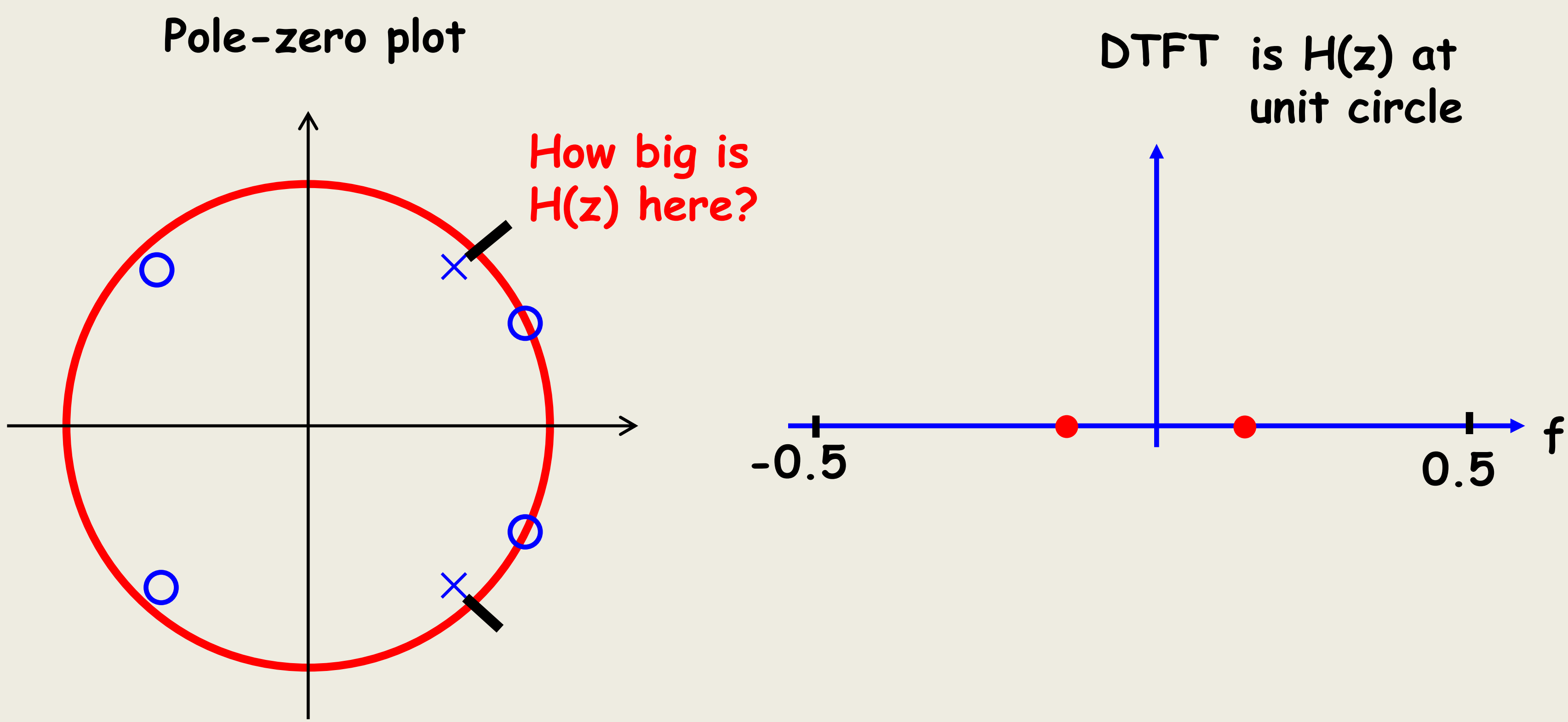




We are at a zero

$$H(z) = \frac{\boxed{(z-z_1)}\cdots(z-z_M)}{(z-p_1)\cdots(z-p_N)}$$

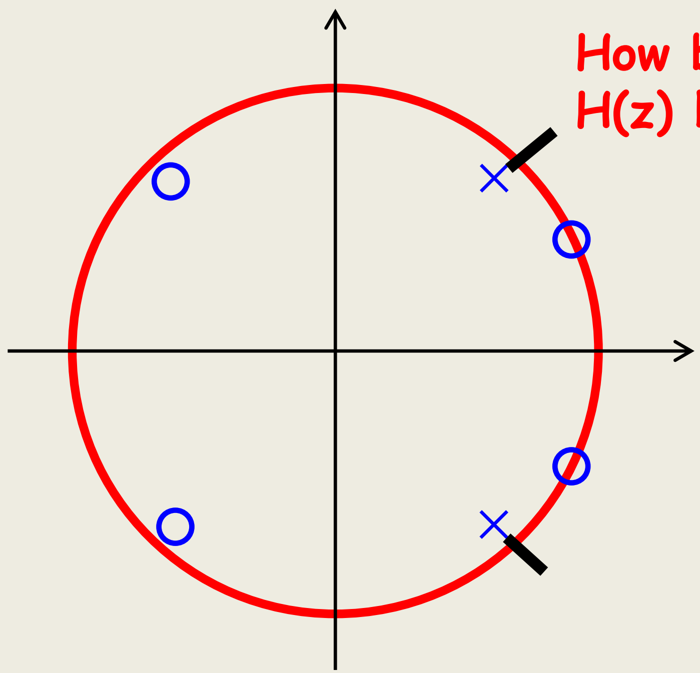
EITF75, DTFT



$$H(z) = \frac{(z - z_1) \cdots (z - z_M)}{(z - p_1) \cdots (z - p_N)}$$

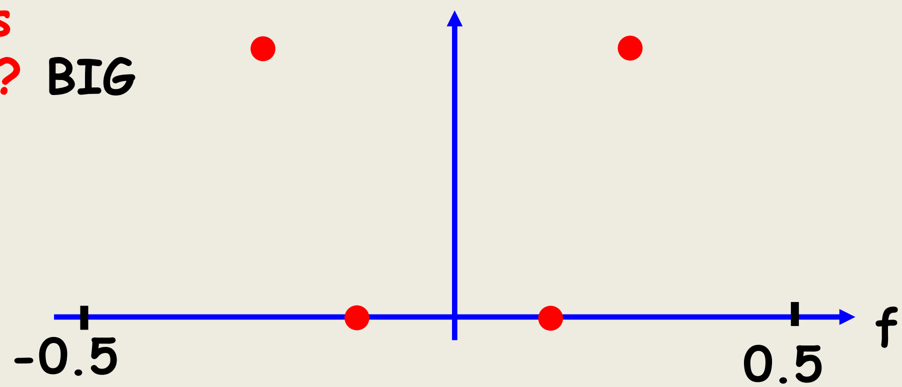
EITF75, DTFT

Pole-zero plot



How big is H(z) here? BIG

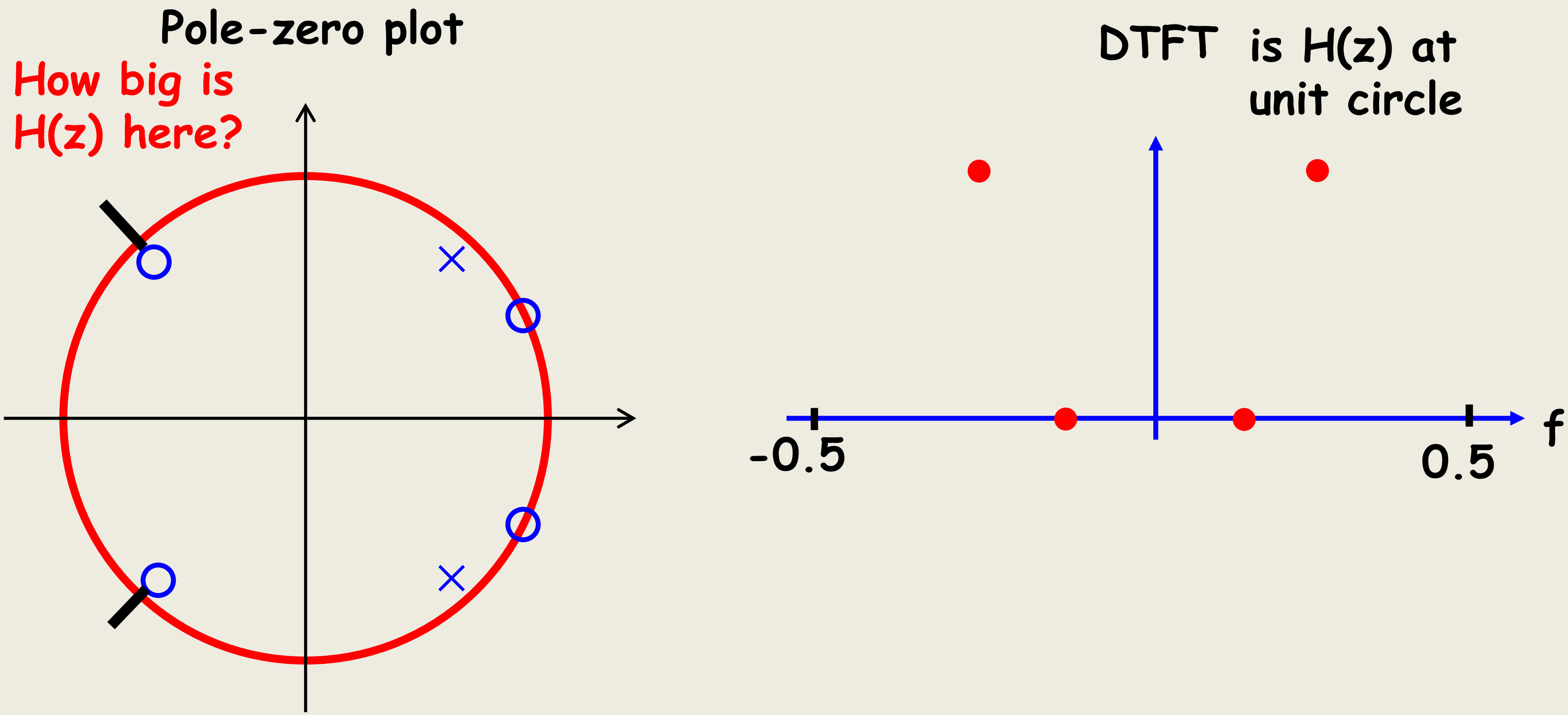
DTFT is H(z) at unit circle



We are close to a pole

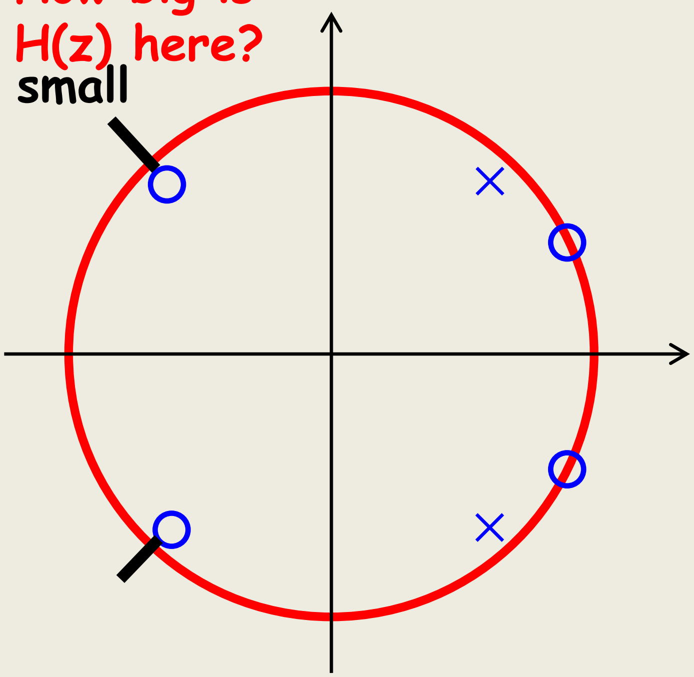
$$H(z) = \frac{(z-z_1)\cdots(z-z_M)}{\boxed{(z-p_1)}\cdots(z-p_N)}$$

