EITF75 Systems and Signals

Lecture 8
More about Filters

Fredrik Rusek

EITF75 Systems and Signals

Data signal

$$x(t) = s(t) + \sin(\Omega_0 t)$$

$$\Omega_0 = 2\pi \cdot 1250$$

Interference

Continuous time signal

Objective: Filter out the interference

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Step 1: Go to discrete time via sampling. More about this next week

$$F_s = 10\,000\,\mathrm{Hz}$$

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$$\omega_0 = 2\pi \cdot \frac{F_0}{F_s} = 2\pi \cdot 0.125$$

Discrete time signal

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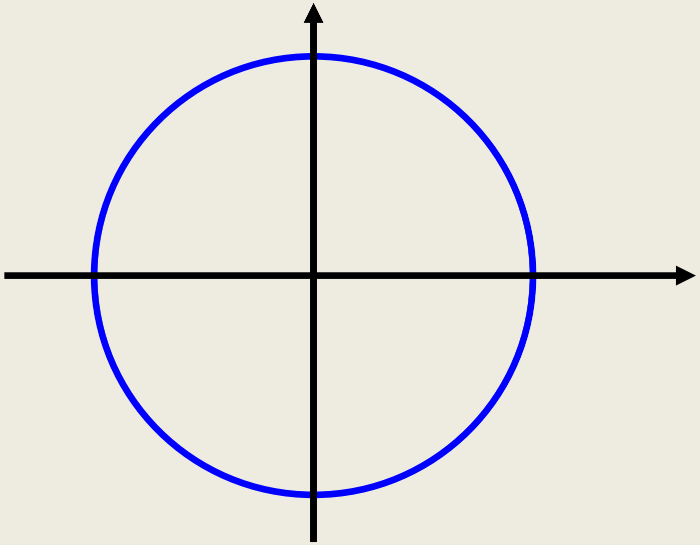
Step 2: Make a pole-zero diagram for filter



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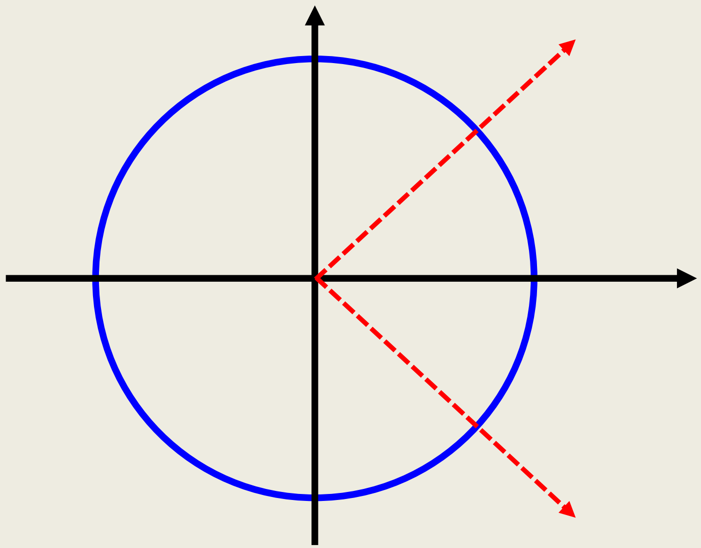

Step 3: Identify interference frequency

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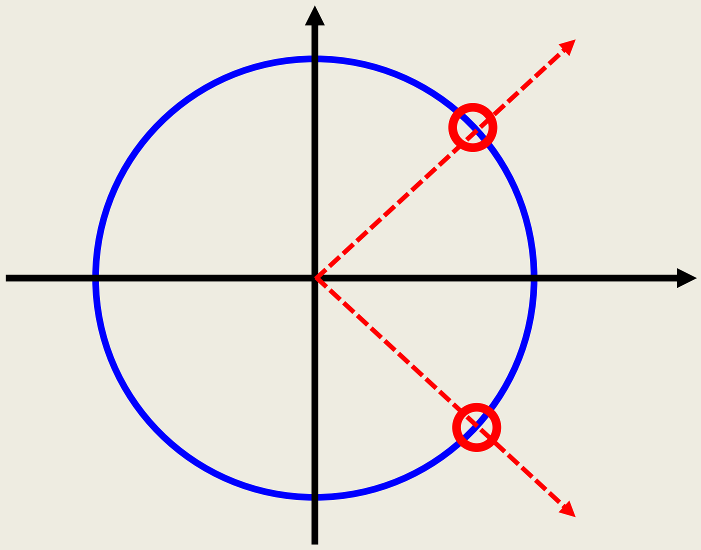

Step 3: Identify interference frequency

Step 4: Try something out. Makes sense to put zeros at unit circle (will cancel interference)

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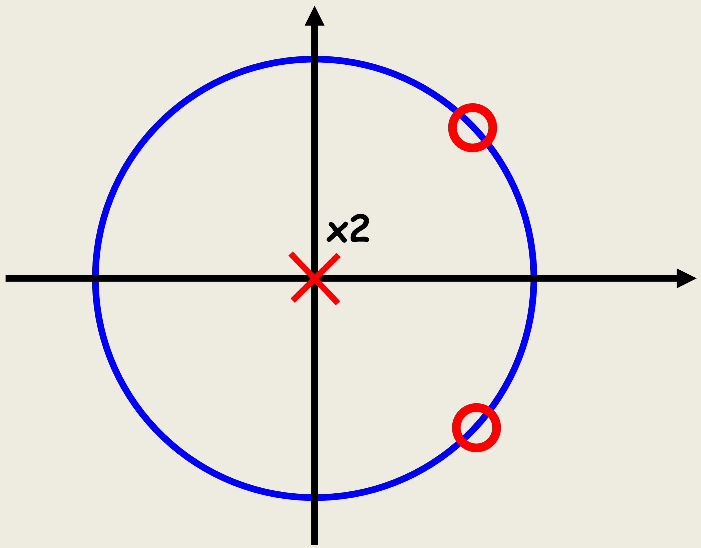

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Do we need any poles? A causal FIR filter has poles in the origin

Data signal

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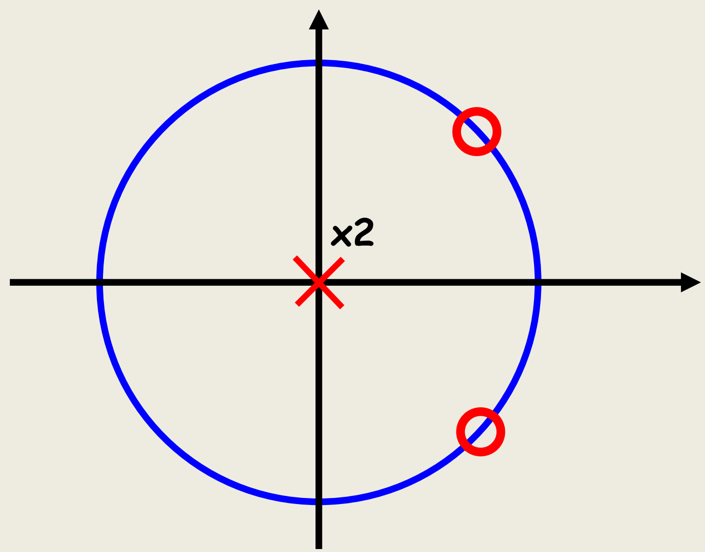

Filter H(z) =

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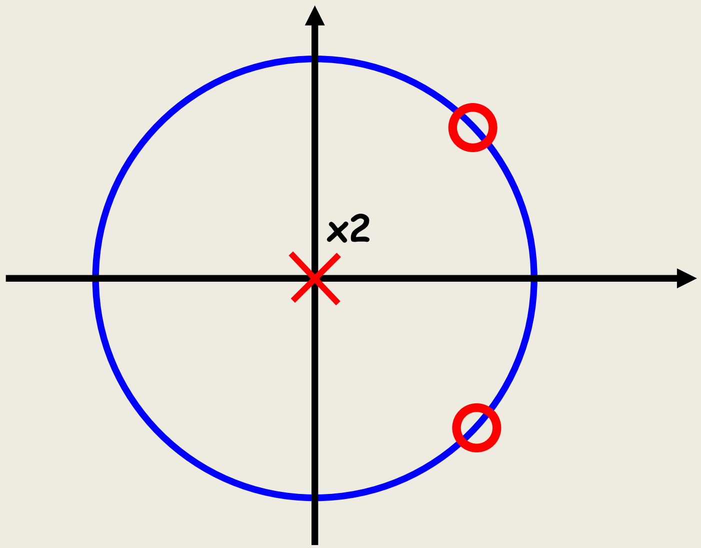
Filter $H(z) = \dfrac{(z-e^{i2\pi/8})(z-e^{-i2\pi/8})}{z^2}$

Step 2: Make a pole-zero diagram for filter



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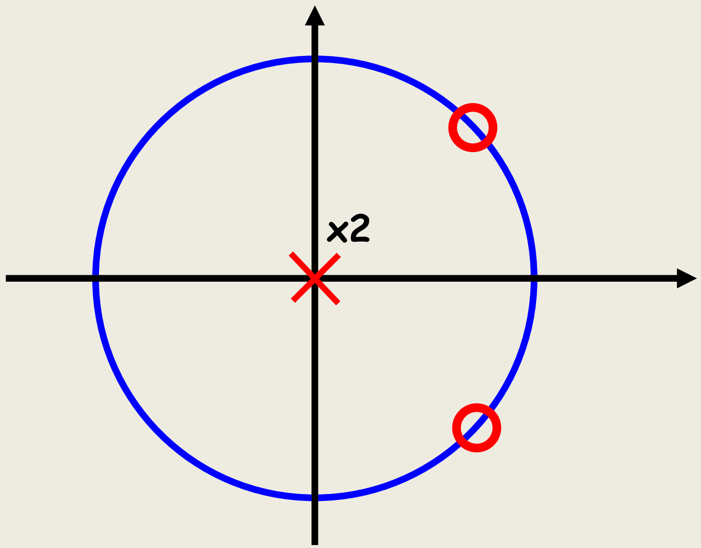

Filter $H(z) = \dfrac{(z-e^{i2\pi/8})(z-e^{-i2\pi/8})}{z^2}$

$= 1 - 2\cos(w_0)z^{-1} + z^{-2}$

$h(n) = \{ \underline{1} \quad -2\cos(w_0) \quad 1\}$ FIR

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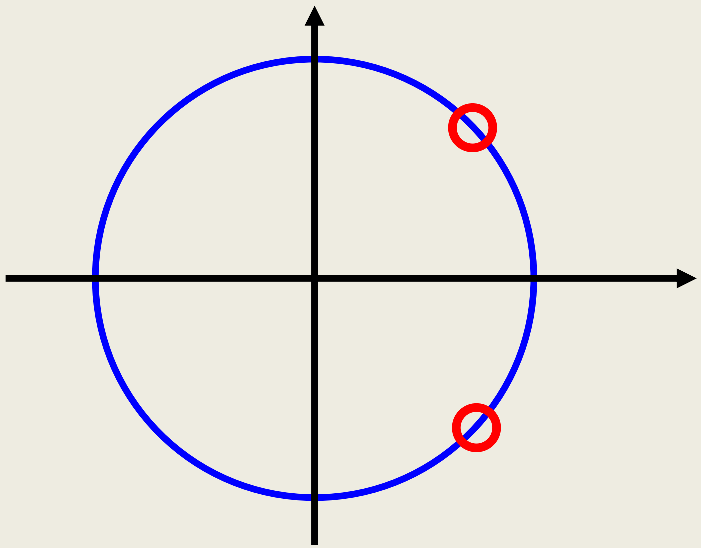

Filter $H(z) = (z-e^{i2\pi/8})(z-e^{-i2\pi/8})$

WHAT IF WE SKIP POLES AT THE ORIGIN ?

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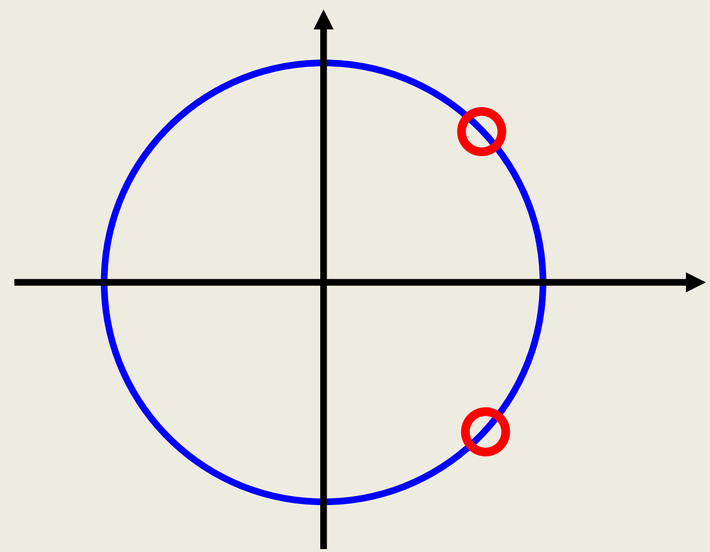

Filter $H(z) = (z - e^{i2\pi/8})(z - e^{-i2\pi/8})$

$= z^2 - 2\cos(w_0)z + 1$

$h(n) = \{\, 1 \quad -2\cos(w_0) \quad \underline{1} \,\}$

FIR

Not Causal

Interlude

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Magnitude response $|H(f)|$

$H(z) = z^2 - 2\cos(w_0)z + 1$	$H(z) = 1 - 2\cos(w_0)z^{-1} + z^{-2}$
$h(n) = \{\ 1 \quad -2\cos(w_0) \quad \underline{1}\ \}$	$h(n) = \{\ \underline{1} \quad -2\cos(w_0) \quad 1\ \}$

Same magnitude response?

Interlude

EITF75 Systems and Signals

Magnitude response |H(f)|

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$\lvert H(f)\rvert = \left\lvert e^{i4\pi f} - 2\cos(\omega_0)e^{i2\pi f} + 1\right\rvert$	$\lvert H(f)\rvert = \left\lvert 1 - 2\cos(\omega_0)e^{-i2\pi f} + e^{-i4\pi f}\right\rvert$

Method 1

$$H(f) = H(z)\Big|_{z=\exp(i2\pi f)}$$

Interlude

EITF75 Systems and Signals

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$$|H(f)| = \left|e^{i4\pi f} - 2\cos(\omega_0)e^{i2\pi f} + 1\right|$$

$$|H(f)| = \left|1 - 2\cos(\omega_0)e^{-i2\pi f} + e^{-i4\pi f}\right|$$

$$= \frac{\left|e^{i4\pi f} - 2\cos(\omega_0)e^{i2\pi f} + 1\right|}{\left|e^{i4\pi f}\right|}$$

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EITF75 Systems and Signals

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Equal

Method 1 $\quad H(f) = H(z)\Big|_{z=\exp(i2\pi f)}$

Interlude

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Magnitude response $|H(f)|$

$H(z) = z^2 - 2\cos(w_0)z + 1$	$H(z) = 1 - 2\cos(w_0)z^{-1} + z^{-2}$
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$$h(n) \leftrightarrow H(f)$$

Method 2

$$h(n - n_0) \leftrightarrow \mathrm{e}^{-i2\pi n_0 f} H(f)$$

Interlude

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Magnitude response $|H(f)|$

$H(z) = z^2 - 2\cos(w_0)z + 1$ h(n) = { 1 -2cos(w_0) $\underline{1}$ }	$H(z) = 1 - 2\cos(w_0)z^{-1} + z^{-2}$ h(n) = { $\underline{1}$ -2cos(w_0) 1 }
h(n) = h(n+2) **Apply the below property**	

$$h(n) \leftrightarrow H(f)$$

Method 2

$$h(n - n_0) \leftrightarrow \mathrm{e}^{-i2\pi n_0 f} H(f)$$

Interlude

EITF75 Systems and Signals

Phase response	$\theta(f)$
$H(z) = z^2 - 2\cos(w_0)z + 1$	$H(z) = 1 - 2\cos(w_0)z^{-1} + z^{-2}$
$h(n) = \{ 1 \quad -2\cos(w_0) \quad \underline{1} \}$	$h(n) = \{ \underline{1} \quad -2\cos(w_0) \quad 1 \}$

Same phase response?

Interlude

EITF75 Systems and Signals

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Same phase response?

No, since

1. Signals not the same
2. Signals have equal magnitude responses
3. Therefore, must have different phase responses

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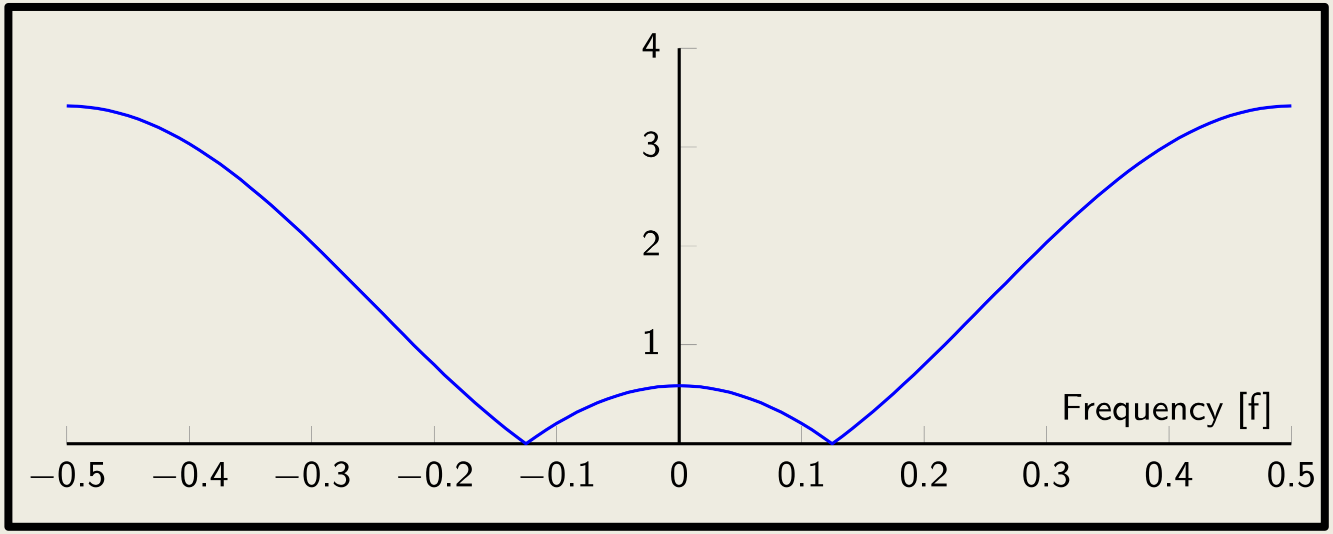
$$|H(f)| = \left|1 - 2\cos(\omega_0)\mathrm{e}^{-i2\pi f} + \mathrm{e}^{-i4\pi f}\right|$$

