Lecture 4 z-transform

Thursday 17 15-17 extra lecture/seminar
solve old hand-ins old exam problems

$$y(n) = 1.05\,y(n-1) + x(n)$$

save money
5% interest
deposit 100 kr/month
start at 0 money
$\Longleftrightarrow$
$y(-1)=0$

z-transf.

$$Y(z) = 1.05\,Y(z)\cdot z^{-1} + X(z)$$

$$Y(z)\cdot\left[1-1.05z^{-1}\right] = X(z)$$

$$Y(z) = \frac{X(z)}{1-1.05z^{-1}}$$

$x(n) = u(n)\cdot 100$

$\longleftrightarrow$

$X(z) = \frac{1}{1-z^{-1}}\cdot 100$

$$= \boxed{\frac{1}{1-1.05z^{-1}}\cdot\frac{1}{1-z^{-1}}\cdot 100}$$

PFE

$$Y(z) = 100\left[\frac{21}{1-1.05z^{-1}} - \frac{20}{1-z^{-1}}\right]$$

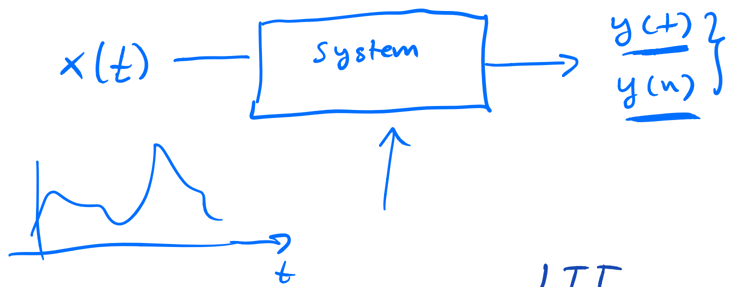
$$\boxed{\frac{1}{1-az^{-1}} \longleftrightarrow a^n u(n)}$$

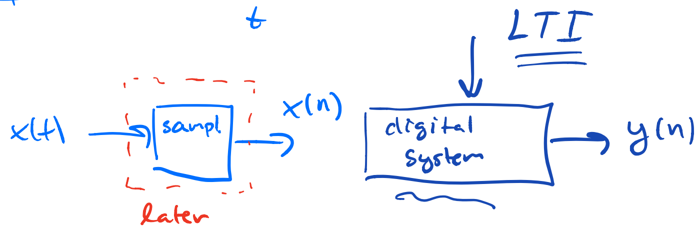
$$\boxed{y(n) = 100\cdot 21\cdot 1.05^n u(n) - 100\cdot 20\, u(n)}$$

Issue 1 $\quad y(-1)\neq 0 \quad y(-1)=1000$

$$Y(z) = \ldots\frac{P(z)}{\boxed{1-Az^{-1}+Bz^{-2}}}\ldots$$

Issue 2 PFE-II





$$y(n) = \begin{cases} 1, & x(n) > 10 \\ 0, & x(n) \leq 10 \end{cases}$$

$\neq LTI$

$x(n)$ temp.

$$y(n) = \begin{cases} 2 \cdot x(n) & \text{winter} \\ 3 \cdot x(n) & \text{summer} \end{cases}$$

$\neq LTI$

LTI

$$y(n) = h(n) * x(n)$$

$$\sum_{k=0}^{N} y(n-k)\, a(k) = \sum_{k=0}^{M} x(n-k)\, b(k)$$

$\boxed{a(0)=1}$ $\Rightarrow$

$$y(n) + \sum_{k=1}^{N} y(n-k)\, a(k) = \sum_{k=0}^{M} x(n-k)\, b(k)$$

Recall

$\boxed{y(n-k) \longleftrightarrow z^{-k} Y(z)}$

apply Z-transf.

