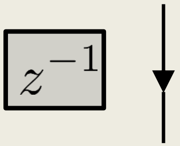
EITF75 Systems and Signals

Recap

In general, a signal $y(n)$ generated from $x(n)$ via $\oplus$ 

can be mathematically described by

$$y(n) + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

For systems at rest: z-transform

$$Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$Y(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} \cdot X(z) = H(z)X(z)$$

System describes a convolution

$$y(n) = h(n) * x(n) = \sum_{k} h(k) x(n-k)$$

FIR filter: $a_k=0$, $k>0$

IIR filter: otherwise

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Recap

For stable h(n) $H(f) = H\left(e^{i2\pi f}\right)$

For input $x(n) = \exp(i2\pi f_0 n)$

We get the output $y(n) = H(f_0)\exp(i2\pi f_0 n)$

Today:

1. $x(n) = \cos(2\pi f_0 n)$ $x(n) = \sin(2\pi f_0 n)$

2. $x(n) = \cos(2\pi f_0 n)u(n)$

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EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find $H(f)$

Find b such that $\max|H(f)|=1$

Find $y(n)$ for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

EITF75 Systems and Signals

EXAMPLE $y(n)=ay(n-1)+bx(n)$ Find H(f)

$Y(z)=az^{-1}Y(z)+bX(z)$ Start by z-transform

$Y(z)(1-az^{-1})=bX(z)$ Minor (standard) manipulation

$$Y(z)=\frac{b}{(1-az^{-1})}X(z)$$ Minor (standard) manipulation

$$H(z)=\frac{b}{(1-az^{-1})}$$

Identify system

Stable if $|a|<1$

EITF75 Systems and Signals

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$$Y(z) = \frac{b}{(1-az^{-1})} X(z)$$

Minor (standard) manipulation

$$H(z) = \frac{b}{(1-az^{-1})}$$

Identify system

Stable if $|a|<1$

$$H(f) = \frac{b}{(1-ae^{-i2\pi f})}$$

Find DTFT

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EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$

Find H(f)

$$1 - ae^{-i2\pi f} = [1 - a\cos(2\pi f)] + ia\sin(2\pi f)$$

$$|1 - ae^{-i2\pi f}| = \sqrt{[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2}$$

$$H(f) = \frac{b}{(1 - ae^{-i2\pi f})}$$

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$$|H(f)| = \frac{|b|}{\sqrt{[1 - a\cos(2\pi f)]^2 + [a\sin(2\pi f)]^2}}$$

$$\theta(f) = -\tan^{-1}\frac{a\sin(2\pi f)}{1 - a\cos(2\pi f)}$$

(assume $b > 0$)

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EXAMPLE $y(n) = a\,y(n-1) + b\,x(n)$

Find b such that $\max|H(f)| = 1$

Assume $a > 0$

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EITF75 Systems and Signals

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EITF75 Systems and Signals

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maximized

$$|H(f)| = \frac{|b|}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

← When minimized

EITF75 Systems and Signals

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EITF75 Systems and Signals

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EITF75 Systems and Signals

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$$\max_f |H(f)| = 1 \rightarrow b = 1 - a$$

maximized

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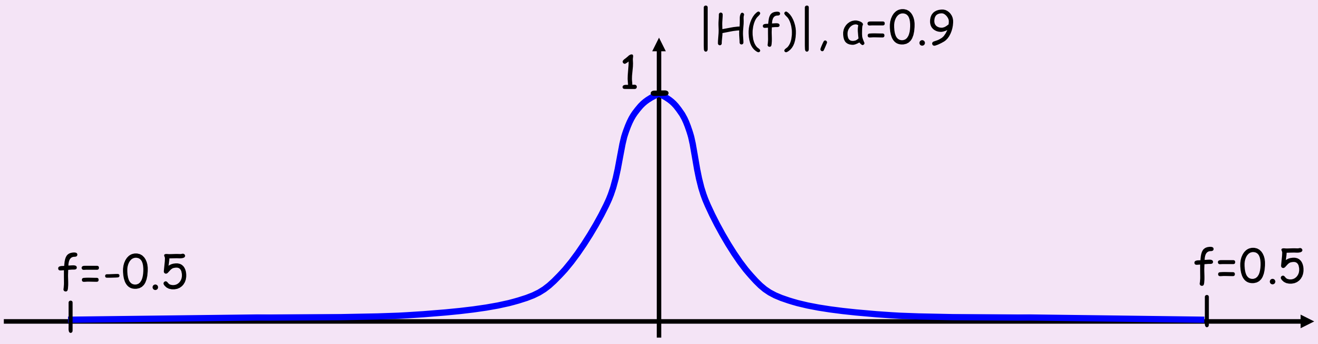

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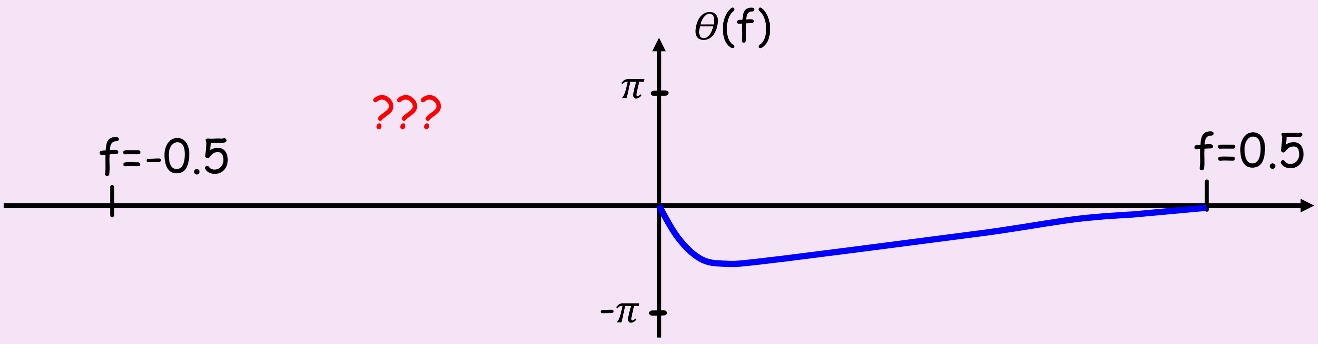

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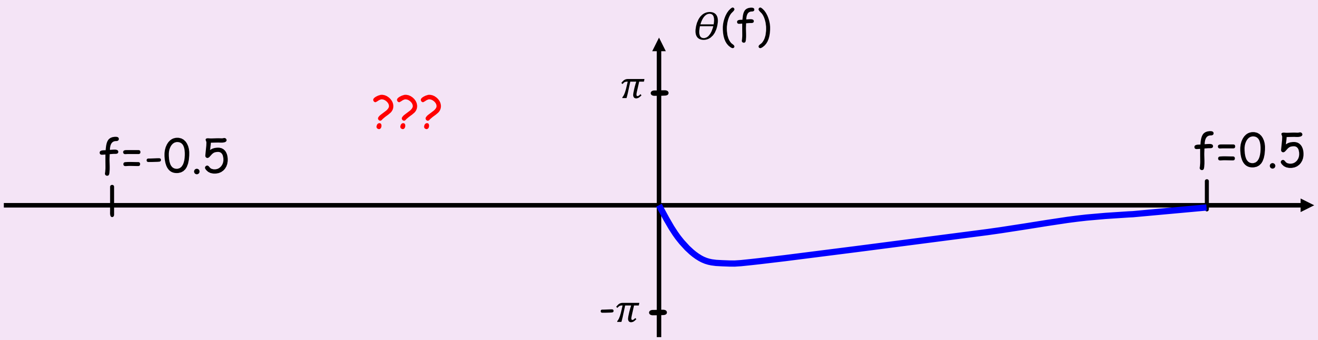

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Relation between $H(f_0)$ **and** $H(-f_0)$: $H(f) = H^*(-f)$

EITF75 Systems and Signals

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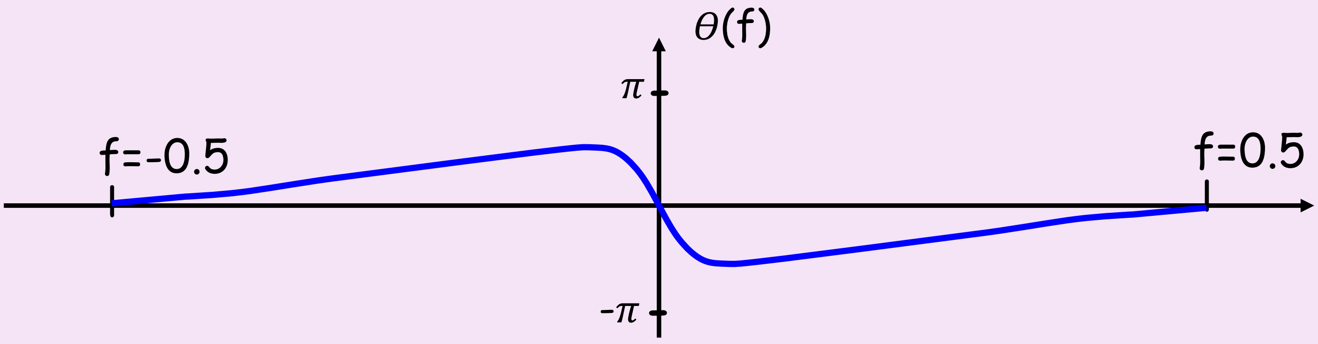

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EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

Find y(n) for $x(n)=5 + 12\sin(\pi n/2) - 20\cos(\pi n + \pi/4)$

Step 1: Identify the frequencies

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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Step 2: How to handle delay in input ?

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EITF75 Systems and Signals

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Step 2: How to handle delay in input ? LTI systems, so delays are remain in output

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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Step 3: Compute $|H(f)|$ for above frequencies. (assume a=0.9)

$|H(0)| = \ldots = 1$ $|H(0.25)| = \ldots = 0.074$ $|H(0.5)| = \ldots = 0.053$

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$\theta(0) = 0$ $\theta(0.25) = \ldots = -42^\circ$ $\theta(0.5) = \ldots = 0$

$$|H(f)| = \frac{1-a}{\sqrt{1 + a^2 - 2a\cos(2\pi f)}}$$

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Step 4: "Modify" x(n) to obtain y(n)

$y(n) = 5 +$

EITF75 Systems and Signals

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$y(n) = 5 + 0.888\sin(\pi n/2 - 42^{\circ}) -$

$12 \times 0.074 = 0.888$

EITF75 Systems and Signals

EXAMPLE $y(n)=a\,y(n-1)+b\,x(n)$

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$y(n) = 5 + 0.888\sin(\pi n/2 - 42^o) - 1.06\cos(\pi n + \pi/4)$

$20 \times 0.053 = 1.06$

EITF75 Systems and Signals

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Step 4: "Modify" $x(n)$ to obtain $y(n)$

$$y(n) = 5 + 0.888\sin(\pi n/2 - 42^o) - 1.06\cos(\pi n + \pi/4)$$

Note: this is due to LTI

Next case to study

$$x(n) = \cos(2\pi f_0 n)u(n)$$

What can we say before starting ?