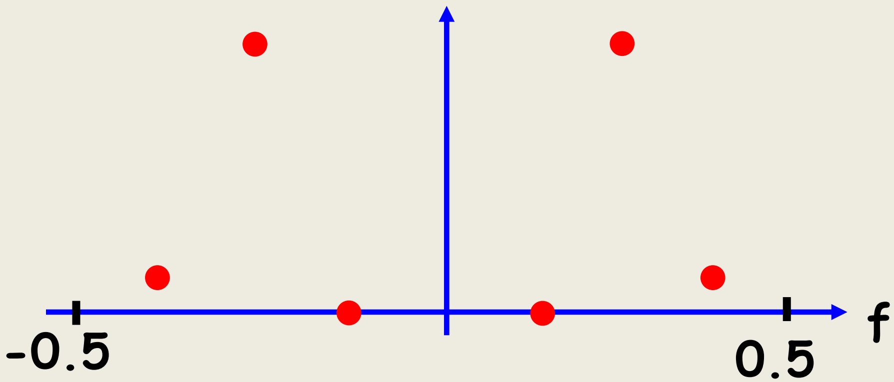
EITF75, DTFT



$$H(z) = \frac{(z - z_1)\cdots(z - z_M)}{(z - p_1)\cdots(z - p_N)}$$

EITF75, DTFT



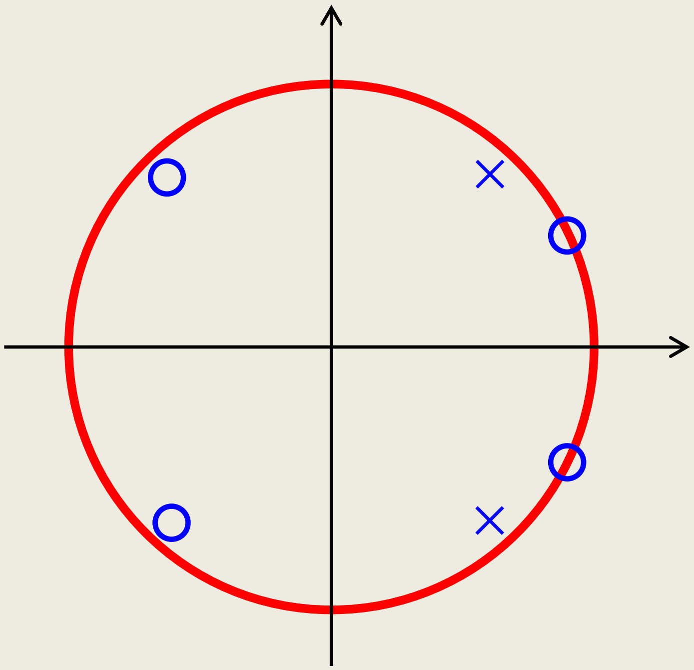


We are close to a zero

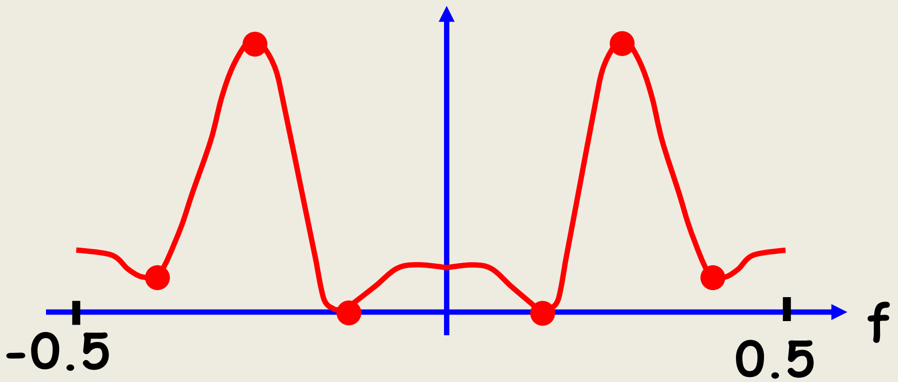
$$H(z) = \frac{(z - z_1) \cdots \boxed{(z - z_M)}}{(z - p_1) \cdots (z - p_N)}$$

EITF75, DTFT

Pole-zero plot

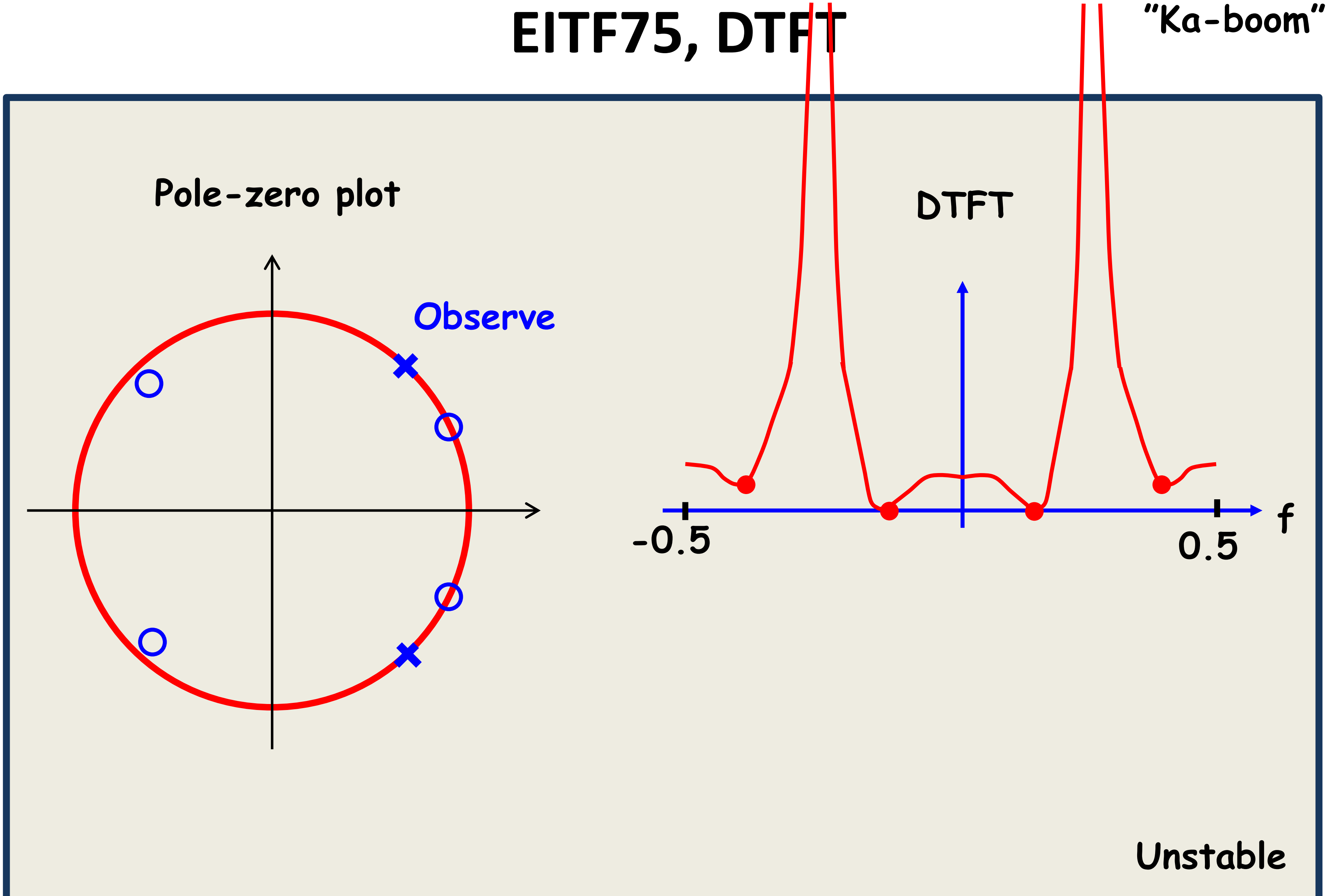


DTFT

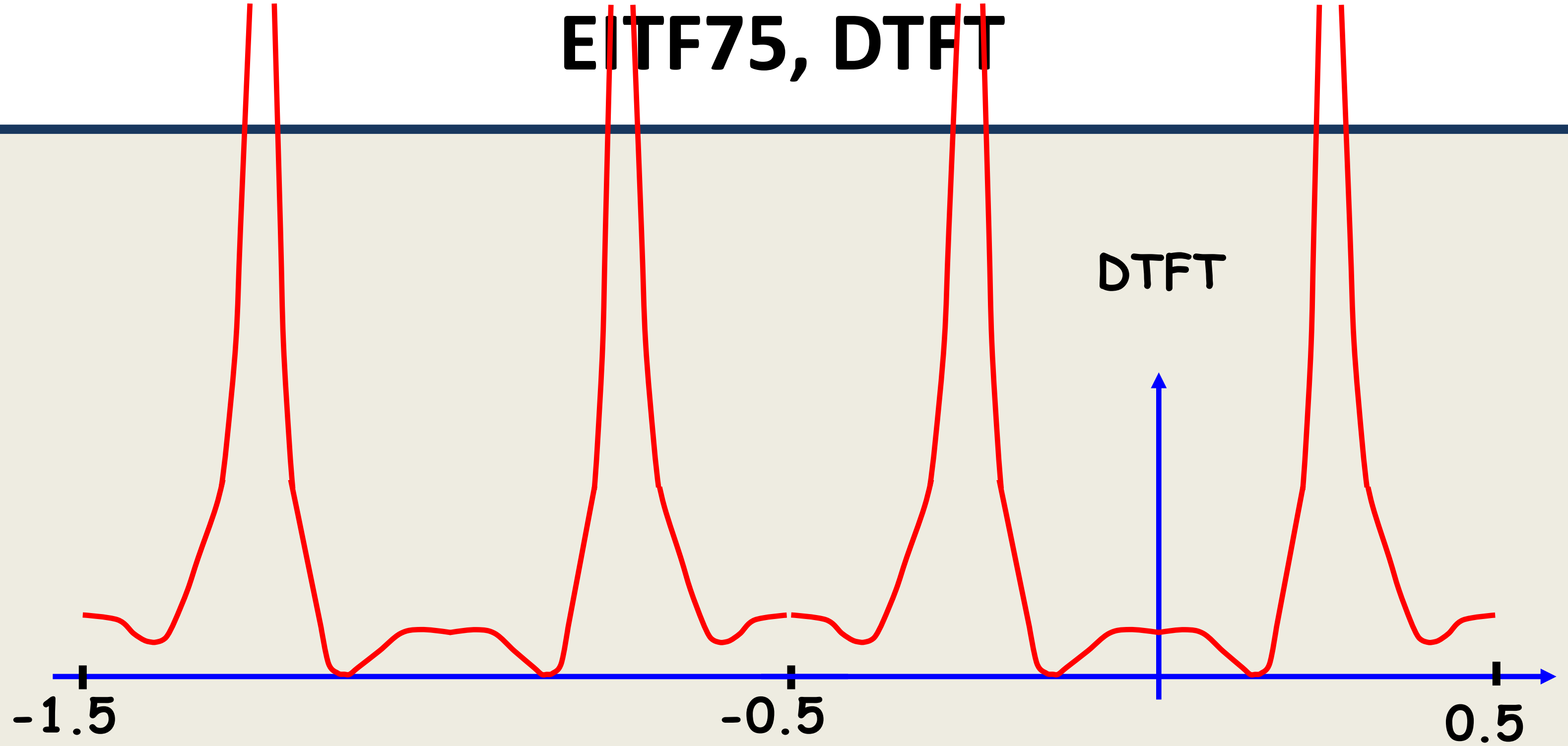


Non-zero everywhere else, since no further zeros at unit circle

EITF75, DTFT



EITF75, DTFT



Final remark: X(f) is periodic

EITF75 Systems and Signals

DTFT Z-transform

For stable h(n) $H(f) = H\left(\mathrm{e}^{i2\pi f}\right)$

For input $x(n) = \exp(i2\pi f_0 n)$

We get the output $y(n) = H(f_0)\exp(i2\pi f_0 n)$

Important: An LTI system cannot create frequencies not present in the input signal

EITF75 Systems and Signals

For cos/sin

Inputs

$$x(n) = \cos(2\pi f_0 n)$$

$$x(n) = \sin(2\pi f_0 n)$$

Outputs (LTI system)

$$y(n) = |H(f_0)|\cos(2\pi f_0 + \Theta(f_0))$$

$$y(n) = |H(f_0)|\sin(2\pi f_0 + \Theta(f_0))$$

EITF75 Systems and Signals

Assume oscillating input, but turned on at n=0

$$x(n) = \cos(\omega_0 n) \cdot u(n) \quad \omega_0 = \frac{2\pi}{16}$$

Steady state solution (i.e., y(n) at big n) is the same as before. At small n, there is a transient behavior.

$$Y(z) = H(z)X(z) = \frac{B(z)}{A(z)}X(z)$$

$$= \frac{B(z)}{A(z)} \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

EITF75 Systems and Signals

Assume oscillating input, but turned on at n=0

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$$\begin{aligned}
Y(z) &= H(z)X(z) = \frac{B(z)}{A(z)}X(z) \\
&= \frac{B(z)}{A(z)} \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} \\
&= \sum_{n=1}^{N} \frac{A_n}{1 - p_n z^{-1}} + \frac{X_1 + X_2 z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}
\end{aligned}$$

EITF75 Systems and Signals

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$$= \sum_{n=1}^{N} \frac{A_n}{1 - p_n z^{-1}} + \frac{X_1 + X_2 z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}}$$