$$Y(z) + \sum_{k=1}^{N} a(k)\, z^{-k}\, Y(z) = \sum_{k=0}^{M} b(k)\, z^{-k} X(z)$$

$$Y(z)\left[1+\sum_{k=1}^{N} a(k)\, z^{-k}\right] = X(z)\left[\sum_{k=0}^{M} b(k)\, z^{-1}\right]$$

$$Y(z) = \frac{b(0)+b(1)z^{-1}+\dots+b(M)z^{-M}}{1+a(1)z^{-1}+\dots+a(N)z^{-N}}\, X(z)$$

all are ok!

$$\boxed{\begin{array}{l} N > M \\ N = M \\ N < M \end{array}}$$

$$= \underbrace{\frac{z^{-M}}{z^{-N}} \frac{b(0)z^{M}+b(1)z^{M-1}+\dots+b(M)}{z^{N}+a(1)z^{N-1}+\dots+a(N)}}_{\triangleq\ H(z)} X(z)$$

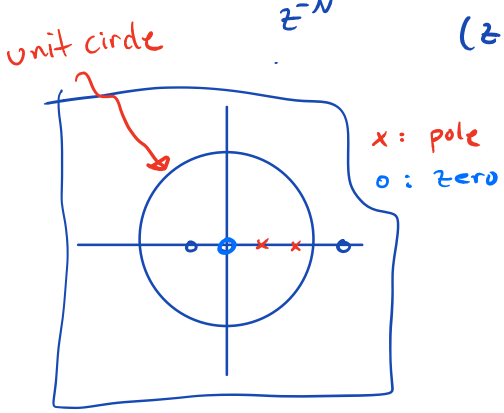
$$\boxed{Y(z) = H(z)\cdot X(z)} \quad \Longleftrightarrow \quad \boxed{y(n) = h(n) * x(n)}$$

$$Y(z) = \underbrace{\frac{z^{-M}}{z^{-N}} \frac{b(0)z^{M}+b(1)z^{M-1}+\dots+b(M)}{z^{N}+a(1)z^{N-1}+\dots+a(N)}}_{\triangleq\ H(z)} X(z)$$

$$= \frac{z^{-M}}{z^{-N}}\, b(0)\, \frac{(z-z_1)\cdot(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)} \cdot X(z)$$

zero

poles



$$\frac{z^{-M}}{z^{-N}} = z^{N-M}$$

$N-M > 0$

$\Rightarrow$ zeros at $z=0$

$N-M < 0$

$\frac{1}{z^{M-N}}$, $M-N>0$, poles at $z=0$

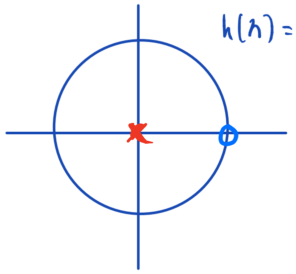
(IIR infinite imp. resp. filter) (FIR: finite impulse response filter)

Ex1 $H(z) = 1 - z^{-1} = \frac{z}{z}\cdot(1-z^{-1}) = \frac{z-1}{z}$

zeros: $z=1$

poles: $z=0$

$h(n) = [\underline{1} \;\; -1]$



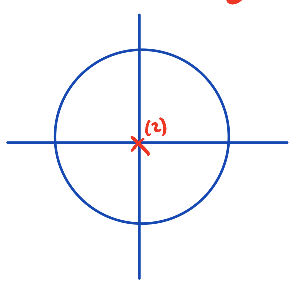
FIR-filter ("does not have any denominator in z^{-1}")

$\Rightarrow$ poles at $z=0$

Ex2

$$H(z) = \underbrace{1 - z^{-1} + z^{-2}}_{L=3} = \frac{z^2}{z^2}\left(1 - z^{-1} + z^{-2}\right) = \frac{z^2 - z + 1}{z^{\underline{\underline{2}}}}$$