1. More realistic, so important
2. For a causal h(n), y(n) also causal

EITF75 Systems and Signals

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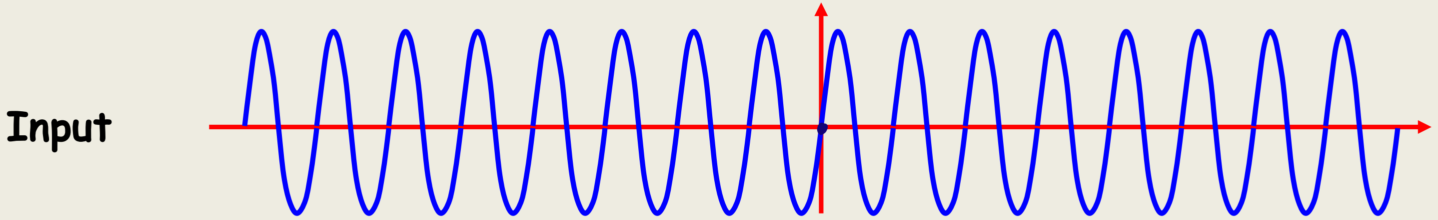

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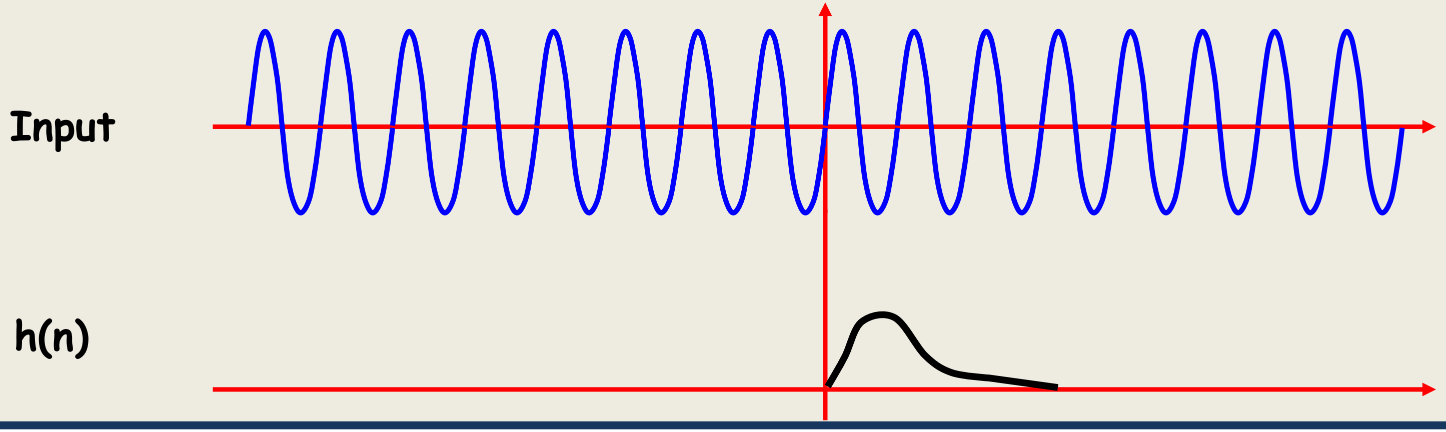

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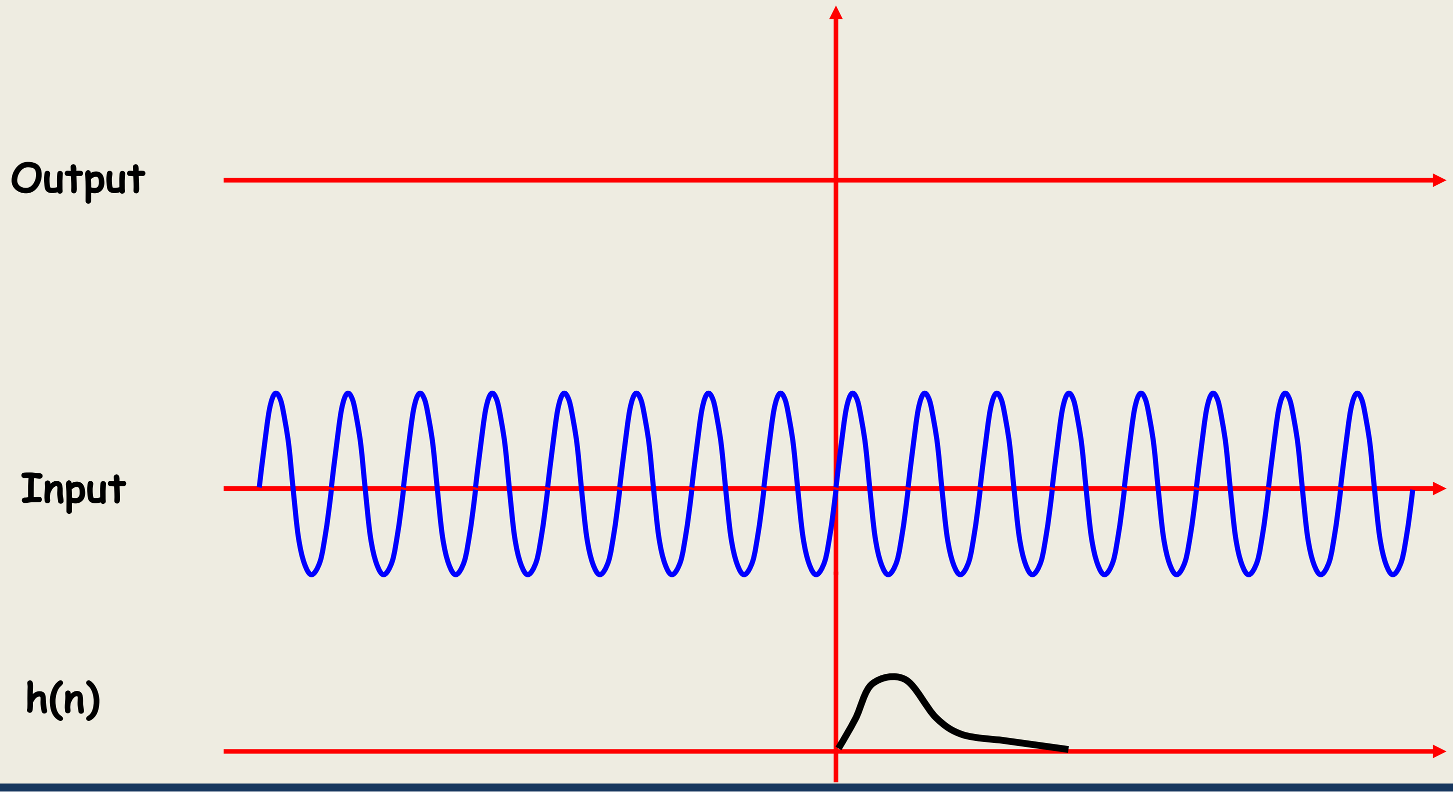

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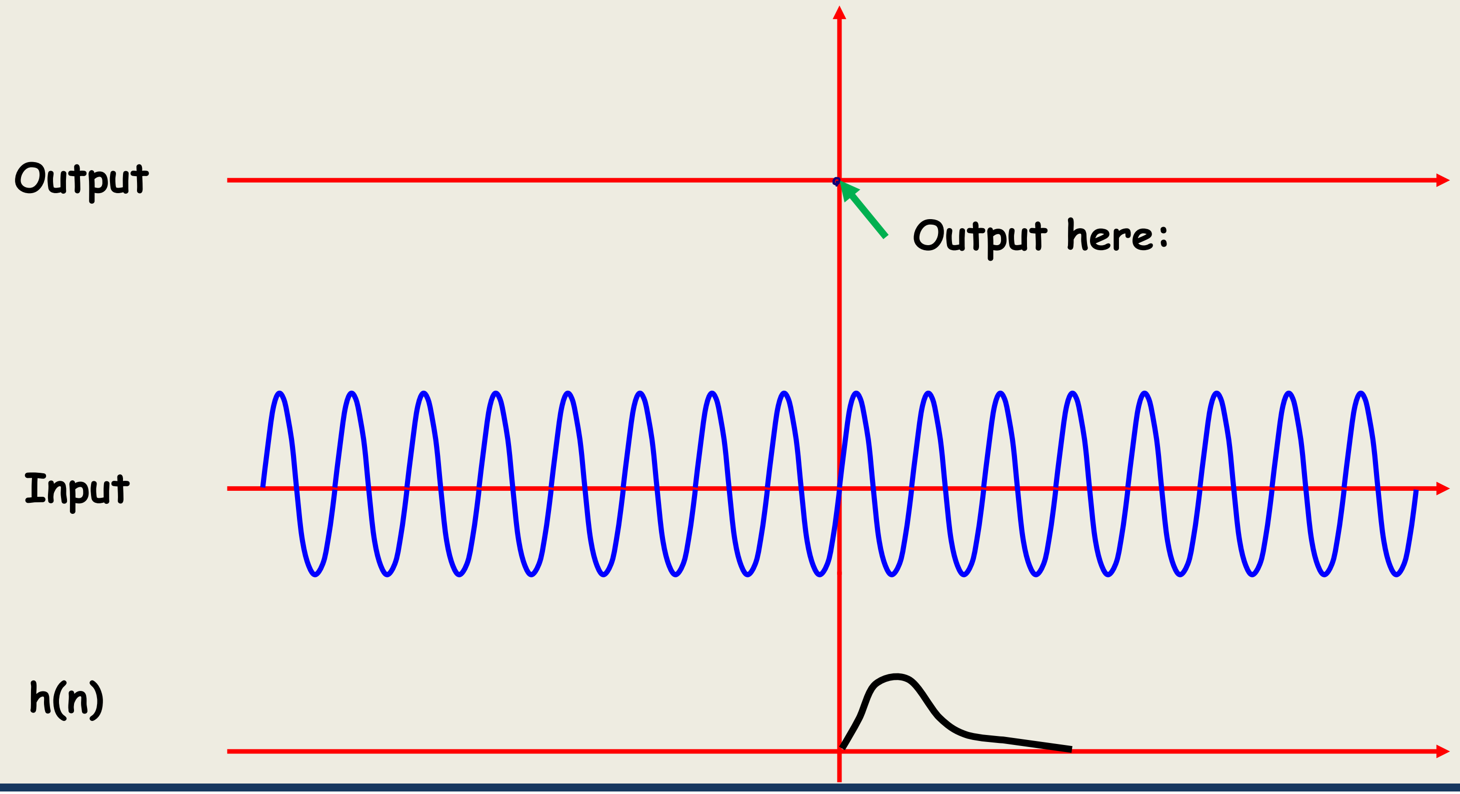