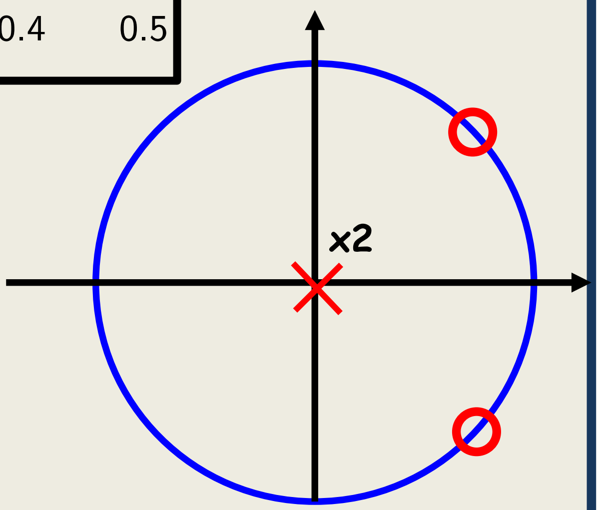


Magnitude response

FIASCO
distorts s(n)

$$x(n) = s(n) + \sin(\omega_0 n)$$

$$\omega_0 = 2\pi \cdot \frac{F_0}{F_s} = 2\pi \cdot \boxed{0.125}$$



$h(n) = \{ \underline{1} \quad -2\cos(w_0) \quad 1 \}$

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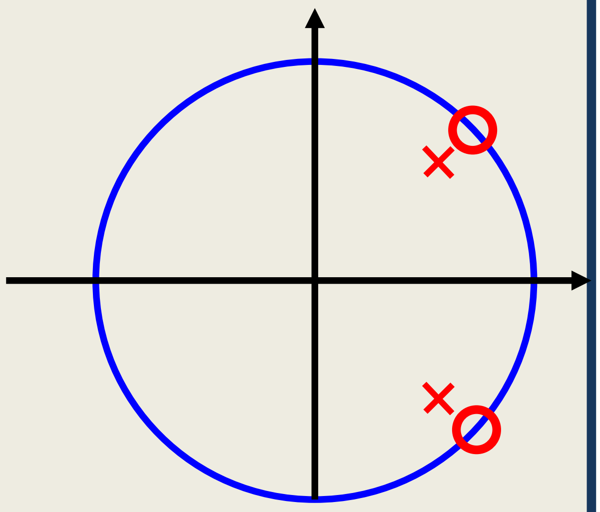
Let us try



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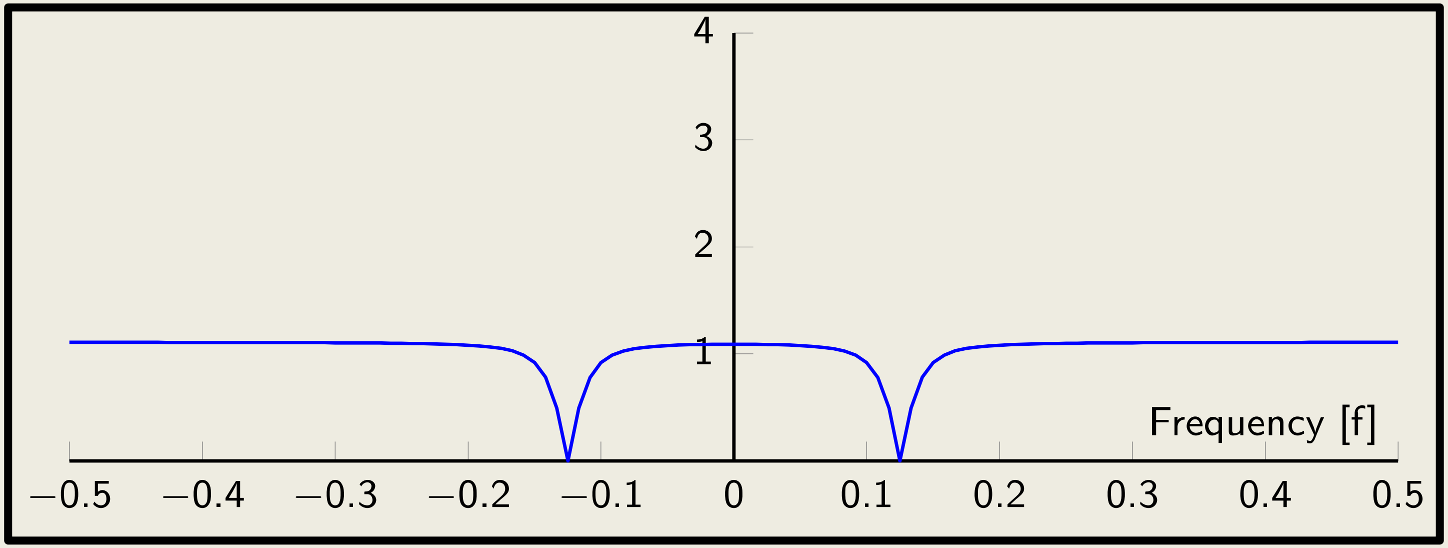
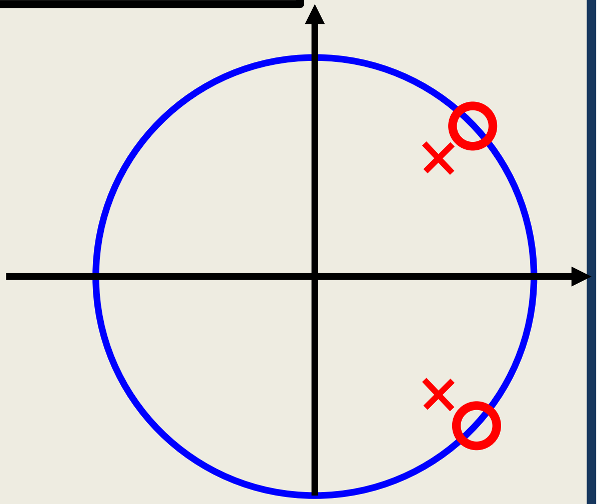


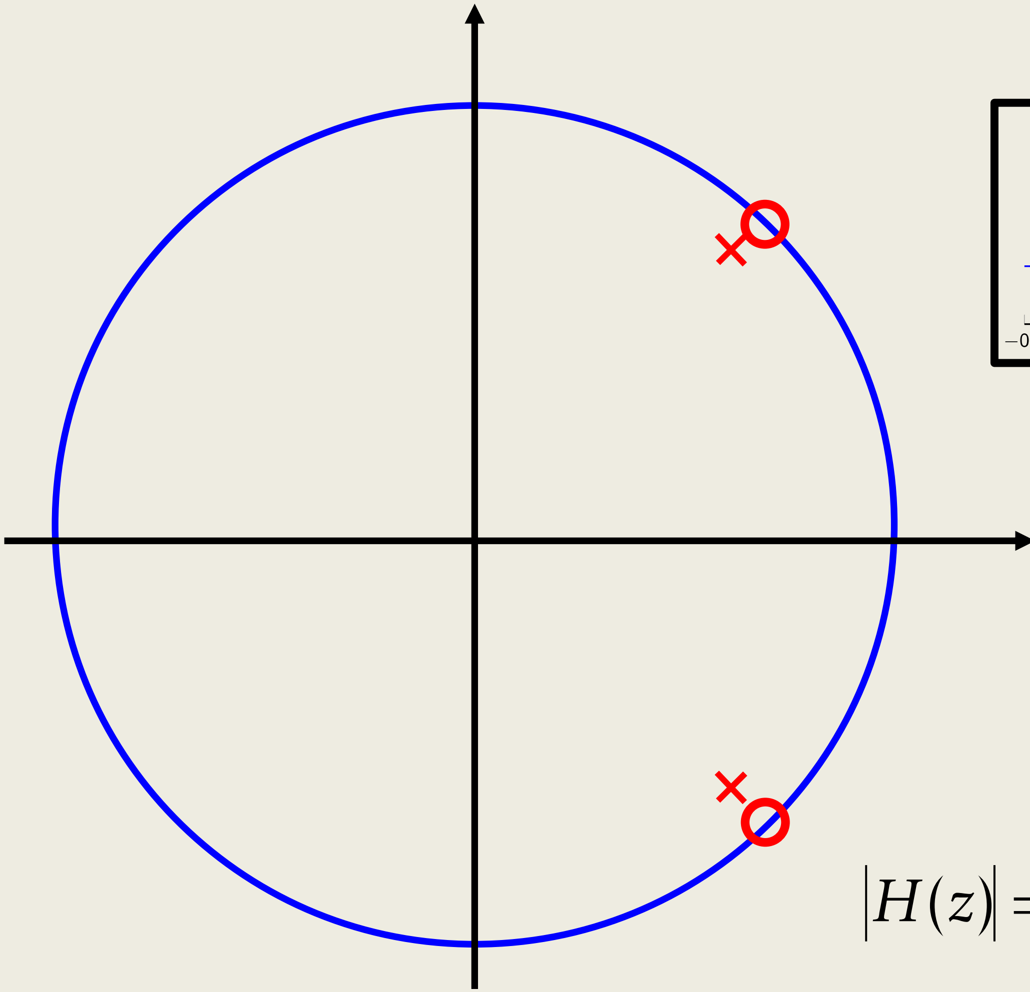
Magnitude response

Much better

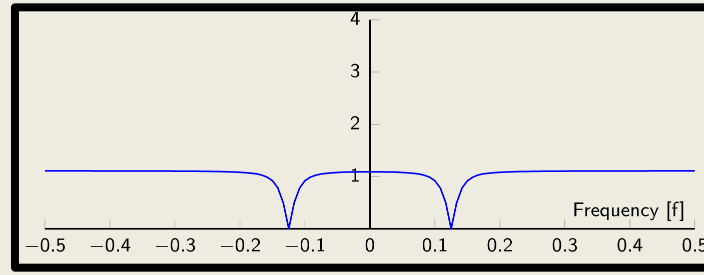
NOTCH filter



EITF75 Systems and Signals



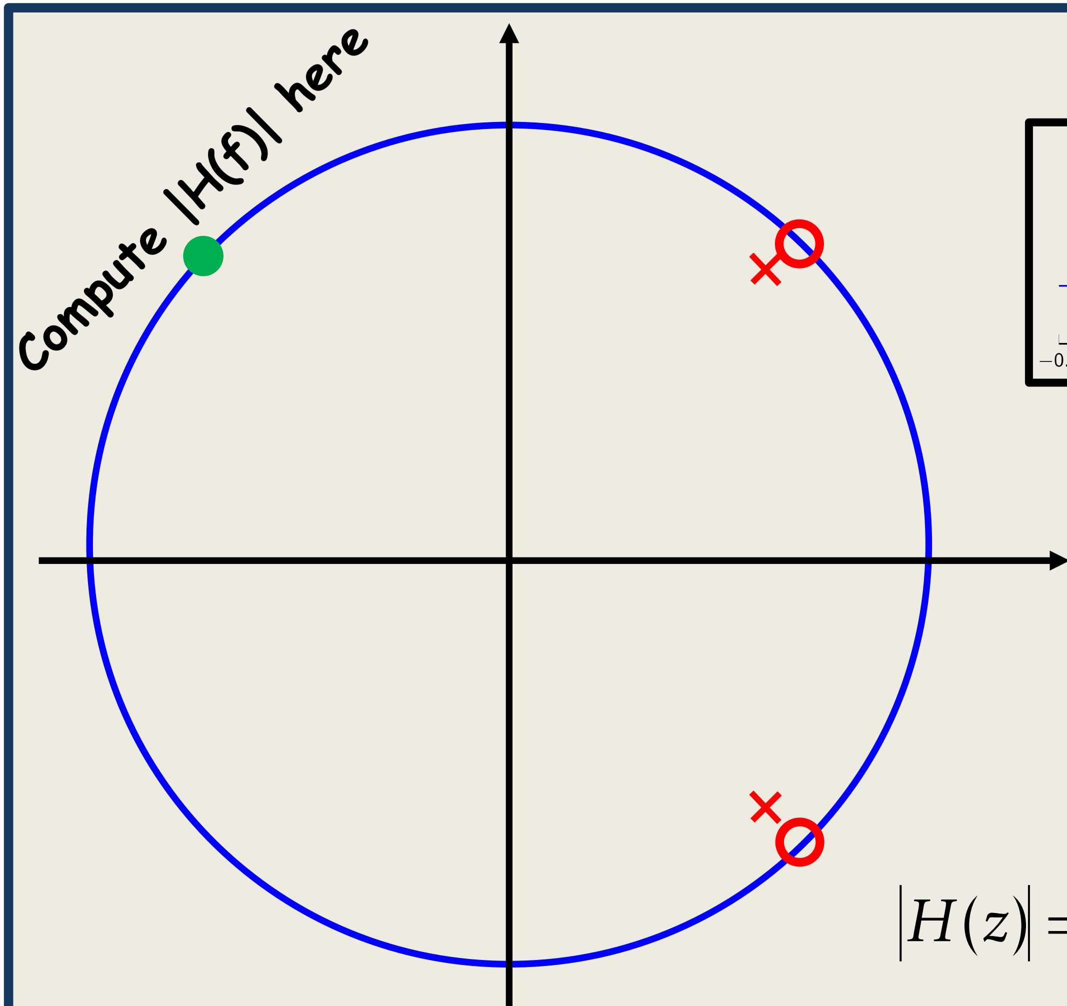
Why is it so much better?



$$|H(z)| = \frac{|z - e^{-j\omega_0}|\,|z - e^{j\omega_0}|}{|z - \alpha e^{-j\omega_0}|\,|z - \alpha e^{j\omega_0}|}$$

$\alpha \approx 1$

EITF75 Systems and Signals



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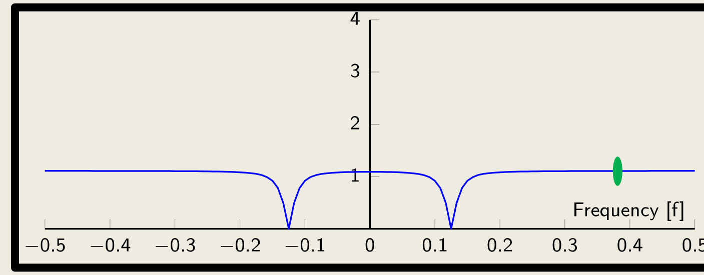

$$\left|H(z)\right| = \frac{\left|z - e^{-j\omega_0}\right| \left|z - e^{j\omega_0}\right|}{\left|z - \alpha e^{-j\omega_0}\right| \left|z - \alpha e^{j\omega_0}\right|}$$

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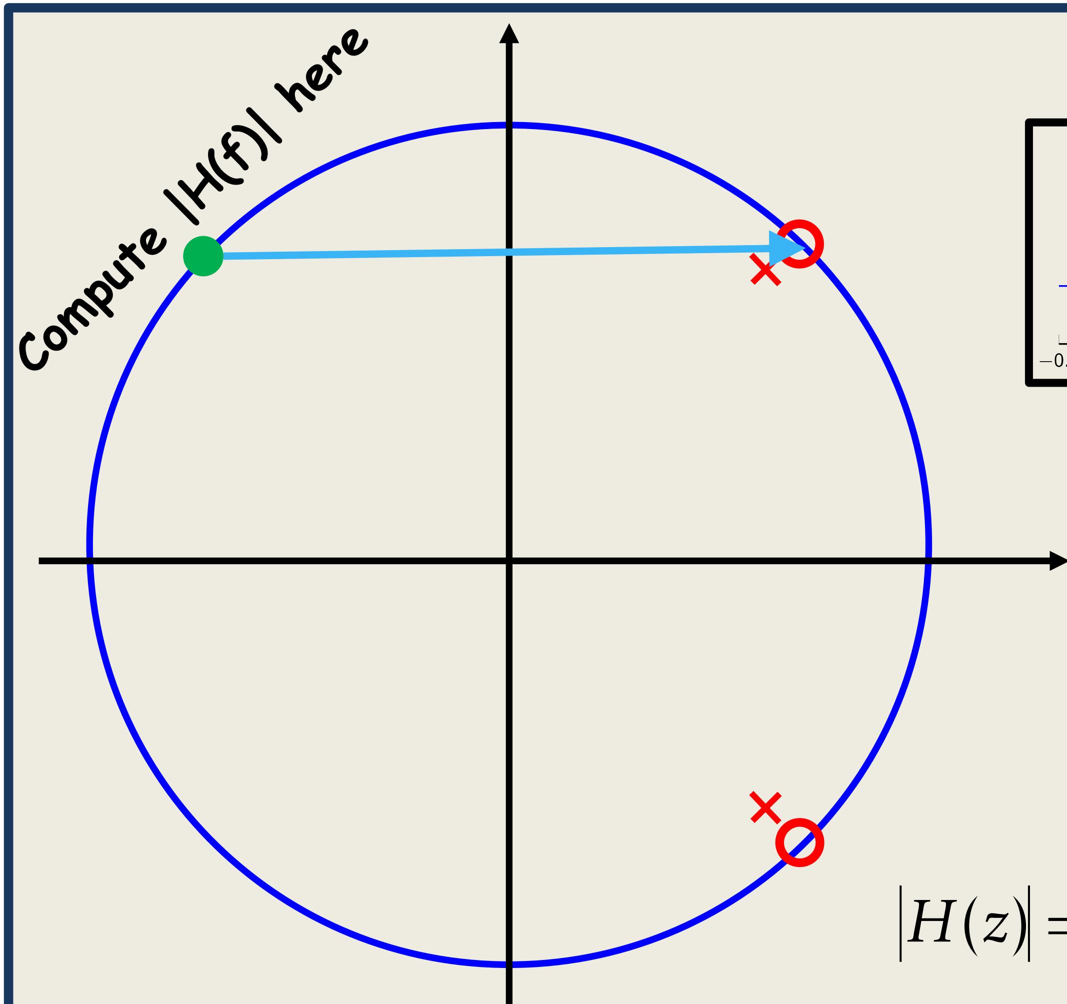
EITF75 Systems and Signals