Transient (if all poles inside unit circle)

Steady state (same as for infinite cos)

EITF75 Systems and Signals

Parseval's formula

$$\sum_{n=-\infty}^{\infty} x(n)y^*(n) = \int_{-0.5}^{0.5} X(f)Y^*(f)\mathrm{d}f$$

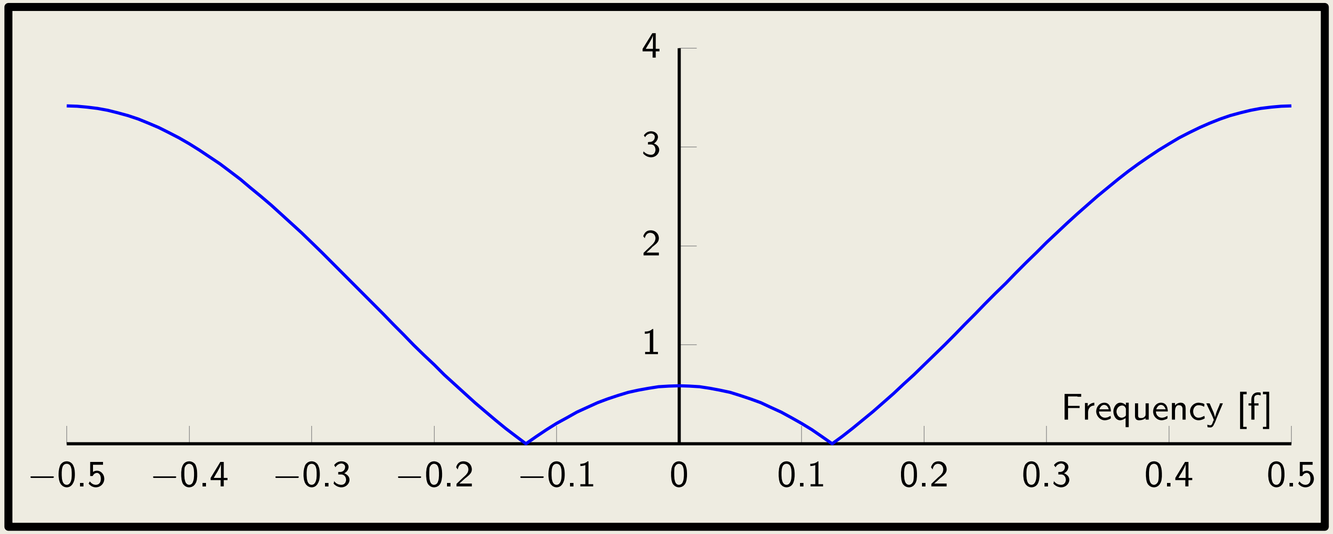
Special case: $y(n) = x(n)$

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \int_{-0.5}^{0.5} |X(f)|^2\mathrm{d}f$$

EITF75 Systems and Signals

Some filter design

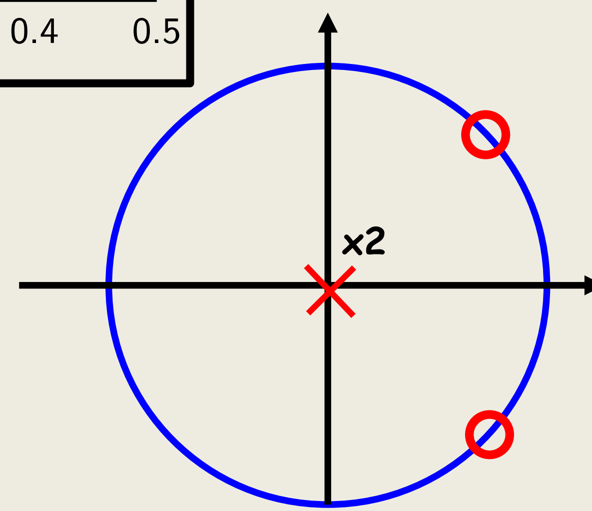
EITF75 Systems and Signals



Magnitude response

FIASCO

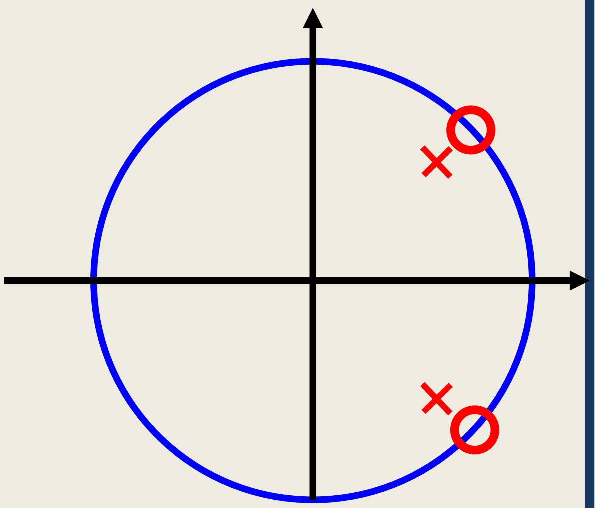
An attempt to cancel f=0.125 by using two zeros



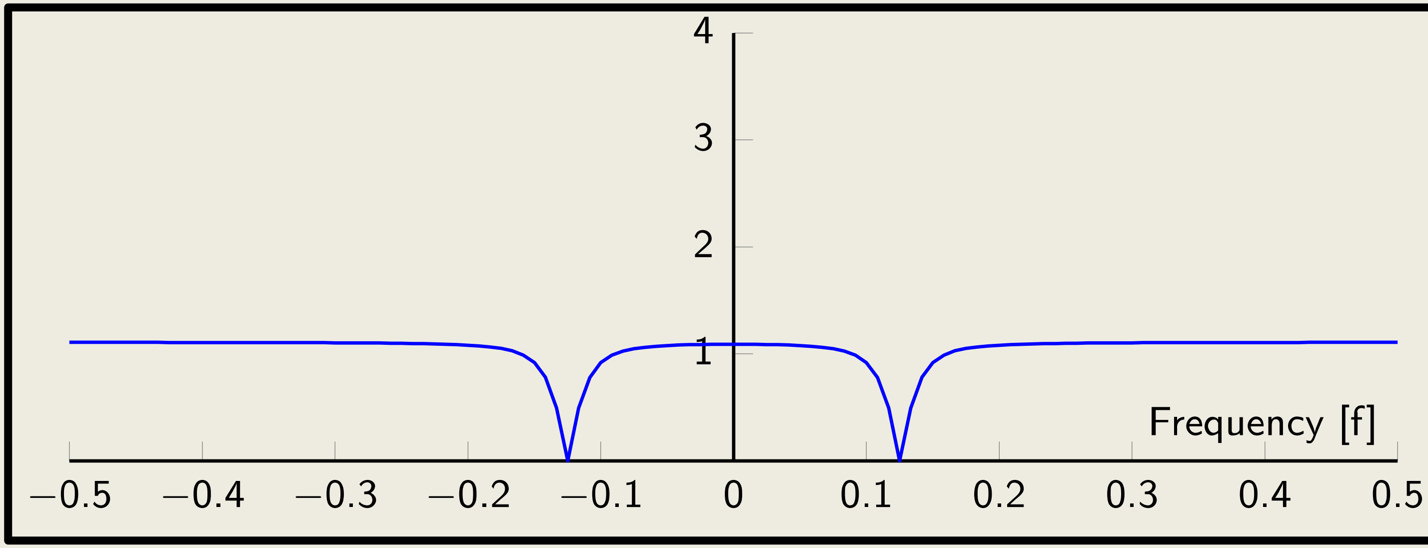
$h(n) = \{ \underline{1} \quad -2\cos(w_0) \quad 1 \}$

EITF75 Systems and Signals

Let us try



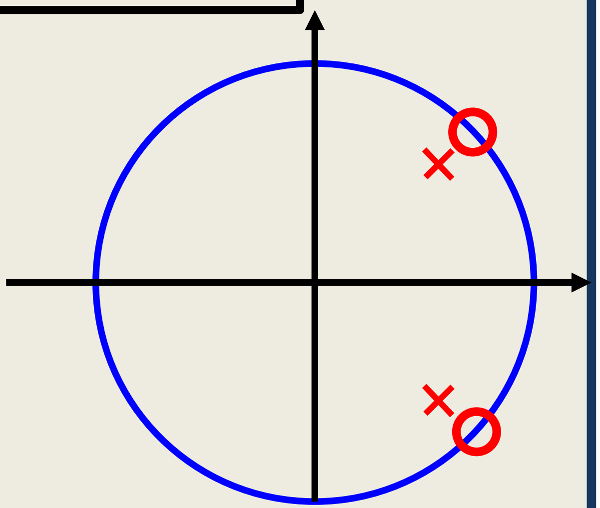
EITF75 Systems and Signals



Magnitude response

Much better

NOTCH filter



FIR filters with linear phase

Linear phase is desirable since it delays all frequencies equally much

Linear phase is defined as $\Theta(\omega) = \kappa\omega + 2\pi\ell$

Whenever there is a phase jump with π, this should be seen as a magnitude response that is negative

EITF75 Systems and Signals

FIR filters with linear phase

Linear phase is desirable since it delays all frequencies equally much

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Whenever there is a phase jump with π, this should be seen as a magnitude response that is negative

$h(n) = h(-n)$	Symmetry around n=0. Not causal
$h(n) = h(N-n)$	Symmetry around n=(N-1)/2.
$h(n) = -h(N-n)$	Anti-symmetry around n=(N-1)/2.

Three types of linear phase filters

EITF75 Systems and Signals

Example TYPE 1

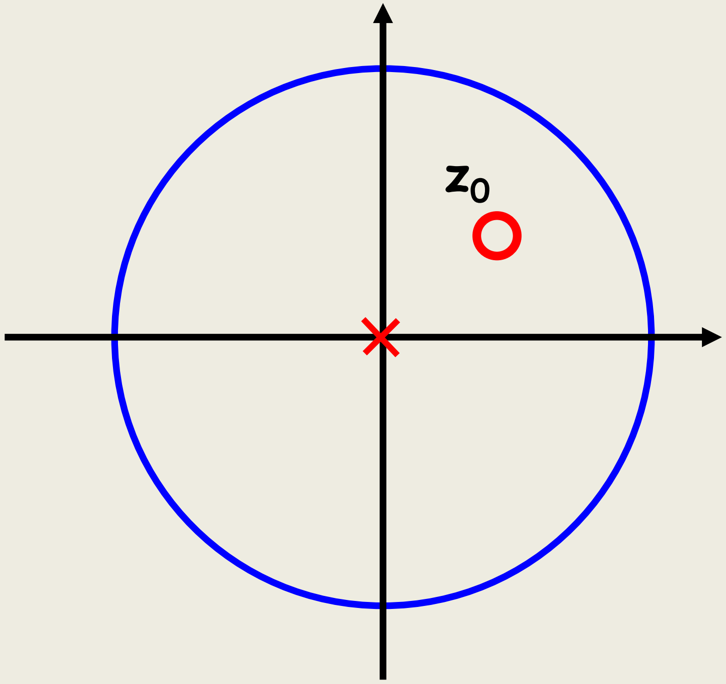
$$h(n) = \left\{ 1 \quad 2 \quad \underline{3} \quad 2 \quad 1 \right\}$$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



EITF75 Systems and Signals

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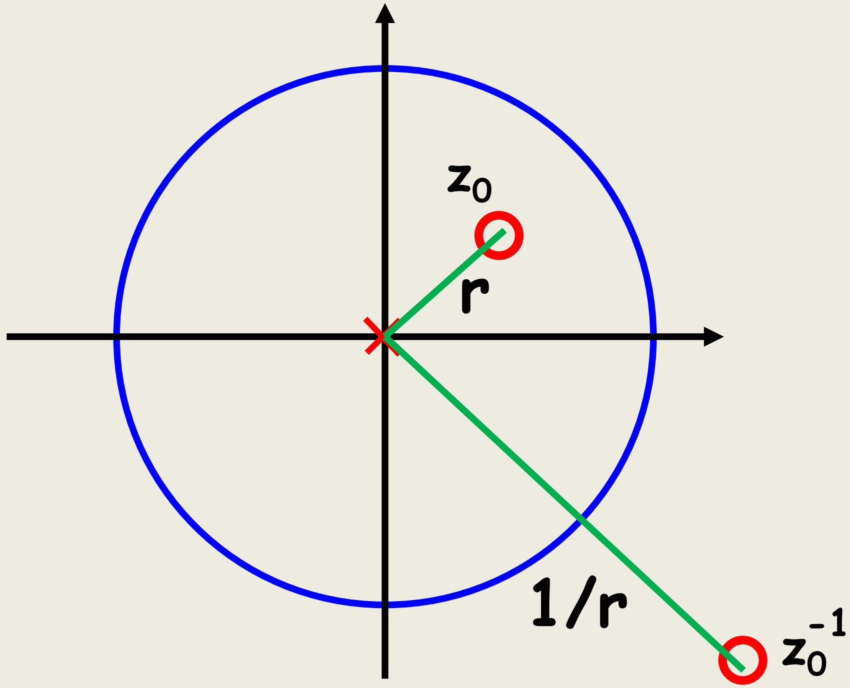