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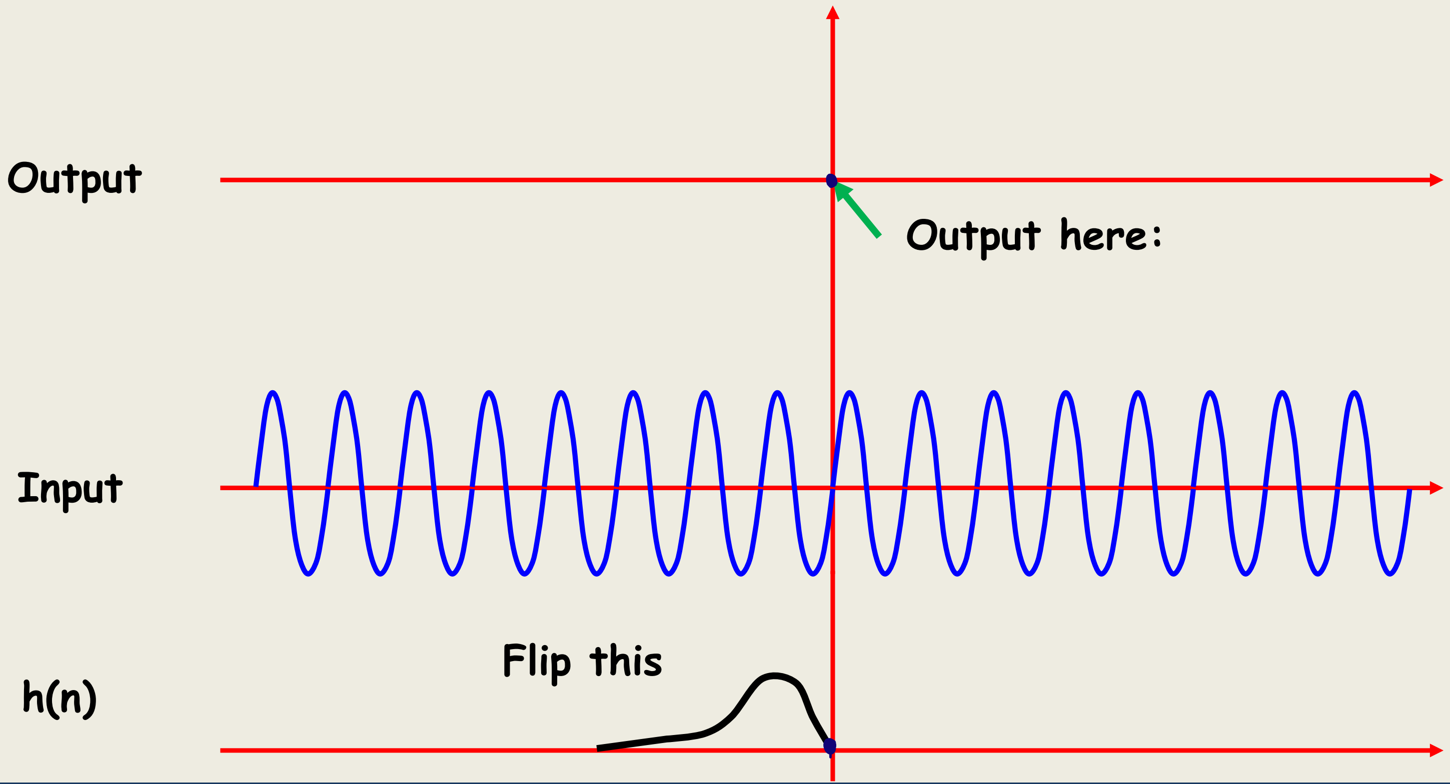

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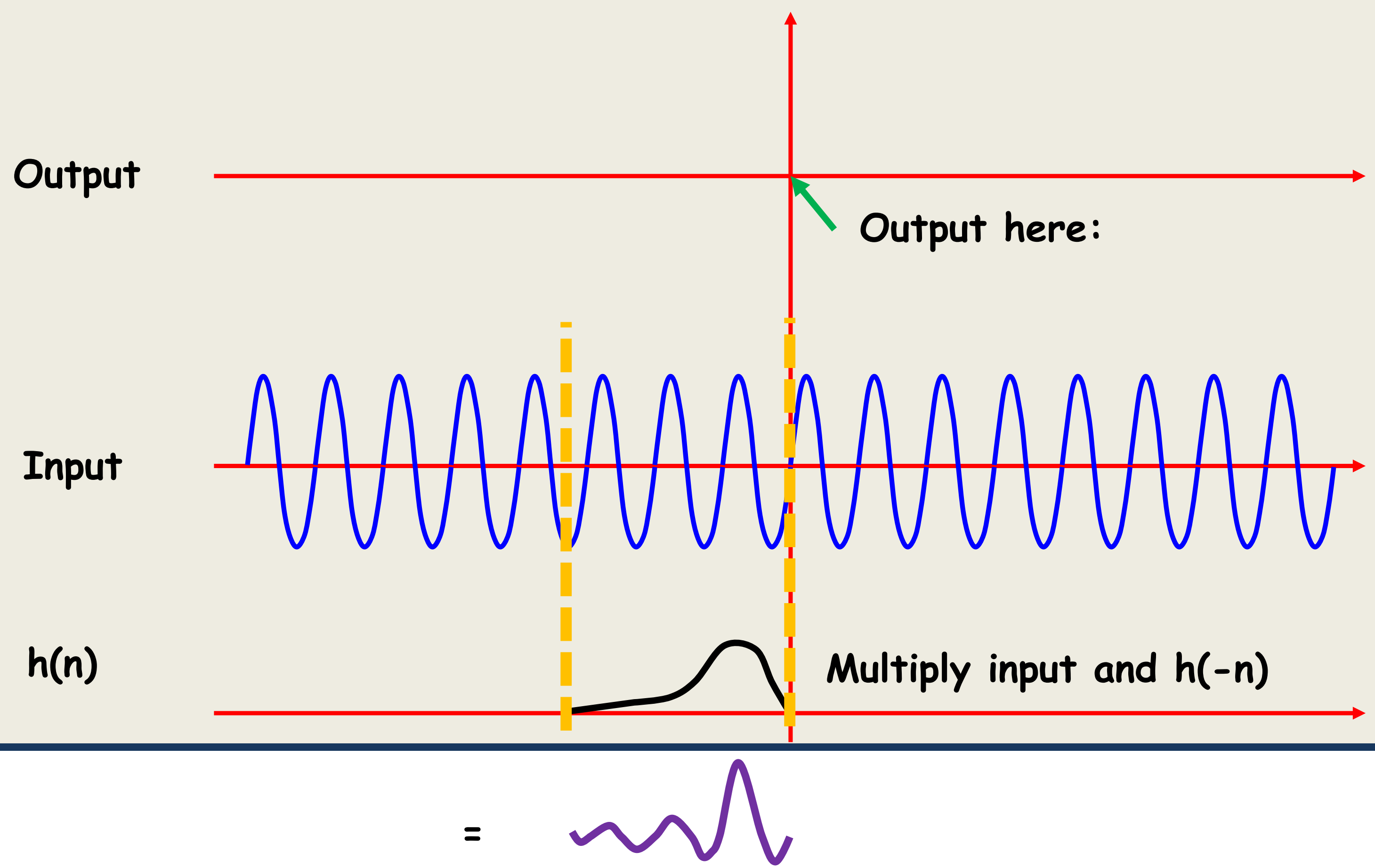

EITF75 Systems and Signals

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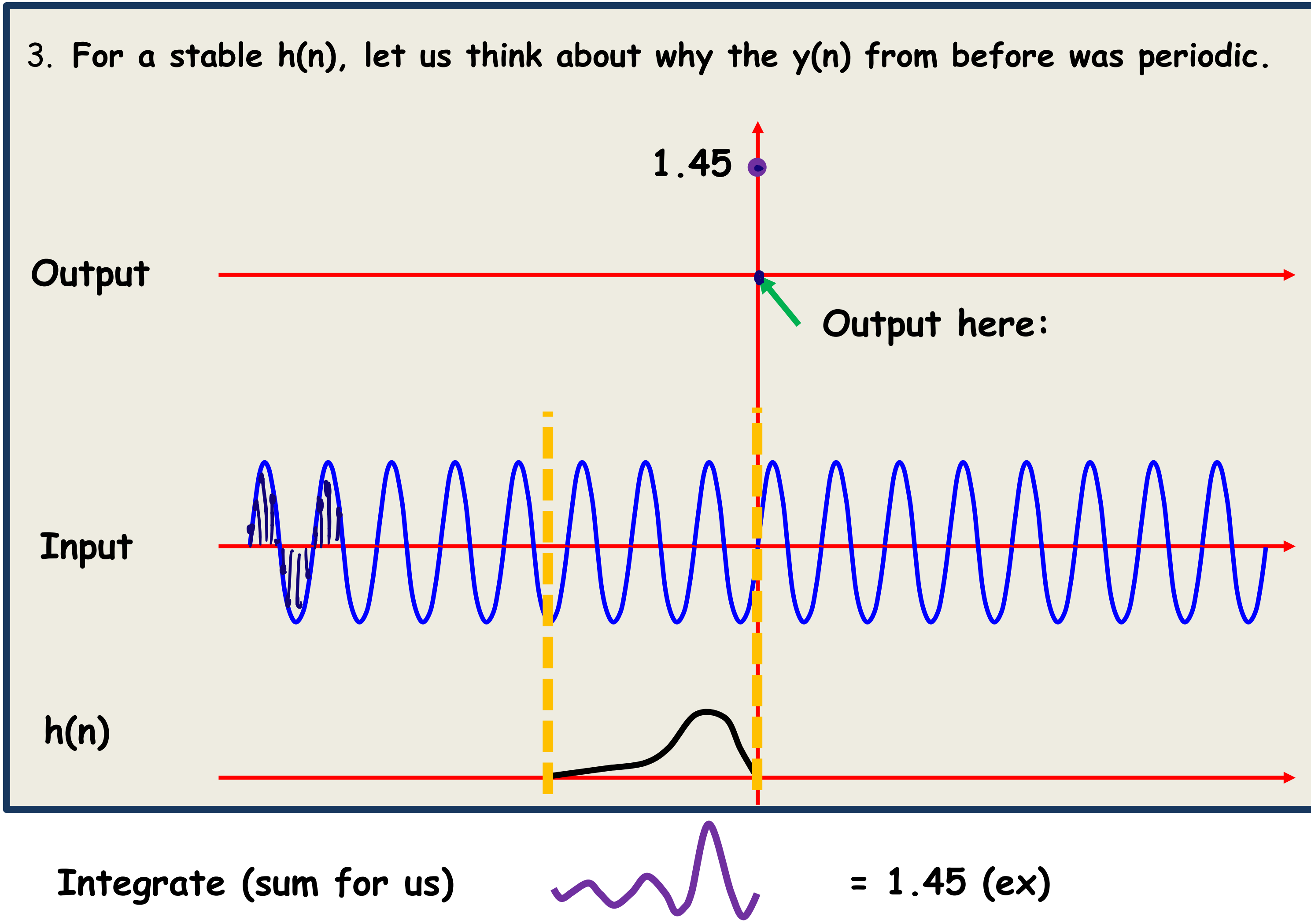

Integrate (sum for us) = 1.45 (ex)