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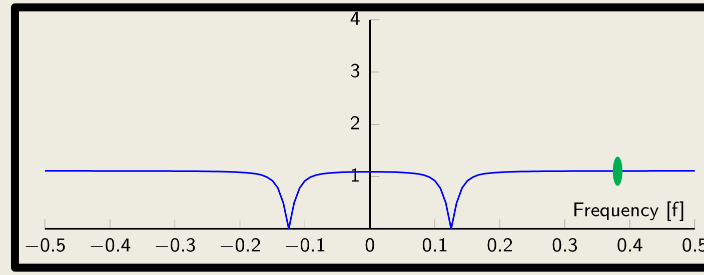
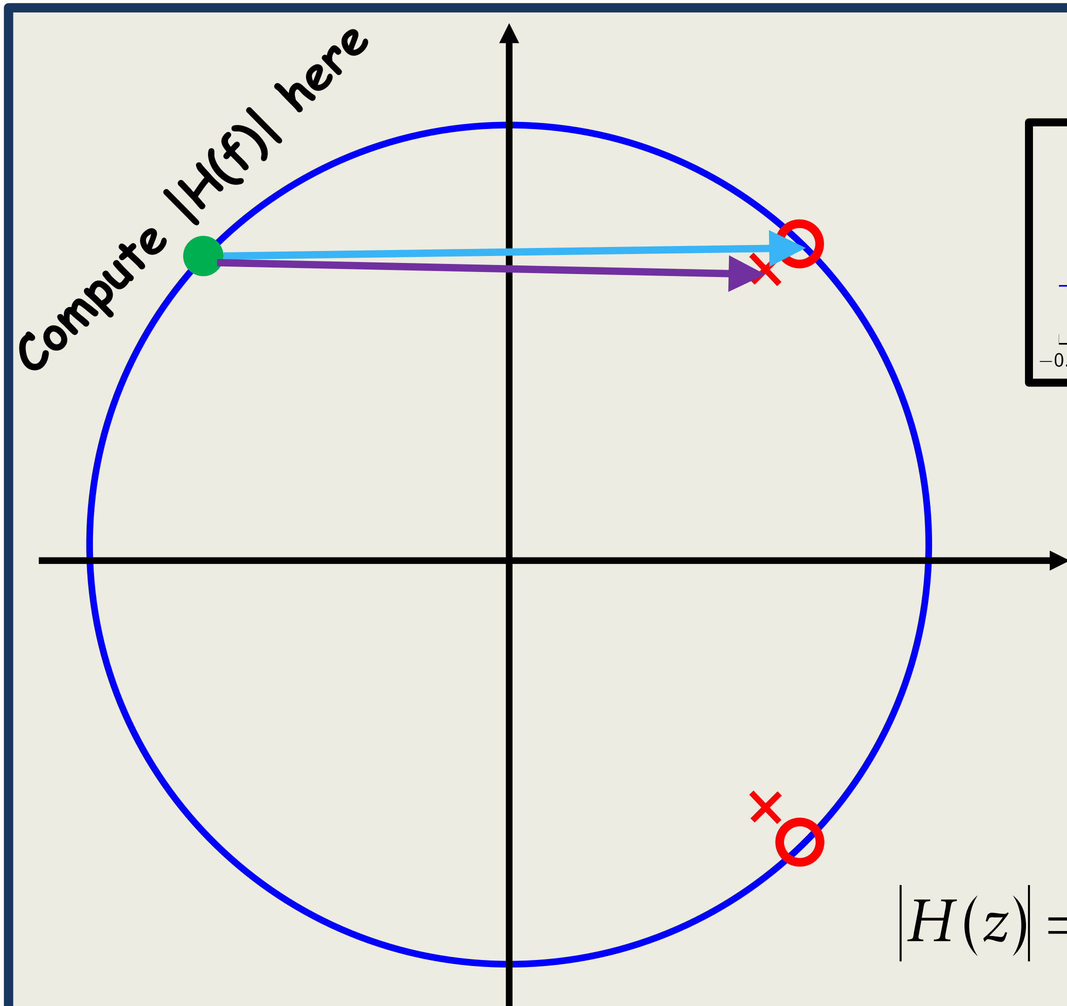


= length of the arrow

$$\left|H(z)\right| = \frac{\left|z-\mathrm{e}^{-\mathrm{j}\omega_0}\right|\left|z-\mathrm{e}^{\mathrm{j}\omega_0}\right|}{\left|z-\alpha\mathrm{e}^{-\mathrm{j}\omega_0}\right|\left|z-\alpha\mathrm{e}^{\mathrm{j}\omega_0}\right|}$$

EITF75 Systems and Signals



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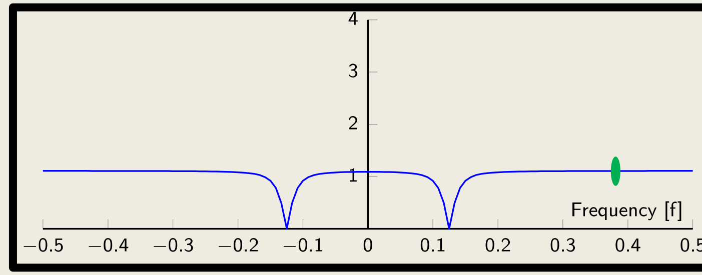

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= length of the other arrow

≈ the same

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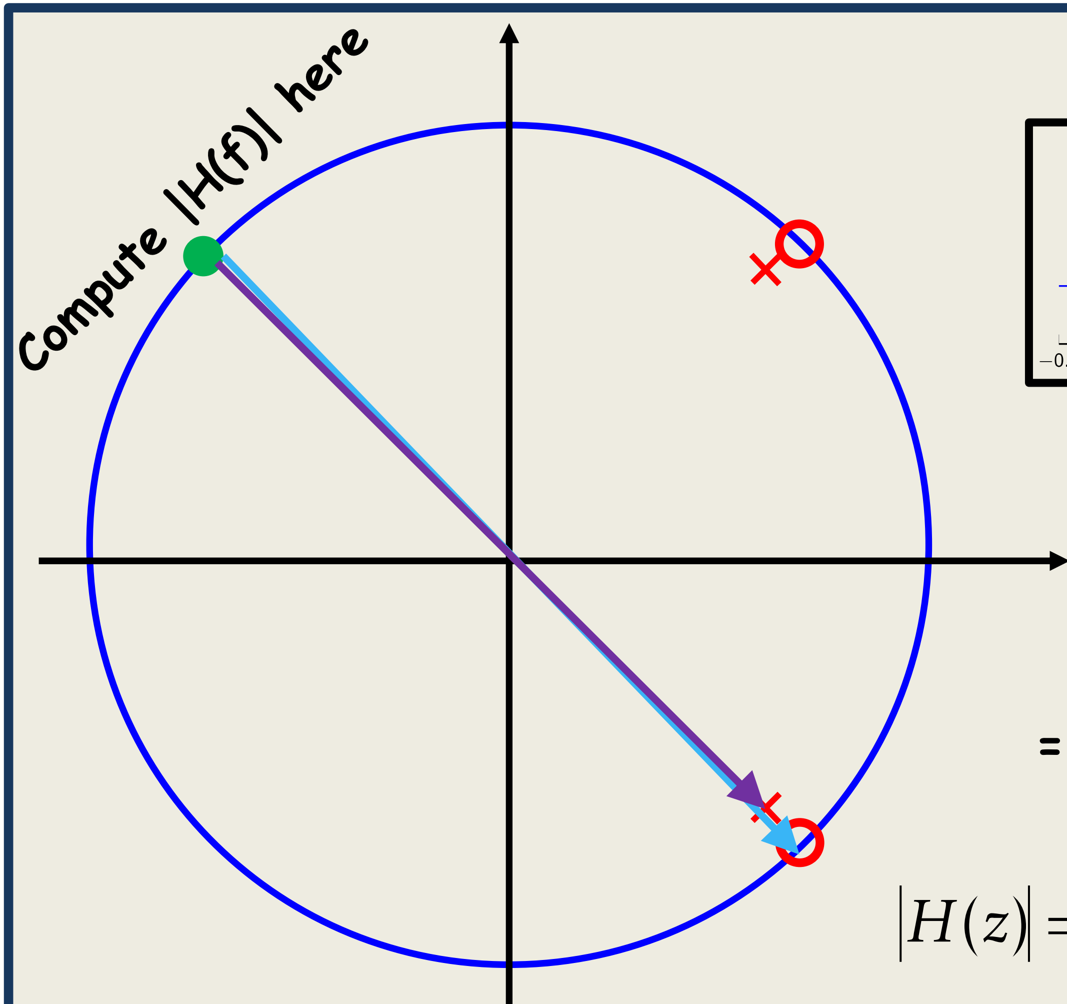

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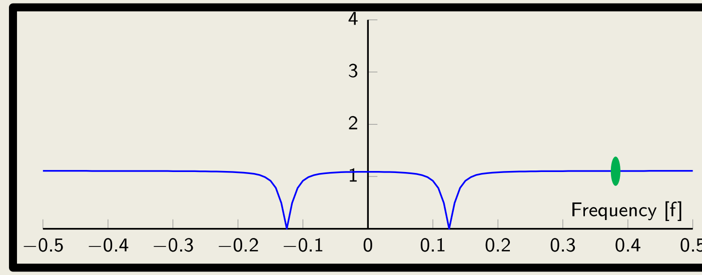

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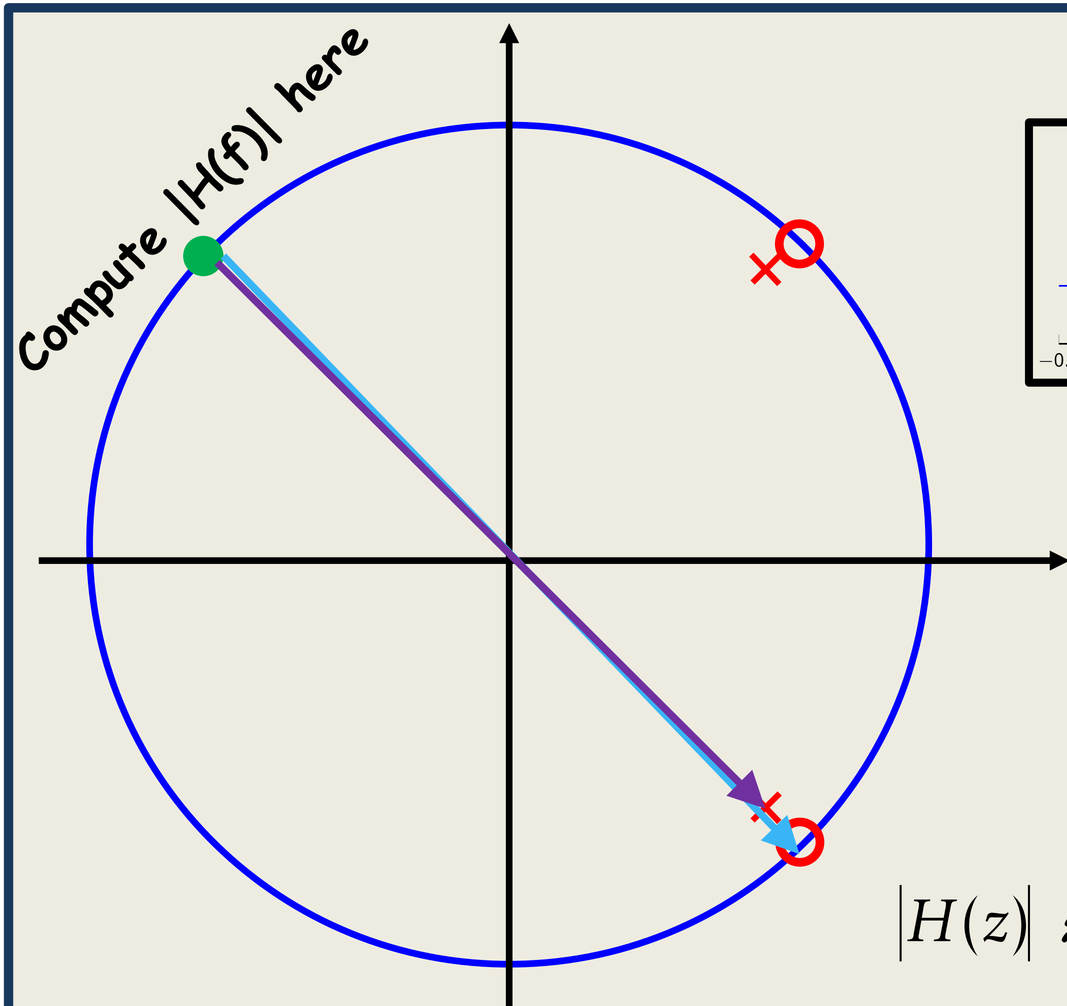
EITF75 Systems and Signals



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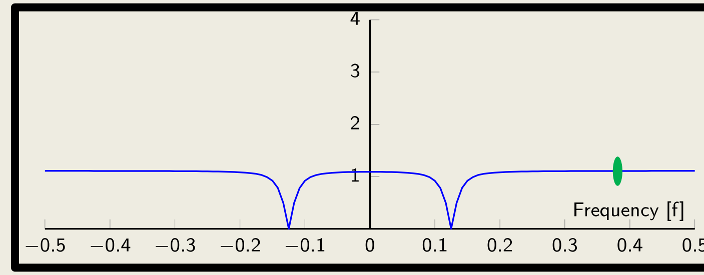
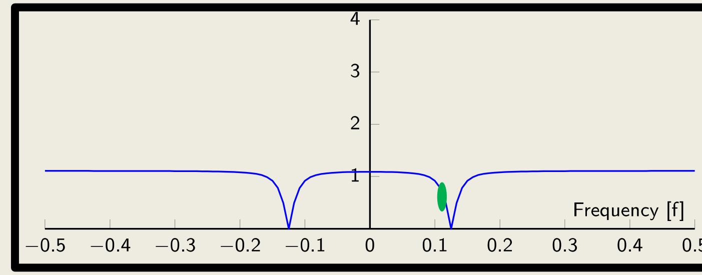
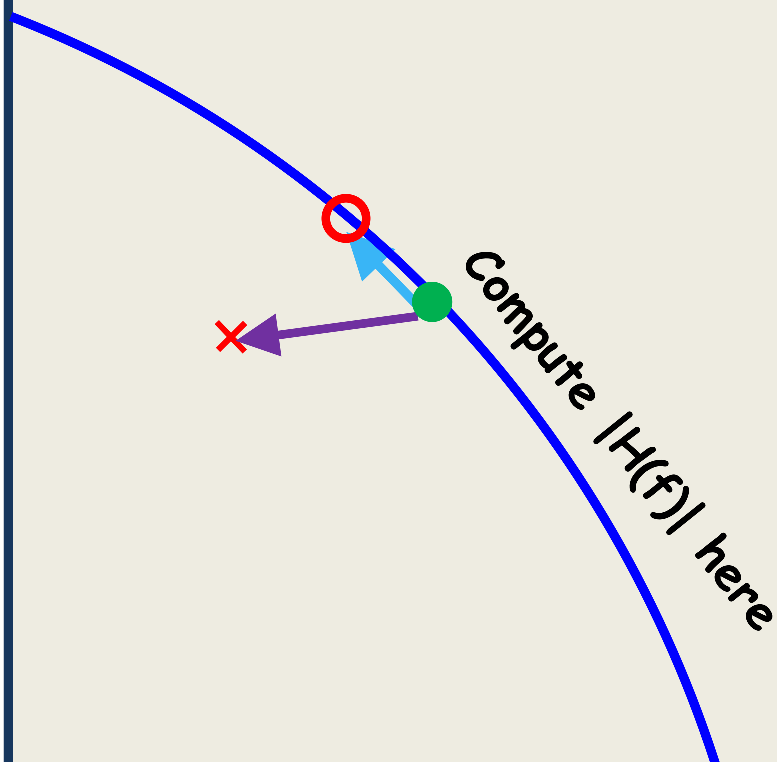


$$\left|H(z)\right| \approx 1$$

EITF75 Systems and Signals

Why is it so much better?





Arrow lengths are different

$$\left|H(z)\right| = \frac{\left|z-\mathrm{e}^{-\mathrm{j}\omega_0}\right|\left|z-\mathrm{e}^{\mathrm{j}\omega_0}\right|}{\left|z-\alpha\mathrm{e}^{-\mathrm{j}\omega_0}\right|\left|z-\alpha\mathrm{e}^{\mathrm{j}\omega_0}\right|}$$

EITF75 Systems and Signals

Summary:

A pole close to a zero "stabilizes" the magnitude response

A causal FIR filter has poles at the origin

If no poles at all, not a causal filter

Indeed possible to remove interference digitally

Comb filters

Assume a FIR filter $$H(z) = \sum_{k=0}^{K} h(k) z^{-k}$$

Comb filters

Assume a FIR filter $$H(z) = \sum_{k=0}^{K} h(k) z^{-k}$$

Construct another filter as $$H_L(z) = \sum_{k=0}^{K} h(k) z^{-kL}$$

EITF75 Systems and Signals

Comb filters

Assume a FIR filter $$H(z) = \sum_{k=0}^{K} h(k) z^{-k}$$

Construct another filter as $$H_L(z) = \sum_{k=0}^{K} h(k) z^{-kL}$$

Fourier transform $$H_L(f) = \sum_{k=0}^{K} h(k) \mathrm{e}^{-i2\pi kLf} = H(fL)$$

Discrete-time Fourier transform

From Wikipedia, the free encyclopedia

Properties of discrete-time Fourier transforms [edit]

Time Expansion	$\begin{cases} x[n/M] & n=\text{multiple of M} \\ 0 & \text{otherwise} \end{cases}$	$X_{2\pi}(M\omega)$

Fourier transform $$H_L(f) = \sum_{k=0}^{K} h(k) e^{-i2\pi kLf} = H(fL)$$

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Discrete-time Fourier transform

From Wikipedia, the free encyclopedia

Properties of discrete-time Fourier transforms [edit]

Time Expansion	$\begin{cases} x[n/M] & n\text{=multiple of M} \\ 0 & \text{otherwise} \end{cases}$	$X_{2\pi}(M\omega)$

Fourier transform $H_L(f) = \sum_{k=0}^{K} h(k)e^{-i2\pi kLf} = H(fL)$

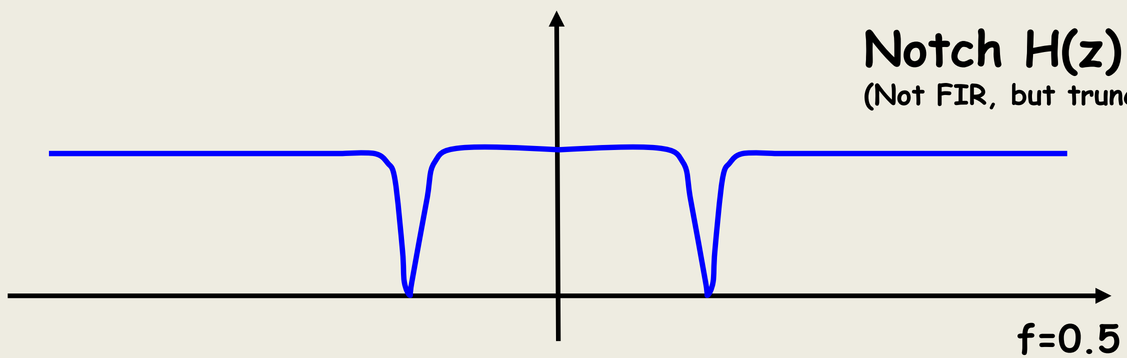
$$h_L(nL) = h(n)$$

Comb filters

Assume a FIR filter $H(z) = \sum_{k=0}^{K} h(k) z^{-k}$

Construct another filter as $H_L(z) = \sum_{k=0}^{K} h(k) z^{-kL}$

Fourier transform $H_L(f) = \sum_{k=0}^{K} h(k) e^{-i2\pi kLf} = H(fL)$



Notch H(z)

(Not FIR, but truncate to large length)

EITF75 Systems and Signals

Comb filters

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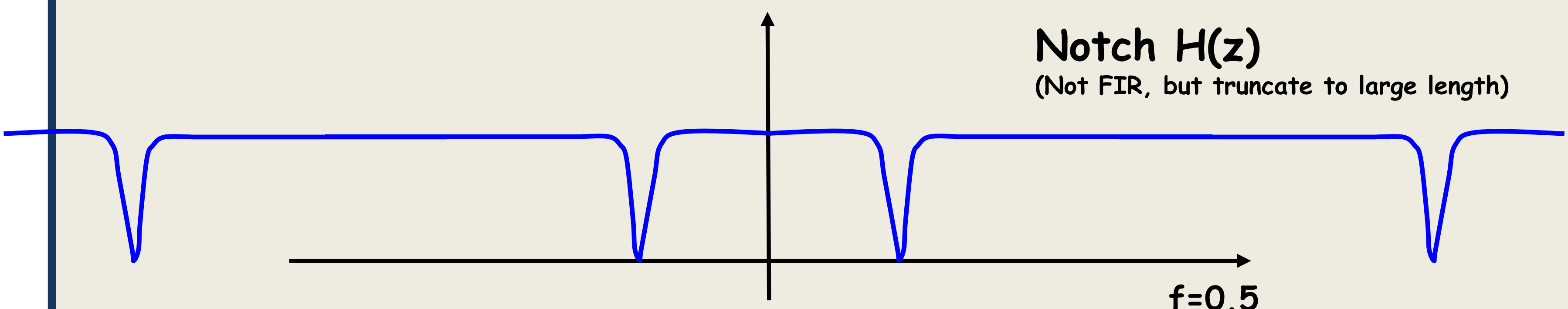

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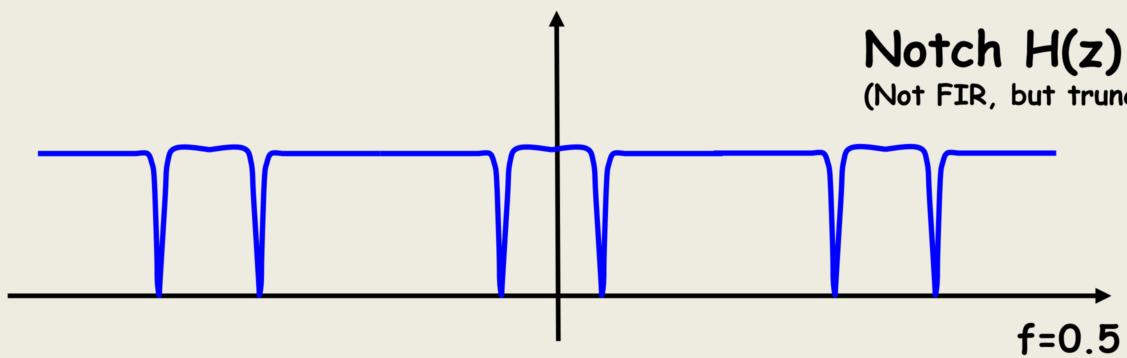

Notch H(z)
(Not FIR, but truncate to large length)

EITF75 Systems and Signals

FIR filters with linear phase

Linear phase is desirable since it delays all frequencies equally much

How to create it?