2 poles at $z=0$



FIR of length L taps

$\Rightarrow$ $(L-1)$ poles at $z=0$

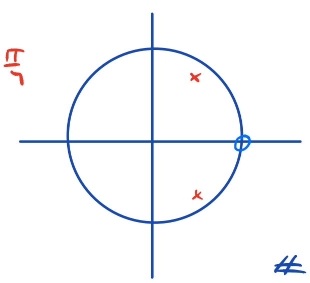
Ex3

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}} = \frac{z^2}{z^2}\cdot\left[\quad \cdots \quad\right]$$

$$= \frac{z-1}{z^2 - 1.27z + 0.81}$$

zeros: $z=1$

poles: $z = 0.9\,e^{\pm i\frac{\pi}{4}}$



#

$$Y(z) = b(0)\frac{(z-z_1)\cdot(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}\,X(z)$$

$$X(z) = \frac{N(z)}{Q(z)} = \frac{N(z)}{(z-q_1)(z-q_2)\cdots(z-q_L)}$$

some polynomial

$$Y(z) = \frac{P(z)}{(z-p_1)(z-p_2)\cdots(z-p_N)\cdot(z-q_1)\cdot\cdots(z-q_L)}$$

PFE

- $\deg P(z) < \deg(\text{denominator})$. if not: apply pol. div.
- all poles $\{p_k, q_k\} \neq 0$ have single mult. and are real

$$\Rightarrow \quad Y(z) = \sum_{k=1}^{N} \frac{A_k}{z-p_k} + \sum_{k=1}^{L} \frac{Q_k}{z-q_k}$$

$\{A_k, Q_k\}$ obtained from PFE

$$= z^{-1} \sum \frac{A_k}{1-p_k z^{-1}} + z^{-1} \sum \frac{Q_k}{1-q_k z^{-1}}$$

time delay

$$\boxed{y(n) = \sum_{k=1}^{N} A_k p_k^{n-1} u(n-1) + \sum Q_k q_k^{n-1} u(n-1)}$$

$\forall |p_k| < 1 \quad \Rightarrow \quad \sum_{k=1}^{N} A_k p_k^{n-1} u(n-1) \rightarrow 0 \quad n \rightarrow \infty$

$\boxed{\text{transient}}$

$\exists k : |p_k| > 1$

$\Rightarrow$ transient $\rightarrow \infty \quad n \rightarrow \infty$

unstable

$\boxed{\text{stability} \quad \text{all poles } q_1 \cdots q_n \text{ inside the unit-circle}}$

$$\boxed{y(n) - \frac{1}{2} y(n-1) = x(n-1) \qquad x(n) = \left(\frac{1}{3}\right)^n u(n)}$$

$x(n) \to 0$
$n \to \infty$

$$X(z) = \frac{1}{1 - \frac{1}{3} z^{-1}}$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = z^{-1} X(z)$$

$$Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = z^{-1} X(z)$$

$$Y(z) = H(z) X(z)$$

$$= \underbrace{\frac{z^{-1}}{1 - \frac{1}{2} z^{-1}}}_{H(z)} X(z) = \frac{z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \frac{1}{3} z^{-1})} \cdot \frac{z^2}{z^2}$$

$$= \frac{z}{(z - \frac{1}{2})(z - \frac{1}{3})}$$

zeros : $z = 0$

poles : $p_1 = \frac{1}{2}$

$q_1 = \frac{1}{3}$

deg (num) < deg (denom) ($\Rightarrow$ no pol. div. needed)

$$= \frac{3}{z - \frac{1}{2}} - \frac{2}{z - \frac{1}{3}} = \frac{3 z^{-1}}{1 - \frac{1}{2} z^{-1}} - \frac{2 z^{-1}}{1 - \frac{1}{3} z^{-1}}$$

$$\Leftrightarrow \boxed{y(n) = 3 \cdot \frac{1}{2}^{n-1} u(n-1) - 2 \left(\frac{1}{3}\right)^{n-1} u(n-1)}$$

trans. $\qquad \to 0 \quad n \to \infty$

$\boxed{\text{Redo, but } x(n) = u(n)}$

$$\boxed{y(n) - \frac{1}{2} y(n-1) = x(n-1) \qquad x(n) = \left(\cancel{\tfrac{1}{3}}\right)^n u(n)}$$