EITF75 Systems and Signals

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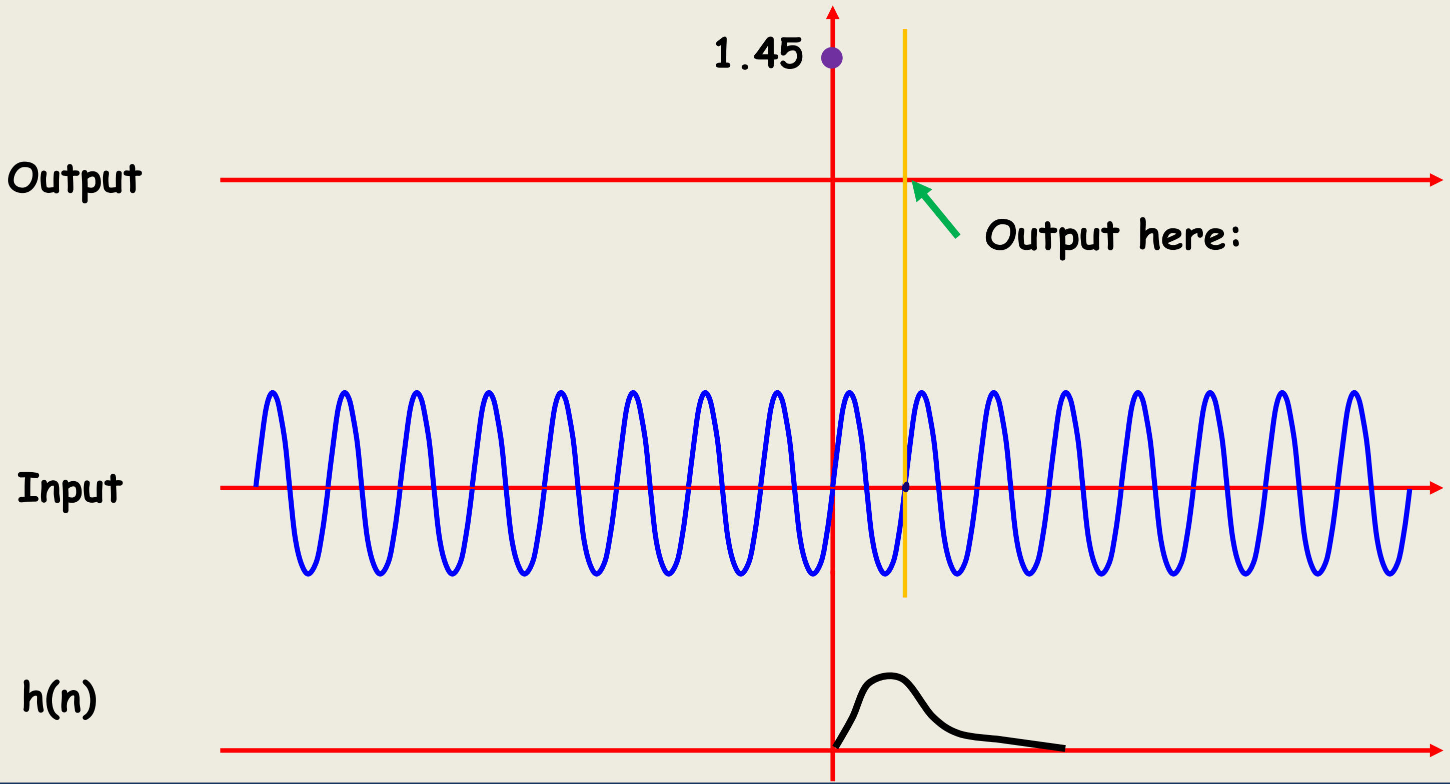

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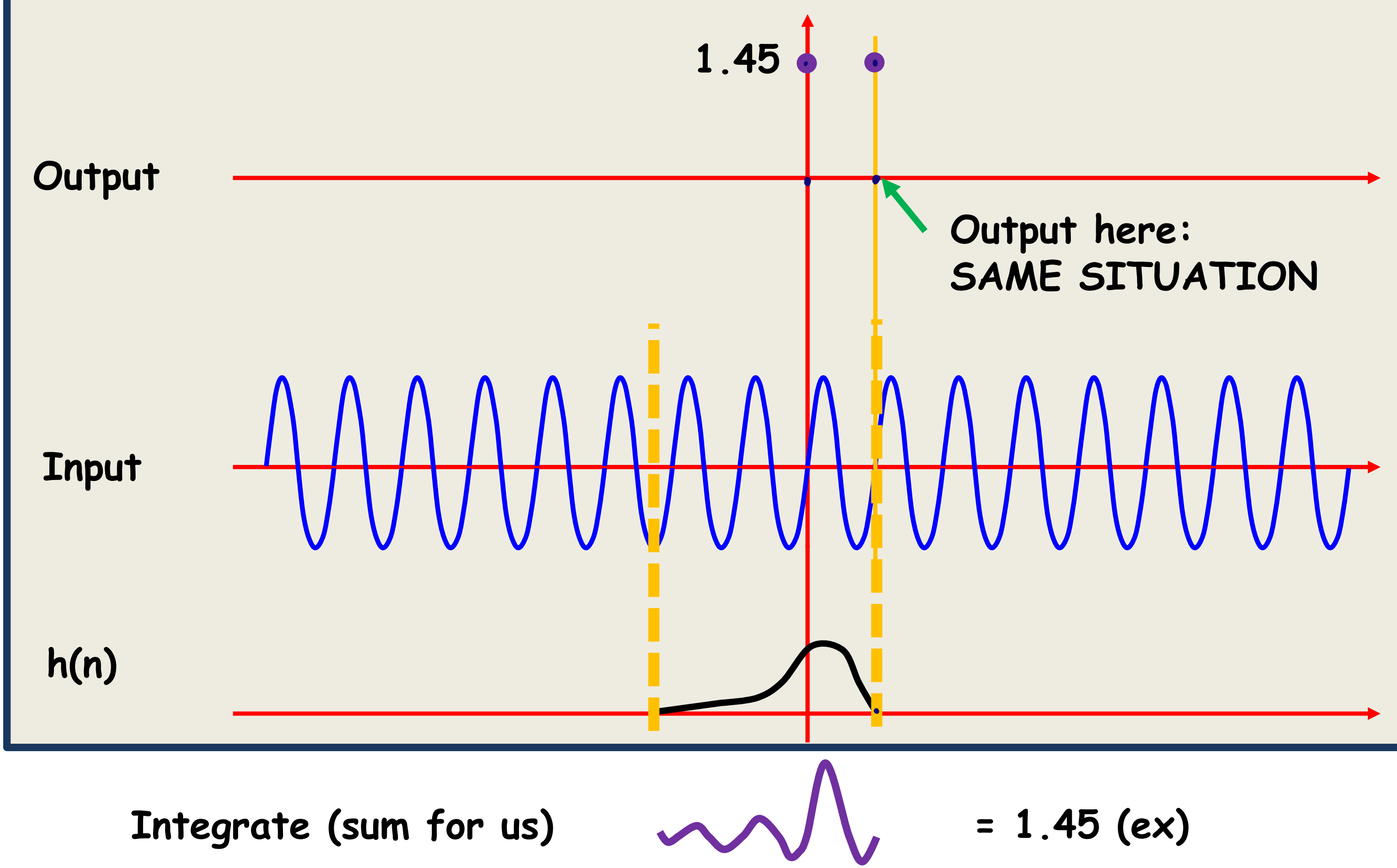

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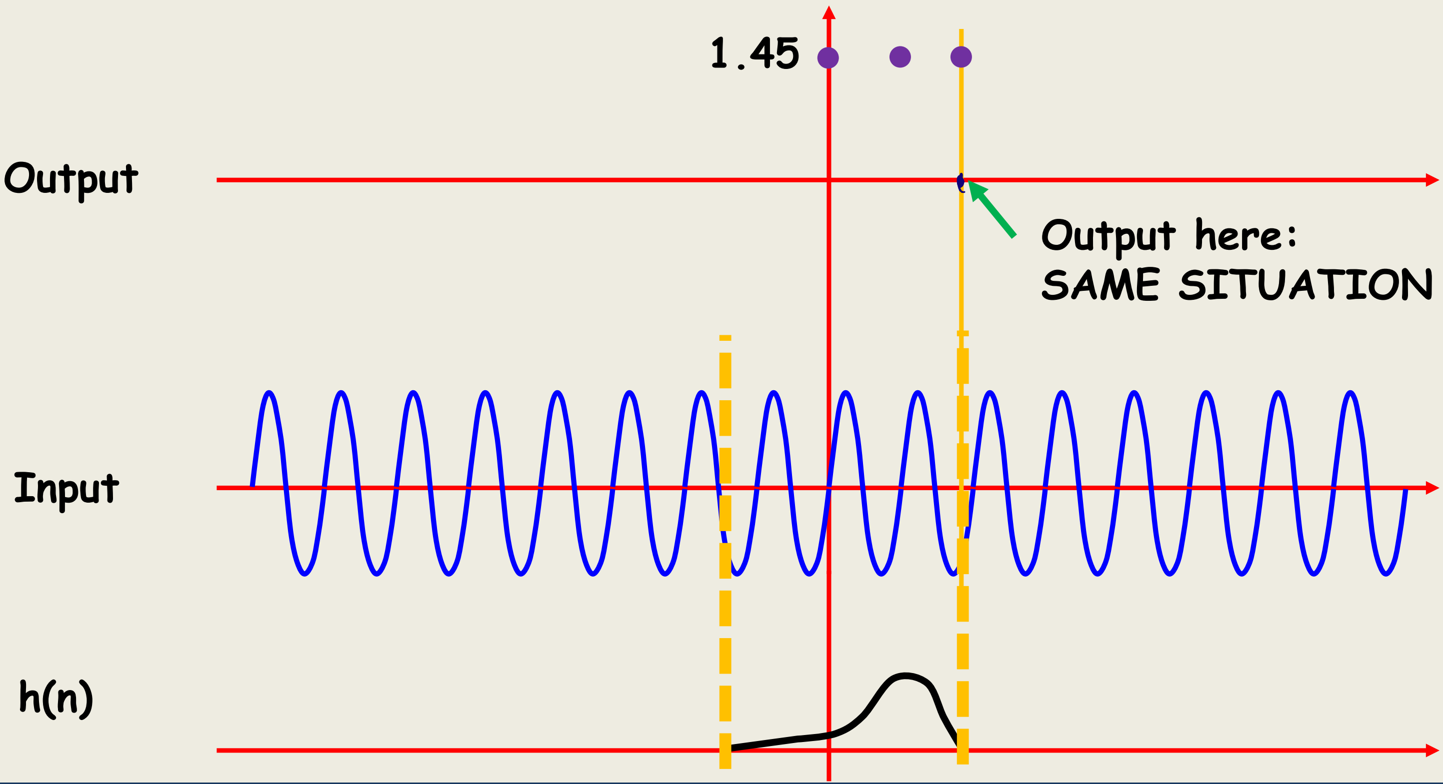

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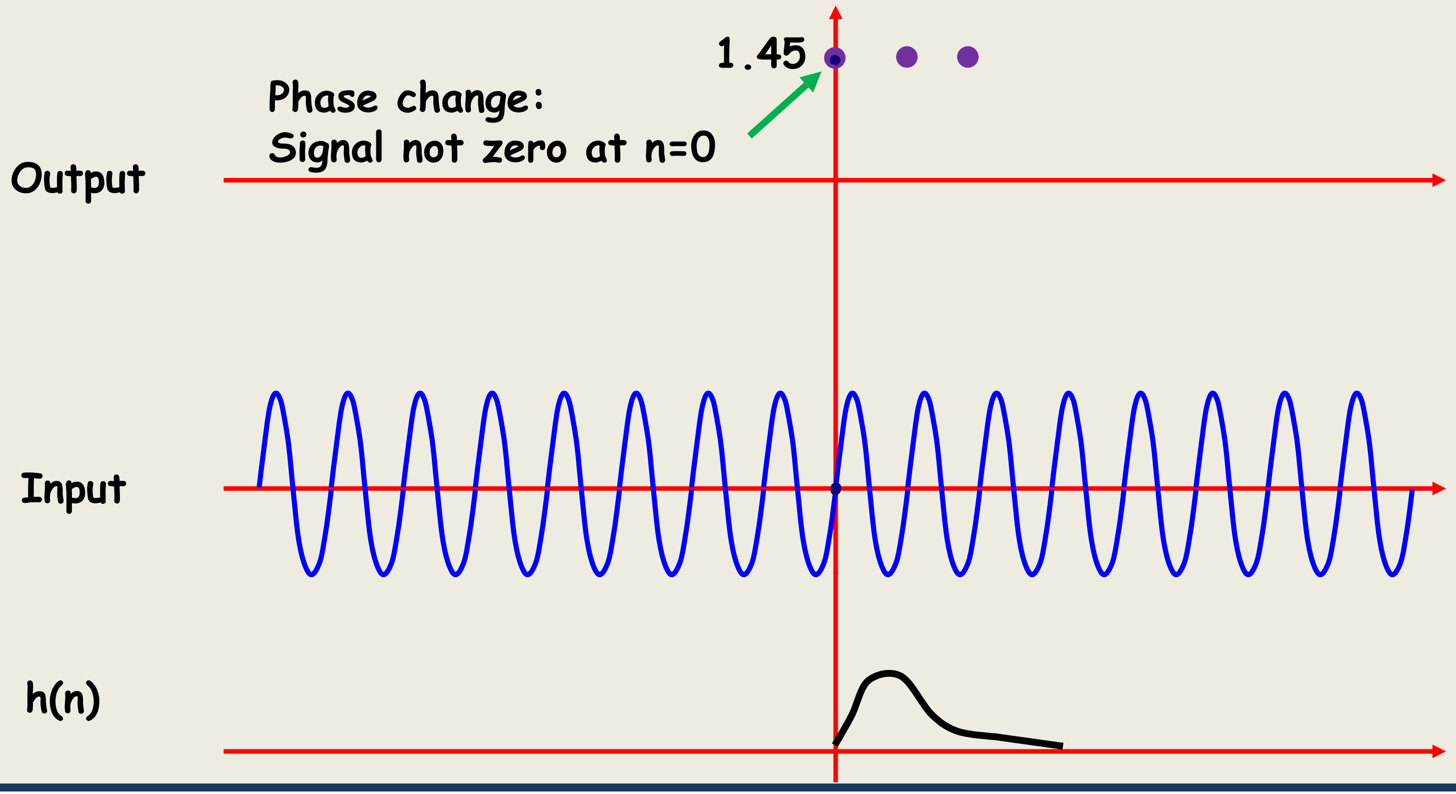