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FIR filters with linear phase

Recall

System frequency response

$$H(\omega) = A(\omega)\mathrm{e}^{i\Theta(\omega)}$$

Linear phase iff

$$\Theta(\omega) = \kappa\omega + 2\pi\ell$$

Constant

Irrelevant

$A(\omega)$ **Non-negative**

EITF75 Systems and Signals

FIR filters with linear phase

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$A(\omega)$ Non-negative

What we know:
"If I send sin(wn), I get sin(wn) out, but with a delay and multiplied by a number. Said number not negative."

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FIR filters with linear phase

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Linear phase iff

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$A(\omega)$ Non-negative

$$A(\omega) \qquad \frac{\Theta(\omega)}{\omega}$$

What we know:

"If I send sin(wn), I get sin(wn) out, but with a delay and multiplied by a number. Said number not negative."

EITF75 Systems and Signals

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$A(\omega)$ Non-negative

Let us accept this:
"If I send sin(wn), I get sin(wn) out, but with a delay and multiplied by a number. ~~Said number not negative.~~"

EITF75 Systems and Signals

FIR filters with linear phase

Recall

System frequency response

$$H(\omega) = A(\omega)\mathrm{e}^{i\Theta(\omega)}$$

Linear phase iff

$$\Theta(\omega) = \kappa\omega + 2\pi\ell$$

$A(\omega)$ Real-valued

Let us accept this:
"If I send sin(wn), I get sin(wn) out, but with a delay and multiplied by a real-valued number."

EITF75 Systems and Signals

FIR filters with linear phase

Recall

System frequency response

$$H(\omega) = A(\omega)\mathrm{e}^{i\Theta(\omega)}$$

Linear phase iff

$$\Theta(\omega) = \kappa\omega + \quad \pi\ell$$

$A(\omega)$ Real-valued

A phase shift with will flip sign of $A(\omega)$

So still claimed as "linear phase"

Let us accept this:
"If I send sin(wn), I get sin(wn) out, but with a delay and multiplied by a real-valued number."

EITF75 Systems and Signals

FIR filters with linear phase

Recall

System frequency response

$$H(\omega) = A(\omega)e^{i\Theta(\omega)}$$

Linear phase iff

$$\Theta(\omega) = \kappa\omega + \pi\ell$$

Summary: Linear phase is defined as $\Theta(\omega) = \kappa\omega + \pi\ell$

Whenever there is a phase jump with π, this should be seen as a magnitude response that is negative

Let us accept this:
"If I send sin(wn), I get sin(wn) out, but with a delay and multiplied by a real-valued number."

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. **TYPE 1**

$$h(n) = h(-n)$$

Symmetry around n=0. Not causal

Ex: $h(n) = [1 \ 2 \ \underline{3} \ 2 \ 1]$

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. **TYPE 1**

$$\boxed{h(n) = h(-n)}$$

Symmetry around n=0. Not causal

$$H(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-i2\pi n f}$$

Need to derive its phase response

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. **TYPE 1**

$$h(n) = h(-n)$$

Symmetry around n=0. Not causal

$$H(f) = \sum_{n=-\infty}^{\infty} h(n)\mathrm{e}^{-i2\pi n f}$$

$$= h(0) + \sum_{n=1}^{\infty} h(n)\left(\mathrm{e}^{i2\pi n f} + \mathrm{e}^{-i2\pi n f}\right)$$

Due to symmetry

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. **TYPE 1**

$$h(n) = h(-n)$$ **Symmetry around n=0. Not causal**

$$H(f) = \sum_{n=-\infty}^{\infty} h(n)e^{-i2\pi n f}$$

$$= h(0) + \sum_{n=1}^{\infty} h(n)\left(e^{i2\pi n f} + e^{-i2\pi n f}\right)$$

$$= h(0) + 2\sum_{n=1}^{\infty} h(n)\cos(2\pi f n)$$

Euler

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FIR filters with linear phase: 3 types. **TYPE 1**

$$\boxed{h(n) = h(-n)}$$ **Symmetry around n=0. Not causal**

$$H(f) = h(0) + 2\sum_{n=1}^{\infty} h(n)\cos(2\pi f n)$$

Phase response

$$\Theta(f) = \tan^{-1}\frac{\mathrm{Im}(H(f))}{\mathrm{Re}(H(f))}$$

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$$H(f) = h(0) + 2\sum_{n=1}^{\infty} h(n)\cos(2\pi f n)$$

real-valued

Phase response

$$\Theta(f) = \tan^{-1}\frac{\mathrm{Im}(H(f))}{\mathrm{Re}(H(f))} = \tan^{-1}\frac{0}{\mathrm{Re}(H(f))}$$

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real-valued

Phase response

$$\Theta(f) = \tan^{-1}\frac{\mathrm{Im}(H(f))}{\mathrm{Re}(H(f))} = \tan^{-1}\frac{0}{\mathrm{Re}(H(f))} = 0$$

$$\tan^{-1} 0 = 0$$

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FIR filters with linear phase: 3 types. **TYPE 1**

$$h(n) = h(-n)$$ **Symmetry around n=0. Not causal**

$$H(f) = h(0) + 2\sum_{n=1}^{\infty} h(n)\cos(2\pi f n)$$ **real-valued**

Phase response

$$\Theta(f) = \tan^{-1}\frac{\mathrm{Im}(H(f))}{\mathrm{Re}(H(f))} = \tan^{-1}\frac{0}{\mathrm{Re}(H(f))} = 0$$

But a problem: If $\mathrm{Re}(H(f))<0$, **then** π **must be added to** $\tan^{-1}$

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 1

$$\boxed{h(n) = h(-n)}$$

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$$H(f) = h(0) + 2\sum_{n=1}^{\infty} h(n)\cos(2\pi f n)$$

real-valued

Phase response

$$\Theta(f) = \pi \cdot \underbrace{\left[\frac{1}{2} - \frac{1}{2}\mathrm{sign}(H(f))\right]}_{\substack{0 \text{ if } H(f) > 0 \\ 1 \text{ if } H(f) < 0}}$$

But a problem: If $\mathrm{Re}(H(f)) < 0$, then π must be added to $\tan^{-1}$

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 1

$$h(n) = h(-n)$$ Symmetry around n=0. Not causal

$$H(f) = h(0) + 2\sum_{n=1}^{\infty} h(n)\cos(2\pi f n)$$ real-valued

Phase response

$$\Theta(f) = \pi \cdot \left[\frac{1}{2} - \frac{1}{2}\text{sign}(H(f))\right]$$

In agreement.
So linear phase

Summary: Linear phase is defined as $\Theta(\omega) = \kappa\omega + \pi\ell$

Whenever there is a phase jump with π, this should be seen as a magnitude response that is negative

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 2

$$h(n) = h(N-n)$$ Symmetry around n=(N-1)/2.

Proof for odd N. Do even at home.

Ex: h(n) = [<u>1</u> 2 3 2 1]

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. **TYPE 2**

$$h(n) = h(N-n)$$ **Symmetry around** n=(N-1)/2.

Proof for odd N. Do even at home.

Ex: h(n) = [<u>1</u> 2 3 2 1]

$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = [1 2 <u>3</u> 2 1]

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FIR filters with linear phase: 3 types. TYPE 2

$$h(n) = h(N-n)$$ Symmetry around n=(N-1)/2.

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Ex: h(n) = [1 2 3 2 1]

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Symmetry around n=0. Type 1

EITF75 Systems and Signals

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Ex: g(n) = [1 2 $\underline{3}$ 2 1]

Symmetry around n=0. Type 1

$$H(f) = \mathrm{e}^{-i\pi f(N-1)}G(f)$$

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FIR filters with linear phase: 3 types. TYPE 2

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Symmetry around n=0. Type 1

$$H(f) = \mathrm{e}^{-i\pi f(N-1)} G(f)$$

$$\Theta_H(f) = -\pi f(N-1) + \Theta_G(f)$$

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FIR filters with linear phase: 3 types. **TYPE 2**

$$\boxed{h(n) = h(N-n) \quad \text{Symmetry around } n=(N-1)/2.}$$

Proof for odd N. Do even at home.

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$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = [1 2 $\underline{3}$ 2 1]

Symmetry around n=0. Type 1

$$H(f) = \mathrm{e}^{-i\pi f(N-1)}G(f)$$

$$\Theta_H(f) = -\pi f(N-1) + \Theta_G(f)$$

$$= -\pi f(N-1) + \pi \cdot \left[\frac{1}{2} - \frac{1}{2}\operatorname{sign}(G(f))\right]$$

Linear phase

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FIR filters with linear phase: 3 types. TYPE 3

$$h(n) = -h(N-n)$$

Anti-symmetry around n=(N-1)/2.

Proof for odd N. Do even at home.

Ex: h(n) = [1 2 0 -2 -1]

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$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = $[1\ 2\ \underline{0}\ -2\ -1]$

$$G(f) = \sum_{n=-\infty}^{\infty} g(n)\mathrm{e}^{-i2\pi nf} = \sum_{n=1}^{\infty} g(n)\left(\mathrm{e}^{-i2\pi nf} - \mathrm{e}^{\ i2\pi nf}\right)$$

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$$= -2i\sum_{n=1}^{\infty} g(n)\sin(2\pi f n)$$

EITF75 Systems and Signals

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Imaginary-valued

EITF75 Systems and Signals

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Imaginary-valued

$$\Theta_G(f) = \tan^{-1}\frac{\mathrm{Im}(G(f))}{\mathrm{Re}(G(f))}$$

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FIR filters with linear phase: 3 types. TYPE 3

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Imaginary-valued

$$\Theta_G(f) = \tan^{-1}\frac{\mathrm{Im}(G(f))}{\mathrm{Re}(G(f))} = \tan^{-1}\pm\infty$$

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. **TYPE 3**

$$h(n) = -h(N-n)$$ **Anti-symmetry around** n=(N-1)/2.

Proof for odd N. Do even at home.

Ex: h(n) = [<u>1</u> 2 0 -2 -1]

$$h(n) = g(n - (N-1)/2)$$

Ex: g(n) = [1 2 <u>0</u> -2 -1]

$$G(f) = -2i \sum_{n=1}^{\infty} g(n) \sin(2\pi f n)$$

Imaginary-valued

$$\Theta_G(f) = \tan^{-1} \frac{\mathrm{Im}(G(f))}{\mathrm{Re}(G(f))} = \tan^{-1} \pm\infty = \pm\frac{\pi}{2}$$

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 3

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Ex: g(n) = [1 2 <u>0</u> -2 -1]

$$G(f) = -2i\sum_{n=1}^{\infty} g(n)\sin(2\pi f n)$$

Imaginary-valued

$$\Theta_G(f) = \tan^{-1}\frac{\mathrm{Im}(G(f))}{\mathrm{Re}(G(f))} = \tan^{-1}\pm\infty = \pm\frac{\pi}{2} = -\frac{\pi}{2} + \pi\ell$$

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FIR filters with linear phase: 3 types. TYPE 3

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Ex: g(n) = [1 2 $\underline{0}$ -2 -1]

$$\Theta_H(f) = -\pi f(N-1) + \Theta_G(f)$$

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EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 3

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$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = [1 2 $\underline{0}$ -2 -1]

$$\Theta_H(f) = -\pi f(N-1) + \Theta_G(f) = -\pi f(N-1) - \frac{\pi}{2} + \pi\ell$$

$$\Theta_G(f) = \tan^{-1}\frac{\mathrm{Im}(G(f))}{\mathrm{Re}(G(f))} = \tan^{-1}\pm\infty = \pm\frac{\pi}{2} = -\frac{\pi}{2} + \pi\ell$$

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$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = [1 2 $\underline{0}$ -2 -1]

$$\Theta_H(f) = -\pi f(N-1) \boxed{-\frac{\pi}{2}} + \pi\ell$$

Not strictly linear phase due to

Sometimes called "affine phase"

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 3

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$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = [1 2 $\underline{0}$ -2 -1]

$$\Theta_H(f) = -\pi f(N-1) - \frac{\pi}{2} + \pi\ell$$

Interpretation. Assume x(n) = sin(wn)

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 3

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Proof for odd N. Do even at home.

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$$\Theta_H(f) = \boxed{-\pi f(N-1)} - \frac{\pi}{2} + \pi\ell$$

Interpretation. Assume x(n) = sin(wn)

Output is delayed equally much for all frequencies f

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 3

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Ex: g(n) = [1 2 0 -2 -1]

$$\Theta_H(f) = -\pi f(N-1) - \frac{\pi}{2} + \boxed{\pi\ell}$$

Interpretation. Assume x(n) = sin(wn)

For some frequencies, the signal is multiplied with "-1"

EITF75 Systems and Signals

FIR filters with linear phase: 3 types. TYPE 3

$$h(n) = -h(N-n)$$

Anti-symmetry around n=(N-1)/2.

Proof for odd N. Do even at home.

Ex: h(n) = [1 2 0 -2 -1]

$$h(n) = g(n-(N-1)/2)$$

Ex: g(n) = [1 2 0 -2 -1]

$$\Theta_H(f) = -\pi f(N-1) \boxed{-\frac{\pi}{2}} + \pi\ell$$

Interpretation. Assume x(n) = sin(wn)

A sin(wn) comes out as a cos(wn), no matter the frequency

EITF75 Systems and Signals

FIR filters with linear phase: 4 types. TYPE 4

$$h(n) = -h(\quad - n)$$

Anti-symmetry around n=0.

Ex: h(n) = [1 2 $\underline{0}$ -2 -1]

$$\Theta_H(f) = -\frac{\pi}{2} + \pi\ell$$

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Example TYPE 1

$$h(n) = \{\ 1 \quad 2 \quad \underline{3} \quad 2 \quad 1\ \}$$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

EITF75 Systems and Signals

Example TYPE 1

$$h(n) = \{\ 1 \quad 2 \quad \underline{3} \quad 2 \quad 1\ \}$$

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$H(\omega) = 1 + 2\mathrm{e}^{-\mathrm{j}\omega} + 3\mathrm{e}^{-\mathrm{j}2\omega} + 2\mathrm{e}^{-\mathrm{j}3\omega} + \mathrm{e}^{-\mathrm{j}4\omega}$$

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Example TYPE 1

$$h(n) = \{\ 1 \quad 2 \quad \underline{3} \quad 2 \quad 1\ \} \qquad H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$H(\omega) = 1 + 2\mathrm{e}^{-\mathrm{j}\omega} + 3\mathrm{e}^{-\mathrm{j}2\omega} + 2\mathrm{e}^{-\mathrm{j}3\omega} + \mathrm{e}^{-\mathrm{j}4\omega}$$

$$= \left(\mathrm{e}^{\mathrm{j}2\omega} + 2\mathrm{e}^{\mathrm{j}\omega} + 3 + 2\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega}\right) \cdot \mathrm{e}^{-\mathrm{j}2\omega}$$

Preparation for Euler

EITF75 Systems and Signals

Example TYPE 1

$$h(n) = \{ \; 1 \quad 2 \quad \underline{3} \quad 2 \quad 1 \; \} \qquad H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$H(\omega) = 1 + 2\mathrm{e}^{-\mathrm{j}\omega} + 3\mathrm{e}^{-\mathrm{j}2\omega} + 2\mathrm{e}^{-\mathrm{j}3\omega} + \mathrm{e}^{-\mathrm{j}4\omega}$$

$$= \left(\mathrm{e}^{\mathrm{j}2\omega} + 2\mathrm{e}^{\mathrm{j}\omega} + 3 + 2\mathrm{e}^{-\mathrm{j}\omega} + \mathrm{e}^{-\mathrm{j}2\omega}\right) \cdot \mathrm{e}^{-\mathrm{j}2\omega}$$

$$= (3 + 4\cos\omega + 2\cos 2\omega) \cdot \mathrm{e}^{-\mathrm{j}2\omega}$$

Application of Euler

EITF75 Systems and Signals

Example TYPE 1

$$h(n) = \{\ 1 \quad 2 \quad \underline{3} \quad 2 \quad 1\ \} \qquad H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$H(\omega) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega} + e^{-j4\omega}$$

$$= \left(e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}\right) \cdot e^{-j2\omega}$$

$$= (3 + 4\cos\omega + 2\cos 2\omega) \cdot e^{-j2\omega}$$

$$= |(3 + 4\cos\omega + 2\cos 2\omega)| \cdot e^{-j2\omega + j\pi \cdot k}$$

EITF75 Systems and Signals

Example TYPE 1

$$h(n) = \{ 1 \quad 2 \quad \underline{3} \quad 2 \quad 1 \} \qquad H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$H(\omega) = 1 + 2e^{-j\omega} + 3e^{-j2\omega} + 2e^{-j3\omega} + e^{-j4\omega}$$

$$= \left(e^{j2\omega} + 2e^{j\omega} + 3 + 2e^{-j\omega} + e^{-j2\omega}\right) \cdot e^{-j2\omega}$$

If negative

$$= (3 + 4\cos\omega + 2\cos 2\omega) \cdot e^{-j2\omega}$$

Activate this (k=1)

$$= |(3 + 4\cos\omega + 2\cos 2\omega)| \cdot e^{-j2\omega + j\pi \cdot k}$$

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$= z^{-4} \cdot \left(z^{4} + 2z^{3} + 3z^{2} + 2z + 1 \right)$$