EITF75 Systems and Signals

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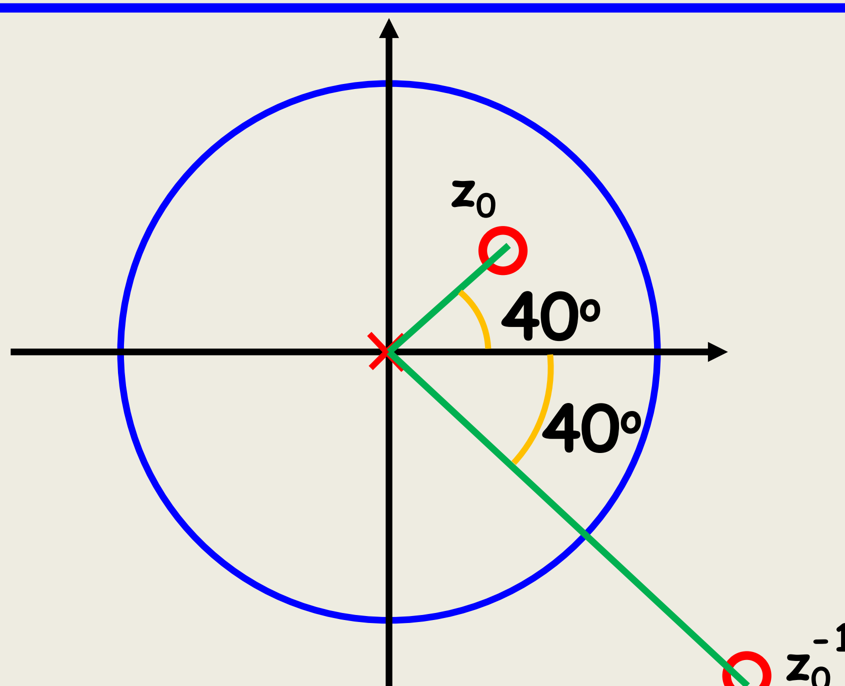

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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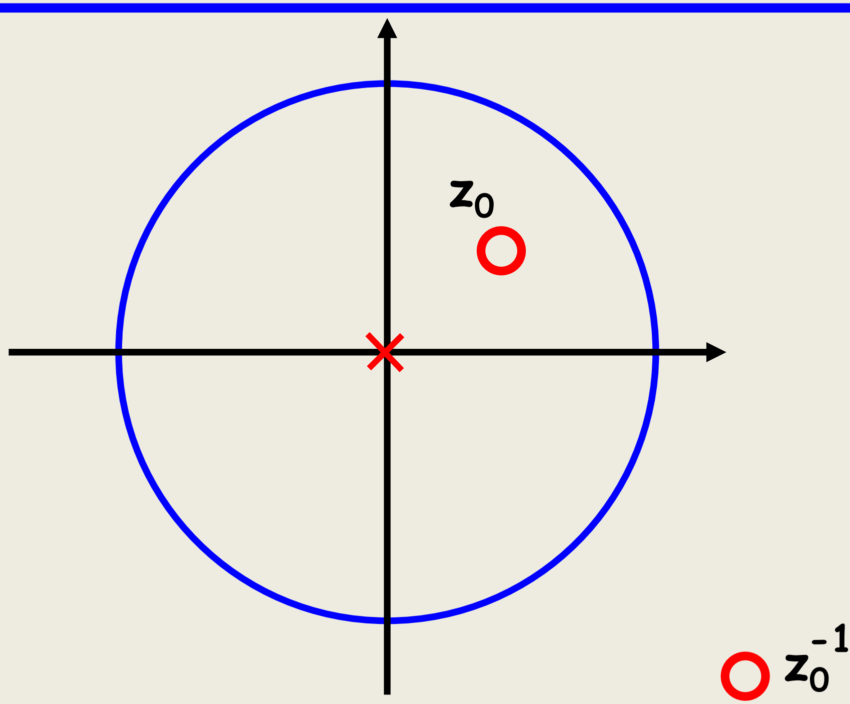
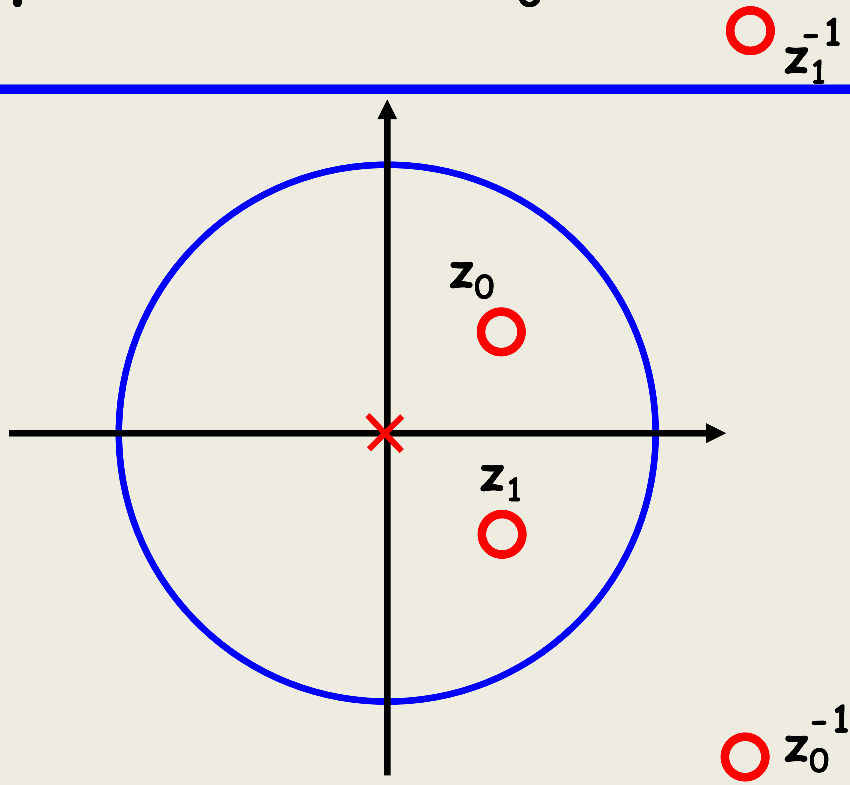


This is not a real-valued h(n)

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



This is

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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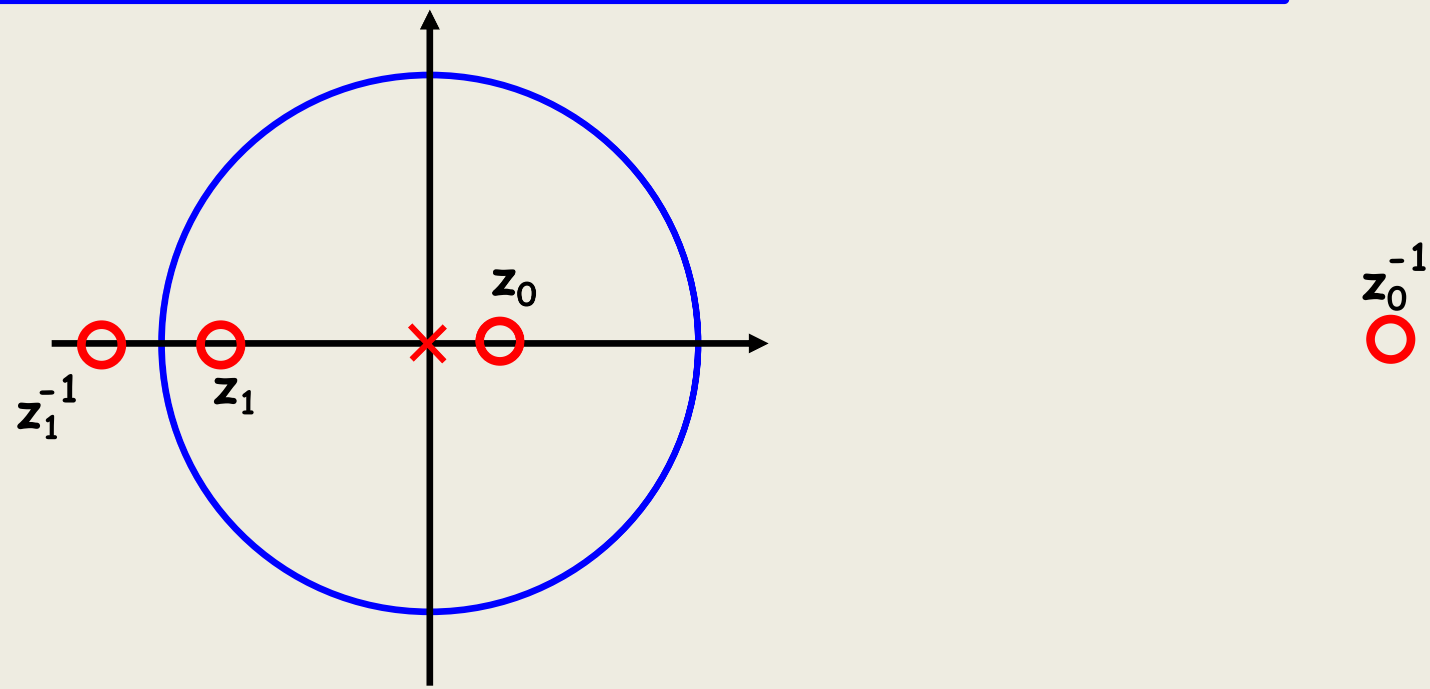


EITF75 Systems and Signals

LTI systems

$$x(n) \rightarrow \boxed{\text{system}} \rightarrow y(n)$$

A system is LTI if-and-only if:

- It is linear
- It is time-invariant

Linear system

$$x(n) = \alpha x_1(n) + \beta x_2(n)$$
$$\Longleftrightarrow$$
$$y(n) = \alpha y_1(n) + \beta y_2(n)$$

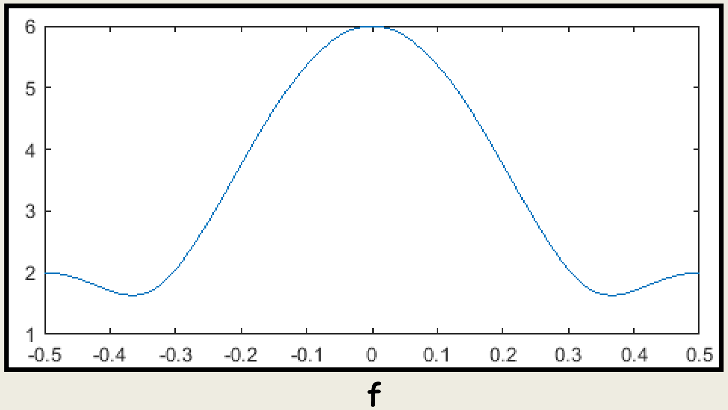
Time invariant system

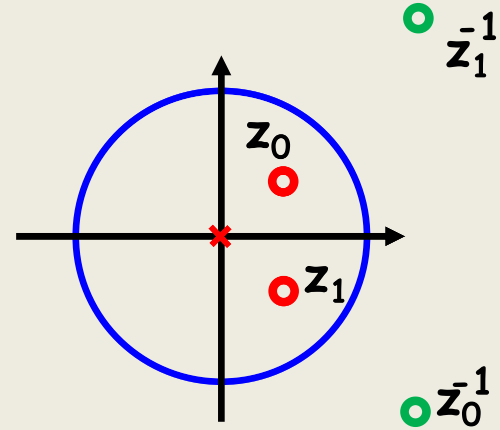
$$x(n) \text{ replaced by } x(n-D)$$
$$\Longleftrightarrow$$
$$y(n) \text{ replaced by } y(n-D)$$