EITF75 Systems and Signals

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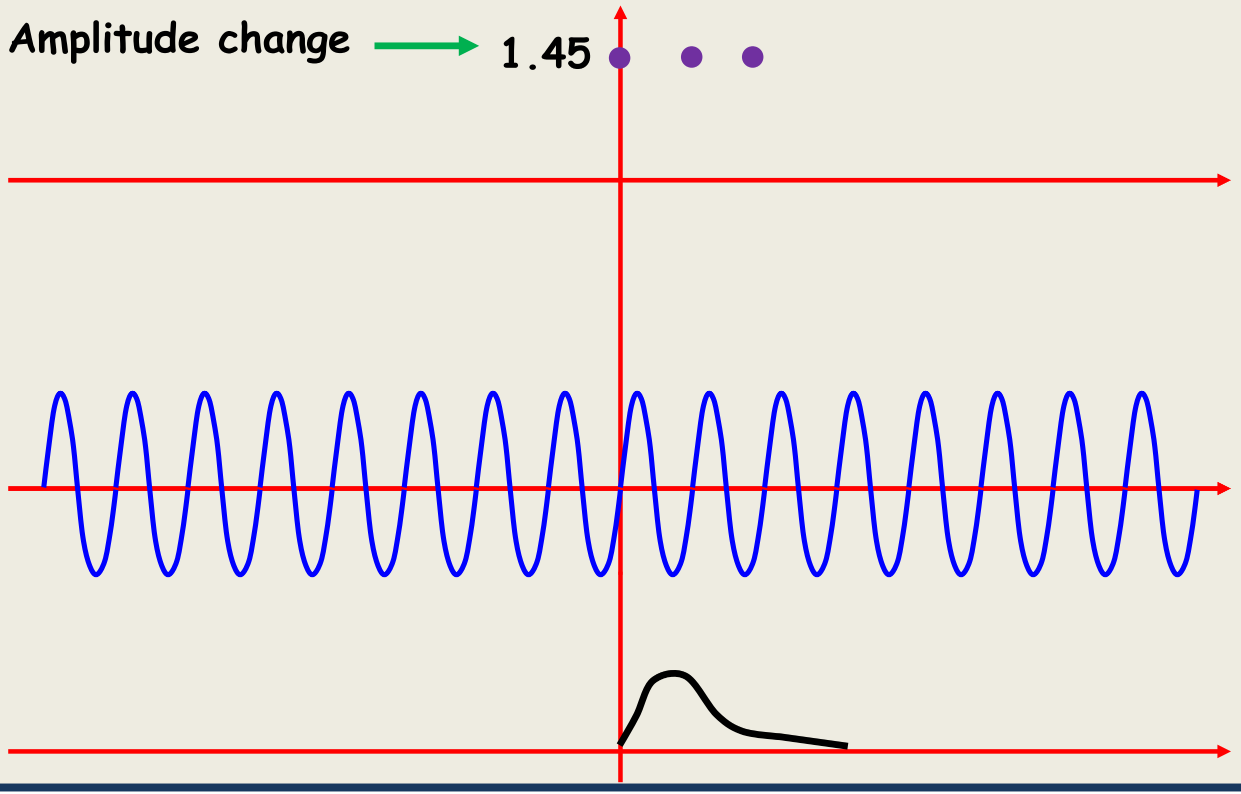

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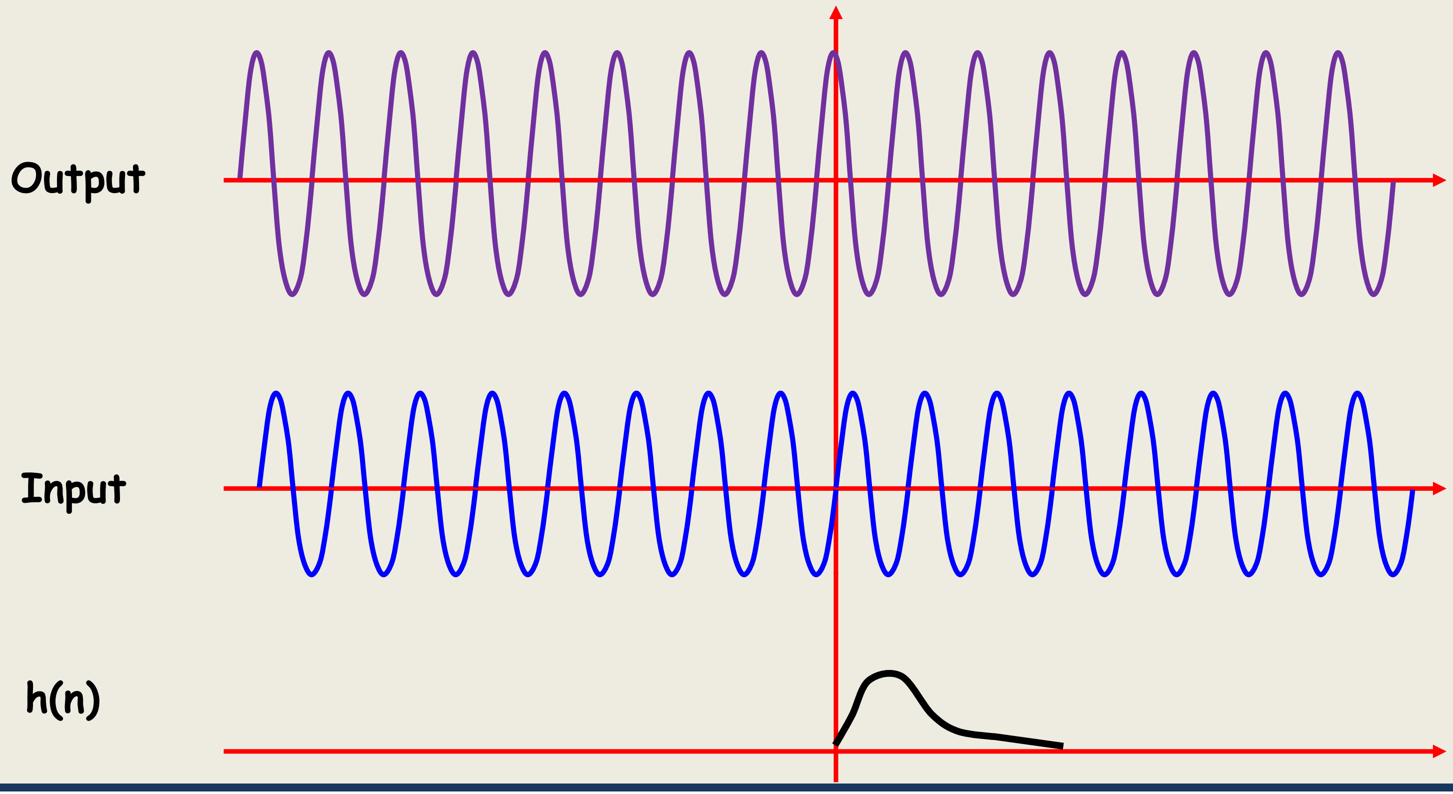

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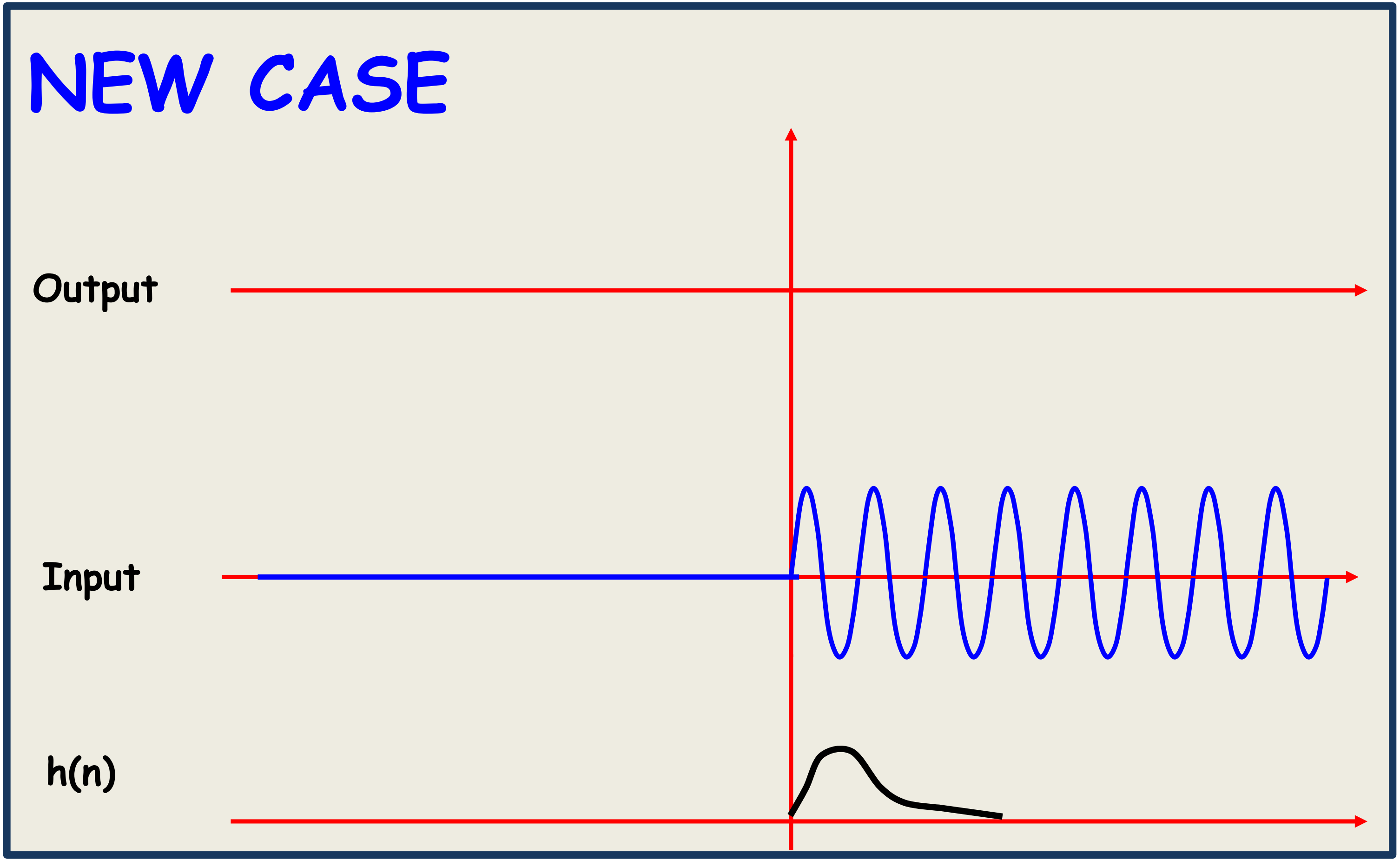