4 poles at the origin

4 zeros

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$= z^{-4} \cdot \left(z^{4} + 2z^{3} + 3z^{2} + 2z + 1\right)$$

$$= z^{-4} \cdot H(z^{-1})$$

Important property

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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$$H(z) = 1 + 2z^{-1} + 3z^{-2} + 2z^{-3} + z^{-4}$$

$$= z^{-4} \cdot \left(z^{4} + 2z^{3} + 3z^{2} + 2z + 1\right)$$

$$= z^{-4} \cdot H(z^{-1})$$

Conclusions

Assume z_0 to be a zero

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z_0) = 1 + 2z_0^{-1} + 3z_0^{-2} + 2z_0^{-3} + z_0^{-4} = 0$$

$$= z^{-4} \cdot \left(z^4 + 2z^3 + 3z^2 + 2z + 1 \right)$$

$$= z^{-4} \cdot H(z^{-1})$$

Conclusions

Assume z_0 to be a zero

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z_0) = 1 + 2z_0^{-1} + 3z_0^{-2} + 2z_0^{-3} + z_0^{-4}$$

$$= z_0^{-4} \cdot \left(z_0^4 + 2z_0^3 + 3z_0^2 + 2z_0 + 1\right)$$

$$= z_0^{-4} \cdot H(z_0^{-1})$$

$$= 0$$

Conclusions

Assume z_0 to be a zero

Last expression must also be zero

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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$$H(z_0) = 1 + 2z_0^{-1} + 3z_0^{-2} + 2z_0^{-3} + z_0^{-4}$$

$$= z_0^{-4} \cdot \left(z_0^{4} + 2z_0^{3} + 3z_0^{2} + 2z_0 + 1 \right)$$

$$= z_0^{-4} H(z_0^{-1})$$

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Conclusions

Assume z_0 to be a zero

Last expression must also be zero
Can it be zero due to the first term

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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$$= 0$$

Conclusions

Assume z_0 to be a zero

Last expression must also be zero
Can it be zero due to the first term
NO

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z_0) = 1 + 2z_0^{-1} + 3z_0^{-2} + 2z_0^{-3} + z_0^{-4}$$

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$$= z_0^{-4} H(z_0^{-1})$$

$$= 0$$

Conclusions

Assume z_0 to be a zero

Last expression must also be zero
Can it be zero due to the first term
NO
Thus, $H(z_0^{-1}) = 0$

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

$$H(z_0) = 1 + 2z_0^{-1} + 3z_0^{-2} + 2z_0^{-3} + z_0^{-4}$$

$$= z_0^{-4} \cdot \left(z_0^{4} + 2z_0^{3} + 3z_0^{2} + 2z_0 + 1 \right)$$

$$= z_0^{-4} \cdot H(z_0^{-1})$$

$$= 0$$

Conclusions

Assume z_0 to be a zero

Last expression must also be zero
Can it be zero due to the first term
NO
Thus, $H(z_0^{-1}) = 0$ $\quad z_0^{-1}$ also a zero

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Let us continue with TYPE 1 (others are similar)

Linear phase <-> If z_0 is a zero, so is z_0^{-1}

Conclusions

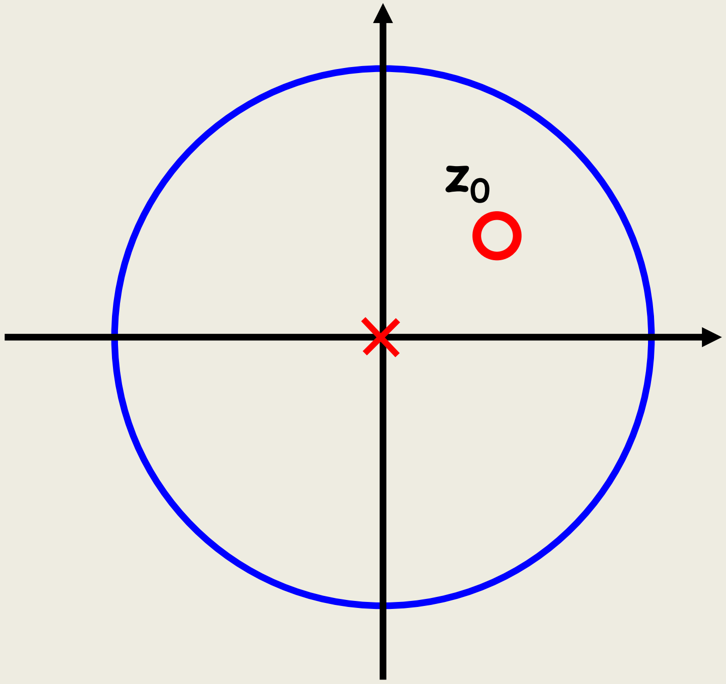
Assume z_0 to be a zero

Last expression must also be zero
Can it be zero due to the first term
NO
Thus, $H(z_0^{-1}) = 0$ $\quad z_0^{-1}$ also a zero

EITF75 Systems and Signals

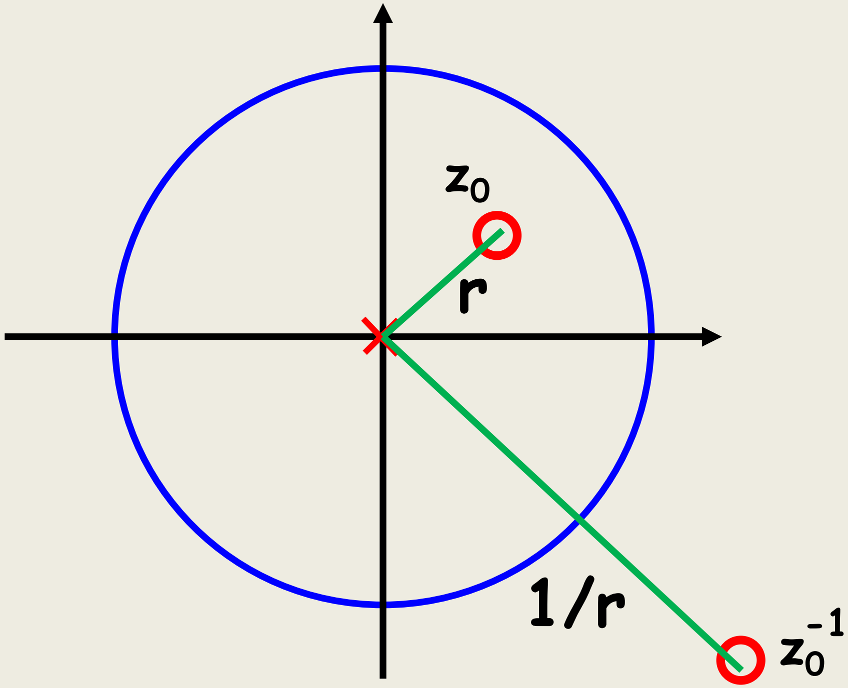
Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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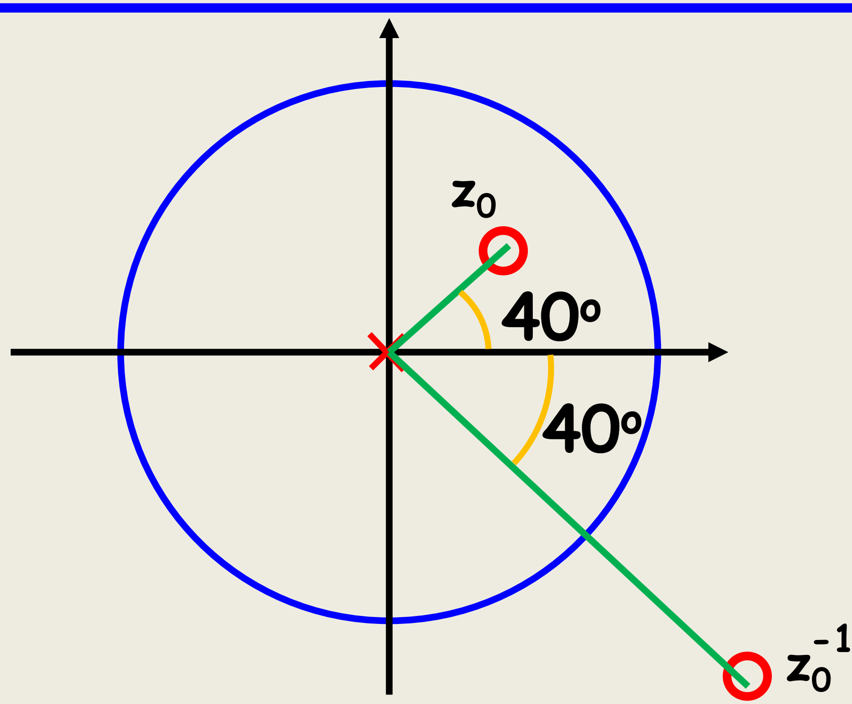

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

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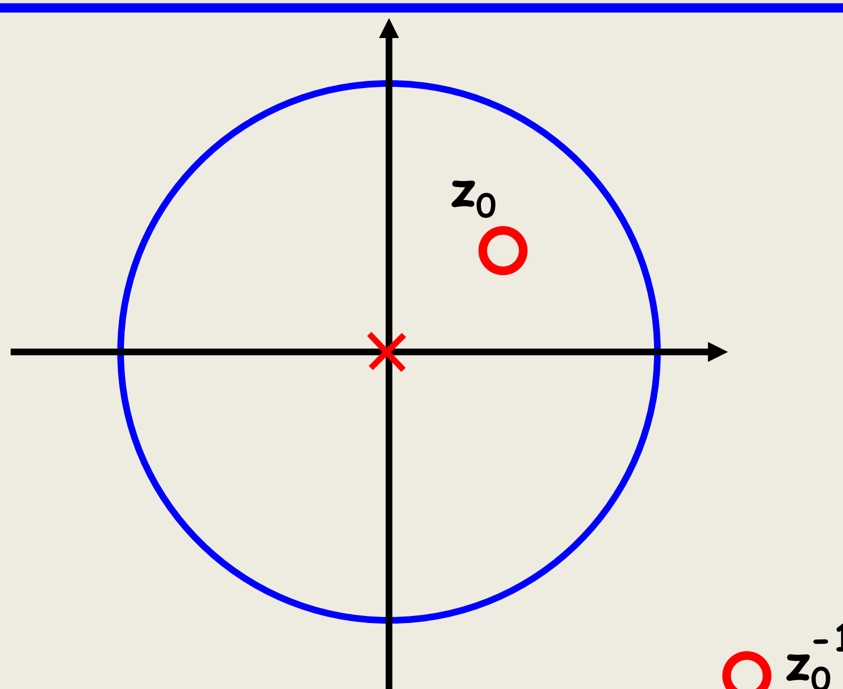
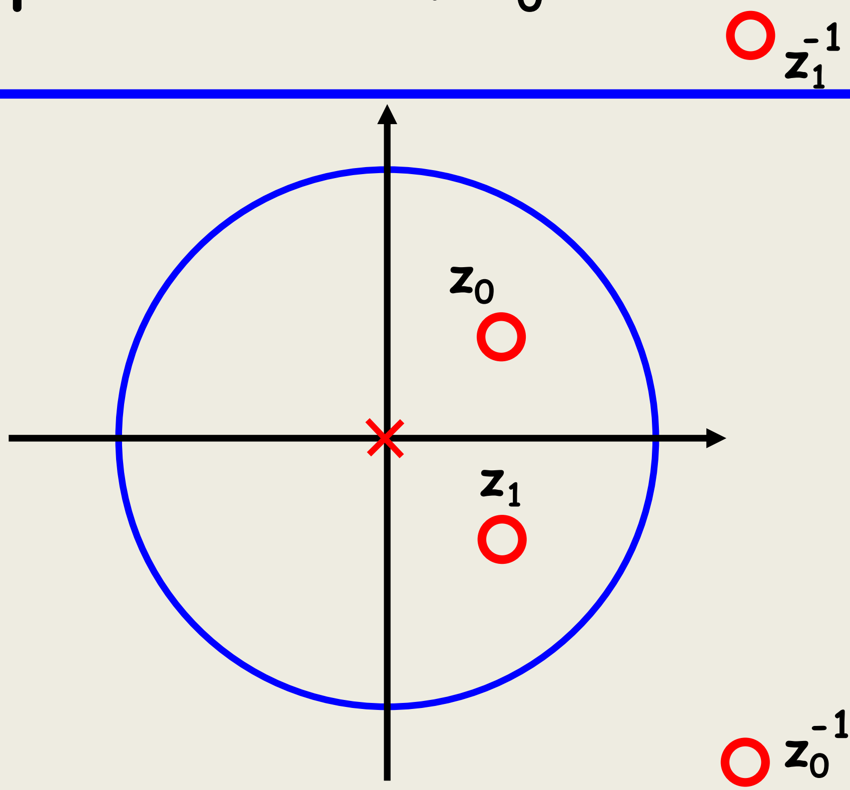


This is not a real-valued h(n)

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}

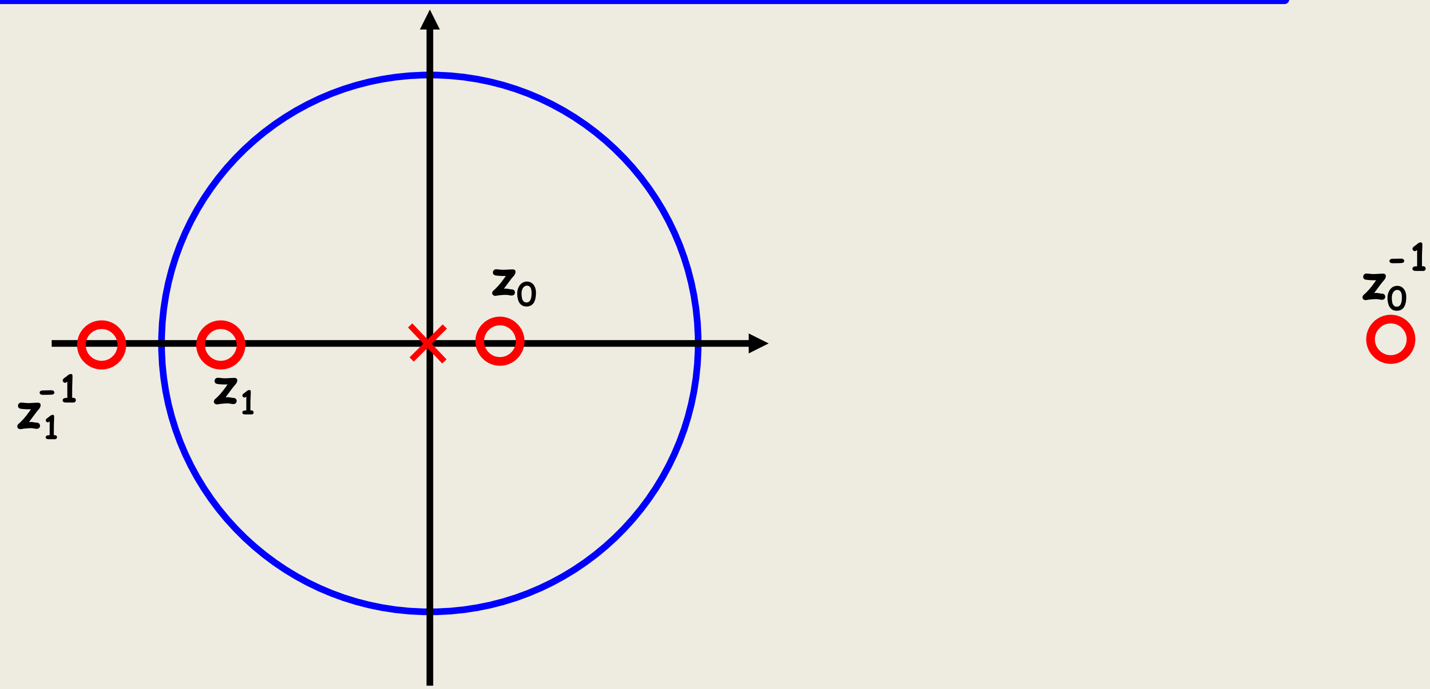


This is

EITF75 Systems and Signals

Pole-Zero diagram for linear phase FIR filters

Linear phase <-> If z_0 is a zero, so is z_0^{-1}



EITF75 Systems and Signals

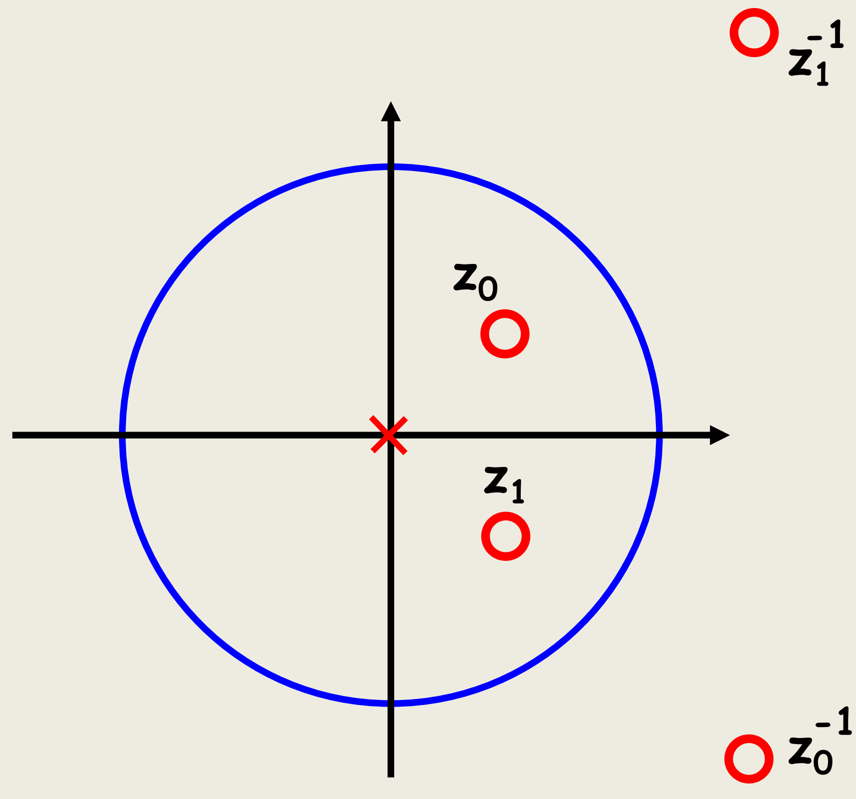
Dimensionality considerations

When one zero is selected (assume not real), three more zeros are automatically placed.

Assume a 5 tap TYPE 1 filter

$h(n) = \{ a \; b \; c \; b \; a \}$

We then have 4 zeros



EITF75 Systems and Signals

Dimensionality considerations

When one zero is selected (assume not real), three more zeros are automatically placed.