EITF75 Systems and Signals

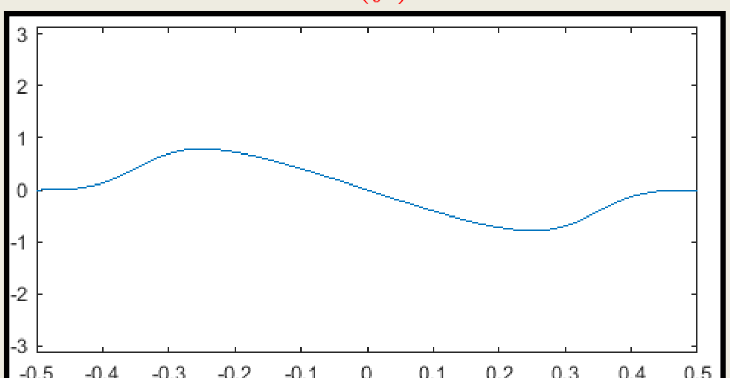
Minimum phase filters

$$|H(f)| = |H^*(f)|$$





$\Theta(f)$



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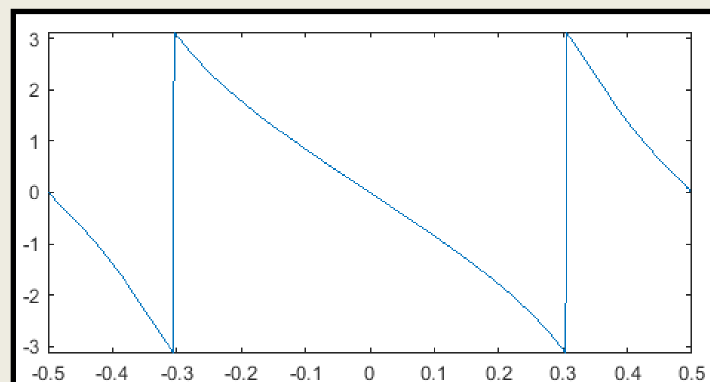
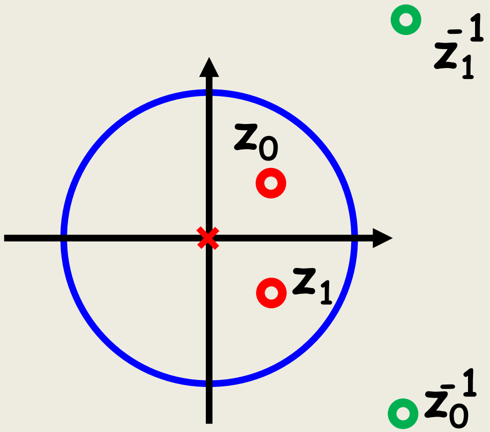


EITF75 Systems and Signals

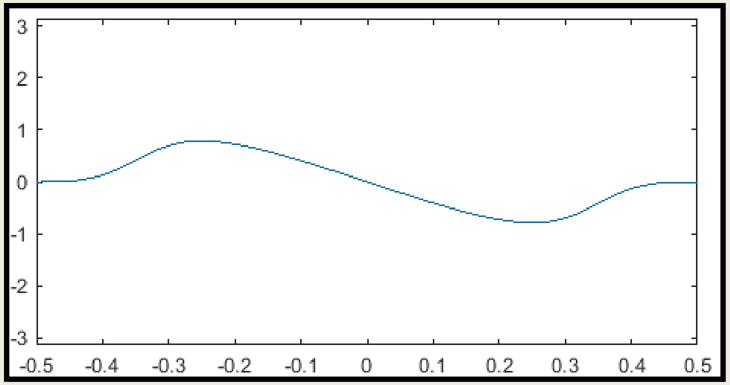
Minimum phase filters

This is a general rule:
A filter with all zeros inside the unit circle has smaller phase.

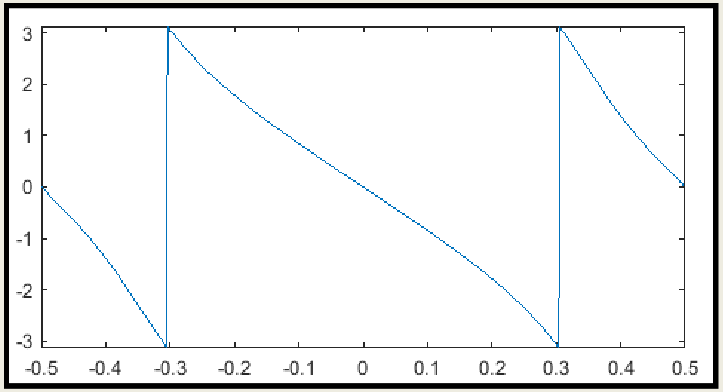
Minimum phase filter
Maximum phase filter



$\Theta(f)$



$\Theta(f)$



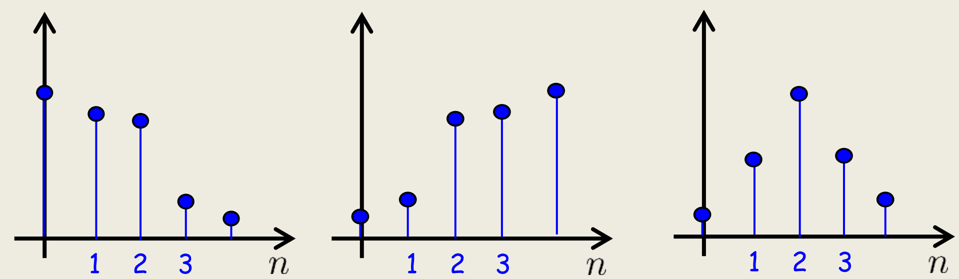
EITF75 Systems and Signals

Minimum phase filters

This is a general rule:
A filter with all zeros inside the unit circle has smaller phase.

Minimum phase filter
Maximum phase filter

Super-important for trunction



Min-phase **Max-phase** **Mix-phase**

Let $h(k)$ be any filter with magnitude response $|H(f)|$

Let $h_{\mathrm{mp}}(k)$ be the minimum phase filter with magnitude $|H(f)|$

Let $h_{\mathrm{maxp}}(k)$ be the maximum phase filter with magnitude $|H(f)|$

Then $\displaystyle\sum_{k=0}^{K} |h_{\mathrm{mp}}(k)|^2 \geq \sum_{k=0}^{K} |h(k)|^2$

Then $\displaystyle\sum_{k=0}^{K} |h_{\mathrm{maxp}}(k)|^2 \leq \sum_{k=0}^{K} |h(k)|^2$

EITF75 Systems and Signals

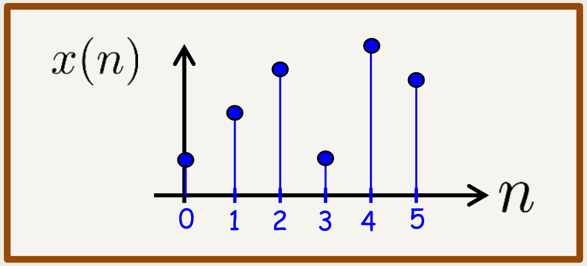
The DFT

EITF75 Systems and Signals

Background and motivation for yet another transform

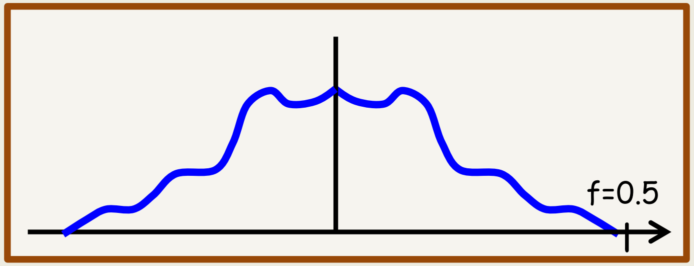
The DTFT is continuous -> Cannot be evaluated by a computer/DSP

Besides, the DTFT is terribly inefficient



These 6 numbers, are in the frequency domain represened by a continuous curve !

It should be possible to Fourier represent x(n) by 6 numbers as well

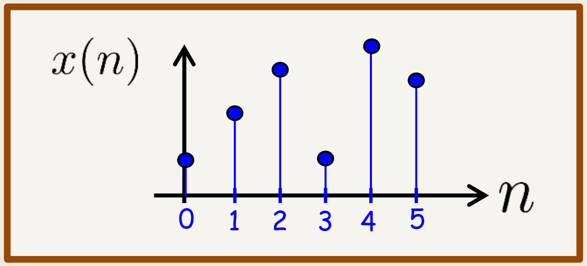


EITF75 Systems and Signals

Background and motivation for yet another transform

The discrete Fourier Transform (DFT) in one sentence:
A Fourier version of x(n) with 6 numbers

Besides, the DTFT is terribly inefficient



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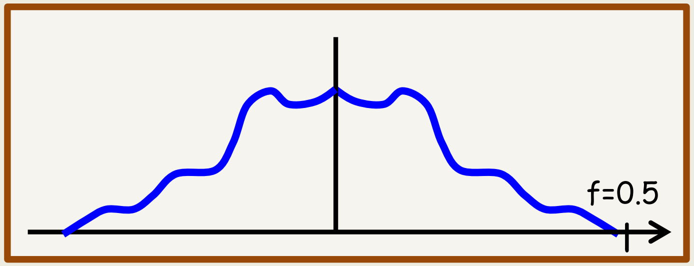


EITF75 Systems and Signals

Formal definition

For a sequence $x(n)$ of arbitrary length, the N-point DFT is defined as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n)\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

and the inverse transform (IDFT) as

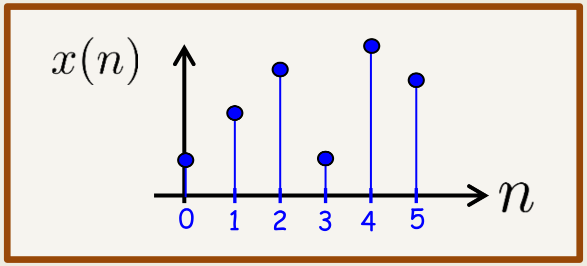
$$x_{\mathrm{IDFT}}(n) = \frac{1}{N}\cdot\sum_{k=0}^{N-1} X(k)\mathrm{e}^{\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n} \qquad \text{for } n = 0, 1, \ldots, N-1$$

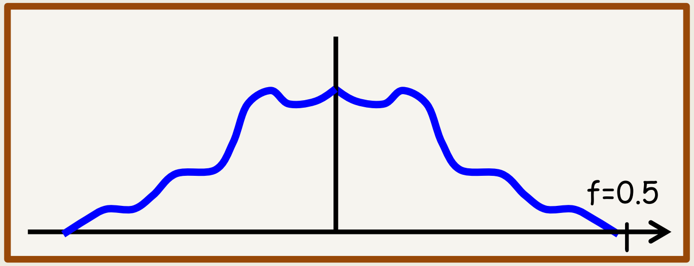
Result

if the length of $x(n)$ is N, then

$x_{\mathrm{IDFT}}(n) = x(n)$ and $X_{\mathrm{DFT}}(k) = X(f \mid f = k/N)$

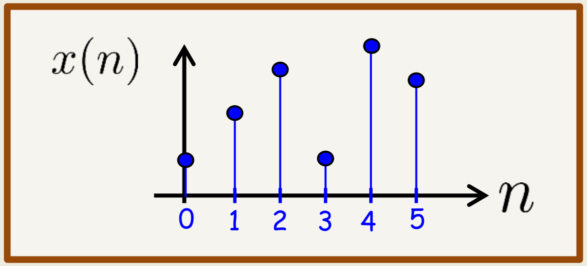
EITF75 Systems and Signals

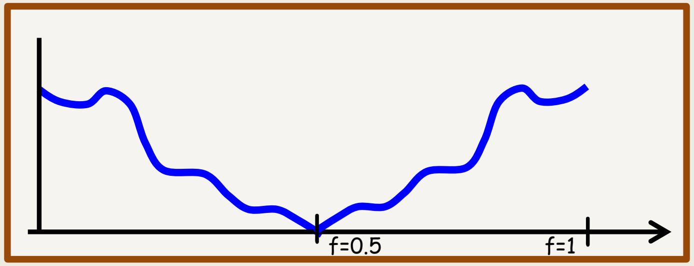




The DTFT is periodic

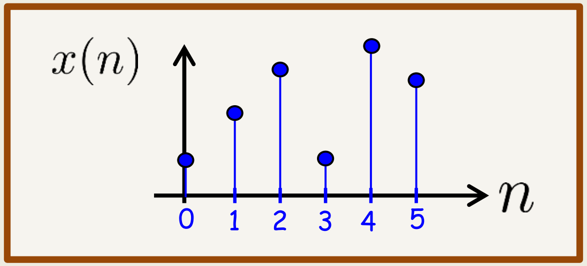
EITF75 Systems and Signals

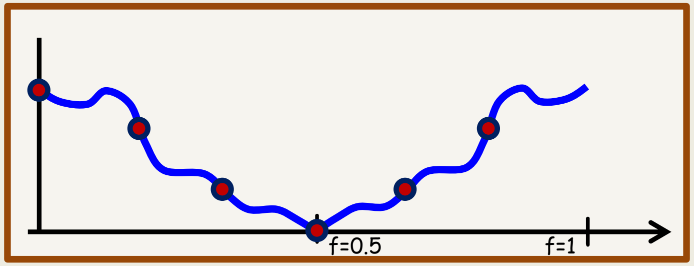




We can represent it like this

EITF75 Systems and Signals

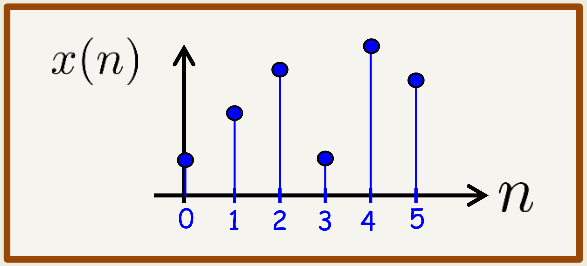


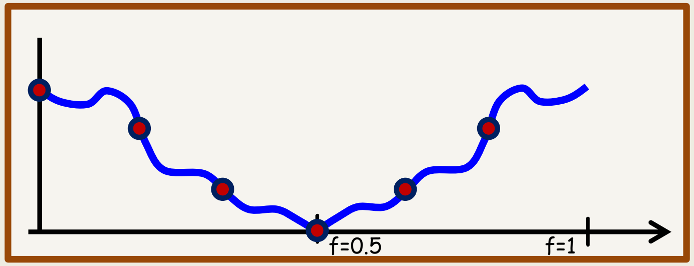


A 6-point DFT would compute the samples of the DTFT

This is sufficient to represent x(n)