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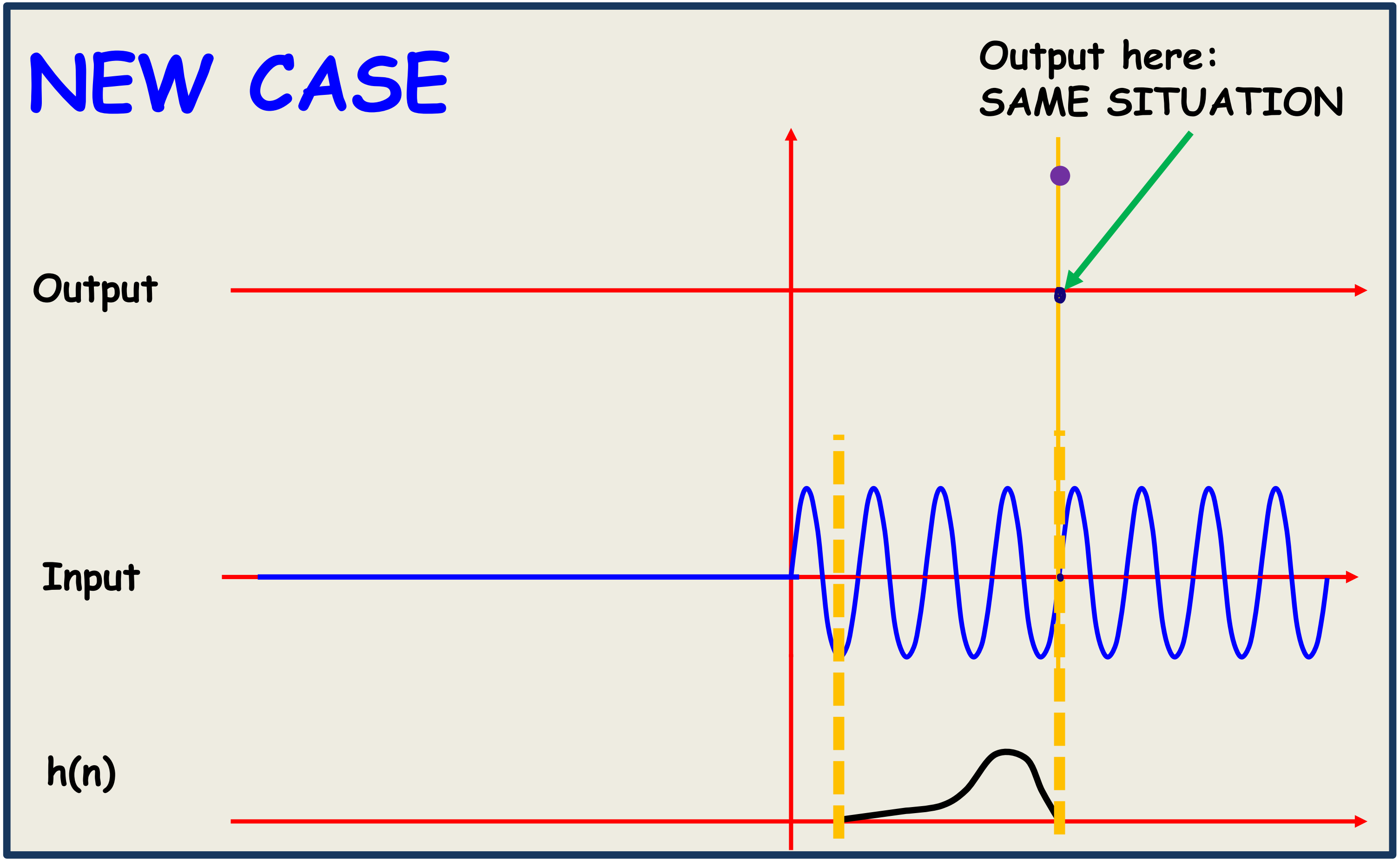

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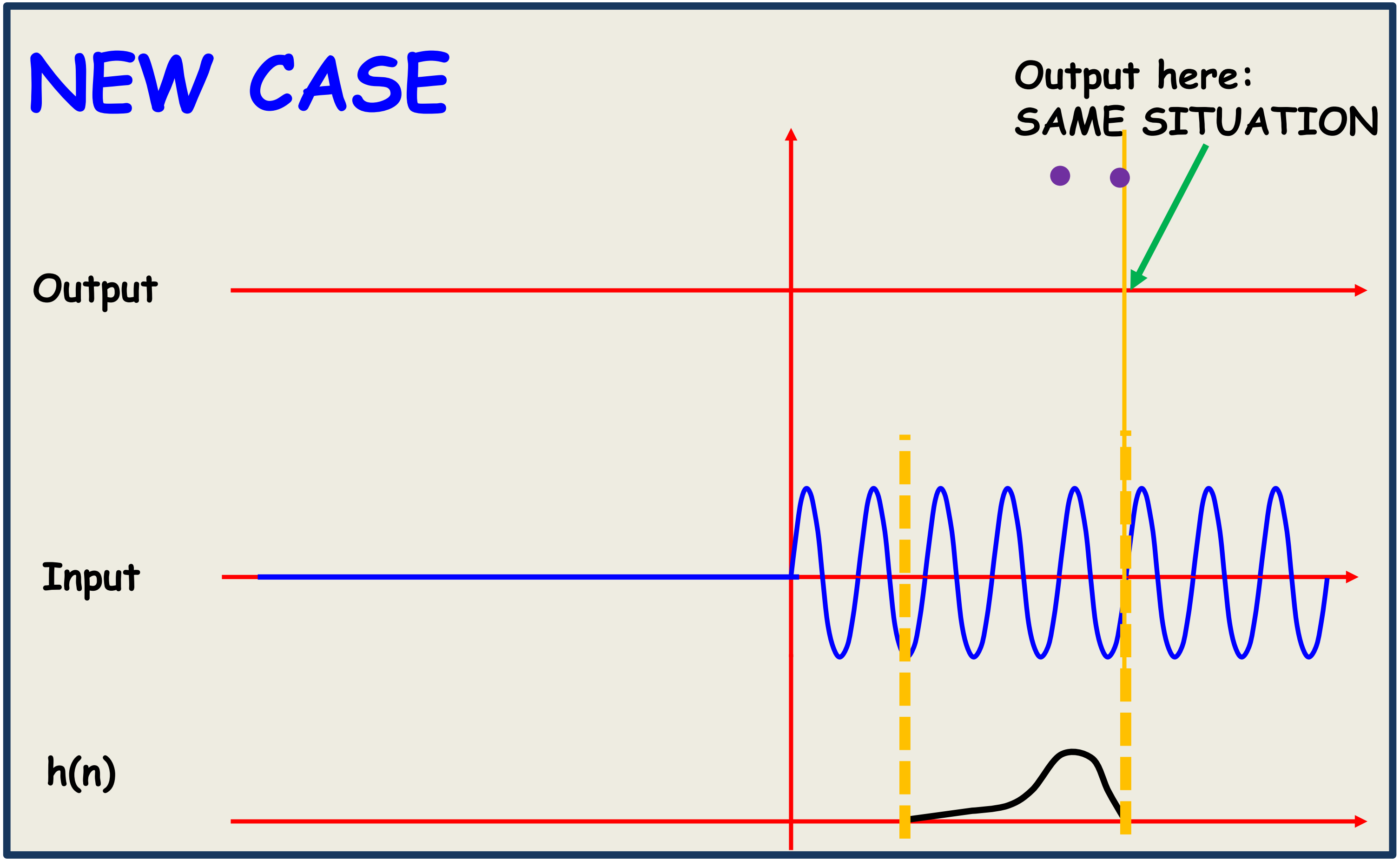

Integrate (sum for us)

= 1.45 (ex)