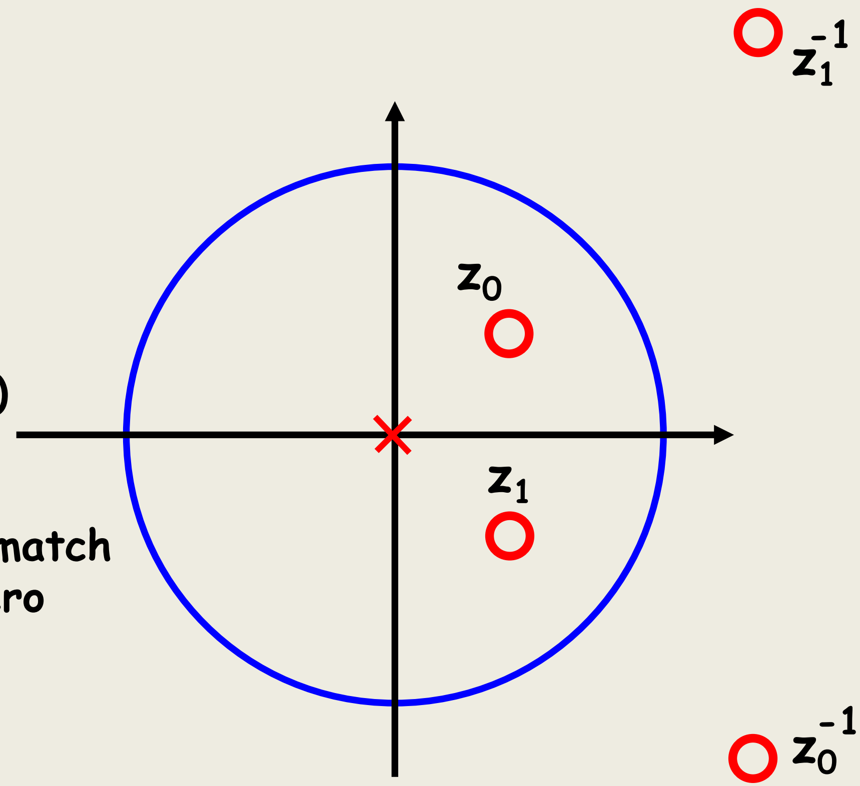
Assume a 5 tap TYPE 1 filter

h(n) = { a b c b a }

We then have 4 zeros

We have 3 real numbers to pick (a,b,c) but only 1 zero to place

Seems as there is a dimensionality mismatch
More DoFs with a,b,c than with one zero



EITF75 Systems and Signals

Dimensionality considerations

When one zero is selected (assume not real), three more zeros are automatically placed.

Assume a 5 tap TYPE 1 filter

h(n) = { a b c b a }

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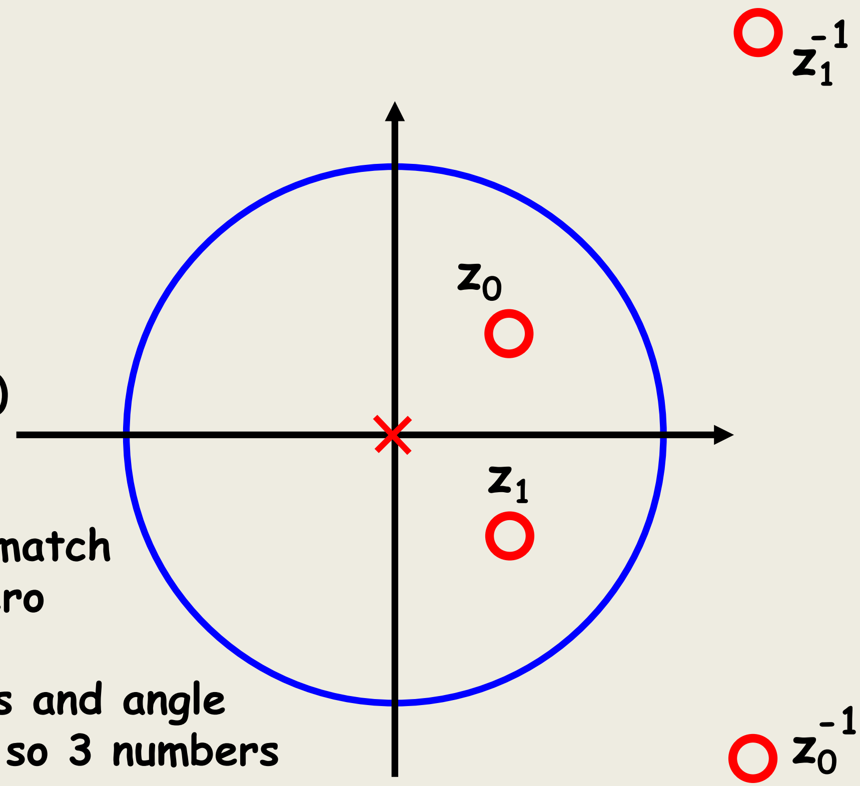
We have 3 real numbers to pick (a,b,c)
but only 1 zero to place

Seems as there is a dimensionality mismatch
More DoFs with a,b,c than with one zero

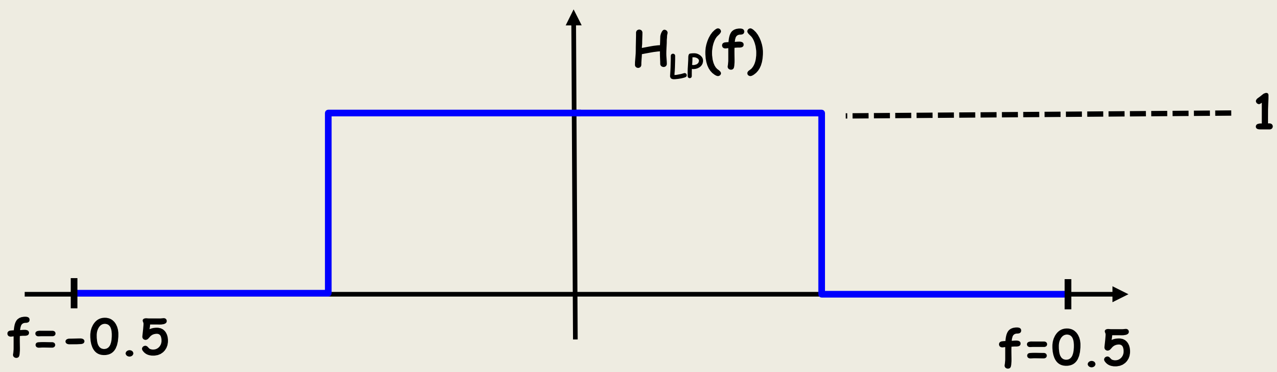
However, one zero is 2 numbers, radius and angle
Then we can scale H(z) by a constant, so 3 numbers