EITF75 Systems and Signals



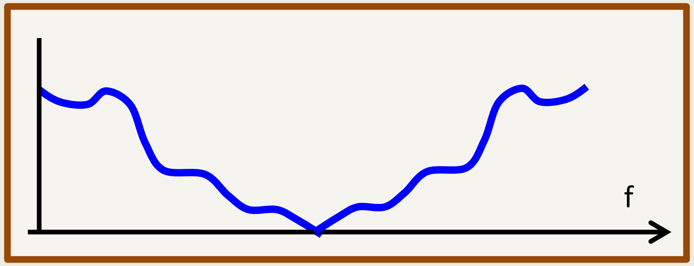


A 6-point DFT would compute the samples of the DTFT

This is sufficient to represent x(n)

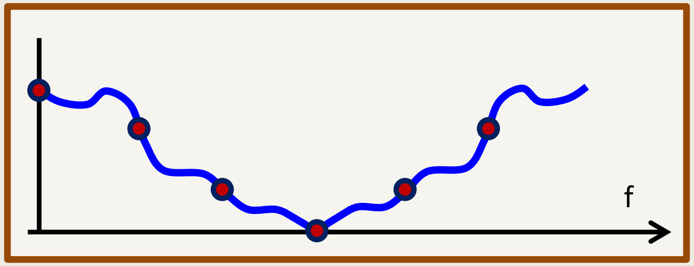
Important: The DFT size must be at least as long as the signal, otherwise there is a loss (aliasing in time)

EITF75 Systems and Signals



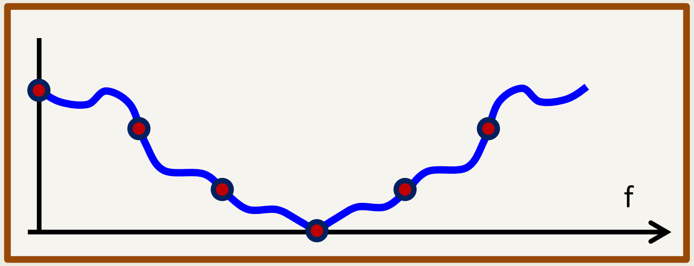
Assume a DTFT of a 6-tap signal

EITF75 Systems and Signals



Sample it

EITF75 Systems and Signals



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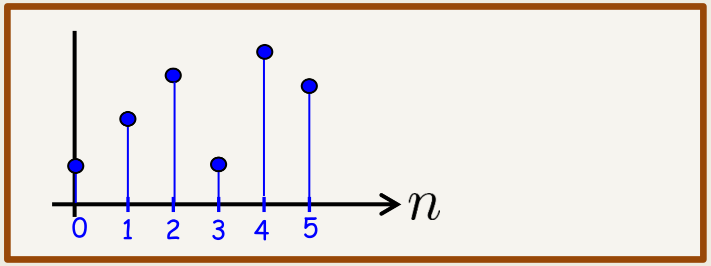
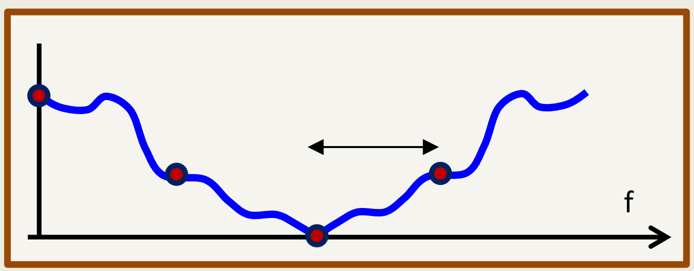
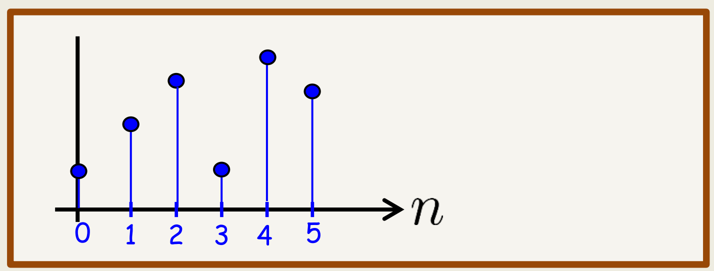


Compute the "other domain" representation from samples. In this case, the time domain

EITF75 Systems and Signals



But if sample
spacing is too small…



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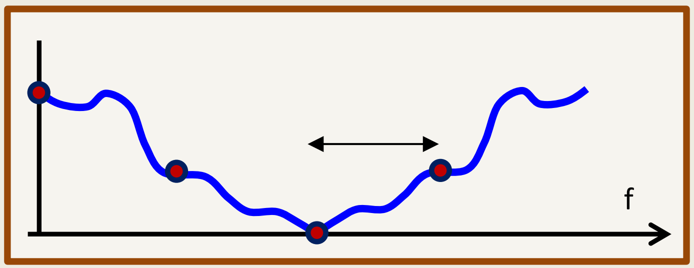
EITF75 Systems and Signals



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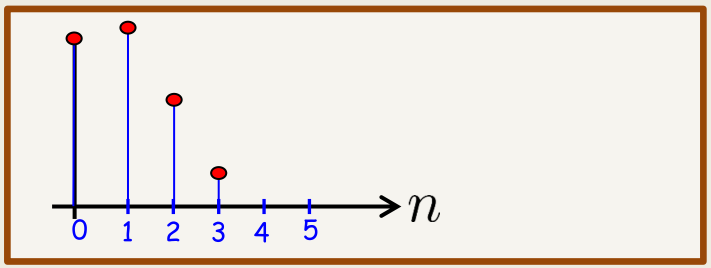
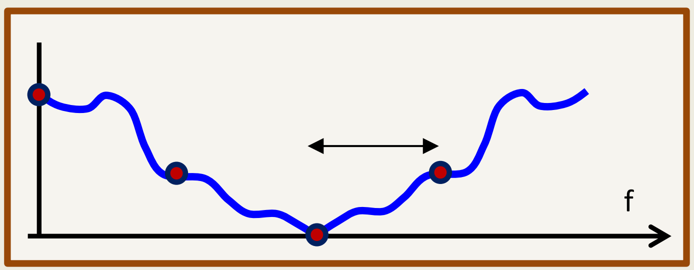


There is aliasing

EITF75 Systems and Signals



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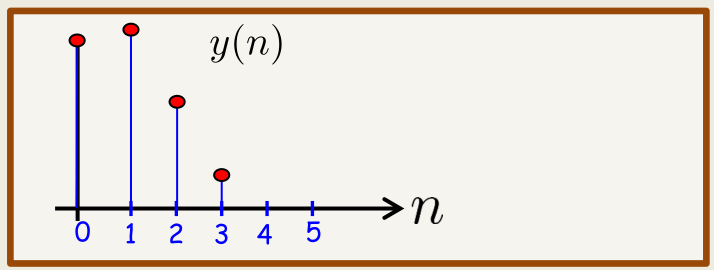


There is aliasing

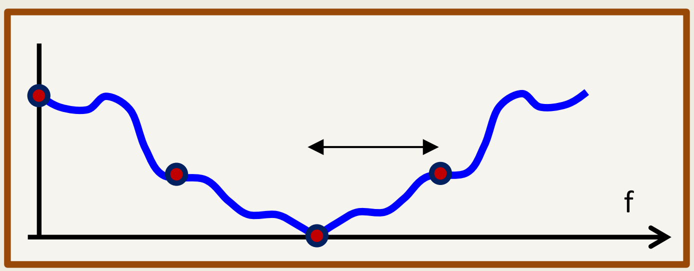
Periodically extended

Aliasing $y(n) = \sum_{m=-\infty}^{\infty} x(n - mN)$

No aliasing $y(n) = x(n)$

EITF75 Systems and Signals

The time-aliasing only occurs if we are not careful with the DFT size. If it is equal or larger than the length of the signal, there is no time-aliasing



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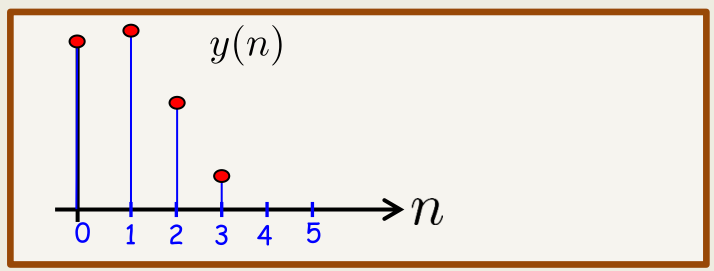

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EITF75 Systems and Signals

Computational complexity

DFT defined as

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n)\mathrm{e}^{-\mathrm{j}2\pi\cdot\frac{k}{N}\cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Number of operations needed:

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EITF75 Systems and Signals

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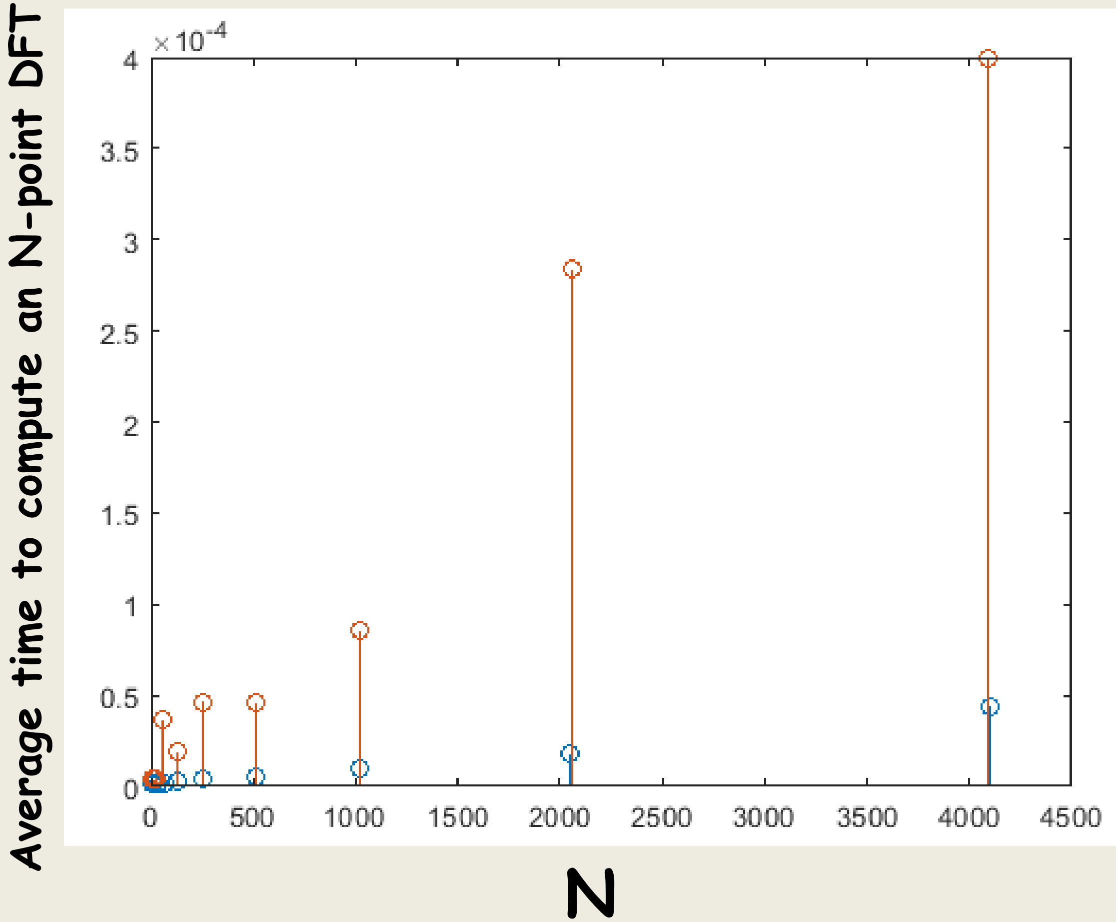
Total complexity N^2