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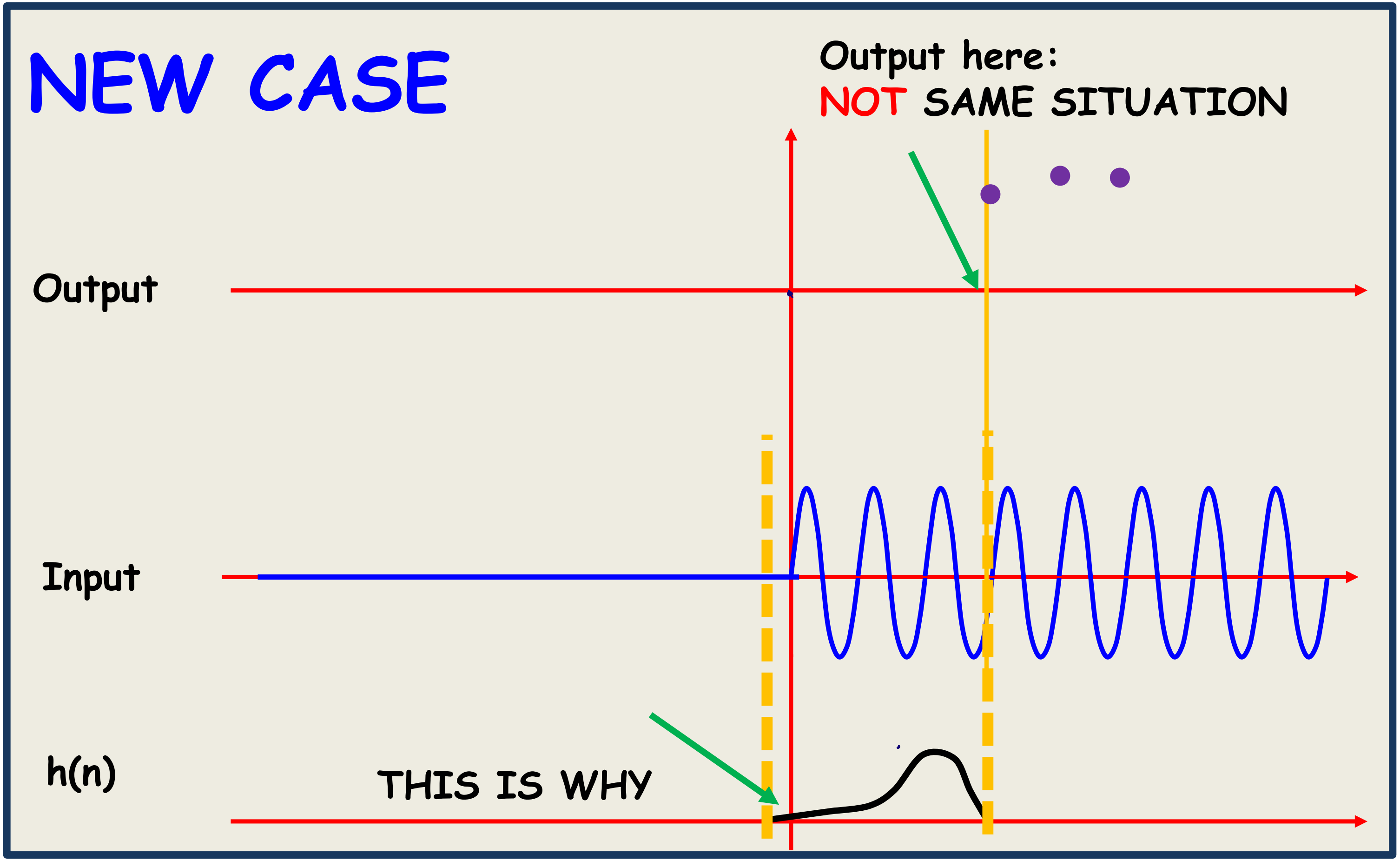
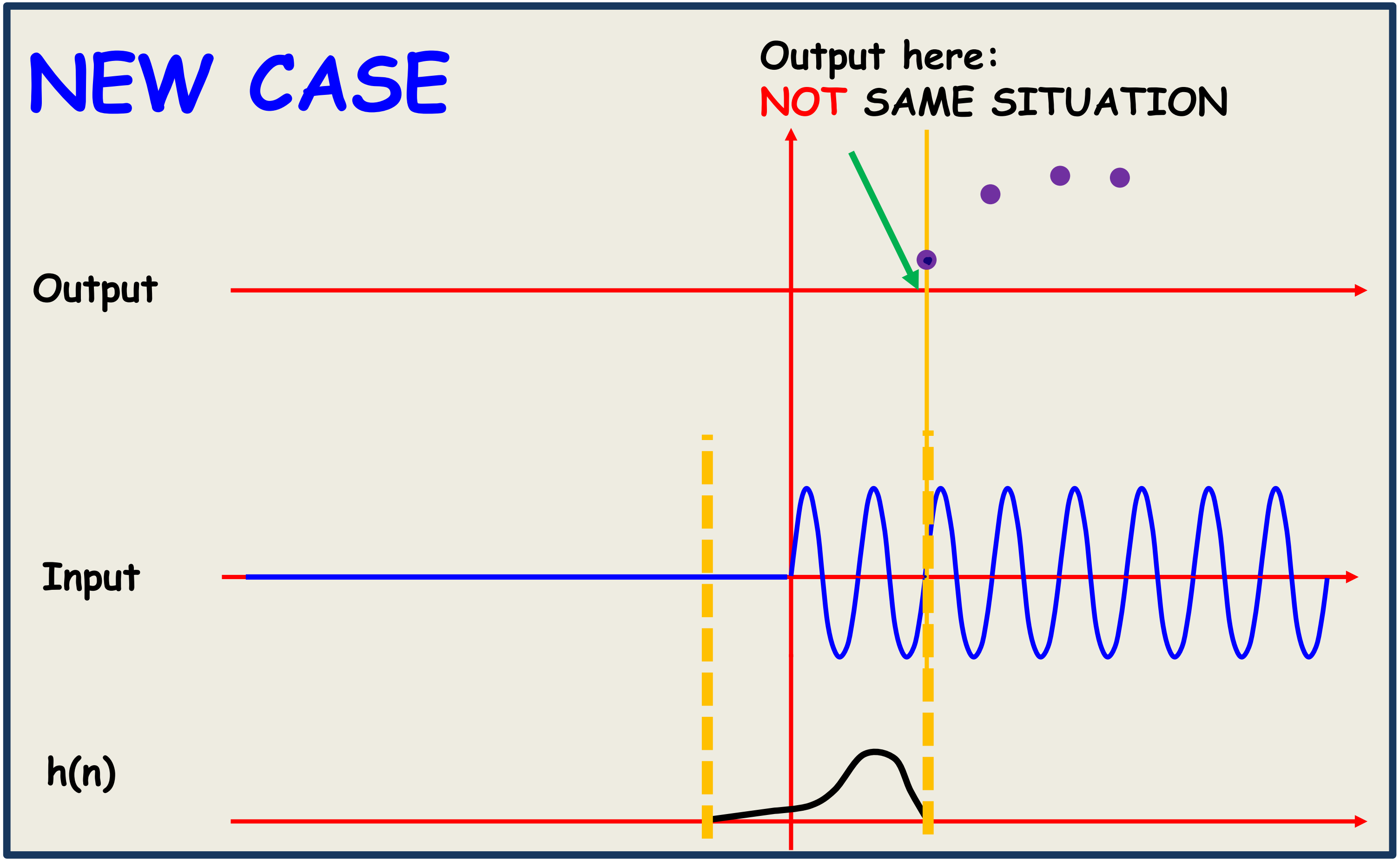


Integrate (sum for us)  = 1.3 (ex)

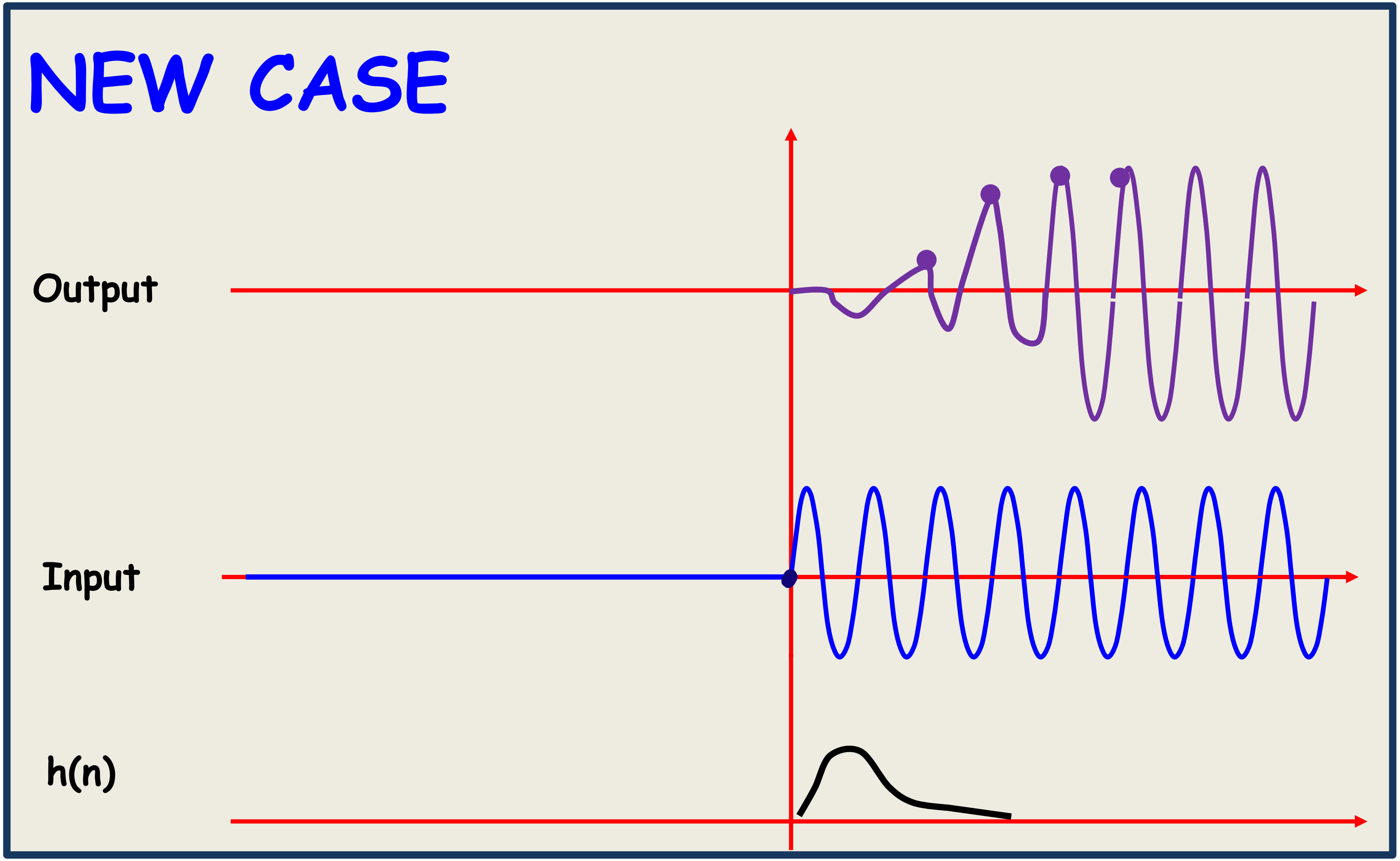
EITF75 Systems and Signals



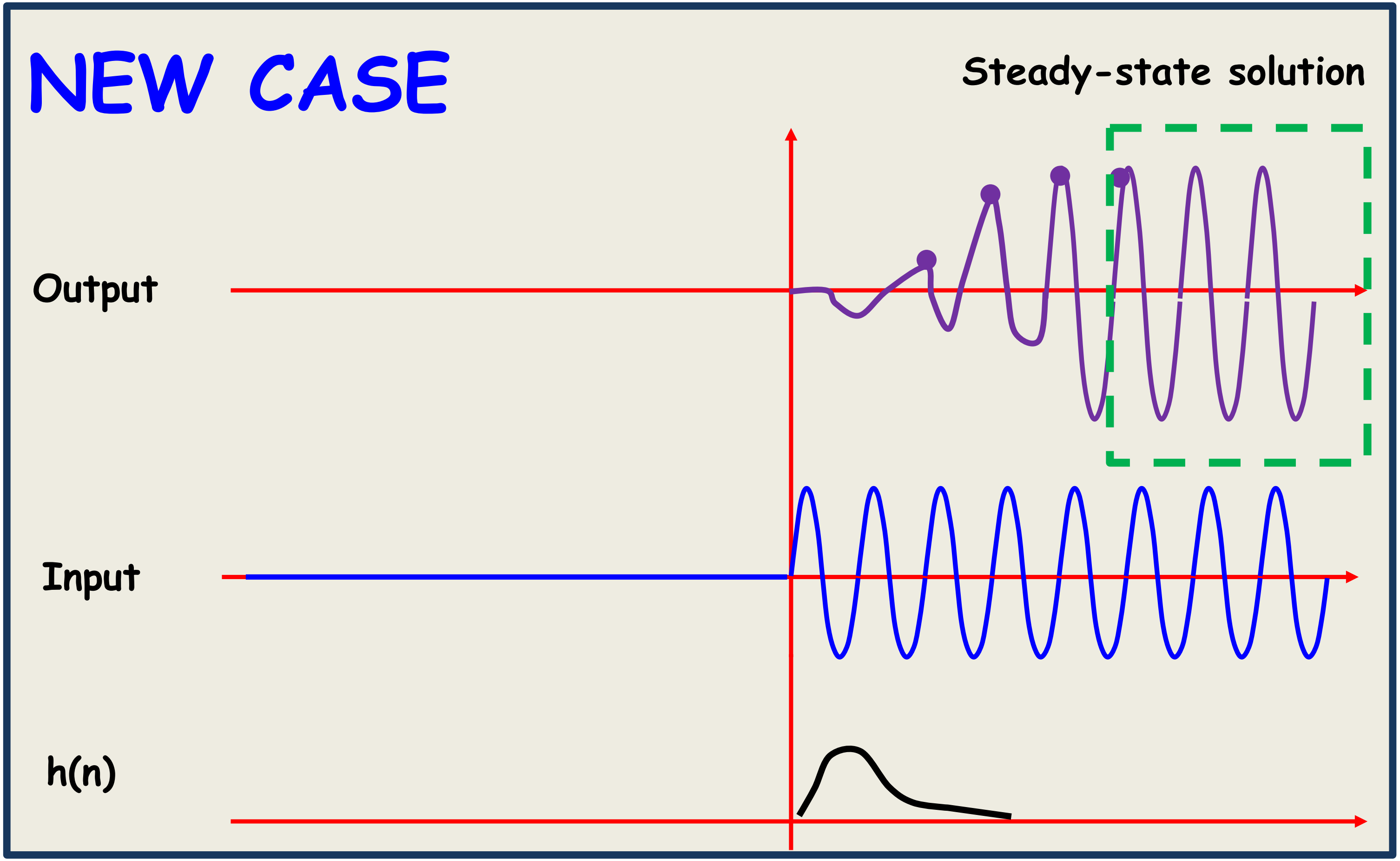
Integrate (sum for us)