EITF75 Systems and Signals

Low-pass to High-pass conversion

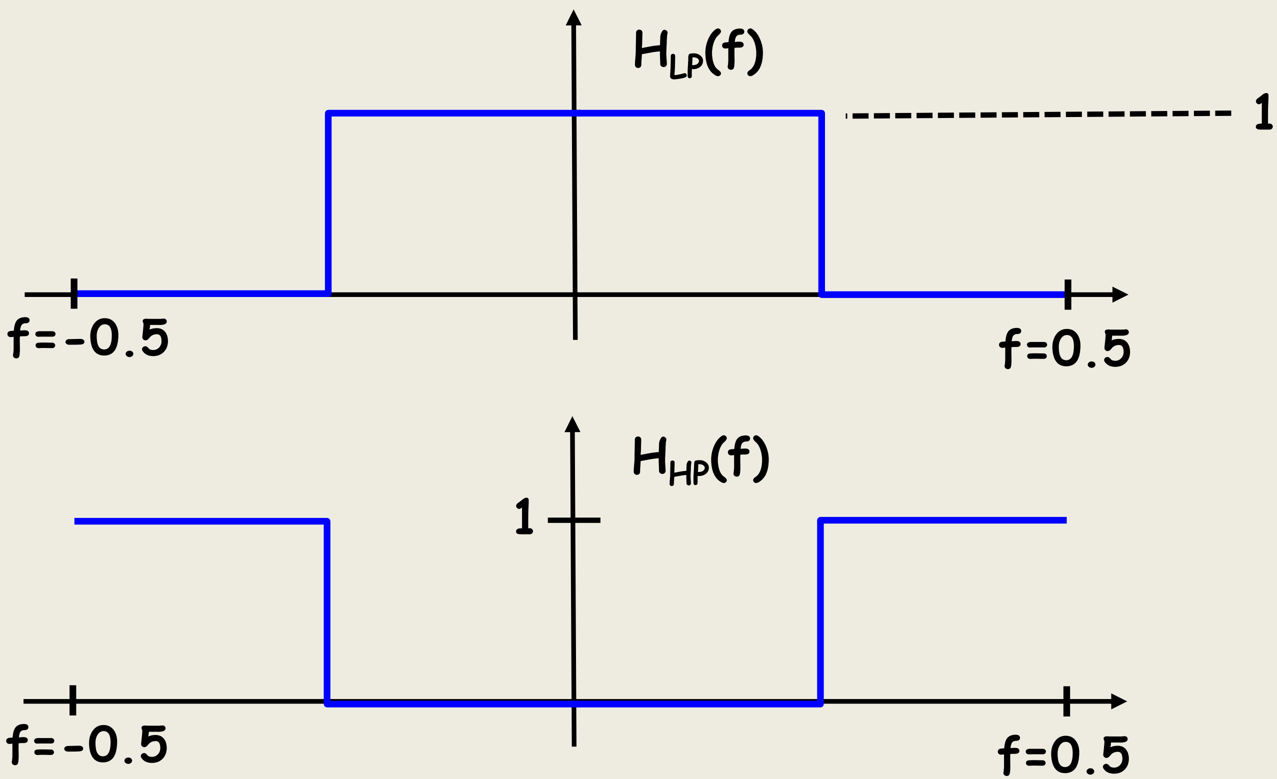
Method A



EITF75 Systems and Signals

Low-pass to High-pass conversion

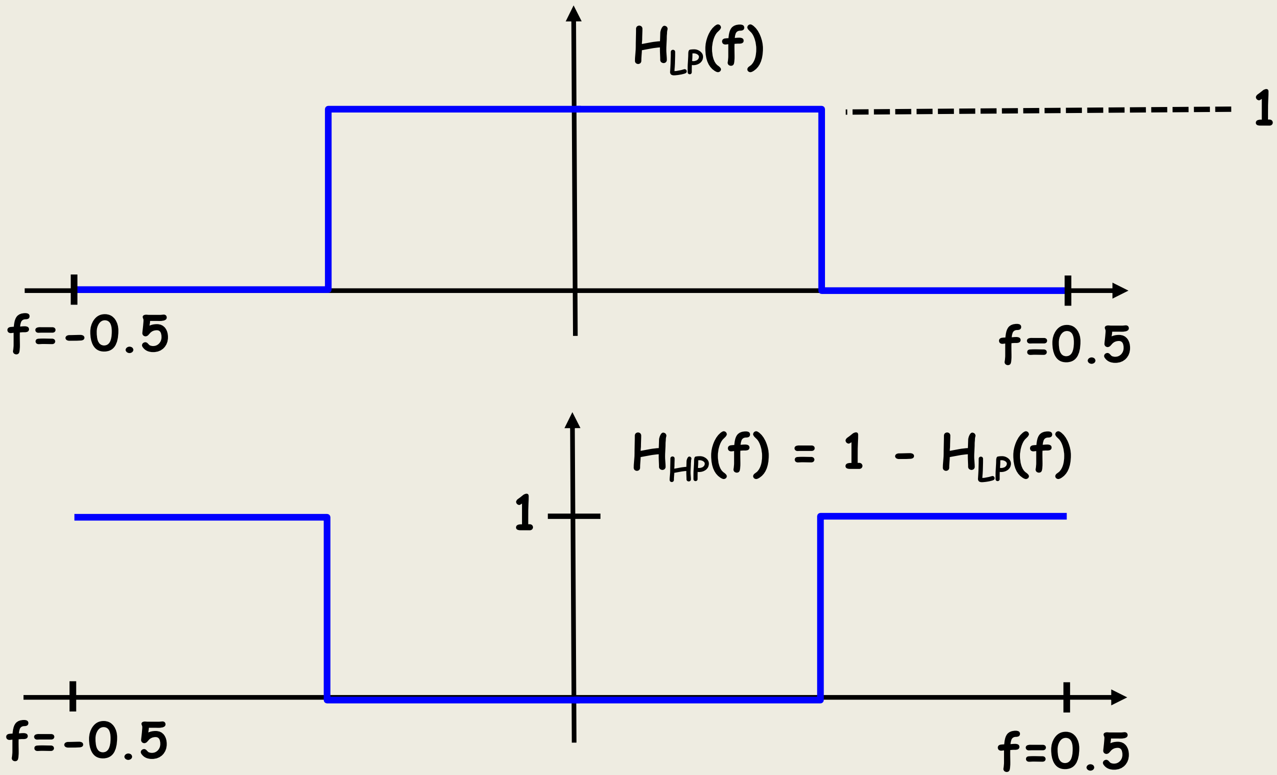
Method A



EITF75 Systems and Signals

Low-pass to High-pass conversion

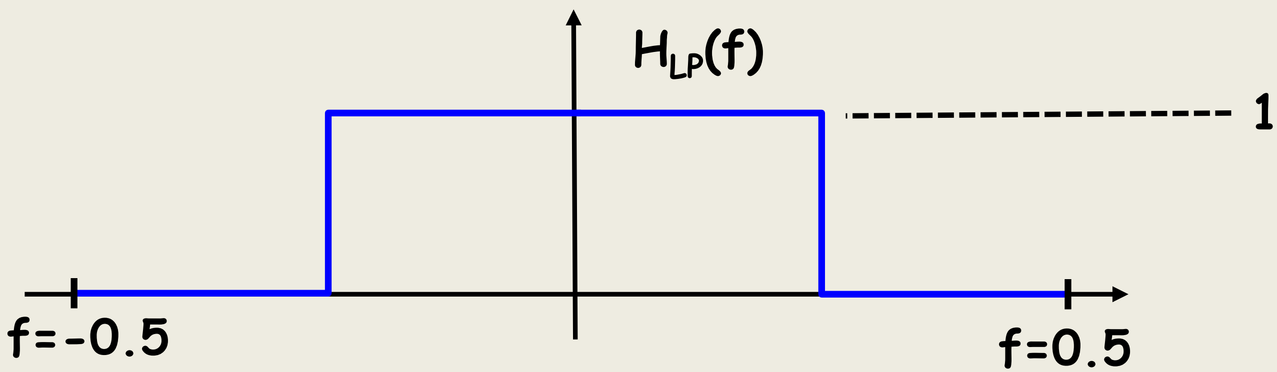
Method A

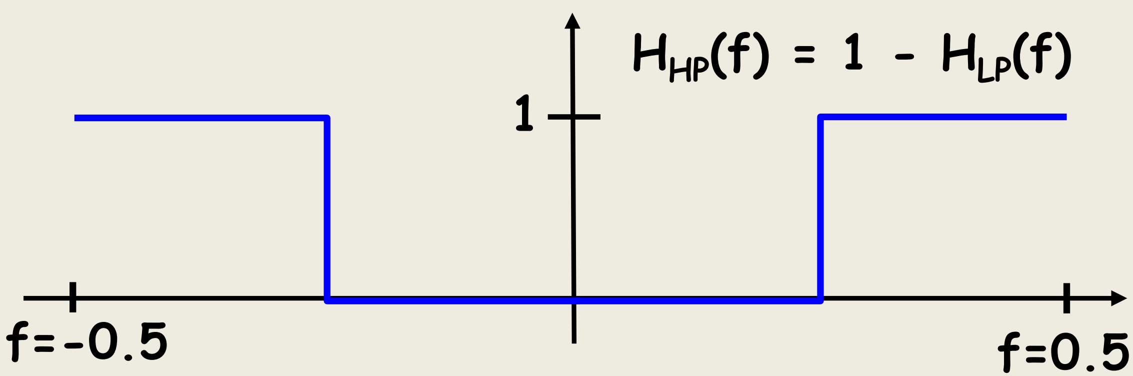


EITF75 Systems and Signals

Low-pass to High-pass conversion

Method A $h_{\mathrm{HP}}(n) = \delta(n) - h_{\mathrm{LP}}(n)$

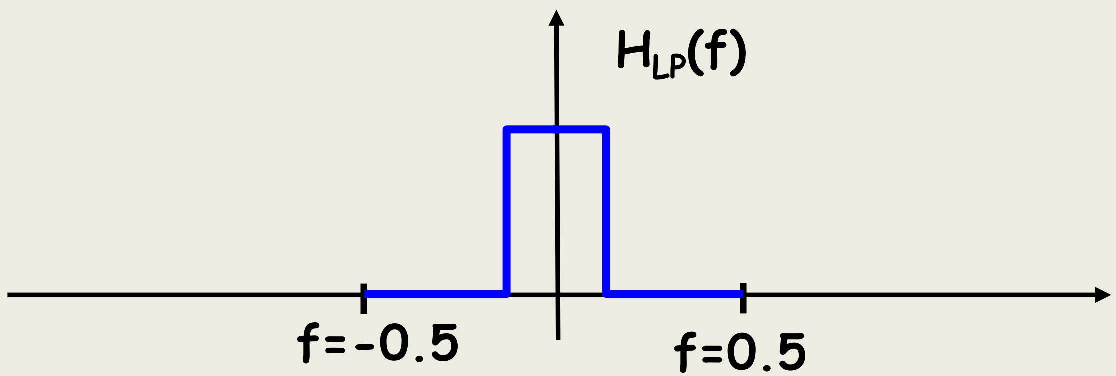




EITF75 Systems and Signals

Low-pass to High-pass conversion

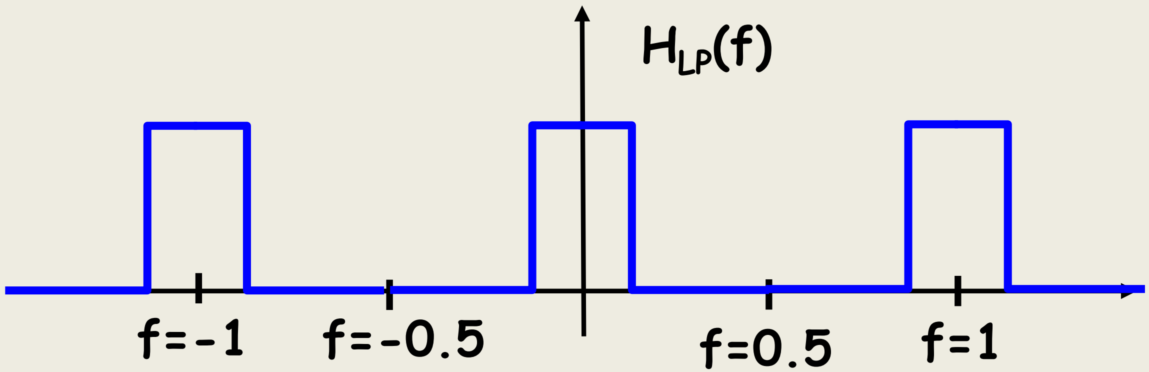
Method B



EITF75 Systems and Signals

Low-pass to High-pass conversion

Method B

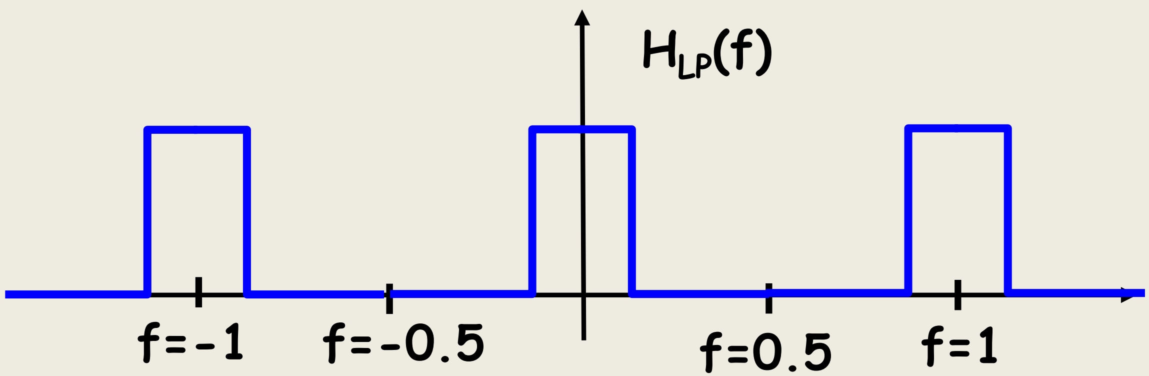


$H_{LP}(f)$ is periodic with period 1

EITF75 Systems and Signals

Low-pass to High-pass conversion

Method B



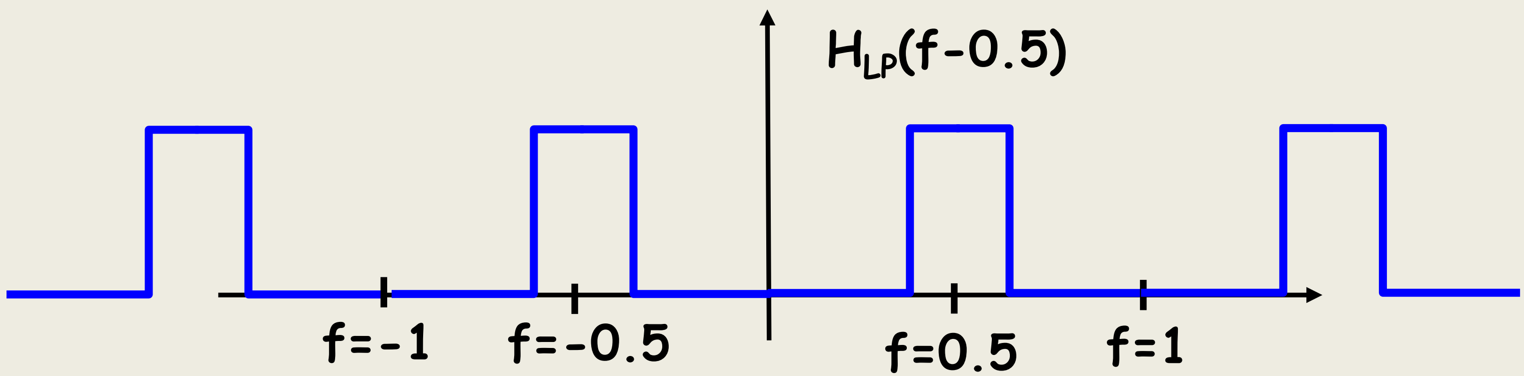
$H_{LP}(f)$ is periodic with period 1

Next slide: Study $H_{LP}(f-0.5)$

EITF75 Systems and Signals

Low-pass to High-pass conversion

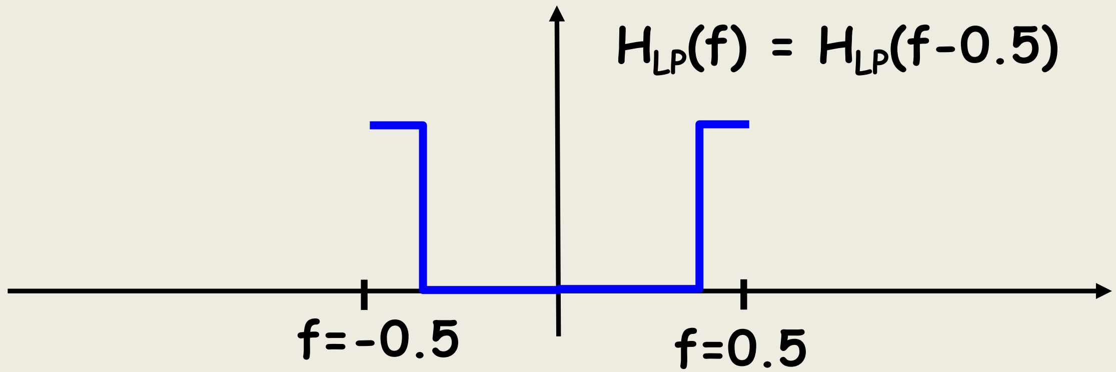
Method B



EITF75 Systems and Signals

Low-pass to High-pass conversion

Method B



This is a high pass filter

EITF75 Systems and Signals

Low-pass to High-pass conversion

Method B

$H_{LP}(f) = H_{LP}(f-0.5)$

Discrete-time Fourier transform

From Wikipedia, the free encyclopedia

WIKIPEDIA
The Free Encyclopedia

Properties of discrete-time Fourier transforms [edit]

Shift in time / Modulation in frequency	$x[n-k]$	$X_{2\pi}(\omega) \cdot e^{-i\omega k}$
Shift in frequency / Modulation in time	$x[n] \cdot e^{ian}$	$X_{2\pi}(\omega - a)$

EITF75 Systems and Signals

Low-pass to High-pass conversion

Method B

$H_{LP}(f) = H_{LP}(f-0.5)$

Discrete-time Fourier transform

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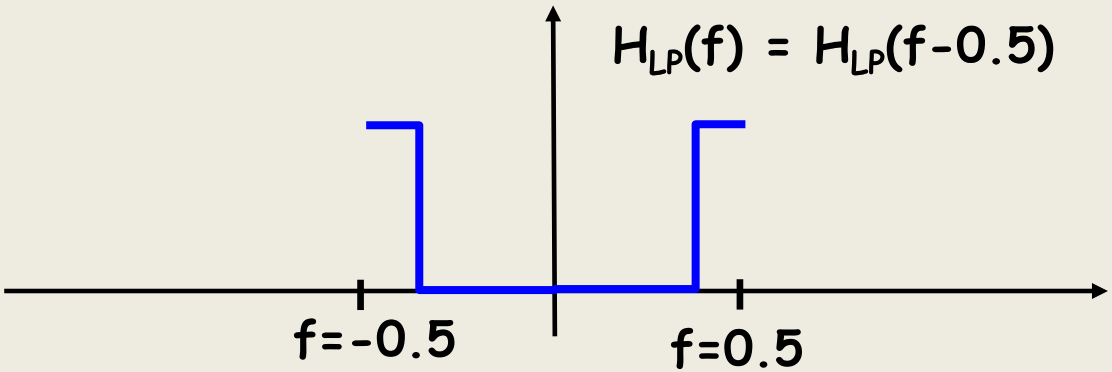
Properties of discrete-time Fourier transforms [edit]

Shift in time / Modulation in frequency	$x[n-k]$	$X_{2\pi}(\omega)\cdot e^{-i\omega k}$
Shift in frequency / Modulation in time	$x[n]\cdot e^{2\pi a n}$	$X_{2\pi}(f-a)$

Low-pass to High-pass conversion

Method B



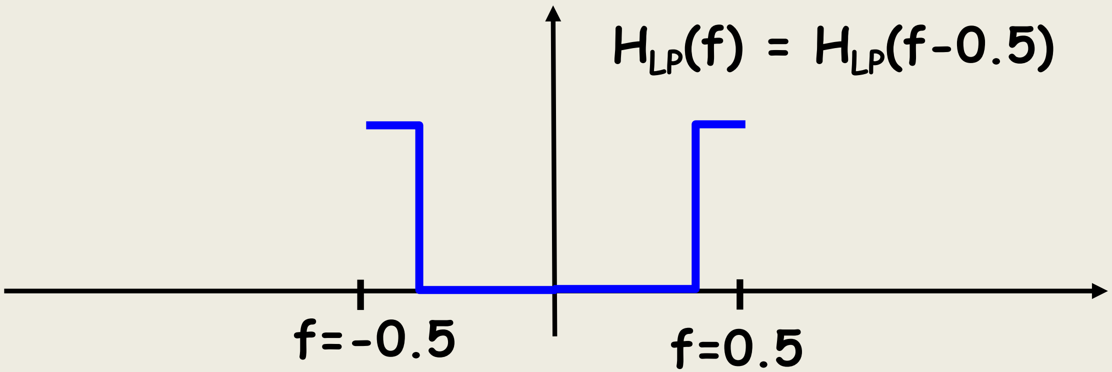
This is a high pass filter

$$h_{\mathrm{HP}}(n) = \mathrm{e}^{i\pi n} h_{\mathrm{LP}}(n)$$

EITF75 Systems and Signals

Low-pass to High-pass conversion

Method B $h_{\mathrm{HP}}(n) = (-1)^n h_{\mathrm{LP}}(n)$



This is a high pass filter

$$h_{\mathrm{HP}}(n) = \mathrm{e}^{i\pi n} h_{\mathrm{LP}}(n)$$

EITF75 Systems and Signals

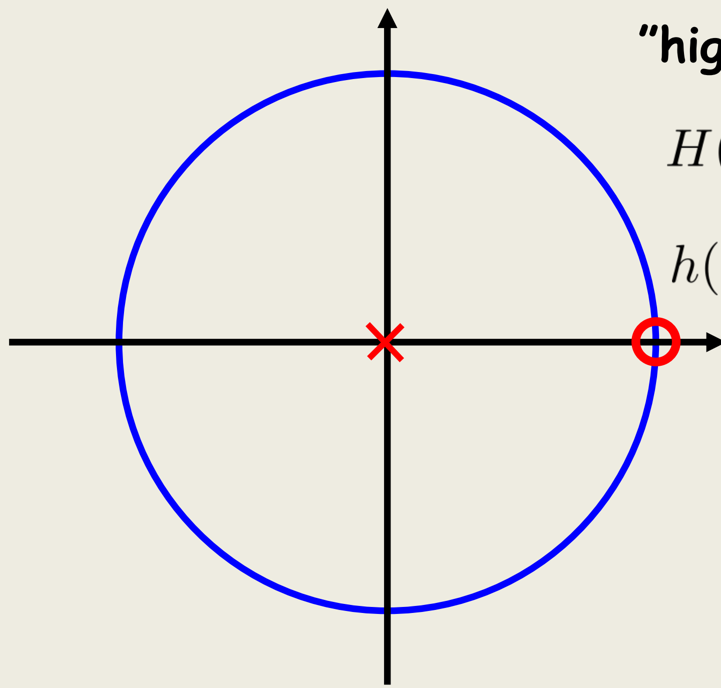
Viewed in another way

If $h_{\mathrm{HP}}(n) = (-1)^n h_{\mathrm{LP}}(n)$ **then** $h_{\mathrm{LP}}(n) = (-1)^n h_{\mathrm{HP}}(n)$

EITF75 Systems and Signals

Viewed in another way

If $h_{\mathrm{HP}}(n) = (-1)^n h_{\mathrm{LP}}(n)$ **then** $h_{\mathrm{LP}}(n) = (-1)^n h_{\mathrm{HP}}(n)$



"high pass" $z_0 = 1$

$$H(z) = (z-1) = \frac{1-z^{-1}}{z^{-1}}$$

$$h(n) = \{1 \quad -1\}$$

EITF75 Systems and Signals

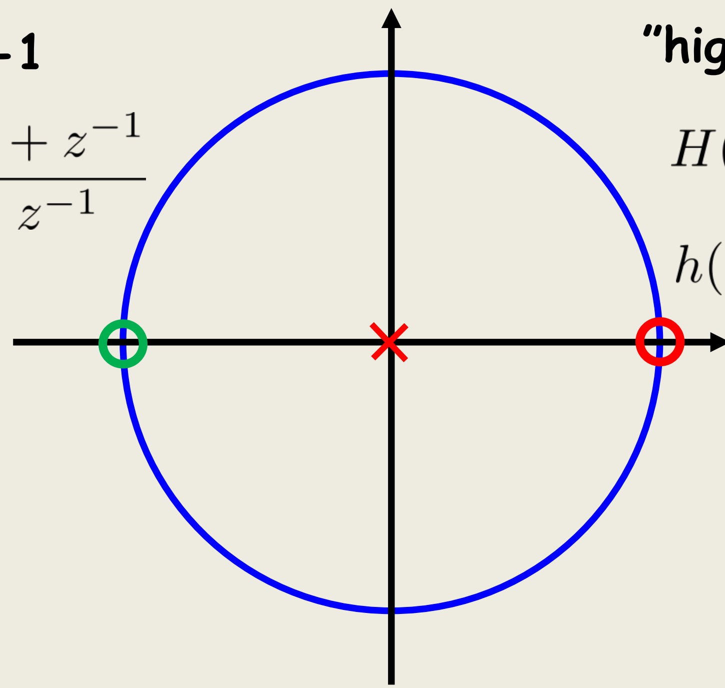
Viewed in another way

If $h_{\mathrm{HP}}(n) = (-1)^n h_{\mathrm{LP}}(n)$ **then** $h_{\mathrm{LP}}(n) = (-1)^n h_{\mathrm{HP}}(n)$

"low pass" $z_0 = -1$

$$H(z) = (z+1) = \frac{1+z^{-1}}{z^{-1}}$$

$$h(n) = \{1 \;\; 1\}$$



"high pass" $z_0 = 1$

$$H(z) = (z-1) = \frac{1-z^{-1}}{z^{-1}}$$

$$h(n) = \{1 \;\; -1\}$$

EITF75 Systems and Signals

Minimum phase filters

Assume some h(n) and compute $|H(f)|^2 = H(f)H^*(f)$

$$|H(f)| = |H^*(f)|$$

EITF75 Systems and Signals

Minimum phase filters

Assume some h(n) **and compute** $|H(f)|^2 = H(f)H^*(f)$

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Discrete-time Fourier transform

From Wikipedia, the free encyclopedia

Properties of discrete-time Fourier transforms [edit]

Time reversal & conjugation	$x[-n]^*$	$X_{2\pi}(\omega)^*$

EITF75 Systems and Signals

Minimum phase filters

So
$$G(f) = |H(f)|^2 = H(f)H^*(f)$$
$$g(n) = h(n) \star h(-n)$$

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Properties of discrete-time Fourier transforms [edit]

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EITF75 Systems and Signals

Minimum phase filters

So $$G(f) = |H(f)|^2 = H(f)H^*(f)$$
$$g(n) = h(n) \star h(-n)$$

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Z-transform

From Wikipedia, the free encyclopedia

Properties [edit]

Time reversal	$x[-n]$	$X(z^{-1})$

EITF75 Systems and Signals

Minimum phase filters

So
$$G(f) = |H(f)|^2 = H(f)H^*(f)$$
$$g(n) = h(n) \star h(-n)$$
$$G(z) = H(z)H(z^{-1})$$
$$|H(z)| = |H(z^{-1})|$$

$$|H(f)| = |H^*(f)|$$

EITF75 Systems and Signals

Minimum phase filters

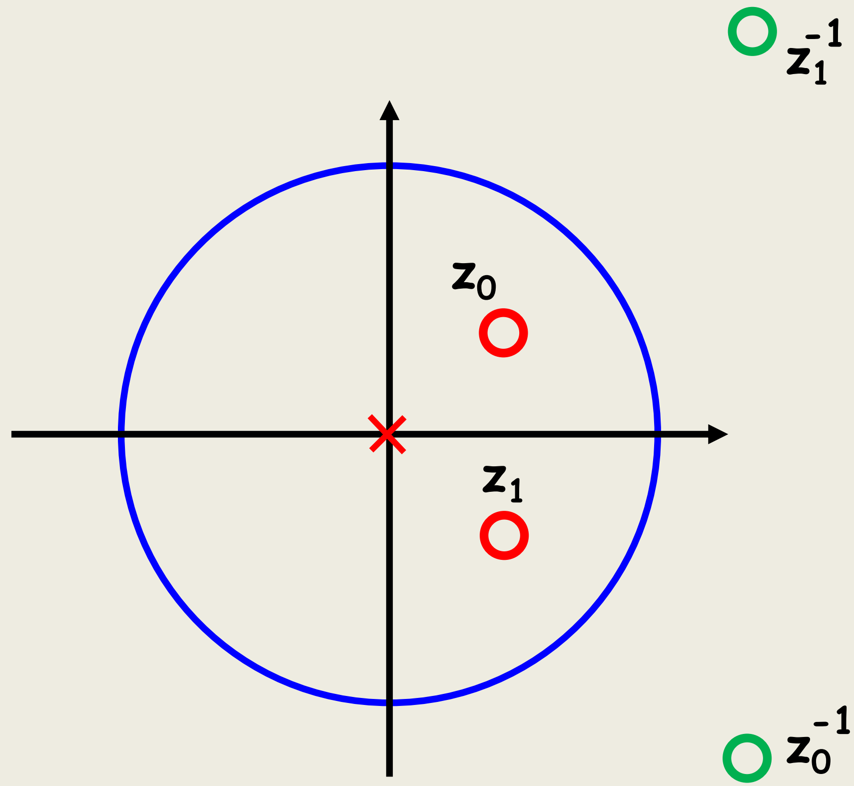
So
$$G(f) = |H(f)|^2 = H(f)H^*(f)$$
$$g(n) = h(n) \star h(-n)$$
$$G(z) = H(z)H(z^{-1})$$
$$|H(z)| = |H(z^{-1})|$$

From before z_0 is a zero z_0^{-1} is a zero

$$|H(f)| = |H^*(f)|$$

EITF75 Systems and Signals

Minimum phase filters



Implication:
A filter using the red zeros have same magnitude response as a fitlter using the green zeros

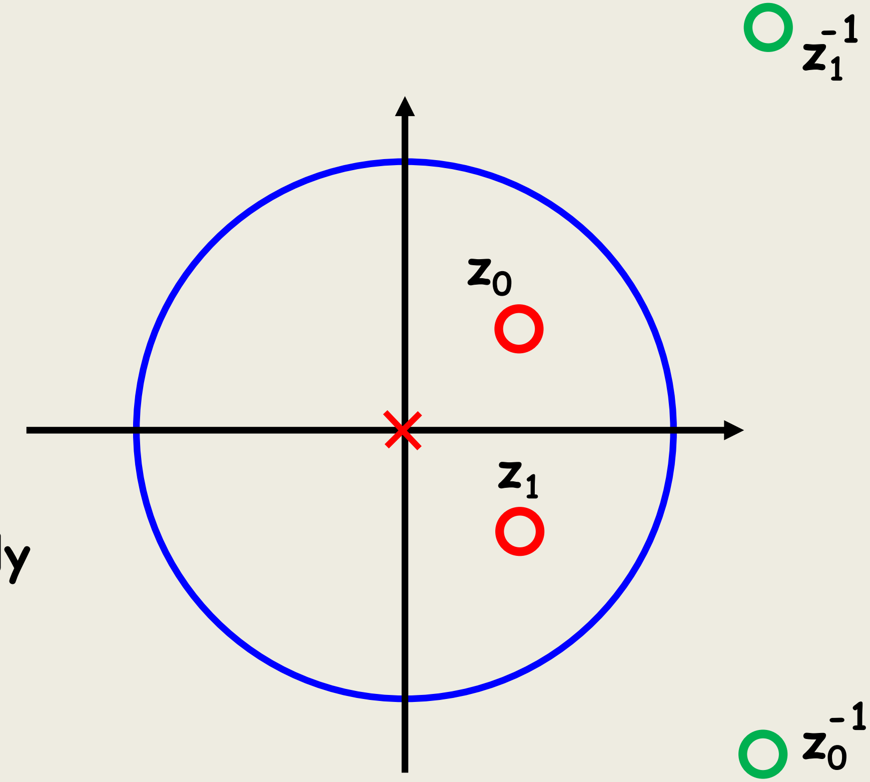
$$|H(z)| = |H(z^{-1})|$$

z_0 is a zero

z_0^{-1} is a zero

EITF75 Systems and Signals

Minimum phase filters



Implication:
A filter using the red zeros have same magnitude response as a fitlter using the green zeros