EITF75 Systems and Signals

Computational complexity

Test in Matlab

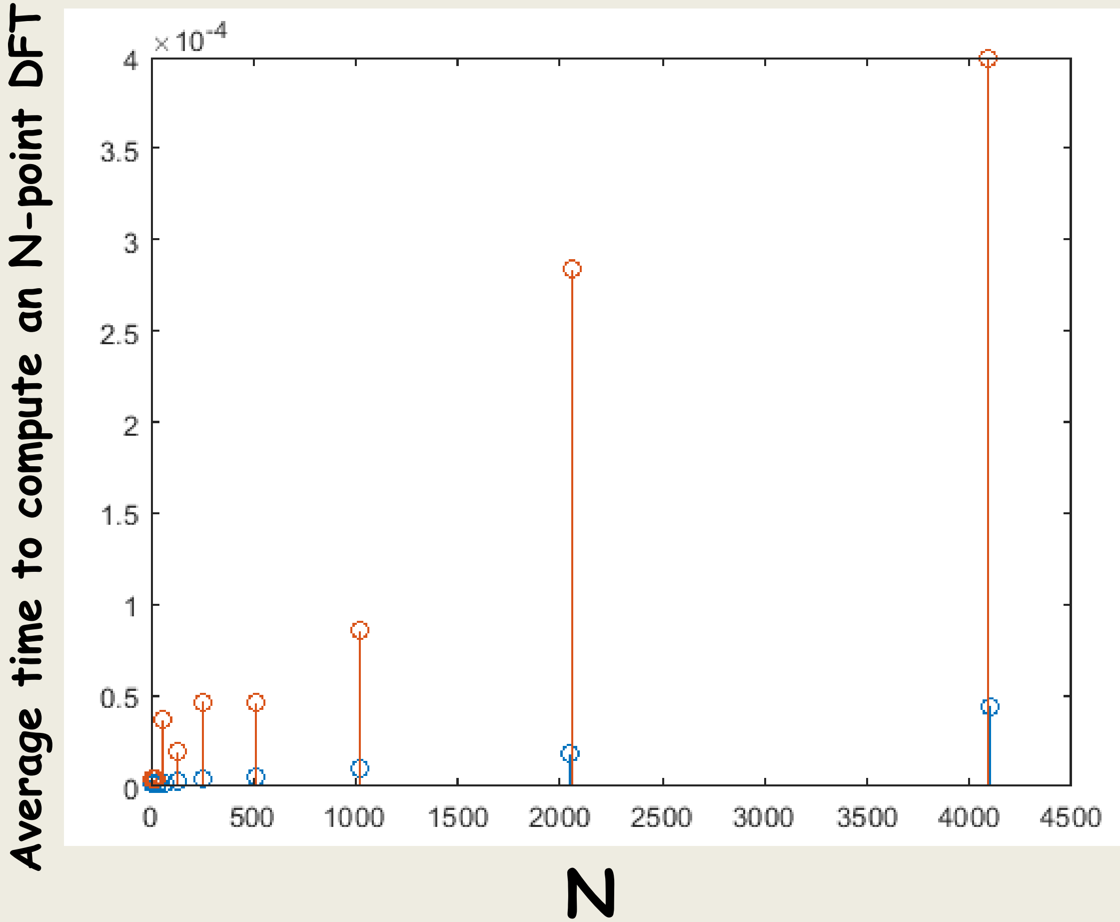


EITF75 Systems and Signals

Computational complexity

Test in Matlab

We see that for some values of N, much less time is needed



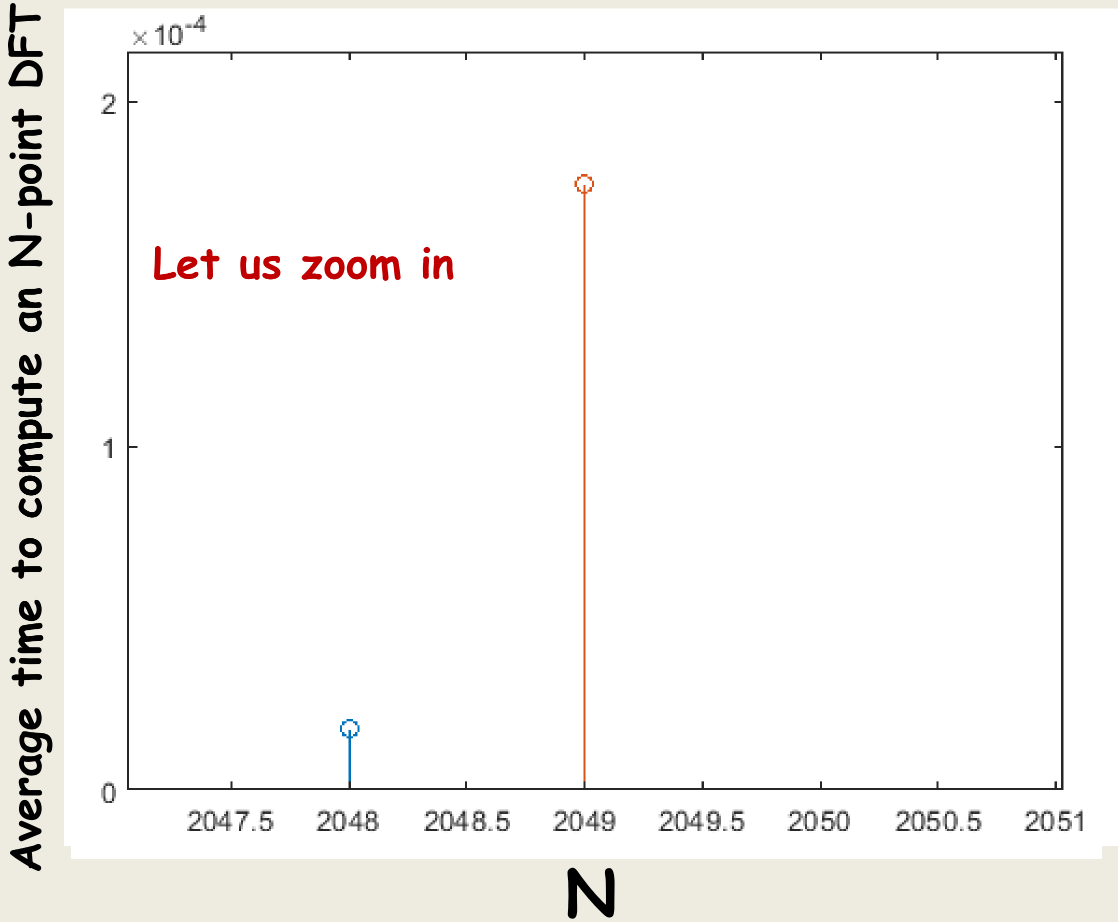
EITF75 Systems and Signals

Computational complexity

Test in Matlab

2048 is 2^{11}

2049 not power of 2



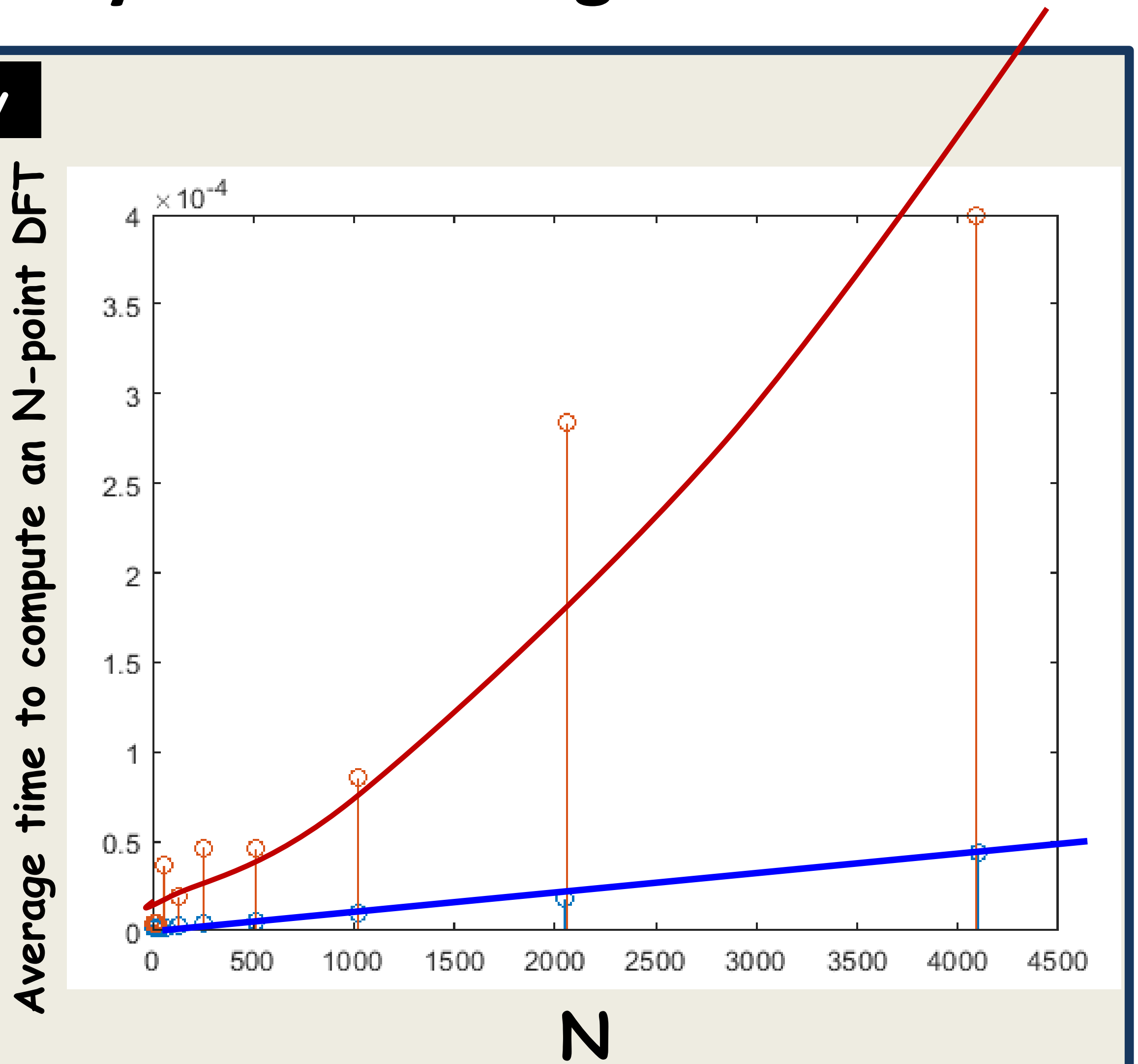
EITF75 Systems and Signals

Computational complexity

Test in Matlab

2048 is 2^{11}

Significant speed-up possible for $N=2^k$



EITF75 Systems and Signals

Computational complexity

FFT not included in course, but good to know about

Test in Matlab

Fast Fourier transform (FFT)

If $N=2^k$, then $N \log_2(N)$ complexity to compute

$$X_{\mathrm{DFT}}(k) = \sum_{n=0}^{N-1} x(n) \mathrm{e}^{-\mathrm{j}2\pi \cdot \frac{k}{N} \cdot n} \qquad \text{for } k = 0, 1, \ldots, N-1$$

Made possible by some algebraic manipulations and tricks.

Cooley and Tukey 1965

Method known to, and used by, Gauss in 1805

EITF75 Systems and Signals

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Made possible by some algebraic manipulations and tricks.