= 0.4 (ex)

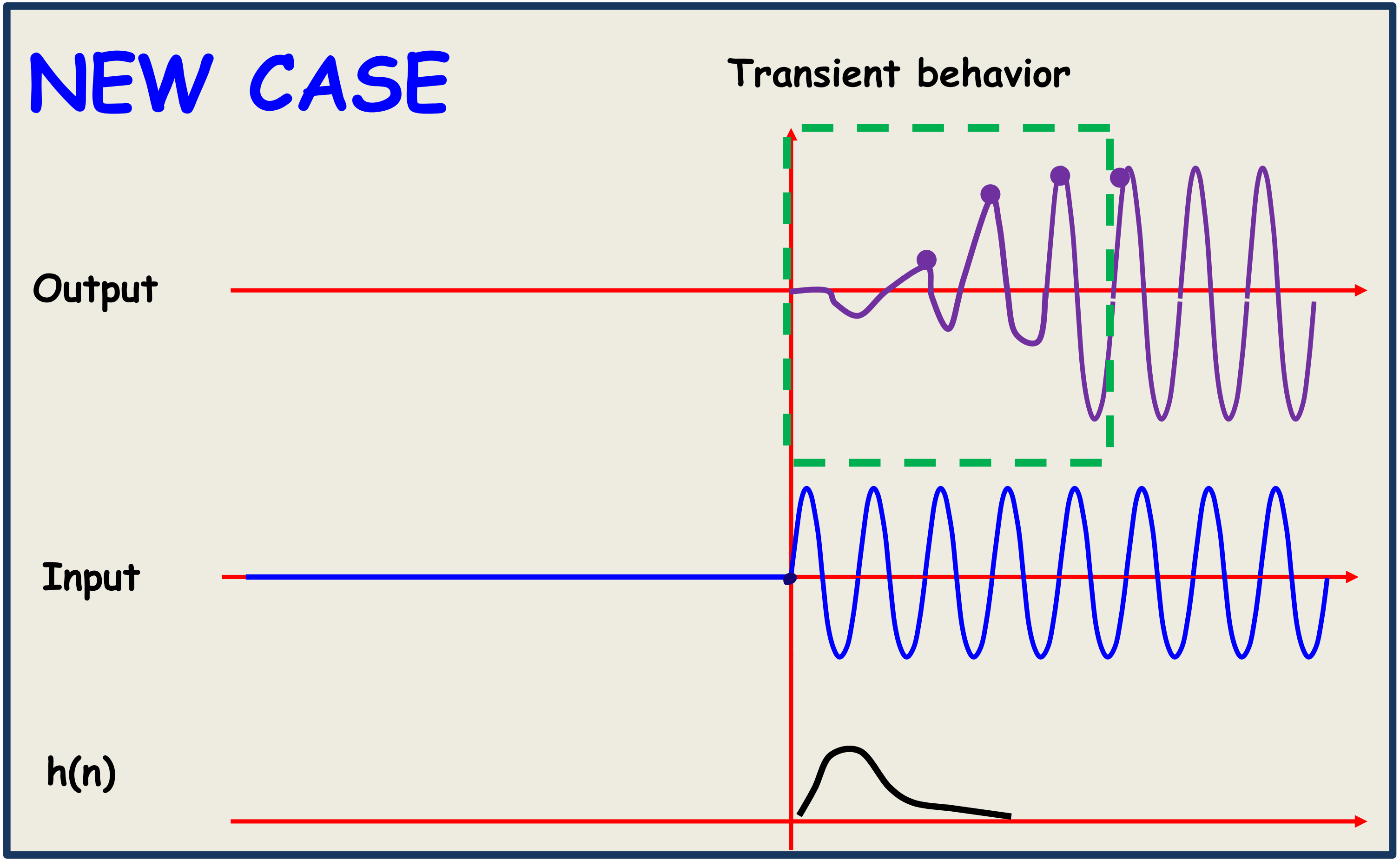
EITF75 Systems and Signals



EITF75 Systems and Signals



EITF75 Systems and Signals

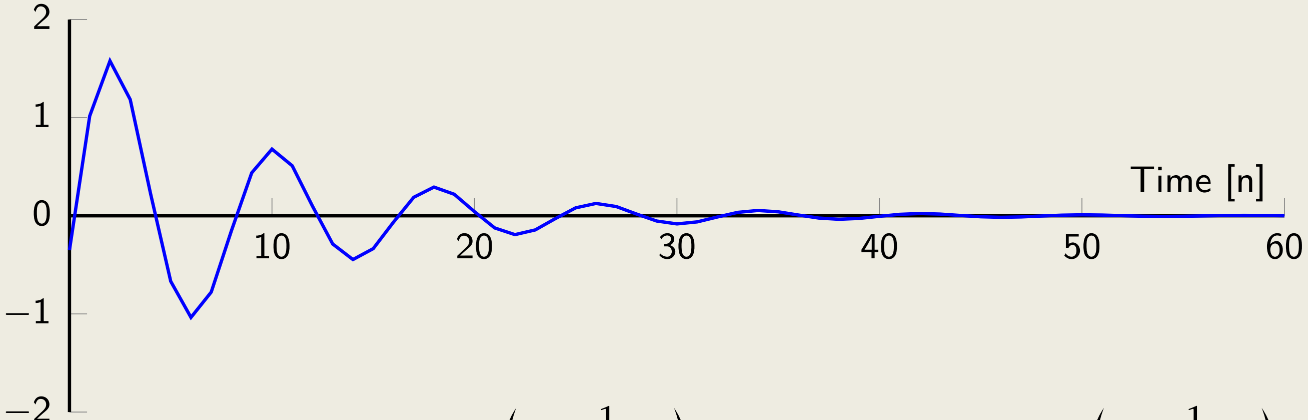


EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Transient



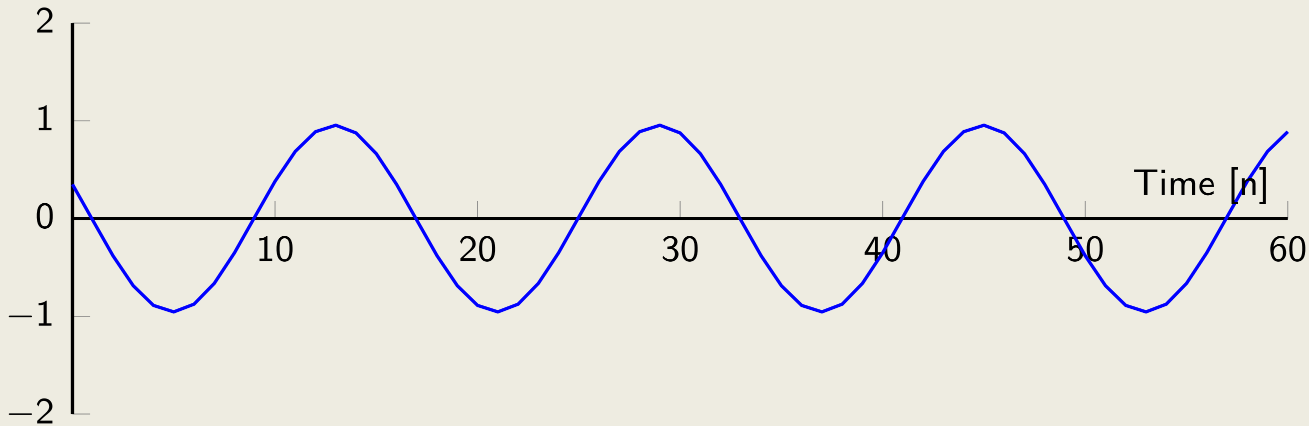
$$-0.35\cdot 0.9^n\cdot\cos\left(2\pi\cdot\frac{1}{8}\cdot n\right)+0.35\cdot 5.562\cdot 0.9^n\sin\left(2\pi\cdot\frac{1}{8}\cdot n\right)$$

EITF75 Systems and Signals

$$X(z) = \frac{1-\cos(\omega_0)z^{-1}}{1-2\cos(\omega_0)z^{-1}+z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1}-z^{-2}}{1-1.27z^{-1}+0.81z^{-2}}$$

Steady state



$$0.35\cdot\cos(\omega_0\cdot n) - 0.35\cdot 2.5392\cdot\sin(\omega_0\cdot n)$$

EITF75 Systems and Signals

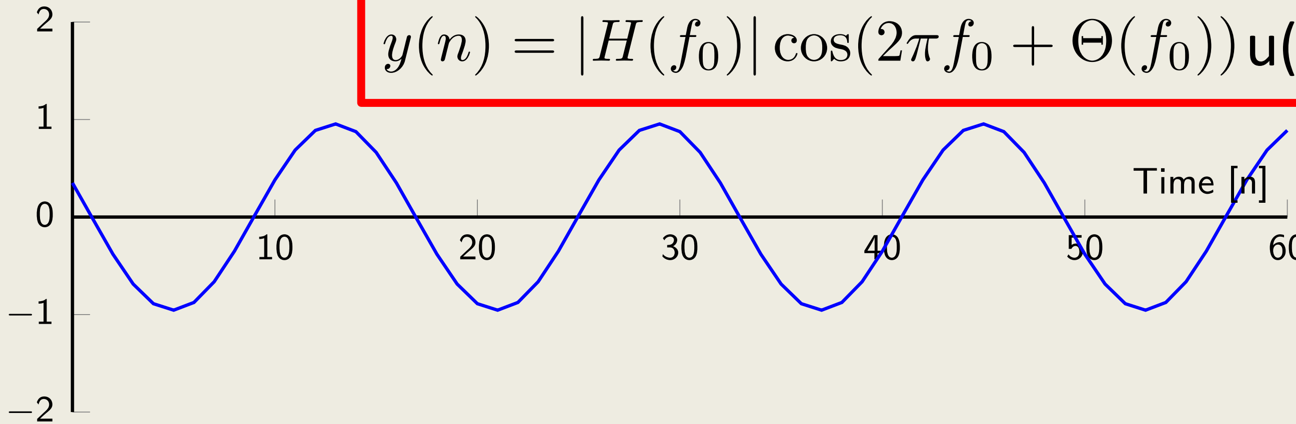
$$X(z) = \frac{1 - \cos(\omega_0)z^{-1}}{1 - 2\cos(\omega_0)z^{-1} + z^{-2}} \quad \omega_0 = \frac{2\pi}{16}$$

$$H(z) = \frac{z^{-1} - z^{-2}}{1 - 1.27z^{-1} + 0.81z^{-2}}$$

Steady state

Important:
The steady state solution can be computed via

$$y(n) = |H(f_0)| \cos(2\pi f_0 + \Theta(f_0))\,\mathrm{u(n)}$$



$$0.35 \cdot \cos(\omega_0 \cdot n) - 0.35 \cdot 2.5392 \cdot \sin(\omega_0 \cdot n)$$