We can choose arbitrarily if we use green or red

$$|H(z)| = |H(z^{-1})|$$

z_0 is a zero

z_0^{-1} is a zero

$$|H(f)| = |H^*(f)|$$

EITF75 Systems and Signals

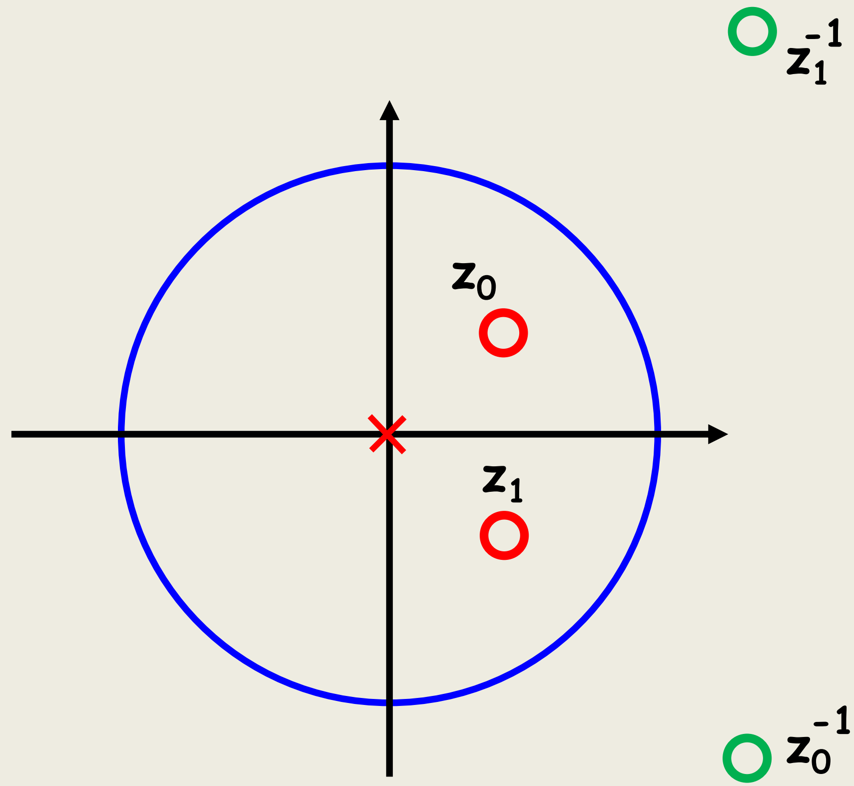
Minimum phase filters

Implication:

So, magnitudes are equal

How about phase

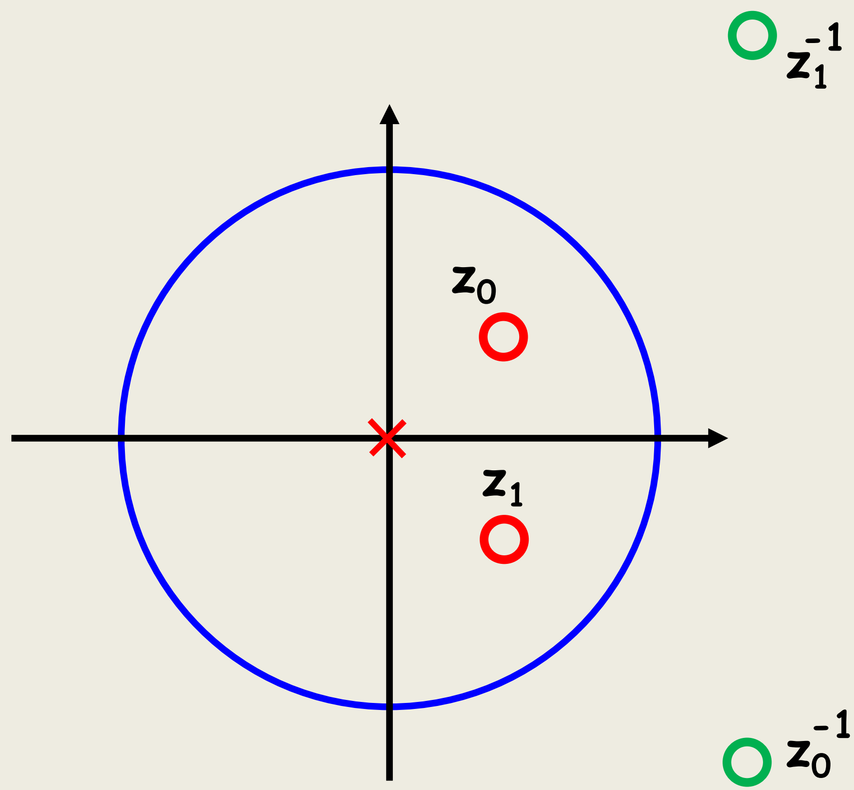
$$\Theta(f) \stackrel{?}{=} \Theta(f)$$



$$|H(f)| = |H^*(f)|$$

EITF75 Systems and Signals

Minimum phase filters



Implication:
So, magnitudes are equal

How about phase

$$\Theta(f) \neq \Theta(f)$$

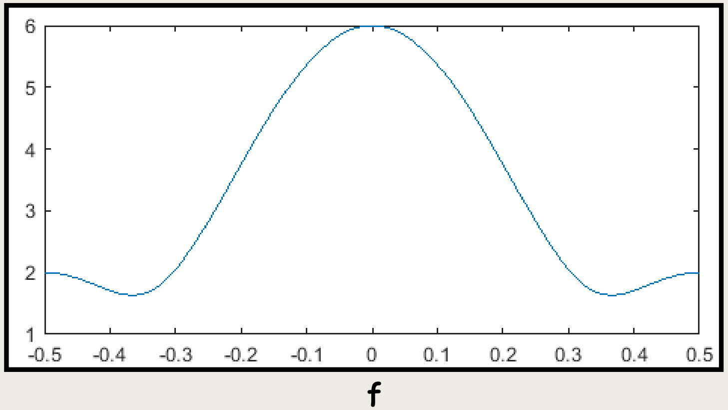
Cannot be, since impulse reponses are different

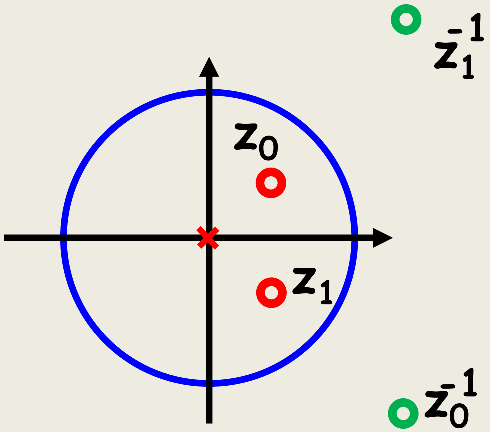
$$|H(f)| = |H^*(f)|$$

EITF75 Systems and Signals

Minimum phase filters

$$|H(f)| = |H^*(f)|$$

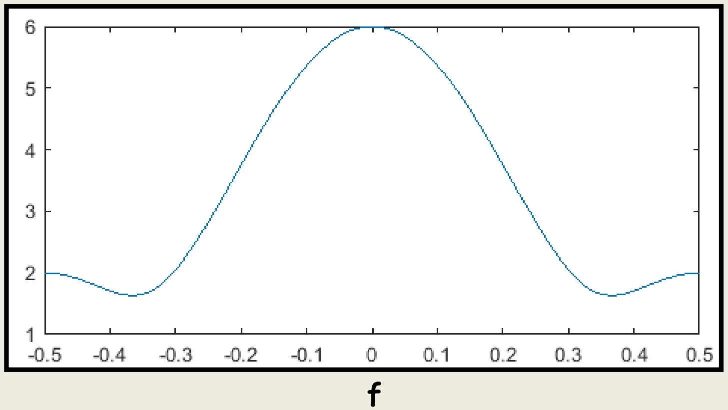


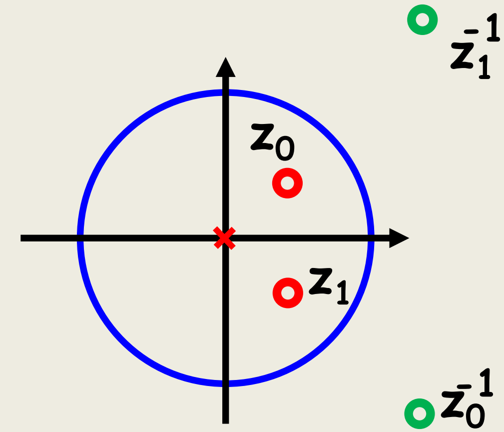


EITF75 Systems and Signals

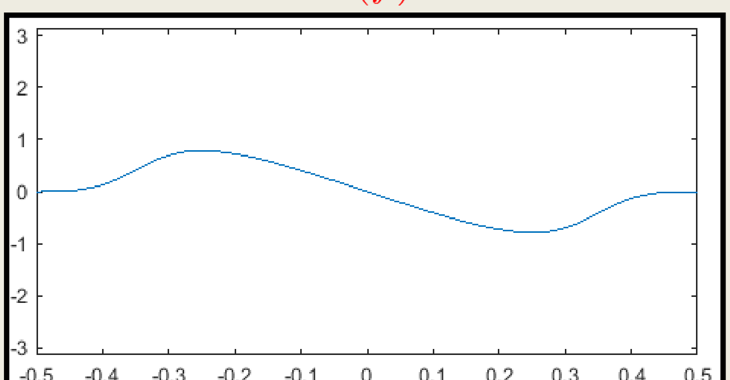
Minimum phase filters

$$|H(f)| = |H^*(f)|$$

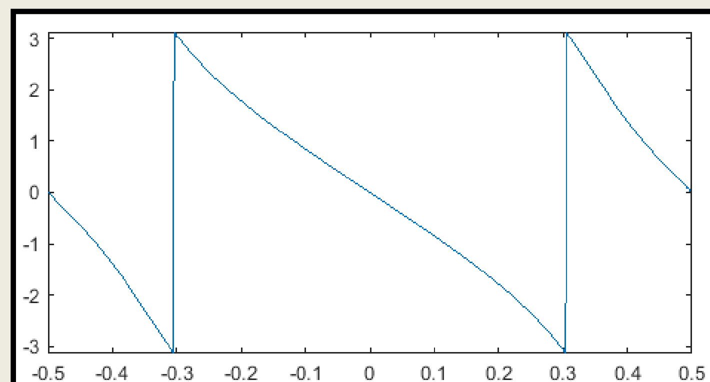




$\Theta(f)$



$\Theta(f)$

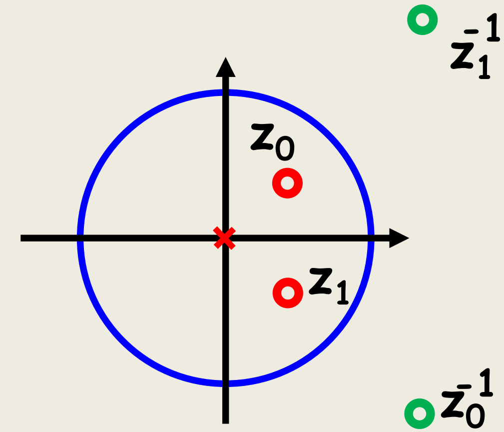


EITF75 Systems and Signals

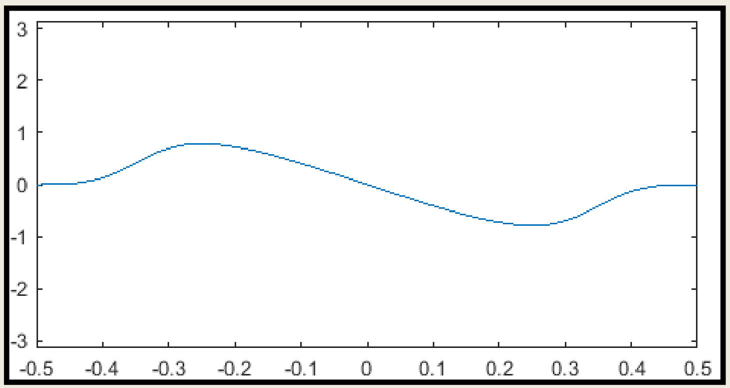
Minimum phase filters

This is a general rule:
A filter with all zeros inside the unit circle has smaller phase.

Minimum phase filter
Maximum phase filter



$\Theta(f)$



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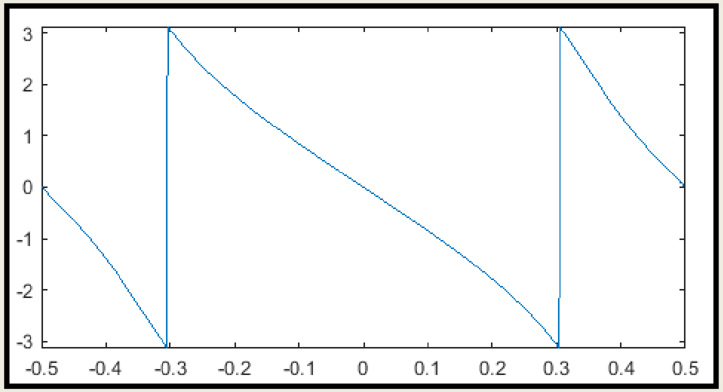

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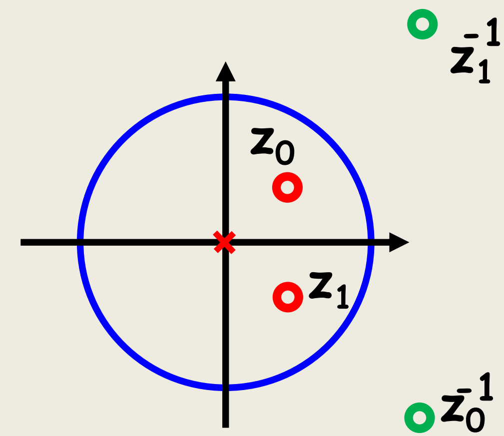

Let $h(k)$ be any filter with magnitude response $|H(f)|$

EITF75 Systems and Signals

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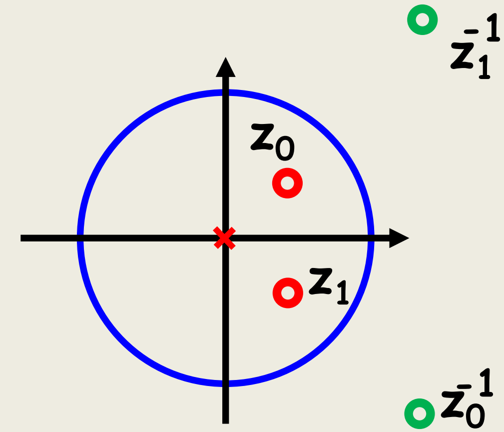

Let $h(k)$ be any filter with magnitude response $|H(f)|$

Let $h_{\mathrm{mp}}(k)$ be the minimum phase filter with magnitude $|H(f)|$

EITF75 Systems and Signals

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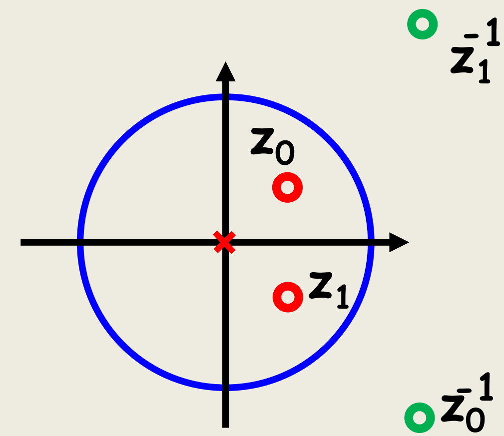

Let $h(k)$ be any filter with magnitude response $|H(f)|$

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Then $\sum_{k=0}^{K} |h_{\mathrm{mp}}(k)|^2 \geq \sum_{k=0}^{K} |h(k)|^2$

EITF75 Systems and Signals

Minimum phase filters

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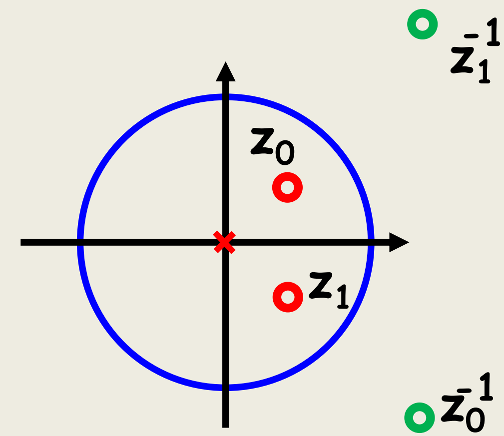

Let $h(k)$ be any filter with magnitude response $|H(f)|$

Let $h_{\mathrm{mp}}(k)$ be the minimum phase filter with magnitude $|H(f)|$

Let $h_{\mathrm{maxp}}(k)$ be the maximum phase filter with magnitude $|H(f)|$

Then $\sum_{k=0}^{K} |h_{\mathrm{mp}}(k)|^2 \geq \sum_{k=0}^{K} |h(k)|^2$

Then $\sum_{k=0}^{K} |h_{\mathrm{maxp}}(k)|^2 \leq \sum_{k=0}^{K} |h(k)|^2$

EITF75 Systems and Signals

Minimum phase filters

Super-important for trunction

This is a general rule:
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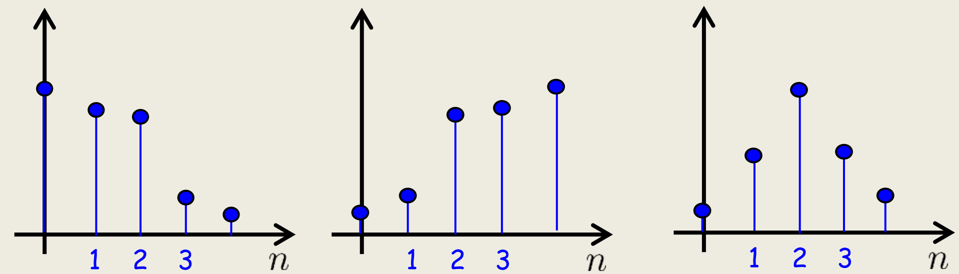

Min-phase Max-phase Mix-phase

Let $h(k)$ be any filter with magnitude response $|H(f)|$

Let $h_{\mathrm{mp}}(k)$ be the minimum phase filter with magnitude $|H(f)|$

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Then $\sum_{k=0}^{K} |h_{\mathrm{mp}}(k)|^2 \geq \sum_{k=0}^{K} |h(k)|^2$

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