The importance of the FFT cannot be underestimated. WIFI and 4G, etc could not been implemented without the FFT

For a computer,

1. It can avoid the continuous DTFT
2. It can compute the DFT extremely fast

EITF75 Systems and Signals

Properties

For DTFTs, we have

$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0}$$

Still true ? I.e.

$$x(n) \star y(n) \leftrightarrow X(k)Y(k)$$

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Still length N

Makes no sense…

EITF75 Systems and Signals

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Still true ? NO

EITF75 Systems and Signals

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$$x(n) \star y(n) \leftrightarrow X(f)Y(f)$$

$$x(n) \leftrightarrow X(f) \qquad x(n-n_0) \leftrightarrow X(f)\mathrm{e}^{-i2\pi f n_0}$$

For DFTs, we have

$$x_1(n) \otimes x_2(n) \leftrightarrow X(k)Y(k)$$

$$x(n-n_0 \bmod N) \leftrightarrow X(k)\mathrm{e}^{-i2\pi k n_0/N}$$

where $x_1(n) \otimes x_2(n) = \sum_{k=0}^{N-1} x_1(k) x_2(n-k \bmod N)$

Circular convolution

Example

Linear convolution computed via DFTs

Given: Two length N sequences, $x(n)$, $y(n)$

Task: Compute their linear convolution by using DFT and its inverse IDFT

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>> x=[1 2 3 4];
>> y=[2 2 1 1];
>> yL=conv(x,y)
yL =
     2     6    11    17    13     7     4
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This is the result, But not computed via DFT

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     2     6    11    17    13     7     4
```

This is the result, But not computed via DFT

```
>> xp=[1 2 3 4 0 0 0 0];
>> yp=[2 2 1 1 0 0 0 0];
>> yL=ifft(fft(xp).*fft(yp))
yL =
    2.0000    6.0000   11.0000   17.0000   13.0000    7.0000    4.0000   -0.0000
```

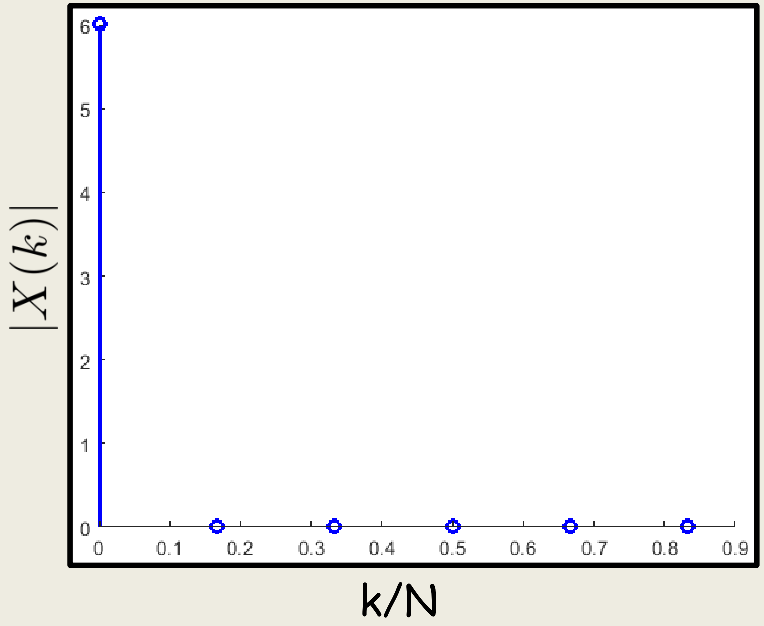
Still a circular convolution carried out, but due to zero-padding, it behaves linear.

EITF75 Systems and Signals

More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\}$$

Compute DFT (N=6)

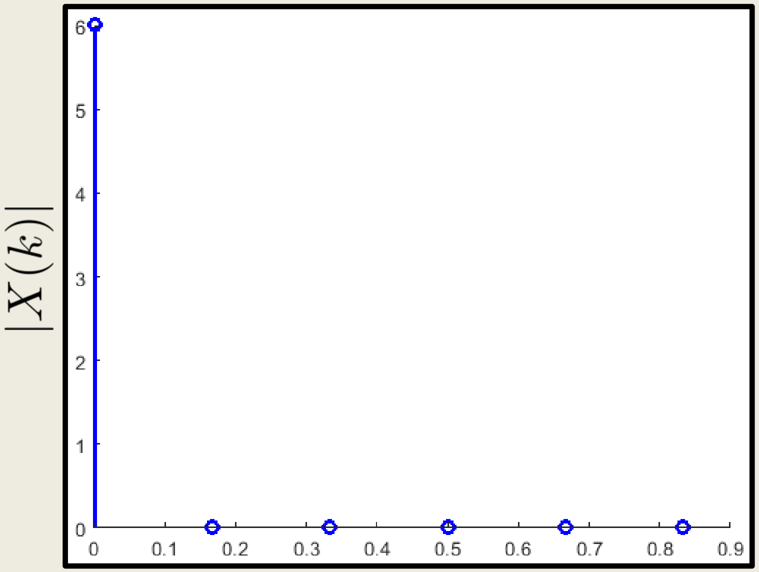


EITF75 Systems and Signals

More examples: Resolution increase

$$x(n) = \{1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 1\ \ 0\ \ 0\}$$

Compute DFT (N=8)



k/N

EITF75 Systems and Signals

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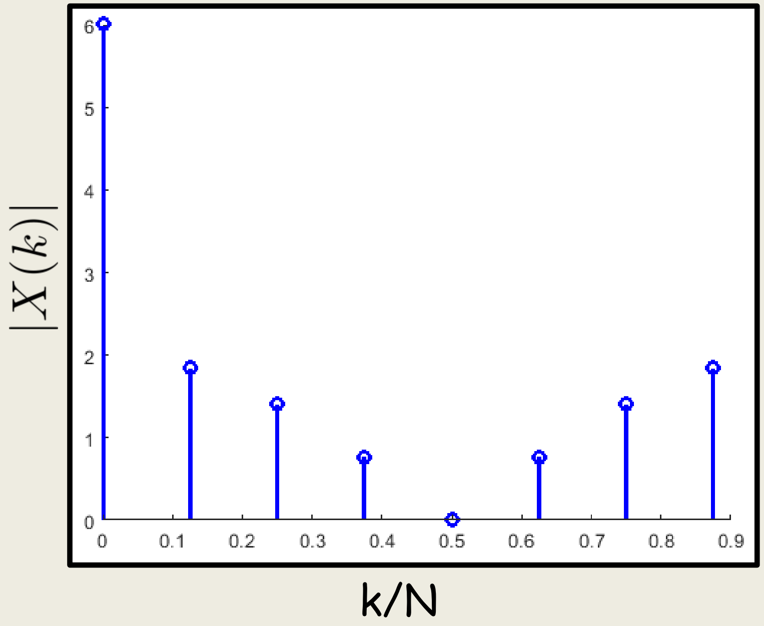
$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\}$$

Compute DFT (N=8)

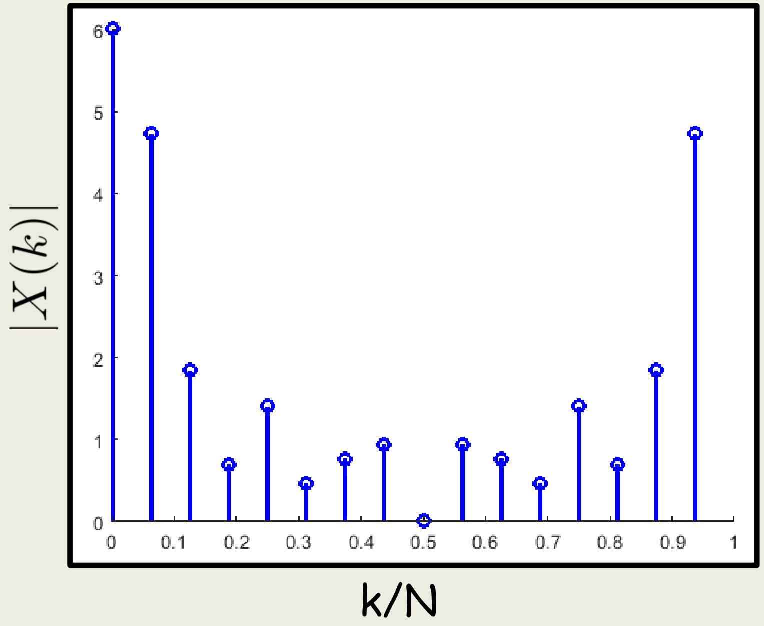


EITF75 Systems and Signals

More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



EITF75 Systems and Signals

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$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

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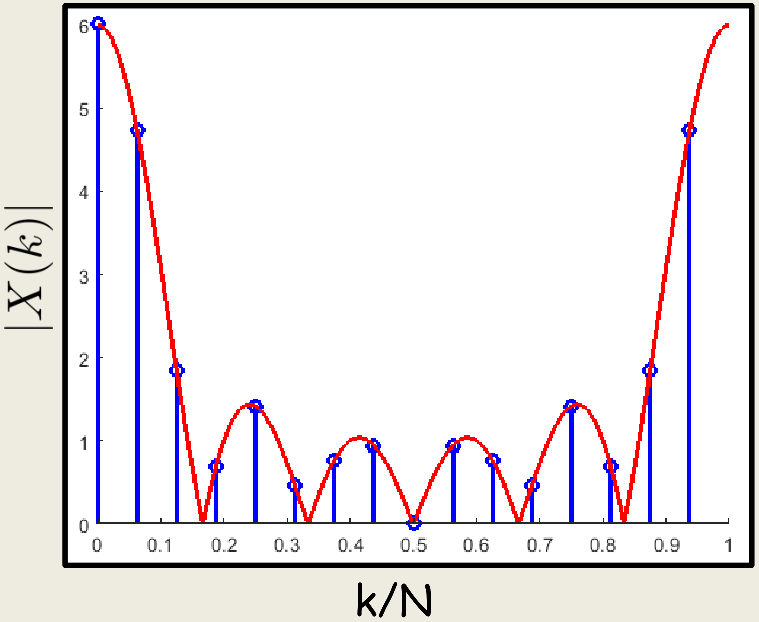


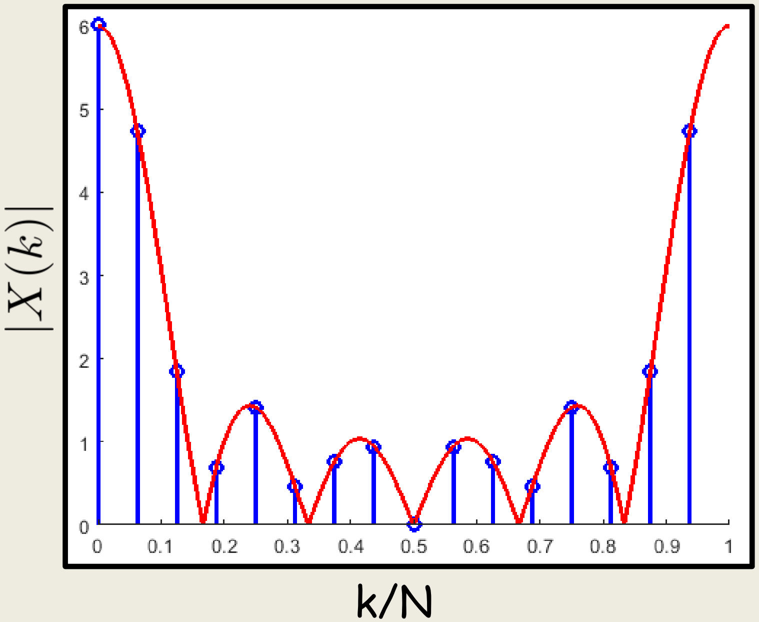
What is this line?

EITF75 Systems and Signals

More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

Compute DFT (N=16)



What is this line?

DFT size larger-or-equal to the length of x(n)

EITF75 Systems and Signals

More examples: Resolution increase

$$x(n) = \{1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \cdots\cdots\}$$

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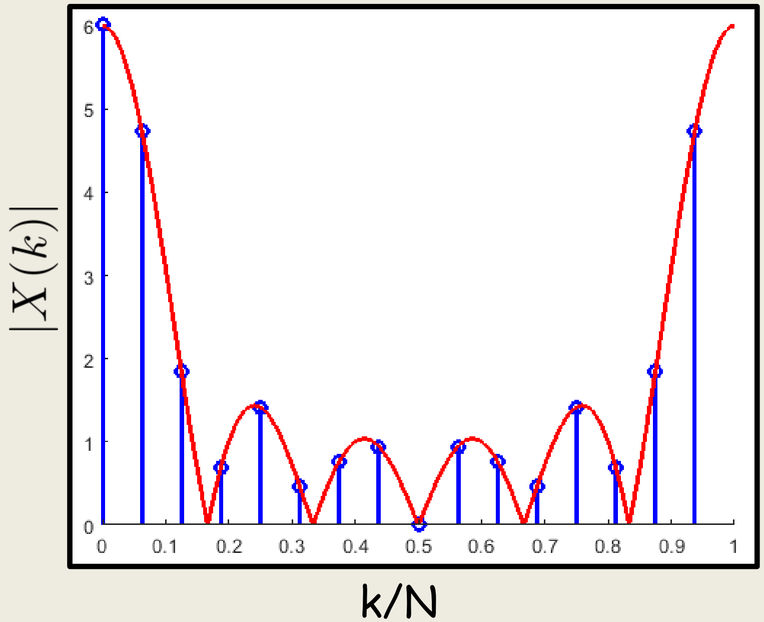

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Therefore, DFT samples of DTFT