~~$x(n) \to 0$, $n \to \infty$~~

$$x(z) = \frac{1}{1 - \cancel{\tfrac{1}{3}} z^{-1}}$$

$$Y(z) - \frac{1}{2} z^{-1} Y(z) = z^{-1} X(z)$$

$$Y(z)\left[1 - \frac{1}{2} z^{-1}\right] = z^{-1} X(z)$$

$$Y(z) = H(z) X(z)$$

$$= \underbrace{\frac{z^{-1}}{1 - \frac{1}{2} z^{-1}}}_{H(z)} X(z) = \frac{z^{-1}}{(1 - \frac{1}{2} z^{-1})(1 - \cancel{\tfrac{1}{3}} z^{-1})} \cdot \frac{z^2}{z^2}$$

$$= \frac{z}{(z - \frac{1}{2})(z - 1)}$$

zeros : $z = 0$

$\boxed{\text{poles} : p_1 = \frac{1}{2}}$

$q_1 = \cancel{\tfrac{1}{3}}\ 1$

deg (num) < deg (denom) (⟹ no pol. div. needed)

$$= \frac{3}{z - \frac{1}{2}} - \frac{2}{z - \cancel{\tfrac{1}{3}}1} = \frac{3 z^{-1}}{1 - \frac{1}{2} z^{-1}} - \frac{2 z^{-1}}{1 - 1 z^{-1}}$$

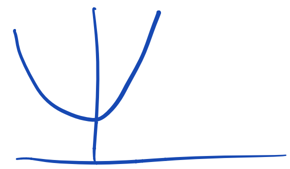
$$\longleftrightarrow \quad \boxed{y(n) = 3 \cdot \underbrace{\tfrac{1}{2}^{n-1} u(n-1)}_{\text{trans.}} - 2 \left(\cancel{\tfrac{1}{3}}\right)^{n-1} u(n-1)}$$

~~$\to 0$ $n \to \infty$~~

$\sim 1 \quad n \to \infty$

$\rightarrow \; Y(z) = \cdots \underbrace{\dfrac{1}{(z-p_1)(z-p_2)}} \cdot \dfrac{1}{(z-p_3)\cdots(z-p_N)}$

$f(x) = x^2+1$



$x_{1,2} = \pm i$

$\rightarrow \; \underline{p_1 = p_2^*} \qquad p_1 = re^{iw}$

$$\frac{1}{(z-p_1)(z-p_2)} = \frac{1}{z^2 + \underbrace{p_1 p_2}_{=r^2} - \underbrace{z(p_1+p_2)}_{2r\cos(w)}}$$

$$= \frac{1}{z^2+r^2-2r\cos(w)\,z}$$

$$= \frac{1}{z^2-2r\cos(w)+r^2}$$

$\rightarrow$ PFE $\rightarrow$

$$Y(z) = \frac{A+zB}{z^2-2r\cos(w)+r^2} + \underline{\sum_{k=3}^{N} \frac{A_k}{z-p_k}}$$

$$\boxed{h(n) = r^n \cos(wn)\, u(n)}$$

$$= r^n \frac{1}{2}\left[e^{jwn} + e^{-jwn}\right] u(n)$$

$$H(z) = \frac{1}{2} \sum_n (z^{-1} r e^{jw})^n u(n) + \frac{1}{2} \sum_n (z^{-1} r e^{-jw})^n u(n)$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} (z^{-1} r e^{jw})^n + \frac{1}{2} \sum_{n=0}^{\infty} (z^{-1} r e^{-jw})^n$$

$$= \frac{1}{2}\left[\frac{1}{1-re^{j\omega}z^{-1}} + \frac{1}{1-re^{-j\omega}z^{-1}}\right] = \cdots =$$

$$\boxed{= \frac{1 - r\cos(\omega)z^{-1}}{1-2r\cos(\omega)z^{-1}+r^2z^{-2}}}$$

function	z-transform
$h(n) = r^n \cos(\omega n)\, u(n)$	$\dfrac{1 - r\cos(\omega)z^{-1}}{1-2r\cos(\omega)z^{-1}+r^2z^{-2}}$